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An elliptic parameterisation of the Zamolodchikov model

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Dedicated to Sasha Zamolodchikov on the occasion of his sixtieth birthday

Abstract

The Zamolodchikov model describes an exact relativistic factorized scattering theory of straight strings in $(2+1)$-dimensional space–time. It also defines an integrable 3D lattice model of statistical mechanics and quantum field theory. The three-string S-matrix satisfies the *tetrahedron equation* which is a 3D analog of the Yang–Baxter equation. Each S-matrix depends on three dihedral angles formed by three intersecting planes, whereas the tetrahedron equation contains five independent spectral parameters, associated with angles of an Euclidean tetrahedron. The vertex weights are given by rather complicated expressions involving square roots of trigonometric function of the spectral parameters, which is quite unusual from the point of view of 2D solvable lattice models. In this paper we consider a particular four-parameter specialisation of the tetrahedron equation when one of its vertices goes to infinity and the tetrahedron itself degenerates into an infinite prism. We show that in this limit all the vertex weights in the tetrahedron equation can be represented as meromorphic functions on an elliptic curve. Moreover we show that a special reduction of the tetrahedron equation in this case leads precisely to an example of the tetrahedral Zamolodchikov algebra, previously constructed by Korepanov. This algebra plays important role for a "layered" construction of the Shastry's R-matrix and the 2D S-matrix appearing in the problem of the ADS/CFT correspondence for $\mathcal{N}=4$ SUSY Yang–Mills theory in four dimensions. Possible applications of our results in this field are briefly discussed.

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1. Introduction

The *tetrahedron equation* [1,2] is a three-dimensional analog of the Yang–Baxter equation. It implies the commutativity of layer-to-layer transfer matrices [3] for three-dimensional lattice models of statistical mechanics and field theory and, thus, generalizes the most fundamental integrability structure of exactly solvable models in two dimensions [4].

The first solution of the tetrahedron equation was proposed by Zamolodchikov [1,2] in the context of an exact relativistic factorized scattering theory of "straight strings" in $(2+1)$-dimensional space–time. Subsequently, Baxter [5] proved that this solution indeed satisfies the tetrahedron equations and then [6] exactly calculated the free energy of the corresponding solvable three-dimensional model in the limit of an infinite lattice. Some further developments in the field of 3D integrability in a more general setting could be learned from [7–12]. Many special properties of the Zamolodchikov model were studied in [13–16]. In this paper we will consider some additional analytic and algebraic problems related to this model.

The spin variables in the Zamolodchikov model take two values. In the original formulation [1,2] the spins are assigned to faces of the lattice. Baxter [6] used a different, but closely related "interaction-round-a-cube" (IRC) approach, where the spins are assigned to 3D cells of the lattice, or, equivalently, vertices of the dual lattice. Here we will use yet another approach, namely the *vertex formulation* with edge spins, found by Sergeev, Mangazeev and Stroganov [17], which is related to the other formulations by the "cube–vertex" correspondence [18]. The most general edge-spin solution of the tetrahedron equation in the model contains five independent spectral parameters associated with angles of an Euclidean tetrahedron. Two different reductions of this solution with a fewer number of parameters, known as the "static" and "planar" limits, were previously found by Korepanov [19] and Hietarinta [20].

In this paper we consider another particular limit when one of the tetrahedron vertices goes to infinity, whereas the tetrahedron turns into an infinite prism. We show that the all Boltzmann weights in this case can be parametrised in terms of meromorphic functions on an elliptic curve. Moreover we show that a special reduction of the tetrahedron equation in this case leads precisely to the example of the Zamolodchikov tetrahedral algebra, constructed by Korepanov [19]. This algebra plays an important role for a "layered" construction [21–23] of the Shastry's R-matrix [24] underlying the integrability of the 1D Hubbard model. Remarkably, the same R-matrix [25–28] appears in the problem of the AdS/CFT correspondence for $\mathcal{N}=4$ SUSY Yang–Mills theory in four dimensions. In conclusion we briefly discuss possible applications of our results to this field.

The organization of the paper is as follows. In Section 2 we briefly review known results on the tetrahedron equation in the Zamolodchikov model, used in this paper. In Section 3 we derive an elliptic parameterisation of the Boltzmann weights. In Section 4 we consider special limits of the tetrahedron equation and the tetrahedral Zamolodchikov algebra.

2. The vertex formulation of the Zamolodchikov model

The Zamolodchikov model can be formulated on any 3D lattice formed by an arbitrary set of intersecting 2D planes in the Euclidean space $\mathbb{R}^3$, such that no four planes intersect at one point. In particular, this definition includes a regular cubic lattice as the simplest possibility. The

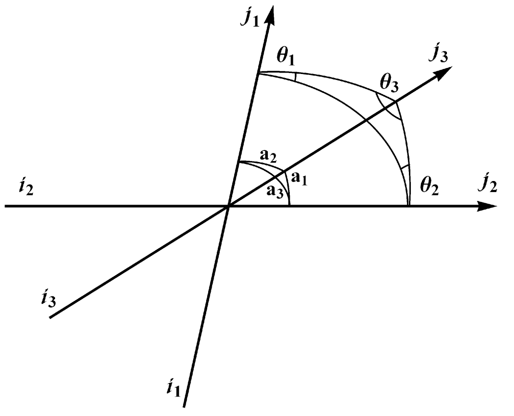

Fig. 1. Arrangement of spins around an elementary vertex formed by three intersecting planes with the dihedral angles $\theta_1, \theta_2, \theta_3$ and the edge angles a_1, a_2, a_3.

fluctuating spin variables in the model take two values, denoted below 0 and 1. Here we will use the vertex formulation of the model found in [17]. The spins in this case are assigned to edges of the lattice while local Boltzmann weights are assigned to vertices at three-plane intersection points. Let $R_{i_1,i_2,i_3}^{j_1,j_2,j_3}$ denote the weight corresponding to a configuration of six edge spins $i_1, i_2, i_3, j_1, j_2, j_3 = 0, 1$, arranged as in Fig. 1. This quantity could be conveniently associated with a linear operator

$$\mathbf{R} : \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, \tag{1}$$

acting in a direct product of three two-dimensional vector spaces, where the pairs of indices (i_1, j_1), (i_2, j_2) and (i_3, j_3) serve as matrix indices in the first, second and third spaces, respectively. The edge states "0" and "1" correspond to the basis vectors v_0 and v_1,

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_0, v_1 \in \mathbb{C}^2. \tag{2}$$

The vertex weights depends on three spectral parameters, $\theta_1, \theta_2, \theta_3$, which are identified with dihedral angles between the three planes forming the vertex, as shown in Fig. 1. To indicate this dependence we will write the weights as $\mathbf{R}(\theta_1, \theta_2, \theta_3)$. Consider a spherical triangle with the angles $\theta_1, \theta_2, \theta_3$ and let a_1, a_2, a_3 denote three sides of this triangle opposite to the angles $\theta_1, \theta_2, \theta_3$, which obey the spherical sine theorem

$$K = \frac{\sin\theta_1}{\sin a_1} = \frac{\sin\theta_2}{\sin a_2} = \frac{\sin\theta_3}{\sin a_3}. \tag{3}$$

Define related variables

$$\begin{aligned} 2\alpha_0 &= \theta_1 + \theta_2 + \theta_3 - \pi, \qquad & \alpha_i &= \theta_i - \alpha_0, \\ 2\beta_0 &= 2\pi - a_1 - a_2 - a_3, \qquad & \beta_i &= \pi - a_i - \beta_0, \end{aligned} \tag{4}$$

for $i = 1, 2, 3$. They satisfy the relations

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \pi, \qquad \beta_0 + \beta_1 + \beta_2 + \beta_3 = \pi. \tag{5}$$

The variables $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are called *excesses* of the spherical triangle. Choose $\theta_1, \theta_2, \theta_3$ so that $\theta_1, \theta_2, \theta_3$, $\alpha_0, \ldots, \alpha_3$ and $\beta_0, \ldots, \beta_3$ are all real and between 0 and π. Further, define four variables (taking positive values of the square roots)

$$t_i = \sqrt{\tan\frac{\alpha_i}{2}}, \quad i = 0, 1, 2, 3, \tag{6}$$

which are constrained by the relation

$$1 - t_0^2t_1^2 - t_0^2t_2^2 - t_0^2t_3^2 - t_1^2t_2^2 - t_1^2t_3^2 - t_2^2t_3^2 + t_0^2t_1^2t_2^2t_3^2 = 0, \tag{7}$$

following from (5). The weights obey a "parity conservation law", such that 32 vertex configurations with an odd number of 0's have the identically vanishing weight

$$R_{i_1i_2i_3}^{j_1j_2j_3} \equiv 0, \quad \text{if } i_1 + i_2 + i_3 \neq j_1 + j_2 + j_3 \pmod 2. \tag{8}$$

The remaining 32 non-vanishing matrix elements of $\mathbf{R}(\theta_1, \theta_2, \theta_3)$ have the form [17][1]

$$\begin{aligned}
R_{0,0,0}^{0,0,0} &= R_{0,1,1}^{0,1,1} = R_{1,0,1}^{1,0,1} = R_{1,1,0}^{1,1,0} = 1,\\
R_{1,1,1}^{1,1,1} &= R_{1,0,0}^{1,0,0} = R_{0,1,0}^{0,1,0} = R_{0,0,1}^{0,0,1} = t_0t_1t_2t_3,\\
R_{0,0,1}^{0,1,0} &= R_{0,1,0}^{0,0,1} = -R_{1,1,1}^{1,0,0} = -R_{1,0,0}^{1,1,1} = t_2t_3,\\
R_{1,1,0}^{1,0,1} &= R_{1,0,1}^{1,1,0} = -R_{0,0,0}^{0,1,1} = -R_{0,1,1}^{0,0,0} = t_0t_1,\\
R_{0,1,0}^{1,1,1} &= R_{0,0,1}^{1,0,0} = -R_{1,0,0}^{0,0,1} = -R_{1,1,1}^{0,1,0} = -\mathrm{i}t_1t_3,\\
R_{1,0,1}^{0,0,0} &= R_{1,1,0}^{0,1,1} = -R_{0,1,1}^{1,1,0} = -R_{0,0,0}^{1,0,1} = \mathrm{i}t_0t_2,\\
R_{1,1,1}^{0,0,1} &= R_{0,0,1}^{1,1,1} = R_{0,1,0}^{1,0,0} = R_{1,0,0}^{0,1,0} = t_1t_2,\\
R_{0,0,0}^{1,1,0} &= R_{1,1,0}^{0,0,0} = R_{1,0,1}^{0,1,1} = R_{0,1,1}^{1,0,1} = t_0t_3.
\end{aligned} \tag{10}$$

In the following we will regard the weights as functions of the three independent variables $\theta_1, \theta_2, \theta_3$, or, equivalently, of the four dependent variables $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, constrained by the relation (5). We would like to stress that the expressions (10) do not contains any other free parameters.

In the original face-spin formulation of Ref. [2] the weights are explicitly symmetric with respect to all spatial symmetry transformations, generated by permutations of the lines 1, 2, 3 and inversions of their directions. Namely, the weights remains unchanged if the associated permutations of spins are accompanied by the corresponding transformations of the dihedral angles. Contrary, to this the operator $\mathbf{R}(\theta_1, \theta_2, \theta_3)$ is not explicitly symmetric. Nevertheless it possesses the same spatial symmetry group of the 3D cube [17,29], but its realisation now involves linear similarity transformations of the weights. This 48-element group is generated just by two transformations, which we choose as

[1] In [17] the weights are given in terms of the variables $\beta_0, \beta_1, \beta_2, \beta_3$. Here they are re-expressed in terms of α's with the help of the identity

$$\tan\frac{\alpha_i}{2}\tan\frac{\alpha_j}{2} = \tan\frac{\beta_k}{2}\tan\frac{\beta_\ell}{2}, \tag{9}$$

where (i, j, k, ℓ) is any permutation of the indices $(0, 1, 2, 3)$.

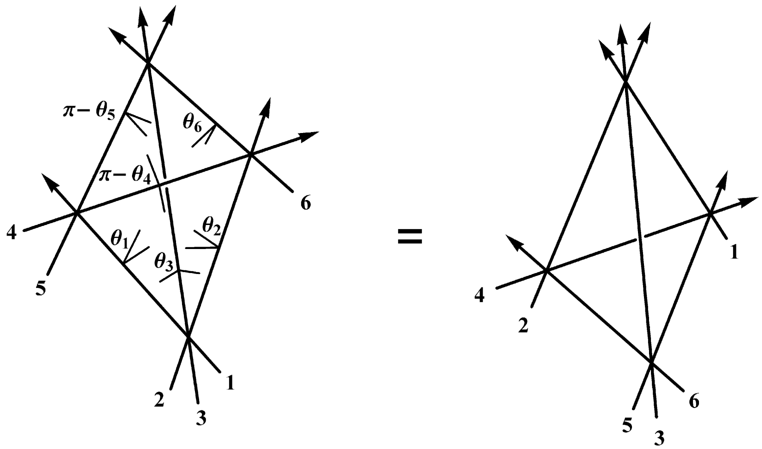

Fig. 2. Graphical representation of the tetrahedron equation (13) showing the arrangement of the *internal* dihedral angles of the tetrahedron.

$$\begin{aligned} \mathbf{R}(\pi-\theta_1,\pi-\theta_2,\theta_3) &= (\sigma_y\otimes\sigma_y\otimes\sigma_z)\,\mathbf{R}^{t_1t_2}(\theta_1,\theta_2,\theta_3)\,(\sigma_y\otimes\sigma_y\otimes\sigma_z),\\ P_{13}\mathbf{R}^{t_1t_2t_3}(\theta_3,\theta_2,\theta_1)P_{13} &= (\tau\otimes\tau\otimes\tau)\,\mathbf{R}(\theta_1,\theta_2,\theta_3)\,(\tau\otimes\tau\otimes\tau)^{-1}, \end{aligned} \tag{11}$$

where the superscripts t_i, $i=1,2,3$, denote the matrix transposition in the i-th space, P_{13} denotes the operator permuting the spaces 1 and 3 and σ_x, σ_y, σ_z denote the Pauli matrices,

$$\tau=\begin{pmatrix}1 & 0\\ 0 & \mathsf{i}\end{pmatrix},\qquad \tau^2=\sigma_z. \tag{12}$$

In the following the symmetry properties will not play any essential role, since the elliptic parameterisation considered below explicitly breaks the spatial symmetry.

The operator $\mathbf{R}$ satisfies the tetrahedron equation,

$$\begin{aligned} &\mathbf{R}_{123}(\theta_1,\theta_2,\theta_3)\,\mathbf{R}_{145}(\theta_1,\theta_4,\theta_5)\,\mathbf{R}_{246}(\pi-\theta_2,\theta_4,\theta_6)\,\mathbf{R}_{356}(\theta_3,\pi-\theta_5,\theta_6)\\ &\quad=\mathbf{R}_{356}(\theta_3,\pi-\theta_5,\theta_6)\,\mathbf{R}_{246}(\pi-\theta_2,\theta_4,\theta_6)\,\mathbf{R}_{145}(\theta_1,\theta_4,\theta_5)\,\mathbf{R}_{123}(\theta_1,\theta_2,\theta_3), \end{aligned} \tag{13}$$

which ensures the integrability of the model. The above equation involves operators acting in a direct product $\mathcal{V}=(\mathbb{C}^2)^{\otimes 6}$, of six identical two-dimensional vector spaces.[2] Geometrically, it is associated with an arbitrary intersection of four planes. The indices in (13) numerate six intersection lines among these planes, which form extended edges of an Euclidean tetrahedron. Each of the operators $\mathbf{R}_{ijk}$ depends on its own set of spectral parameters, determined by the dihedral angles corresponding to the lines i, j and k. The geometric arrangement of these angles is shown in Fig. 2 (note that it is exactly the same as in [17], but different from that used in Eq. (2.2) in [5]). Altogether there are six dihedral angles corresponding to six edges of the tetrahedron. At this point it is worth mentioning that these angles are not independent; they satisfy one non-linear constraint,

[2] We follow usual notations, where R_{123} acts non-trivially in the first three spaces and coincides with the identity operator in all the remaining components of the product. The other operators in (13) are defined similarly.

$$\det\begin{pmatrix} 1 & \cos\theta_1 & \cos\theta_2 & -\cos\theta_4 \\ \cos\theta_1 & 1 & \cos\theta_3 & -\cos\theta_5 \\ \cos\theta_2 & \cos\theta_3 & 1 & \cos\theta_6 \\ -\cos\theta_4 & -\cos\theta_5 & \cos\theta_6 & 1 \end{pmatrix} = 0, \tag{14}$$

which follows from the vanishing of the Gram determinant of four unit normal vectors to faces of a tetrahedron in the Euclidean 3-space. Therefore, Eq. (13) contains only five independent parameters. In matrix form it reads

$$\begin{aligned} &\sum_{\substack{k_1k_2k_3,\\ k_4k_5k_6}} R(\theta_1,\theta_2,\theta_3)_{i_1i_2i_3}^{k_1k_2k_3} R(\theta_1,\theta_4,\theta_5)_{k_1i_4i_5}^{j_1k_4k_5} R(\pi-\theta_2,\theta_4,\theta_6)_{k_2k_4i_6}^{j_2j_4k_6} R(\theta_3,\pi-\theta_5,\theta_6)_{k_3k_5k_6}^{j_3j_5j_6} \\ &\quad= \sum_{\substack{k_1,k_2,k_3,\\ k_4,k_5,k_6}} R(\theta_3,\pi-\theta_5,\theta_6)_{i_3i_5i_6}^{k_3k_5k_6} R(\pi-\theta_2,\theta_4,\theta_6)_{i_2i_4k_6}^{k_2k_4j_6} R(\theta_1,\theta_4,\theta_5)_{i_1k_4k_5}^{k_1j_4j_5} R(\theta_1,\theta_2,\theta_3)_{k_1k_2k_3}^{j_1j_2j_3}. \end{aligned} \tag{15}$$

Altogether there are 2^{12} distinct equations, half of which is non-trivial (for the other half both sides vanish identically due to the parity conservation (8)). There are several proofs that the weights (10) indeed satisfy these equations. Firstly, thanks to the “cube–vertex” correspondence of Ref. [18], this follows from the Baxter’s proof [5] of the tetrahedron equation for the interaction-round-a-cube (IRC) formulation of the model. Baxter used various symmetry relations and a computer-aided analysis of a pattern of signs in some equations to reduce all non-trivial equations to just two equations, which were then proven with the help of spherical trigonometry. Secondly, there exists a completely algebraic proof of (15) given in [17]. This proof (as well as its earlier variant [9] for the IRC formulation) is based on a repeated application of the so-called *restricted star-triangle relation*, invented in [8]. It is worth mentioning, that this relation was later interpreted as the five-term identity for a cyclic version [30] of the quantum dilogarithm [31]. Finally, an alternative (and, in fact, simpler) algebraic proof of (15), based on the auxiliary linear problem, was obtained in [32].

To conclude this section, mention a simple fact that Eq. (13) is unaffected by arbitrary linear similarity transformations in any of the six vector spaces therein,

$$\mathbf{R}_{ijk} \to (\mathbf{G}_i\otimes\mathbf{G}_j\otimes\mathbf{G}_k)\mathbf{R}_{ijk}(\mathbf{G}_i\otimes\mathbf{G}_j\otimes\mathbf{G}_k)^{-1}, \tag{16}$$

where $\mathbf{G}_i$, $i=1,2,\dots,6$ are arbitrary non-degenerate matrices. We will use this freedom to bring the expressions for matrix elements of $\mathbf{R}(\theta_1,\theta_2,\theta_3)$ to the most convenient form.

3. Elliptic parameterisation of the weights

3.1. Linear transformations of the weights

For the following analysis it is convenient to introduce the operator

$$\mathbf{L}_{123} = \big(\mathbf{D}(\xi)\otimes\mathbf{F}\otimes\mathbf{F}\big)\mathbf{R}_{123}\big(\mathbf{D}(\xi)\otimes\mathbf{F}\otimes\mathbf{F}\big)^{-1}, \tag{17}$$

where

$$\mathbf{F} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{D}(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix} \tag{18}$$

which differs from $\mathbf{R}_{123}$ by a simple similarity transformation of the type (16). The parameter ξ will be specified later on. Evidently, the transformation breaks the symmetry between the three spaces in (17), since the first space is distinguished from the other two. The matrix elements of $\mathbf{L}_{123}$ can be arranged into a set of 4×4 matrices L_0^0, L_0^1, L_1^0 and L_1^1, acting in the tensor product of the second and third spaces, while the indices 0, 1, labelling these matrices, refer to the first space,

$$\big(L_0^0\big)_{2,3}=\begin{pmatrix} a & 0 & 0 & d\\ 0 & b & c & 0\\ 0 & c & b & 0\\ d & 0 & 0 & a\end{pmatrix},\qquad \big(L_1^1\big)_{2,3}=\begin{pmatrix} b & 0 & 0 & -c\\ 0 & a & -d & 0\\ 0 & -d & a & 0\\ -c & 0 & 0 & b\end{pmatrix},\tag{19}$$

where

$$\begin{aligned} a&=1-t_0t_1+t_2t_3+t_0t_1t_2t_3, &\qquad b&=1+t_0t_1-t_2t_3+t_0t_1t_2t_3,\\ c&=1+t_0t_1+t_2t_3-t_0t_1t_2t_3, &\qquad d&=1-t_0t_1-t_2t_3-t_0t_1t_2t_3,\end{aligned}\tag{20}$$

and

$$\big(L_0^1\big)_{2,3}=\xi^{-1}\begin{pmatrix} -a' & 0 & 0 & -d'\\ 0 & -b' & -c' & 0\\ 0 & c' & b' & 0\\ d' & 0 & 0 & a'\end{pmatrix},\qquad \big(L_1^0\big)_{2,3}=\xi\begin{pmatrix} -b' & 0 & 0 & c'\\ 0 & -a' & d' & 0\\ 0 & -d' & a' & 0\\ -c' & 0 & 0 & b'\end{pmatrix},\tag{21}$$

where

$$\begin{aligned} a'&=-t_1t_2-t_0t_3+it_0t_2+it_1t_3, &\qquad b'&=-t_1t_2-t_0t_3-it_0t_2-it_1t_3,\\ c'&=-t_1t_2+t_0t_3+it_0t_2-it_1t_3, &\qquad d'&=-t_1t_2+t_0t_3-it_0t_2+it_1t_3.\end{aligned}\tag{22}$$

We regard these matrices as two by two block matrices with two-dimensional blocks. The indices (i_2, j_2) numerate the blocks while the indices (i_3, j_3) numerate matrix elements inside the blocks. From (7) it follows that

$$ab=a'b',\qquad cd=c'd',\qquad a^2+b^2-c^2-d^2=0,\qquad a'^2+b'^2-c'^2-d'^2=0,\tag{23}$$

therefore each of the above matrices can be viewed as an R-matrix of the eight-vertex free-fermion model, which satisfy the condition

$$\mathcal{R}^{(\mathrm{FFM})}=\begin{pmatrix} \omega_1 & 0 & 0 & \omega_7\\ 0 & \omega_3 & \omega_5 & 0\\ 0 & \omega_6 & \omega_4 & 0\\ \omega_8 & 0 & 0 & \omega_2\end{pmatrix},\qquad \omega_1\omega_2+\omega_3\omega_4-\omega_5\omega_6-\omega_7\omega_8=0.\tag{24}$$

3.2. *Elliptic parameterisation of the weights*

Here we will show that the weights (10) can be parametrised in terms of Jacobi elliptic functions. First, note that the form of the matrices (19) and (21) and the relations (23) suggest to

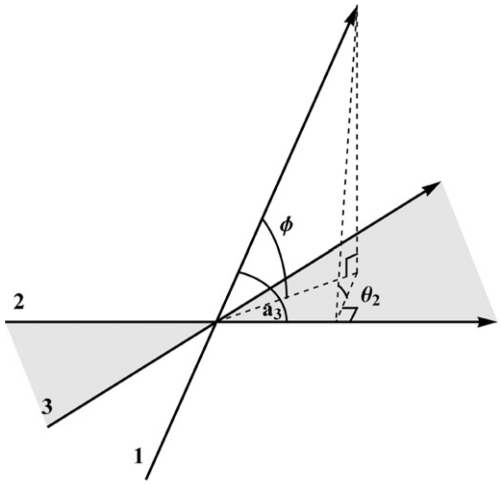

Fig. 3. This figure clarifies the geometric interpretation for the angle ϕ, given by (28), as an angle between the line 1 and the plane containing the lines 2 and 3.

use Baxter's parameterisation of the eight-vertex model [33] (the latter contains the symmetric free-fermion model as a particular case). To do this define the elliptic modulus

$$k = \frac{cd}{ab} \equiv \frac{c'd'}{a'b'}, \tag{25}$$

similarly to that in the symmetric eight-vertex model with the weights a, b, c, d (cf. Eq. (5.7) of [33]). With an account of (20) and (22) the RHS of Eq. (25) is a function of the dihedral angles $\theta_1, \theta_2, \theta_3$. After elementary simplifications, one obtains

$$k = \frac{1-\sin\phi}{1+\sin\phi}, \quad \sin\phi = \frac{4t_0t_1t_2t_3}{(1-t_0^2t_1^2)(1-t_2^2t_3^2)} = 2\frac{\sqrt{\sin\alpha_0\sin\alpha_1\sin\alpha_2\sin\alpha_3}}{\sin\theta_1}. \tag{26}$$

Then using the relations (see Section 132 of [34])

$$\sin\alpha_0 = 2K\sin\frac{a_1}{2}\sin\frac{a_2}{2}\sin\frac{a_3}{2}, \qquad \sin\alpha_i = 2K\sin\frac{a_i}{2}\cos\frac{a_j}{2}\cos\frac{a_k}{2}, \tag{27}$$

where (i, j, k) is any permutation of $(1, 2, 3)$ and K is defined in (3), one obtains

$$\sin\phi = K^2\frac{\sin a_1}{\sin\theta_1}\sin a_2\sin a_3 = \sin\theta_2\sin a_3 \tag{28}$$

which shows that ϕ is the angle between the line 1 and the plane containing the lines 2 and 3, as illustrated in Fig. 3. Note that value $k = 0$ corresponds to the right angle $\phi = \pi/2$.

Further, let $\operatorname{sn} x = \operatorname{sn}(x, k)$, $\operatorname{cn} x = \operatorname{cn}(x, k)$, $\operatorname{dn} x = \operatorname{dn}(x, k)$ and $\operatorname{cd} x = \operatorname{cd}(x, k)$ denote Jacobi elliptic functions of the argument x and modulus k (we follow the notations of [35]). Now let us parametrise the angles θ_2 and θ_3 in terms of new parameters w_1 and w_2,

$$e^{-i\theta_2} = -\frac{(1+\operatorname{sn}(2w_1))(1-k\operatorname{sn}(2w_1))}{\operatorname{cn}(2w_1)\operatorname{dn}(2w_1)}, \qquad e^{i\theta_3} = \frac{(1+\operatorname{sn}(2w_2))(1-k\operatorname{sn}(2w_2))}{\operatorname{cn}(2w_2)\operatorname{dn}(2w_2)}. \tag{29}$$

Given θ_2, θ_3 each of these equations have two solutions in the periodicity rectangle (we assume $0 < k < 1$)

$$0 \leqslant w_1, w_2 < 2\mathbf{K}, \quad 0 \leqslant \operatorname{Im} w_1, \operatorname{Im} w_2 < i\mathbf{K}'/2, \tag{30}$$

where $\mathbf{K} = \mathbf{K}(k)$ and $\mathbf{K}' = \mathbf{K}(\sqrt{1-k^2})$ are the complete elliptic integral of the first kind. Now substitute (29) back into (26) and solve the resulting equation for θ_1 in terms k, w_1, w_2. There are four solutions of the form

$$e^{i\theta_1} = \frac{\operatorname{cd}(2w_2)}{\operatorname{cd}(2w_1)}, \tag{31}$$

corresponding to four possible choices of w_1, w_2 in (29). Only one of them[3] lead to the original value of θ_1. This allows one to uniquely fix the required solution of (29).

From now on we regard k, w_1, w_2 as new independent variables instead of $\theta_1, \theta_2, \theta_3$. The latter are now defined by the formulae (29) and (31). The weights in Eq. (10) are expressed through the square roots of tangents of halves of spherical excesses $\alpha_0, \alpha_1, \alpha_2, \alpha_3$. Surprisingly enough, after lengthy calculations we found that these tangents have rather simple expressions in the new variables,

$$\begin{aligned} t_0 &= \big(-if(w_-)/f(\mathbf{K}-w_+)\big)^{\frac{1}{2}}, \qquad & t_1 &= \big(if(w_-)f(\mathbf{K}-w_+)\big)^{\frac{1}{2}}, \\ t_2 &= \big(-if(\mathbf{K}-w_-)f(w_+)\big)^{\frac{1}{2}}, \qquad & t_3 &= \big(if(\mathbf{K}-w_-)/f(w_+)\big)^{\frac{1}{2}}, \end{aligned} \tag{32}$$

where

$$w_\pm = w_1 \pm w_2, \qquad k' = \sqrt{1-k^2}. \tag{33}$$

The function $f(w)$ is defined as

$$f(w) = \frac{k'^2 \operatorname{sn} w}{(\operatorname{cn} w + \operatorname{dn} w)(k \operatorname{cn} w + \operatorname{dn} w)}. \tag{34}$$

It follows that the products appearing in (20) and (22) simplify to

$$\begin{aligned} t_0 t_1 &= f(w_-), \qquad & t_2 t_3 &= f(K - w_-), \\ t_0 t_3 &= \left(\frac{f(w_-)f(\mathbf{K}-w_-)}{f(w_+)f(\mathbf{K}-w_+)}\right)^{\frac{1}{2}}, \qquad & t_1 t_2 &= f(w_+)f(\mathbf{K}-w_+)t_0 t_3. \end{aligned} \tag{35}$$

Then using addition theorems for elliptic functions, one obtains

$$\begin{aligned} a &= \rho_- \operatorname{cd} w_-, \quad & b &= \rho_- \operatorname{sn} w_-, \quad & c &= \rho_-, \quad & d &= \rho_- k \operatorname{cd} w_- \operatorname{sn} w_-, \\ a' &= \rho_+ \operatorname{cd} w_+, \quad & b' &= \rho_+ \operatorname{sn} w_+, \quad & c' &= \rho_+, \quad & d' &= \rho_+ k \operatorname{cd} w_+ \operatorname{sn} w_+, \end{aligned} \tag{36}$$

where

$$\begin{aligned} \rho_- &= 1 + t_0 t_1 + t_2 t_3 - t_0 t_1 t_2 t_3 = \frac{4(1-\operatorname{sn} w_-)\operatorname{dn} w_-}{\operatorname{cn} w_-(\operatorname{cn} w_- \operatorname{dn} w_- + (1-\operatorname{sn} w_-)(1+k\operatorname{sn} w_-))}, \\ \rho_+ &= t_0 t_3 - t_1 t_2 + i(t_0 t_2 - t_1 t_3) = -\rho_- \left(\frac{\operatorname{cd} w_- \operatorname{sn} w_-}{\operatorname{cd} w_+ \operatorname{sn} w_+}\right)^{\frac{1}{2}}. \end{aligned} \tag{37}$$

[3] Calculating θ_1 from (31) for all four solutions of (29) one obtains the following values $\theta_1 = \pm\theta_1^{(true)}, \pm\theta_1'$ where $\theta_1^{(true)}$ is the original value of θ_1, while θ_1' is a completely different value, corresponding to a different spherical triangle, $(\theta_1', \theta_2, \theta_3)$ which, nevertheless, leads to the same modulus k in Eq. (26).

Next, we specify the parameter ξ in (17), which so far remained at our disposal,

$$\xi = -\frac{\rho_+}{\rho_-} \equiv h(w_1, w_2), \qquad h(x, y) = \sqrt{\frac{\operatorname{cd}(x-y)\operatorname{sn}(x-y)}{\operatorname{cd}(x+y)\operatorname{sn}(x+y)}}. \tag{38}$$

Finally, introduce the R-matrix of the symmetric eight-vertex model [33] specialized to the free-fermion case,

$$\mathbb{R}(w) = \begin{pmatrix} \operatorname{cd} w & 0 & 0 & k\operatorname{cd} w \operatorname{sn} w \\ 0 & \operatorname{sn} w & 1 & 0 \\ 0 & 1 & \operatorname{sn} w & 0 \\ k\operatorname{cd} w \operatorname{sn} w & 0 & 0 & \operatorname{cd} w \end{pmatrix}. \tag{39}$$

Using this notation, the matrices (19), (21) can be written in a uniform way

$$\begin{aligned} \left(L_0^0\right)_{23} &= \rho_- \mathbb{R}_{23}(w_-), & \left(L_0^1\right)_{23} &= -\rho_+ \xi^{-1} \sigma_z^{(2)} \mathbb{R}_{23}(w_+), \\ \left(L_1^0\right)_{23} &= i\rho_+ \xi \sigma_x^{(2)} \mathbb{R}_{23}(w_+) \sigma_y^{(2)}, & \left(L_1^1\right)_{23} &= \rho_- \sigma_y^{(2)} \mathbb{R}_{23}(w_-) \sigma_y^{(2)}, \end{aligned} \tag{40}$$

where the Pauli matrices $\sigma_x^{(2)}, \sigma_y^{(2)}, \sigma_z^{(2)}$ act in the space 2. Taking into account (37) and (38) is easy to see that all expressions for the matrix elements in (40) are *meromorphic double periodic functions* of the variables w_1 and w_2.

To summarize, we have shown that all matrix elements the operator

$$\begin{aligned} \mathcal{L}_{123}(w_1, w_2|k) &= \mathbf{L}_{123}(\theta_1, \theta_2, \theta_3) \\ &\stackrel{\text{def}}{=} \big(\mathbf{D}(\xi) \otimes \mathbf{F} \otimes \mathbf{F}\big) \mathbf{R}_{123}(\theta_1, \theta_2, \theta_3) \big(\mathbf{D}(\xi) \otimes \mathbf{F} \otimes \mathbf{F}\big)^{-1}, \end{aligned} \tag{41}$$

are meromorphic functions of w_1 and w_2, provided the two sets of variables, entering different sides of this equation, are related by (26), (29) and (31) and the parameter ξ is defined by (38).

We would like to stress the above elliptic parametrisation was obtained for a generic case without any special requirements for the values of $\theta_1, \theta_2, \theta_3$. By construction, this parametrisation breaks the symmetry of the weights (the directions 1 is distinguished). Moreover, the modulus k will, *a priori*, be different for different vertices of the tetrahedron. Therefore, in general, one cannot parametrise all weights in the tetrahedron equation by elliptic functions of the same modulus. Nonetheless, there exists an important four-parameter reduction of this equation, where such parametrisation is possible. This is the "prismatic limit" considered in the next section.

4. Reductions of the tetrahedron equation

In this section we consider certain reductions of the tetrahedron equation for the solution (10) and its connection to an example of the *tetrahedral Zamolodchikov algebra* constructed in [19].

4.1. The "static limit"

The term "static limit" stems from the original Zamolodchikov's work [1] where it was related to the case of slowly moving or "non-relativistic" straight strings in $2+1$ dimensions. On the level of parameters this corresponds to a configuration where the sum of dihedral angles is equal to π for every vertex of the tetrahedron. Namely, for Eq. (15) this means

$$\theta_1+\theta_2+\theta_3=\pi,\qquad \theta_1+\theta_4+\theta_5=\pi,$$
$$\theta_4+\theta_6-\theta_2=0,\qquad \theta_3+\theta_6-\theta_5=0. \tag{42}$$

The linear angles between edges at the vertices all become equal to 0 or π and the tetrahedron degenerates into four planes intersecting along the same line. Their relative orientation is fixed by three angles only, so the tetrahedron equation in this case contains three independent parameters (instead of five in the general case).

We will use a special notation for the operator $\mathbf{R}(\theta_1,\theta_2,\theta_3)$ specialized to this case

$$\mathbf{S}(\theta_1,\theta_2,\theta_3)=\mathbf{R}(\theta_1,\theta_2,\theta_3),\quad \theta_1+\theta_2+\theta_3=\pi. \tag{43}$$

Its matrix elements are determined by (10) where one sets

$$t_0=0,\qquad t_i=T_i\equiv\sqrt{\tan\frac{\theta_i}{2}},\quad i=1,2,3. \tag{44}$$

There are only sixteen non-vanishing elements,

$$\begin{aligned}
S_{0,0,0}^{0,0,0} &= S_{0,1,1}^{0,1,1} = S_{1,0,1}^{1,0,1} = S_{1,1,0}^{1,1,0} = 1,\\
S_{0,0,1}^{0,1,0} &= S_{0,1,0}^{0,0,1} = -S_{1,1,1}^{1,0,0} = -S_{1,0,0}^{1,1,1} = T_2T_3,\\
S_{0,1,0}^{1,1,1} &= S_{0,0,1}^{1,0,0} = -S_{1,0,0}^{0,0,1} = -S_{1,1,1}^{0,1,0} = -\mathrm{i}T_1T_3,\\
S_{1,1,1}^{0,0,1} &= S_{0,0,1}^{1,1,1} = S_{0,1,0}^{1,0,0} = S_{1,0,0}^{0,1,0} = T_1T_2.
\end{aligned} \tag{45}$$

This solution of the tetrahedron equation (with a slightly different parameterisation, see Section 4.5 below) was first obtained in [19]. Two years later [17] it was understood as the vertex form of the original Zamolodchikov's solution in the static limit [1].

The operator $\mathbf{S}$ satisfies the relation

$$\mathbf{S}^2=1. \tag{46}$$

It has a block-diagonal form with two 4×4 blocks, a trivial one, which coincides with the identity matrix, and a non-trivial one which has two eigenvalues $+1$ and two eigenvalues -1.

The operator $\mathbf{S}$ possesses left and right "bare vacuum" eigenvectors

$$(v_0\otimes v_0\otimes v_0)^t\mathbf{S}=(v_0\otimes v_0\otimes v_0)^t,\qquad \mathbf{S}(v_0\otimes v_0\otimes v_0)=v_0\otimes v_0\otimes v_0,\quad v_0=\begin{pmatrix}1\\0\end{pmatrix}. \tag{47}$$

Note that this property does not hold for the general case (10) with $t_0\neq0$.

4.2. The planar limit

Geometrically the "planar limit" corresponds to the case when all four vertices of the tetrahedron lie in one plane, i.e. when the tetrahedron becomes "squashed" into a plane. An edge-spin solution of (13) corresponding to this case was first obtained by Hietarinta [20]. Subsequently it was understood as the planar limit [36] of the Zamolodchikov model. The details of this reduction are rather complicated and not immediately related to the main topic of this paper. Therefore, we refer interested readers to the original publication [36] where this limit is thoroughly studied.

4.3. The prismatic limit

Here we consider yet another limiting case of Eq. (13), when one of the tetrahedron vertices goes to infinity (we choose it to be the vertex corresponding to $\mathbf{R}_{123}$). Then the sum of dihedral angles at this vertex will satisfy an additional constraint,

$$\theta_1 + \theta_2 + \theta_3 = \pi, \tag{48}$$

whereas the whole tetrahedron turns into an infinite prism. The number of independent angles reduces from five to four. The edges 1, 2 and 3 become parallel and therefore, have the same angle to the plane, containing the edges 4, 5 and 6, forming the base of the prism. Remembering the geometric definition (26) of the elliptic modulus k in the previous section, we conclude that all weights corresponding to the vertices $(1,4,5)$, $(2,4,6)$ and $(3,5,6)$ can be parametrised by elliptic functions of the *same modulus*.

Eqs. (26), (29) and (31) define a generic change of variables from three dihedral angles $(\theta_1,\theta_2,\theta_3)$ to new parameters (k,w_1,w_2). Here we want to apply this substitution for three different sets of dihedral angles corresponding to the three vertices at the base of the prism

$$\begin{aligned}&(\theta_1,\theta_4,\theta_5)\to(k,u_1,u_2); \qquad (\pi-\theta_2,\theta_4,\theta_6)\to(k,u_1,u_3);\\ &(\theta_3,\pi-\theta_5,\theta_6)\to(k,u_2,u_3).\end{aligned} \tag{49}$$

As explained above from the geometric considerations the elliptic modulus k will automatically be the same for all three sets. For instance, let (b_1,b_2,b_3) be the sides of the spherical triangle with angles $(\theta_1,\theta_4,\theta_5)$, then we set

$$k=\frac{1-\sin\phi}{1+\sin\phi}, \qquad \sin\phi=\sin\theta_4\sin b_3. \tag{50}$$

Next define the variables u_1,u_2,u_3 by the relations

$$\begin{aligned} e^{-i\theta_4} &= -\frac{(1+\operatorname{sn}(2u_1))(1-k\operatorname{sn}(2u_1))}{\operatorname{cn}(2u_1)\operatorname{dn}(2u_1)},\\ e^{i\theta_5} &= \frac{(1+\operatorname{sn}(2u_2))(1-k\operatorname{sn}(2u_2))}{\operatorname{cn}(2u_2)\operatorname{dn}(2u_2)},\\ e^{i\theta_6} &= \frac{(1+\operatorname{sn}(2u_3))(1-k\operatorname{sn}(2u_3))}{\operatorname{cn}(2u_3)\operatorname{dn}(2u_3)}\end{aligned} \tag{51}$$

and

$$e^{i\theta_1}=\frac{\operatorname{cd}(2u_2)}{\operatorname{cd}(2u_1)}, \qquad e^{i\theta_2}=-\frac{\operatorname{cd}(2u_1)}{\operatorname{cd}(2u_3)}, \qquad e^{i\theta_3}=\frac{\operatorname{cd}(2u_3)}{\operatorname{cd}(2u_2)} \tag{52}$$

which are obtained by a simple specialisation of (29) and (31) for the three substitutions (49). Note that the above formulae (51) and (52) give a parameterisation of six angles of a triangular prism in terms of elliptic functions. In particular, it is easy to check that the three expressions in (52) are consistent with (48).

Let us now rewrite the tetrahedron equation (13) for this case in the new variables. We will do this in several steps. First, guided by the formula (41) define three operators

$$\begin{aligned}\mathcal{L}_{145}(u_1,u_2)&=\big(\mathbf{D}(\xi_1)\otimes\mathbf{F}\otimes\mathbf{F}\big)\mathbf{R}_{145}(\theta_1,\theta_4,\theta_5)\big(\mathbf{D}(\xi_1)\otimes\mathbf{F}\otimes\mathbf{F}\big)^{-1},\\ \mathcal{L}_{246}(u_1,u_3)&=\big(\mathbf{D}(\xi_2)\otimes\mathbf{F}\otimes\mathbf{F}\big)\mathbf{R}_{246}(\pi-\theta_2,\theta_4,\theta_6)\big(\mathbf{D}(\xi_2)\otimes\mathbf{F}\otimes\mathbf{F}\big)^{-1},\\ \mathcal{L}_{356}(u_2,u_3)&=\big(\mathbf{D}(\xi_3)\otimes\mathbf{F}\otimes\mathbf{F}\big)\mathbf{R}_{145}(\theta_3,\pi-\theta_5,\theta_6)\big(\mathbf{D}(\xi_3)\otimes\mathbf{F}\otimes\mathbf{F}\big)^{-1}.\end{aligned} \tag{53}$$

Here $\mathbf{F}$ and $\mathbf{D}(\xi)$ are defined in (18) and the parameters ξ_1, ξ_2, ξ_3 have been chosen in agreement with (38) in each of the three cases,

$$\xi_1 = h(u_1, u_2), \qquad \xi_2 = h(u_1, u_3), \qquad \xi_3 = h(u_2, u_3), \tag{54}$$

where $h(x, y)$ is defined in (38). Therefore, according to the result of Section 3, all operators (53) are meromorphic double periodic functions of u_1, u_2, u_3, which, of course, implicitly depend on the elliptic modulus k.

Next, we need to express in the new variables the weights, corresponding to the infinitely distant vertex $(1, 2, 3)$. It is convenient to define a new operator,

$$\mathcal{S}_{123}(u_1, u_2, u_3) = \big(\mathbf{D}(\xi_1) \otimes \mathbf{D}(\xi_2) \otimes \mathbf{D}(\xi_3)\big) \mathbf{S}_{123}(\theta_1, \theta_2, \theta_3) \big(\mathbf{D}(\xi_1) \otimes \mathbf{D}(\xi_2) \otimes \mathbf{D}(\xi_3)\big)^{-1}, \tag{55}$$

which differs from (43) by a diagonal similarity transformation.

Using (52) it is not difficult to show that

$$\tan\frac{\theta_1}{2} = -ig(u_1, u_2), \qquad \tan\frac{\theta_2}{2} = \frac{i}{g(u_1, u_3)}, \qquad \tan\frac{\theta_3}{2} = -ig(u_2, u_3), \tag{56}$$

where

$$g(x, y) = k'^2 \frac{\operatorname{sd}(x-y)\operatorname{sd}(x+y)}{\operatorname{cn}(x-y)\operatorname{cn}(x+y)}. \tag{57}$$

Combining this with (45), one obtains the following expressions for the non-vanishing matrix elements of $\mathcal{S}(u_1, u_2, u_3)$

$$\mathcal{S}_{000}^{000} = \mathcal{S}_{110}^{110} = \mathcal{S}_{101}^{101} = \mathcal{S}_{011}^{011} = 1,$$

$$\begin{aligned}
\mathcal{S}_{010}^{100} &= U(u_1, u_2, u_3), & \mathcal{S}_{010}^{001} &= U(u_3, u_2, u_1), & \mathcal{S}_{100}^{001} &= V(u_1, u_2, u_3),\\
\mathcal{S}_{100}^{010} &= U(u_1, -u_2, -u_3), & \mathcal{S}_{001}^{010} &= U(u_3, -u_2, -u_1), & \mathcal{S}_{001}^{100} &= V(-u_1, u_2, -u_3),\\
\mathcal{S}_{111}^{001} &= U(u_1, -u_2, u_3), & \mathcal{S}_{111}^{100} &= U(u_3, -u_2, u_1), & \mathcal{S}_{111}^{010} &= V(u_1, u_2, -u_3),\\
\mathcal{S}_{001}^{111} &= U(u_1, u_2, -u_3), & \mathcal{S}_{100}^{111} &= U(u_3, u_2, -u_1), & \mathcal{S}_{010}^{111} &= V(-u_1, u_2, u_3),
\end{aligned} \tag{58}$$

where

$$U(x, y, z) = \frac{\operatorname{sd}(x+y)\operatorname{cn}(x-z)}{\operatorname{cn}(x-y)\operatorname{sd}(x+z)}, \qquad V(x, y, z) = -k'^2 \frac{\operatorname{sd}(y-x)\operatorname{sd}(y+z)}{\operatorname{cn}(y+x)\operatorname{cn}(y-z)}. \tag{59}$$

It is easy to see that all above matrix elements are meromorphic functions of u_1, u_2, u_3, implicitly depending on the elliptic modulus k. Note that even though the parameters ξ_1, ξ_2, ξ_3, entering (55), were fixed by the analyticity requirements for the other three operators, defined in (53), the matrix elements of $\mathcal{S}(u_1, u_2, u_3)$ have automatically became meromorphic functions on the elliptic curve.[4] Note also that the similarity transformation in (55) introduces two additional parameters into (58) and breaks the symmetry relations between different matrix elements

[4] Eq. (61) below implies that ratios of matrix elements of $\mathcal{S}_{123}(u_1, u_2, u_3)$ are meromorphic functions on the elliptic curve. However, some of these matrix elements in (58) are equal to one, therefore the remaining elements should be meromorphic.

exhibited in (45). Remind that the original static limit weights (45) depend on only two independent angles.

To complete our analysis of the prismatic limit apply the similarity transformation with the matrix

$$\mathbf{G} = \mathbf{D}(\xi_1) \otimes \mathbf{D}(\xi_2) \otimes \mathbf{D}(\xi_3) \otimes \mathbf{F} \otimes \mathbf{F} \otimes \mathbf{F} \tag{60}$$

to both sides of the tetrahedron equation (13). Then using the definitions (53) and (55) one obtains

$$\begin{aligned} &\mathcal{S}_{123}(u_1, u_2, u_3)\, \mathcal{L}_{145}(u_1, u_2)\, \mathcal{L}_{246}(u_1, u_3)\, \mathcal{L}_{356}(u_2, u_3) \\ &\quad = \mathcal{L}_{356}(u_2, u_3)\, \mathcal{L}_{246}(u_1, u_3)\, \mathcal{L}_{145}(u_1, u_2)\, \mathcal{S}_{123}(u_1, u_2, u_3). \end{aligned} \tag{61}$$

As required, this equation contains exactly four independent parameters, namely, u_1, u_2, u_3, which are indicated explicitly, and the elliptic modulus k which is implicitly assumed.

4.4. *Elliptic parameterisation of the static limit*

Consider the tetrahedron equation (13) in the static limit, i.e., when the angles satisfy the constraints (42). In the previous subsection we saw that upon the change of variables (52) and the similarity transformation (55) the vertex weights $\mathcal{S}_{123}(u_1, u_2, u_3)$, corresponding to the vertex $(1, 2, 3)$, become meromorphic functions on the elliptic curve. Remarkably, it is possible to do this for the other three vertices as well, using the elliptic functions of the same modulus. First, we need to parameterise the angles, satisfying (42), in term of the elliptic functions. To avoid confusions with the notations of the previous subsection we denote these angles $\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_6$. Let us parameterise $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$ by the same formulae as (52),

$$e^{i\bar{\theta}_1} = \frac{\mathrm{cd}(2u_2)}{\mathrm{cd}(2u_1)}, \qquad e^{i\bar{\theta}_2} = -\frac{\mathrm{cd}(2u_1)}{\mathrm{cd}(2u_3)}, \qquad e^{i\bar{\theta}_3} = \frac{\mathrm{cd}(2u_3)}{\mathrm{cd}(2u_2)} \tag{62}$$

and assume that

$$e^{i\bar{\theta}_4} = -\frac{\mathrm{cd}(2u_1)}{\mathrm{cd}(2u_4)}, \qquad e^{i\bar{\theta}_5} = \frac{\mathrm{cd}(2u_4)}{\mathrm{cd}(2u_2)}, \qquad e^{i\bar{\theta}_6} = \frac{\mathrm{cd}(2u_4)}{\mathrm{cd}(2u_3)}. \tag{63}$$

This implies the expressions (56) with $\theta_1, \theta_2, \theta_3$ replaced by $\bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$ and

$$\tan\frac{\bar{\theta}_4}{2} = \frac{i}{g(u_1, u_4)}, \qquad \tan\frac{\bar{\theta}_5}{2} = -i\,g(u_2, u_4), \qquad \tan\frac{\bar{\theta}_6}{2} = -i\,g(u_3, u_4), \tag{64}$$

where $g(u, w)$ is defined in (57). Note, that these formulae contains five independent parameters: u_1, u_2, u_3, u_4 and k, whereas there are only three independent angles, satisfying (42). So the above parameterisation contains two spurious parameters.

Next, assume that ξ_1, ξ_2, ξ_3 are given by (54) and

$$\xi_4 = h(u_1, u_4), \qquad \xi_5 = h(u_2, u_4), \qquad \xi_6 = h(u_3, u_4), \tag{65}$$

where $h(u, w)$ is defined in (38). Applying now the similarity transformation with the matrix

$$\bar{\mathbf{G}} = \mathbf{D}(\xi_1) \otimes \mathbf{D}(\xi_2) \otimes \mathbf{D}(\xi_3) \otimes \mathbf{D}(\xi_4) \otimes \mathbf{D}(\xi_5) \otimes \mathbf{D}(\xi_6) \tag{66}$$

to both sides of the tetrahedron equation (13) and taking into account the above parameterisation, one obtains

$$\begin{aligned}&\mathcal{S}_{123}(u_1,u_2,u_3)\,\mathcal{S}_{245}(u_1,u_2,u_4)\,\mathcal{S}_{246}(u_1,u_3,u_4)\,\mathcal{S}_{356}(u_2,u_3,u_4)\\&\quad=\mathcal{S}_{356}(u_2,u_3,u_4)\,\mathcal{S}_{246}(u_1,u_3,u_4)\,\mathcal{S}_{245}(u_1,u_2,u_4)\,\mathcal{S}_{123}(u_1,u_2,u_3),\end{aligned}\tag{67}$$

where the matrix elements of $\mathcal{S}_{ijk}(u,v,w)$ are defined by (58). It appears, that this equation naturally complements Eq. (61). As noted in Section 4.1 there are only three independent angles in the tetrahedron in the static limit. However, Eq. (67) contains five independent parameters (four u's and the modulus k). Two additional parameters are not spurious in this case, as they cannot be removed by a re-parameterisation. They were introduced via the similarity transformation with the matrix (66).

4.5. Tetrahedral Zamolodchikov algebra

Here we show that some additional reduction of the tetrahedron equation (61) leads precisely to the example of the tetrahedral Zamolodchikov algebra constructed in [19]. It is convenient to introduce two 4×4 matrices, acting in the product of two spaces $\mathbb{C}^2\otimes\mathbb{C}^2$ as

$$\mathcal{R}^a(u,w)=\sum_{i_2,i_3,j_2,j_3}\mathcal{L}^{aj_2j_3}_{0i_2i_3}(u,w)\big(E^{j_2}_{i_2}\otimes E^{j_3}_{i_3}\big),\quad a=0,1,\tag{68}$$

where $(E^j_i)_{ab}=\delta_{ia}\delta_{jb}$ denotes the 2×2 matrix unit. According to (40), (41) these new matrices are simply related to the R-matrix of the free-fermion model (39),

$$\mathcal{R}^0(u,w)=\rho_-\mathbb{R}(u-w),\qquad \mathcal{R}^1(u,w)=-\rho_+\xi^{-1}(\sigma_z\otimes 1)\,\mathbb{R}(u+w).\tag{69}$$

Now calculate matrix elements of tetrahedron equation (61) sandwiched between the fixed vectors

$$\langle L|=\big(v_0^t\otimes v_0^t\otimes v_0^t\big)_{123},\qquad |R\rangle=(v_a\otimes v_b\otimes v_c)_{123}\tag{70}$$

in the first three spaces, where the basis vectors v_0, v_1 are defined (2) and the indices a,b,c take the values 0, 1 (in the notations of (15) this corresponds to the case when $i_1=i_2=i_3=0$ and $j_1=a$, $j_2=b$, $j_3=c$). The resulting equation considerably simplifies, since the matrix $\mathcal{S}$ in the LHS of (61) drops out due to (47). Writing this equation in an abbreviated form, one gets

$$\big(\mathcal{L}^a_0\big)_{45}\big(\mathcal{L}^b_0\big)_{46}\big(\mathcal{L}^c_0\big)_{56}=\sum_{d,e,f}\mathcal{S}^{abc}_{def}\big(\mathcal{L}^f_0\big)_{56}\big(\mathcal{L}^e_0\big)_{46}\big(\mathcal{L}^d_0\big)_{45},\tag{71}$$

where, for example

$$\big(\mathcal{L}^a_0\big)_{45}=\big(v_0^t\otimes 1\otimes 1\big)\mathcal{L}_{145}\,(v_a\otimes 1\otimes 1)\tag{72}$$

and similarly for the other $\mathcal{L}$'s. For fixed values of a,b,c, both sides of (71) still remain operators acting in the spaces 4, 5, 6. At this point it is convenient to relabel these spaces as 1, 2, 3 (the former spaces 1, 2, 3 are no longer required, so there should be no confusions) and use the definition (68). In this way one obtains

$$\begin{aligned}&\mathcal{R}^a_{12}(u_1,u_2)\mathcal{R}^b_{13}(u_1,u_3)\mathcal{R}^c_{23}(u_2,u_3)\\&\quad=\sum_{d,e,f}\mathcal{S}^{abc}_{def}(u_1,u_2,u_3)\mathcal{R}^f_{23}(u_2,u_3)\mathcal{R}^e_{13}(u_1,u_3)\mathcal{R}^d_{12}(u_1,u_2).\end{aligned}\tag{73}$$

This relation is known as the definition the *tetrahedral Zamolodchikov algebra*. It was introduced in [19]. The coefficients $\mathcal{S}^{abc}_{def}(u_1,u_2,u_3)$ can be considered as the structure constants of the algebra. They satisfy the tetrahedron equations (67), which play the role of the associativity condition for the defining relations (73). This relationship is analogues to that for the Zamolodchikov–Faddeev algebra [37] where the Yang–Baxter equation plays the role of the associativity condition.

In [19] Korepanov constructed a representation of the algebra (73). He essentially postulated the expressions (68), without any connection to the 3D Zamolodchikov model and then calculated the coefficients $\mathcal{S}^{abc}_{def}(u_1, u_2, u_3)$, solving the linear equations (73). He used the normalization, where $\xi_1 = \xi_2 = \xi_3 = 1$, so his expression for $\mathcal{S}_{123}$ (given before Eq. (2.32) in [19]) contains square roots and, upon a change of variables, exactly coincides with our expression for $\mathbf{S}(\theta_1, \theta_2, \theta_3)$, given by (45). More precisely, Korepanov used the variables $\lambda_1, \lambda_2, \lambda_3$ (which exactly correspond to our variables u_1, u_2, u_3, respectively) and the variables $\varphi_1, \varphi_2, \varphi_3$, defined by Eq. (2.32) of [19],

$$\tanh\varphi_i = \frac{1-\mathrm{cd}(2u_i)}{1+\mathrm{cd}(2u_i)}, \quad i = 1, 2, 3. \tag{74}$$

Using addition theorems for elliptic and trigonometric functions it is not difficult to show that

$$g(u_j, u_k) = \tanh(\varphi_j - \varphi_k), \quad j, k = 1, 2, 3, \tag{75}$$

where $g(x, y)$ is given by (57). Then it follows from (56) that

$$\theta_1 = 2i(\varphi_2 - \varphi_1), \qquad \theta_2 = \pi + 2i(\varphi_1 - \varphi_3), \qquad \theta_3 = 2i(\varphi_3 - \varphi_2). \tag{76}$$

Subsequently, Shiroishi and Wadati [22] reproduced the calculations of [19]. They worked in essentially the same normalization and notations as this work and their expressions for the matrix elements of $\mathcal{S}_{123}$ (given by their Eq. (5.4)) exactly coincide with our Eq. (58) with interchanged u_1 and u_2. In addition, Eq. (73) was re-checked once again in [23] in the trigonometric limit ($k = 0$), see Section 12.A.1 in [23].

5. Conclusion

In this paper we have considered a special limit of the tetrahedron equation in the 3D Zamolodchikov model, where one of its vertices goes to infinity and the tetrahedron turns into an infinite prism. We explicitly showed that all vertex weights in the tetrahedron equation (61) in this case are meromorphic functions depending on three points on an elliptic curve. The parameterisation of the weights are given by (40) and (58). As a byproduct we obtained a parameterisation of angles of a triangular prism in terms of elliptic functions (a problem, which is, certainly, too hard for school geometry, but could have taken a comfortable place among exercises in the classic textbook by Whittaker and Watson [35]). The corresponding formulae are given in (51) and (52).

Next, we have shown that a further reduction of this, already special, case of the tetrahedron equation leads precisely to the "tetrahedral Zamolodchikov algebra", originally invented in [19] without any connection to the 3D Zamolodchikov model. Interestingly, this algebra (more precisely, its trigonometric variant, corresponding to $k = 0$) has been used by Shiroishi and Wadati [21] for a two-layer construction of the Shastry's solution [24] of the Yang–Baxter equation connected to the Hubbard model [38]. Recently, the interest to this solution was renewed due its remarkable appearance [25–28] in the problem of the AdS/CFT correspondence for $\mathcal{N} = 4$ SUSY Yang–Mills theory in four dimensions.

Actually, one of motivations for this work was to unravel an algebraic origin of the Shastry's R-matrix, which so far has not been included into the standard quantum group scheme [39].

This R-matrix has long been suspected to have a hidden 3D structure. Indeed, it has an evident two-layer structure implied by the construction of [21], mentioned above. Moreover, it does not possess the "difference property", which is another indication of a 3D origin. To enhance the last argument it is worth mentioning that the chiral Potts model [40,41], which is the most notable example among solved models without the difference property, is just a two-layer case of the multi-state generalisation of the 3D Zamolodchikov model [7,42,43].

Our results suggest that the Shastry's R-matrix is closely related to the 3D Zamolodchikov model, which was originally formulated as an exact relativistic factorized scattering theory of "straight strings" in $(2+1)$-dimensional space–time. This 3D model has an extremely rich algebraic structure, previously studied in [5–9,13–17,32]. It would be interesting to see how this connection will help to understand real origins of the Shastry's R-matrix. We postpone these considerations to a separate publication [44].

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