Chapter 4

Acoustic source localisation

The basic sequence of operation of the relative localisation system can be explained as follows: an AUV (sender) emits two acoustic ‘pings’ in sequence, first from the bow (front) end and next from the stern (rear) end. These two pings constitute one sending event. AUVs (observers) which have the sender within their sensing range would attempt to estimate the angle and the distance to each of the source upon receiving the acoustic pings. These angles (sub-azimuths) and distances (sub-ranges) are then used to estimate the compound azimuth, range and heading of the sending AUV, relative to each of the observers. These azimuth, range and heading estimates are later used as components to assemble a pose vector which contains the position information of the localised AUV.

The specific measurements and estimations carried out by an observing vehicle in this process are explained in the next sections, starting with the angle and distance estimation for a single acoustic source. The methodology and basic measurement schemes are described in detail along with identification of different classes of errors affecting the estimated quantities. An analysis of how the uncertainties associated with the basic measurements propagate towards uncertainties in the estimated quantities is given and a strategy for minimising the random errors of the estimates is also presented which contributes towards improving the precision of the relative
localisation system. Later sections describe how the sub-azimuths and sub-ranges are used to derive the compound estimates for azimuth, range and heading in addition to an alternative scheme of estimating heading and range independent of implicit sender-observer synchronisation. Formulae showing the relationship of the component quantities and their uncertainties with the uncertainty of the compound quantities are also derived and plotted. These are used to analyse the behaviour of the relative localisation estimates with regard to resolution and upper bounds for errors.

4.1 Angle estimation

As mentioned in chapter 2, TDOA measurement is the basis of hyperbolic localisation and navigation schemes as well as many bearing only tracking systems. Bellingham et al. (1992) presents such a self-localisation system for multiple AUVs using TDOA measurements with respect to multiple acoustic beacons. TDOA, as the term suggests, is the difference of arrival times at two receiver locations, of a signal transmitted from a third location. This quantity is then converted to an angle, from which the source signal arrive towards the two receivers. In the context of this research, the source signal consist of an acoustically transmitted MLS signal and transmitters and receivers are projectors and hydrophones as described in the previous chapter.

4.1.1 Mapping path length difference to an angle

Figure 4.1 depicts a source at $P_1$ and two receivers at $H_1$ and $H_2$. The signal from $P_1$ takes two different paths to reach $H_1$ and $H_2$, traversing path lengths of $d_1$ and $d_2$ respectively. This difference in path lengths as a distance is denoted by $\delta$ and given as:

$$\delta = d_1 - d_2$$

(4.1)
With the spacing between the receivers denoted by \( d \) and following the standard definition of a hyperbola, the locus of \( P_1 \) can be defined as lying on a branch of a hyperbola with its foci at \( H_1, H_2 \) and eccentricity \( d/|\delta| \).

If the distance to \( P_1 \) from the midpoint of the two receivers (\( O_1 \)) is denoted by \( r_1 \) and considering the polar coordinates of \( P_1 \) as \( (r_1, \beta) \) where the polar axis lies along \( H_1H_2 \) with the pole at \( O_1 \), the polar coordinate equation of the hyperbola yields the angle of arrival of a signal originating from \( P_1 \) as:

\[
\beta = \pm \tan^{-1}\left( \frac{\sqrt{(4r_1^2 - \delta^2)(d^2 - \delta^2)}}{\delta \sqrt{4r_1^2 + d^2 - \delta^2}} \right)
\]  

(4.2)

Usually, the quantity \( r_1 \) is not available at the time of measuring the angle. When the source is located sufficiently far away from the receivers compared to the spacing between the receivers\(^1\), the same angle can be expressed using the polar coordinate equation of the asymptotes of the hyperbola as follows:

\[
\beta = \pm \tan^{-1}\left( \frac{\sqrt{d^2 - \delta^2}}{\delta} \right)
\]  

(4.3)

where \( \tan\beta \) is the gradient of the asymptotes.

As can be seen from the diagram in figure 4.1 as well as by inspecting the formulae, the path length difference \( \delta \) varies between \( -d \) and \( +d \). The sign of \( \delta \) determines which branch of the hyperbola contains \( P_1 \). However, as seen from figure 4.2, a given value of \( \delta \) yielding a value for \( \beta \) still holds an ambiguity, referred to as the front-back ambiguity. Any location of \( P_1 \) and its reflection about \( H_1H_2 \) would give the same path length difference. The \( \pm \) sign in the formulae (4.2) and (4.3) can be considered to represent this ambiguity. In the absence of extra information from additional sensors, omnidirectional receivers cannot usually resolve this ambiguity. The relative localisation system presented in this thesis uses non-omnidirectional hydrophones as receivers and consequently avoids this problem\(^2\). Therefore, the \( \pm \) sign will be dropped from the formulae in the subsequent discussions. Furthermore, the signal source is assumed to be on the same plane containing the two receivers and their main axes of directivity, hence the localisation system is restricted to two dimensions. However, the principle of path length difference extends to the third dimension where the source lie outside the aforementioned plane. In this case, the branches of the hyperbola are replaced by sheets of a three dimensional hyperboloid and the asymptote lines are replaced by cones. Strategies for extending the localisation system into the

\(^1\) In practise, this condition is satisfied when \( r_1 > 2d \)

\(^2\) The experimental results presented in chapter 7 demonstrates the use of non-omnidirectional hydrophones.
third dimension by fusing additional sensor information (relative depth) and utilising additional hydrophones, are proposed and experimentally validated in chapter 8.

4.1.2 TDOA measurement

While the path length difference described earlier is given as a distance, the speed of propagation of the signal relates it to a time delay $t_D$ as:

$$t_D = \frac{\delta}{v}$$  \hspace{1cm} (4.4)

where $v$ is the speed of propagation of the source signal. Since it is not possible to explicitly measure the path length difference, the actual measurement is of this delay $t_D$ which is referred to as the time difference of arrival (TDOA) of the signal.

When $s(t)$ denotes the source signal at $P_1$, the two signal channels at receivers $H_1, H_2$ represented by $s_1(t), s_2(t)$ respectively, can be modelled as:

$$s_1(t) = s(t) + n_1(t)$$
$$s_2(t) = a \times s(t + t_D) + n_2(t)$$  \hspace{1cm} (4.5)

where $a$ is an attenuation coefficient, $n_1(t)$ and $n_2(t)$ being the uncorrelated noise present in each channel and $t_D$ being the time difference of arrival between the two channels corresponding to the difference in path length $\delta$. Transformation of the signal due to receiving and transmitting transducer characteristics and the propagation medium is not explicitly modelled. These transformations do not affect the measurement of $t_D$ when they are assumed to be common to

Figure 4.2: A given path length difference, while restricting the locus of $P_1$ to one branch of the hyperbola, introduces a front-back ambiguity when using omnidirectional receivers at $H_1$ and $H_2$. 
both channels. Knapp and Carter (1976) gives the estimate for $t_D$ as the argument $\tau$ that maximise the following cross-correlation function:

$$R_{s_1s_2}(\tau) = E[s_1(t)s_2(t - \tau)]$$  \hspace{1cm} (4.6)

where $E$ denotes the expectation.

A continuous-time representation of the cross-correlation function of the time-domain signals $s_1(t)$ and $s_2(t)$ can be expressed as:

$$R_{s_1s_2}(\tau) = \int_{-\infty}^{\infty} s_1(t)s_2(t - \tau)dt$$  \hspace{1cm} (4.7)

However, in the relative localisation system being discussed, the signals received by the hydrophone channels are two discretely-sampled time-series signals of finite length $N$ which can be denoted by $s_1(n)$ and $s_2(n)$ which gives the cross-correlation function using (3.1) given in chapter 3 as follows:

$$R_{s_1s_2}(\tau) = \sum_{n=i}^{N-|k|-1} s_1(n)s_2(n - \tau)$$  \hspace{1cm} (4.8)

where $i = \tau$, $k = 0$ for $\tau \geq 0$, and $i = 0$, $k = \tau$ for $\tau < 0$. When considering the full-range cross-correlation described in section 3.1, which includes both positive and negative lags, the resulting cross-correlation function is of length $2N + 1$ with the sample index spanning $-N$ to $+N$. The sample index which corresponds to the maximum value of $R_{s_1s_2}$ denoted by $\tau_0$ can be expressed as:

$$\tau_0 = \min \{ n_0 \in \{-N \ldots N \} \text{ s.t. } \forall n \in \{-N \ldots N \}, x(n_0) \geq x(n) \}$$  \hspace{1cm} (4.9)

where

$$x(n) \in \{x_{-N} \ldots x_N\} = R_{s_1s_2}$$  \hspace{1cm} (4.10)

Here $R_{s_1s_2}$ denotes the full-range cross-correlation containing values corresponding to both negative and positive lags. $\tau_0$ relates to the TDOA as follows:

$$t_D = \frac{\tau_0}{f_s}$$  \hspace{1cm} (4.11)

where $f_s$ is the sampling frequency of the analogue to digital converters used. In order to measure the angle of arrival of the signal using (4.3), $\tau_0$ is related to $\delta$ by (4.4) and (4.11) as:

$$\delta = \frac{\tau_0 v}{f_s}$$  \hspace{1cm} (4.12)
The sequence of angular measurements carried out by the relative localisation system using the methodology explained above to obtain the azimuth angle of the sender AUV is explained later in this chapter.

4.2 Distance estimation

Time of flight (TOF) of a signal is usually used to estimate distance between the signal source and the receiver. While sonar and radar systems (Nielsen, 1991; Waite, 2002; Ricker, 2003) estimate distance to targets by measuring the round-trip time of a signal reflected off the target, the method used in this system does not involve a reflected signal. The direct-path range estimation using TOF of acoustic signals employed in this work is related to the spherical positioning schemes used in acoustic beacon based underwater localisation (Liang, 1999; Larsen, 2000; Olson et al., 2004).

To measure the travel time of a signal, knowledge of the exact time at which the signal was emitted is required. When localising relative to acoustic beacons, this is achieved by maintaining explicitly synchronised clocks at both the receiver and transmitter locations. The logical time-step\(^1\) concept used in the relative localisation system achieves synchronisation in an implicit manner compared to maintaining clocks synchronised with absolute time. The long-wave radio communication system on-board each Serafina Mk II AUV which transmits according to a distributed omnicast routing schedule (Schill and Zimmer, 2006b; Schill and Zimmer, 2007) is coupled with the relative localisation system such that each acoustic sending event is initiated simultaneously\(^2\) with the start of a long-wave radio transmission from a sender AUV which also marks the start of a logical time-step. Upon receiving the long-wave radio signal, each of the observer AUVs within communication range increments their respective logical clocks marking the start of their logical time-steps.

Due to the sufficient difference in speed of propagation for electromagnetic and acoustic signals underwater, it can be safely assumed that the long-wave radio signal reaches an observer AUV earlier than the acoustic signal. The consequent start of the logical time-step on the observer triggers the acoustic receivers which begins to await the acoustic signal. If it takes time \(t\) for the acoustic signal to arrive since the start of the logical time-step, the distance \(r\) between the sender and observer can be given by:

\[
r = \frac{tv_v}{(v_v - v)} \quad (4.13)
\]

\(^1\) Logical time increments with each update and not necessarily on a fixed absolute time related to a real-time clock as explained by Lamport (1978)

\(^2\) The time jitter involved in this process is taken into consideration in the uncertainty analysis of the range estimates and later discussed in chapter 5.
4.2 Distance estimation

where \( v_e \) and \( v \) are the speeds of propagation for electromagnetic and acoustic signals underwater. However, with the relatively short distances between AUVs in a local neighbourhood and comparing the magnitudes of the quantities\(^1\) \( v_e \) and \( v \), it can be assumed that the long-wave radio signals are transmitted between the neighbouring vehicles instantaneously, which reduces (4.13) to:

\[
\text{r} = tv
\]

(4.14)

where \( t \) can be expressed as the TOF of the acoustic signal.

A similar in-air distance estimation approach which does not rely on explicitly synchronised clocks on the sender and the receiver is used on the Cricket indoor location system developed by Priyantha (2005) and discussed by Balakrishnan et al. (2003) and Smith et al. (2004). The beacon nodes deployed in the Cricket system simultaneously transmit a radio frequency message packet with a narrow ultrasound pulse. The receiving nodes then use the difference of arrival time of the electromagnetic and acoustic signals to estimate the distance between the beacon and the receiver.

### 4.2.1 Modified matched filter for TOF extraction

A common approach used in echo signal detection is the use of a matched filter. In one of the earliest and most comprehensive contributions, Turin (1960) introduces the concept of matched filter processing as a means for recovering a known waveform from a noisy signal. In its conventional form, a noise-free replica of the original signal is cross-correlated with the received signal channel to locate the return signal and thus extract the TOF. Hermand and Roderick (1993) introduces an improved ‘model based’ matched filter which uses a copy of the original signal which is convolved with the impulse response of the transmission medium as the reference signal instead of a noise-free replica. This approach requires some \textit{a priori} knowledge about the characteristics of the underwater environment in which the AUVs operate in order to construct the impulse response.

The proposed relative localisation system uses an actual received signal channel as the reference for cross-correlation. The system initialises with a pre-recorded reference channel consisting of the MLS signal (which is used by the localisation system as the source signal waveform) which has been transmitted and received underwater via the transducers used in the system. This technique compensates for the frequency distortions introduced by the transducers as explained in chapter 3 as well as the transmission characteristics of the underwater medium. As the operation progresses (when the swarm moves to areas with different underwater channel characteristics),

\(^1\) \( v_e \approx 3.0 \times 10^4 \text{ms}^{-1} \) and \( v \approx 1.5 \times 10^4 \text{ms}^{-1} \)
4.2 Distance estimation

This initial reference signal can be replaced by a newly received signal which encompasses more up to date characteristics of the transmission medium if and when the performance of the range estimation system drops below a pre-set threshold value. This methodology presents a technique which can cope with changing underwater channel characteristics without explicit measurements or *a priori* information about the transmission medium.

The cross-correlation scheme described in the previous section is used for this ‘modified’ match filter as well, where one signal channel is replaced by the reference signal in (4.8). Two cross-correlations are performed to extract the TOF to each of the two receivers for each signal received from the sender AUV. The diagram given in figure 4.3 shows the four TOFs associated with the two pings. The TOFs \( t_{11} \) and \( t_{12} \) are related to the front (bow) ping and the TOFs \( t_{21} \) and \( t_{22} \) are related to the rear (stern) ping. As mentioned in the previous section, the long-wave radio signal emitted from the sender simultaneously with the first acoustic ping is assumed to be received instantaneously by the observer, triggering the start of the sending event. The subsequent TOFs are measured from this starting trigger. When implemented in hardware, there is finite latency and timing jitter associated with detection of the long-wave signal and the synchronised sending of the acoustic pings. If this synchronisation latency of the receiving hardware is denoted by \( t_L \), then the sample-domain latency \( \tau_L \) is given by:

\[
\tau_L = f_L t_L
\]
where $f_s$ is the sampling rate. The variation in latency, which is the timing jitter is denoted by $\Delta t_L$. From (4.15), its sample domain counterpart $\Delta \tau_L$ is given by:

$$\Delta \tau_L = f_s \Delta t_L$$ (4.16)

The quantity $\Delta \tau_L$ will be included in the uncertainty analysis of the range estimates in the following sections.

Without loss of generality, the range estimation will be explained in the next section using only the two TOFs where the front (bow) ping is the source signal. For this purpose, the two sample-domain delays obtained from (4.9) corresponding to the receivers $H_1, H_2$ are denoted by $\tau_{01}^{11}$ and $\tau_{02}^{12}$ with regard to a signal transmitted from $P_1$, (figure 4.4) the two TOFs $t_{11}$ and $t_{12}$ can be calculated as:

$$
\begin{align*}
t_{11} &= \frac{\tau_{01}^{11} - \tau_L}{f_s} \\
t_{12} &= \frac{\tau_{02}^{12} - \tau_L}{f_s}
\end{align*}
$$ (4.17)

where $\tau_L$ is the sample-domain latency given in (4.15) and $f_s$ is the sampling frequency of the analogue to digital converters used.

Figure 4.4: Intersecting two circles centred at $H_1$ and $H_2$ with the line defining angle of arrival of the signal uniquely localises the source position $P_1$. 
4.2.2 Distance estimation using TOF

As explained by Deffenbaugh et al. (1996a), in spherical localisation, measuring the travel time of an acoustic signal emitted by a single synchronised beacon with a known location defines a sphere centred at the beacon on which the receiver must lie. Conversely, if the position of the beacon is unknown, the signal source must lie on a sphere centred at the receiver. In the planar case considered in the relative localisation system, the said sphere reduces to a circle. However, instead of the multiple spheres (or circles in the planar case) needed to uniquely identify a position as done in traditional spherical localisation, using a single circle and its intersection with the line defining the angle of arrival of the signal which was described in the previous sections, uniquely defines the position of the signal source.

In order to achieve better accuracy, two circles centred at the two receivers at \( H_1 \) and \( H_2 \) and the line defining the angle of arrival is used as shown in figure 4.4. The radii of the circles centred at \( H_1 \) and \( H_2 \) are denoted by \( r_{11} = \) \( P_1H_1 \) and \( r_{12} = \) \( P_1H_2 \) are obtained by substituting (4.17) in (4.14). The distance \( r_1(=P_1O_1) \) needed to complete the polar coordinates \( (r_1, \beta) \) of signal source \( P_1 \) can be derived by:

\[
r_1 = \sqrt{\frac{r_{11}^2 + r_{12}^2}{2} - \left(\frac{d}{2}\right)^2}
\]

(4.18)

where \( d (=H_1H_2) \) is the spacing between the receivers.

4.3 Source localisation and uncertainty of estimates

The TDOA of the source signal between the receivers at the point of observation \( O_1 \) (mid point between receivers) defines an angle of arrival \( \beta \), while the TOFs of the source signal to the two receivers combined as explained in the previous section defines a radius \( r_1 \), centred at the point of observation. The intersection of the line along the angle of arrival \( \beta \) through \( O_1 \) and the semi-circle (considering the directivity of the receivers) with this radius \( r_1 \) would uniquely define the position of the source in two dimensions. The equation in (4.3) gives the angle of arrival and (4.18) gives the radius mentioned above.

The previous sections described and explained the methodology used in localising an acoustic source by measuring TDOAs and TOFs in the context of the relative localisation system being presented. During the estimation process, these measurements are done twice per sending event, once for the front ping, then again for the rear ping yielding two TDOAs and four TOFs. These are then used for deriving two angles and two distances - the sub-azimuths and sub-ranges, corresponding to the front and rear of the sender relative to the observing submersible.
Subsequently in section 4.4.3, these quantities are used as the basis for deriving compound estimates based on the geometrical relationship of the transducer positions.

The angle and distance estimates derived earlier have uncertainties associated with them. Since these quantities will later be used to derive compound estimates which constitute the pose vector\(^1\) describing the azimuth, range and heading of the sender vehicle, it is important to identify and analyse the possible sources of errors and the effect of uncertainty associated with the component quantities on the derived estimates. The following subsections classify errors based on the sources and their effect on the estimated quantity. In section 4.3.1, error formulae are derived to show the relationship between the estimated quantities and the uncertainties of the component measurements. While this does not constitute a comprehensive error model for the system, it provides a statistical basis to analyse the behaviour of the system in the presence of random errors and to provide theoretical bounds to the precision of the estimates. As compound localisation estimates are derived from these component estimates later in this chapter, similar error propagation formulae will be presented with respect to the uncertainties associated with these compound estimates.

**Random errors**

The primary measurements in each of the estimates are sample-domain delays, measured by detecting the peak position after cross-correlation of pairs of discretized time-domain signal waveforms. By virtue of this discretization, the position of the peak has an uncertainty of 0.5 samples which manifests itself as a form of quantization error with an assumed uniform distribution. Apart from this, which affects the TDOA measurement, the TOF measurement is also affected by the uncertainty introduced by synchronisation time-jitter mentioned earlier. The time-jitter is assumed to have a Gaussian distribution. Both these sources of uncertainty lead to random errors, effects of which are non-deterministic, and contributes to the lowering of precision of the estimated quantities. A strategy for reducing the impact of these random errors is presented and discussed in section 4.3.2. There also are random measurement errors associated with the constant quantities such as the speed of sound in water \(v\), sampling rate \(f_s\) and the base distance between hydrophones \(d\) on the observer (and the separation between projectors \(l\) on the sender, introduced later in section 4.4.3) as well. Considering typical values for the upper bounds of these errors, section 4.3.1 discusses the contributions of these quantities to the final estimates.

**Systematic errors**

In addition to these random errors, biases and variations associated with the constant quantities such as the speed of sound in water \(v\), sampling rate \(f_s\), the base distance between hydrophones

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\(^1\) The structure of the pose vector is described in section 5.2.1 of chapter 5.
4.3 Source localisation and uncertainty of estimates

$d$ on the observer (and the separation between projectors $l$ on the sender) as well as the synchronisation latency $t_1$, manifest themselves as systematic errors in the estimation process. These errors affect the accuracy of the estimated quantities. While it's extremely difficult to individually identify the contributions of these different errors to the final estimates, most of these systematic errors can be compensated by careful calibration of the system and applying compound corrections.

**Errors due to low SNR**

Apart from the two forms of errors, another class of errors arise due to the deterioration of the signal-to-noise ratio (SNR). Loss of signal could occur due to the source being out of the sensing range (radial or angular) of the utilised transducers. In such situations, in the absence of a distinct waveform, the cross-correlation is between channels containing ambient noise, which in the ideal cases would act as uncorrelated noise. As a result, both forms of the cross-correlation peak detection (For TDOA and TOF measurement) would return uniformly distributed random positions, affecting the accuracy of the consequent estimates. In most real cases, the ambient noise appears weakly correlated on the two received channels yielding a discernible peak near '0' for the TDOA measurement, resulting in an angular estimate in the vicinity of 90° (or 0° after the measuring conventions are applied as described later in section 4.4.1). The TOF measurement however is uninfluenced, by correlated noise in the absence of the signal of interest, due to the matched filter processing and would continue to return random estimates.

Operating in highly cluttered, reverberant environments result in interference of the direct path signals by reflected (multipath) signals. A similar form of interference occurs in the presence of multiple sending events within the (1-hop) neighbourhood of an observer due to colliding sending schedules. In such situations, without explicit identification and handling of the situation, both the TDOA and TOF measurements would respond to the louder signal source which gives a higher peak in the cross-correlogram (not necessarily the first arriving signal) which could lead to inaccurate estimates for the source angle and range. With respect to the intended/ direct-path signal of interest, the other signals (with the same wave form) due to colliding sending events and multipath can be considered noise, which contributes to the lowering of the effective SNR. Though not always\footnote{Due to the directivity pattern of receivers, for some source positions, a reflected signal can appear louder than the direct path signal in the presence of multipath arrivals.}, in most operating conditions, due to short-range propagation loss caused by acoustic spreading (Urick, 1983), the first arriving signal would indeed be louder, yielding a higher peak.

Yet another form of interference could occur due to extraneous acoustic sources present in the environment. Detrimental effects due to this last form of interference is largely avoided by the
choice of MLS signals as the source waveform due to their robustness against most such noise sources as discussed in the previous chapter. However in the presence of intense broadband noise, due to the deterioration of the SNR, the TDOA measurement could yield estimates which corresponds to the angular position of the noise source rather than the signal source. The TOF measurements are usually unaffected by such extraneous noise but could result in random errors caused by loss of cross-correlation peak height due to severe deterioration of SNR. These errors due to deterioration of the SNR usually appear as transient effects when considering mobile platforms operating in a dynamic environment. The time history of previous estimates, the transient nature of these errors and additional information derived from the underlying long-wave radio communication system can be utilised to handle these errors. Such strategies are presented in chapter 5 and further experimentally evaluated in chapter 7.

4.3.1 Propagation of errors
Inspecting (4.3), (4.12) by which angle $\beta$ is derived from and (4.14), (4.17), (4.18) which are used to derive range $r$, the sources of random errors can be identified as the uncertainties associated with the sample-domain delay estimations $\tau_0$, $\tau_0^{11}$, $\tau_0^{12}$ and the sample-domain sender-observer synchronisation latency $\tau_L$. If these uncertainties for the quantities $\tau_0, \tau_0^{11}$ and $\tau_0^{12}$ are denoted by $\Delta \tau_0, \Delta \tau_0^{11}$ and $\Delta \tau_0^{12}$ respectively, using the general error propagation formula\(^1\) gives the uncertainty for $\beta$ as:

$$\Delta \beta = \pm \Delta \tau_0 \frac{\partial \beta}{\partial \tau_0}$$

\(^1\) See Bock and Krischer (1998) for a basic introduction to the error propagation formula or Figliola and Beasley (2005) for a comprehensive treatment of uncertainty analysis and error propagation.
and the uncertainty for $r_1$ as:

$$
\Delta r_1 = \pm \sqrt{(\Delta r_0^1)^2 \left( \frac{\partial r_1}{\partial r_0^1} \right)^2 + (\Delta r_0^2)^2 \left( \frac{\partial r_1}{\partial r_0^2} \right)^2 + (\Delta r_L)^2 \left( \frac{\partial r_1}{\partial r_L} \right)^2} \tag{4.20}
$$

where $\Delta r_L$ is the synchronisation timing jitter mentioned in section 4.2.1. As depicted in figure 4.5 these uncertainties associated with the angle and distance estimates places the source in an error bounded area rather than a point.

Since the sample-domain delay estimates are obtained from the same cross-correlation peak detection process, the associated uncertainties for the TDOA and TOF measurements are assumed to be similar ($\Delta r_0 = \Delta r_0^1 = \Delta r_0^2$) and represented by $\Delta r$. The random measurement errors associated with the quantities $d$, $v$ and $f_s$ are denoted by $\Delta d$, $\Delta v$ and $\Delta f_s$ respectively. Evaluating (4.19) and (4.20) gives:

$$
\Delta \beta = \pm \frac{\sqrt{\left( \frac{v}{f_s d \sin \beta} \right)^2 \left( \frac{\partial x}{\partial \beta} \right)^2 + \left( \frac{1}{\tan^2 \beta} \right)^2 \left( \frac{\partial x}{\partial \beta} \right)^2 + \left( \frac{\partial x}{\partial f_s} \right)^2}}{\sqrt{2(\Delta r_0^2 + \Delta r_L^2) \left( \frac{d}{2} \right)^2 + X}} \tag{4.21}
$$

$$
\Delta r_1 = \pm \frac{v}{2r_1 f_s} \sqrt{2(\Delta r_0^2 + \Delta r_L^2) \left( \frac{d}{2} \right)^2 + X} \tag{4.22}
$$

$$
X = \left( \frac{d \Delta d}{2} \right)^2 + 4\left( \frac{d}{2} \right)^2 \left( \frac{\Delta v}{v} \right)^2 + \left( \frac{\Delta f_s}{f_s} \right)^2 \tag{4.23}
$$

The uncertainty for $d$ arises directly from the instrument error of the measuring device (standard measuring tape), which in this case has a minimum measurement of 0.001 m. Hence the maximum random uncertainty associated with the base distance is $\Delta d = 0.0005$ m. It is assumed $d$ remains constant$^1$ at 0.3 m throughout the estimation process due to the mechanical mounting of the hydrophones. By inspecting the structure of the above equations, the contribution of terms containing $\Delta d^2$ are negligible compared to the other contributing components.

The sampling rate of 96000Hz which is used by the experimental system is provided by an Edirol FA-101 sampling device (Roland, 2004). Once again, considering typical values for $\Delta f_s$ caused by clock drifts, the contribution of terms with this quantity to the uncertainty equations are negligible.

The speed of sound in water $v$, is not a direct measurement. While initial explicit measurements were done during experimental calibration described later in chapter 6, the empirical formula

$^1$, however, any minor variations of this base distance will manifest itself as a systematic error in the final estimates as discussed earlier and is not accounted for by $\Delta d$ mentioned here.
4.3 Source localisation and uncertainty of estimates

Given by Coppens (1981) was used for uncertainty calculations. The formula for speed of sound in water which uses the depth \( D \), temperature \( T \) and salinity \( S \) as input parameters is as follows:

\[
v = 1449.05 + 4.57 T - 0.0521 T^2 + 2.3 \times 10^{-4} T^3 + \\
(1.333 + 0.0126 T + 9.0 \times 10^{-5} T^3)(S - 35) + (0.1623 + 2.53 \times 10^{-4} T) D + \\
(2.13 + 0.1 T - 5 D^2 + (0.016 + 2.0 \times 10^{-7} T)(S - 35))(S - 35) \times 10^{-4} TD
\]

(4.24)

The speed of sound is obtained in meters per second when units for temperature, depth and salinity are degrees Celsius (°C), meters (m) and parts per thousand (ppt) respectively. The standard error for speed of sound calculated using this formula is 0.1 ms\(^{-1}\). While the nine term equation given by Mackenzie (1981) has a lower standard error of 0.07 ms\(^{-1}\), it is only valid for salinities between 25 ppt to 40 ppt. The formula given above is valid from 0 ppt to 42 ppt and this range includes the nominal salinity of 0.5 ppt for freshwater in which the ensuing experiments were carried out. By considering the value of \( \Delta v \) to be 0.1 ms\(^{-1}\) and the calculated\(^1\) speed of sound in water to be 1497 ms\(^{-1}\), the contribution of terms associated with \( \Delta v \) in the uncertainty formulae can therefore be considered negligible as well. Ignoring the terms with \( \Delta d \), \( \Delta f \), and \( \Delta v \) the uncertainty for \( \beta \) can be expressed as follows:

\[
\Delta \beta = \pm \frac{v \Delta \tau}{f_d d \sin \beta}
\]

(4.25)

\(\text{1. The input parameters used were } D = 1.0 \text{ m}, \ T = 25 \text{ °C and } S = 0.5 \text{ ppt.}\)
4.3 Source localisation and uncertainty of estimates

while the uncertainty for \( r_1 \) can be expressed as follows:

\[
\Delta r_1 = \pm \frac{v}{2f_s r_1} \sqrt{2(\Delta \tau^2 + \Delta \tau_L^2)\left(r_1^2 + \left(\frac{d}{2}\right)^2\right)}
\]  

(4.26)

These formulae shows how the uncertainties associated with the basic sample-domain delay measurements and timing jitter propagates to the estimated quantity. They also show the relationship between uncertainties associated with estimates and the estimates themselves and other constant quantities\(^1\). The terms \( \Delta d \), \( \Delta f_s \) and \( \Delta v \) that were ignored due to their negligible contributions to (4.21) and (4.22), represented the random measurement errors and do not account for any bias in these quantities assumed to be constant. Furthermore, it shows both estimate uncertainties reduce in magnitude with a higher sampling rate \( f_s \), the only constant that can be arbitrarily chosen within constraints\(^2\).

The uncertainties given by (4.26) and (4.25) were plotted against distance and angle respectively by substituting typical values for the constant parameters \( d \), \( f_s \), \( v \), setting \( \Delta \tau = 0.5 \) and \( \Delta \tau_L = 9.6 \). This value for sample domain synchronisation timing jitter (equivalent to a

\(^1\) The constant assumptions hold for the speed of sound in water and the sampling rate considering the relatively short distances travelled by the acoustic signals and the short sampling duration of the pings.

\(^2\) Lower limit is governed by the required frequency bandwidth of the source signals and the upper limit governed by the available processing capacity.
4.3 Source localisation and uncertainty of estimates

Synchronisation timing jitter of $\Delta t_L = 0.001$ s) is considered nominal based upon experimentally measured values given later in section 6.2 of chapter 6.

The angular uncertainty is independent of the source distance but as seen from figure 4.6, it changes with angle giving the smallest uncertainty close to $90^\circ$. The errors rapidly increase as the angle approach either $0^\circ$ or $180^\circ$. This corresponds to $P_1$ being aligned with the two receivers at $H_1, H_2$.

By observing the plot in figure 4.7, it can be seen that the distance uncertainty does not depend on the distance when $r_1 > d$. This condition is satisfied in the physical implementation of the localisation system and therefore it can be considered that the uncertainty in distance is invariant with the distance measured within the sensing range of the system.

Figure 4.8 plots the variation of the distance uncertainty against synchronisation timing jitter for $r_1 = 5.0$ m. Beyond extremely small jitter values ($\Delta t_L \geq 5.0 \times 10^{-6}$ s) the relation between these quantities tends to be linear.

4.3.2 Sub-sample interpolation

Both the uncertainties for distance and angle estimates derived in the previous sections depend on the quantity $\Delta r$. This measurement error in the sample-domain remains at 0.5 regardless of the sampling frequency used. A reduction of this value is desirable to minimise its contribution.
4.3 Source localisation and uncertainty of estimates

To the random errors in the estimates which ultimately leads to improving the precision of the estimation system.

A sub-sample interpolation scheme\(^1\) is used in the relative localisation system where the process of finding \(\tau_0\), the value which maximise the cross-correlation function according to (4.9) is done in two steps. First, a simple search finds the maximum value of the sample sequence \(R_{r_1r_2}\) and its corresponding sample index. This index is then used to do a more refined search in the neighbourhood on either side of this point using a cubic spline interpolation routine, which subdivides each sample interval in to \(n_{\text{int}}\) segments. This has the same effect as increasing the sampling frequency by a factor of \(n_{\text{int}}\), where \(n_{\text{int}} = 1\) means no sub-sample interpolation. This term relates to \(\Delta r\) as:

\[
\Delta r = \frac{1}{2n_{\text{int}}}
\]  

(4.27)

Just as for the sampling frequency \(f_s\), increasing the number of sub-sample interpolation segments \(n_{\text{int}}\) improves the performance of the estimation. Plots in figure 4.9 show how the angular uncertainty \(\Delta \beta\) varies with an increasing number of sub-sample interpolation points at multiple values for \(\beta\). Figure 4.10 shows plots of \(\Delta r_1\) varying with \(n_{\text{int}}\) for multiple values of \(r_1\). Nevertheless, this parameter cannot be arbitrarily increased due to the higher processing overheads it adds to the relative localisation system. As a compromise, \(n_{\text{int}} = 10\) is used by the

\(^1\) Sub-sample interpolation to avoid sample-interval round-off when detecting the cross-correlation peak using various interpolation routines is mentioned in work presented by Reeder et al. (2004) and Baker et al. (2005a).
4.3 Source localisation and uncertainty of estimates

relative localisation system to improve the performance of the system. According to the plots shown in figures 4.9 and 4.10, increasing beyond 10 does not yield a substantial improvement in the uncertainty to justify the additional processing cost. By changing the value of from  to , with a sampling frequency \( f_s = 96000 \) kHz, the best and worst average\(^1\) absolute uncertainty for angle estimation is improved from \( 1.49^\circ \) to \( 0.14^\circ \) and \( 4.85^\circ \) to \( 1.07^\circ \). The best values correspond to angles close to \( 90^\circ \) and worst is in the vicinity of \( 0^\circ \) and \( 180^\circ \).

The resolution of an estimate is the smallest measurable change, i.e. the minimum ‘reading’ of the estimation system which can also be defined as twice the measurement error. The absolute angular resolution \( \rho \beta \) can also be calculated as follows:

\[
\rho \beta = \left| \beta - \tan^{-1} \left( \frac{1 + \tan^2 \beta}{\sqrt{1 + \tan^2 \beta/f_s d}} \right) \right| \tag{4.28}
\]

where

\[
\tan^{-1} \left( . \right) = \begin{cases} 
\tan^{-1} \left( . \right), & \beta < 90^\circ \\
180^\circ - \tan^{-1} \left( . \right), & \beta \geq 90^\circ 
\end{cases} \tag{4.29}
\]

\(^1\)The average for minimum (best) errors was calculated using four points around \( 90^\circ \) with \( \tau_0 = 0, 1, 2, 3 \) for \( n_{lat} = 1 \) and with \( \tau_0 = 0, 0.1, 0.2, 0.3 \) for \( n_{lat} = 10 \) using (4.3), (4.12) and (4.25). Due to symmetry around \( 90^\circ \), only positive lags were considered. The average for maximum (worst) errors was calculated using four points around \( 180^\circ \) with \( \tau_0 = 16, 17, 18, 19 \) for \( n_{lat} = 1 \) and with \( \tau_0 = 18.8, 18.9, 19.0, 19.1 \) for \( n_{lat} = 10 \).
This $\tan^{-1}(.)$ function expands the range of the inverse tangent function from $0^\circ \to 90^\circ$ to $-90^\circ \to 90^\circ$.

The angular resolution as calculated by (4.28) can be explained as the difference between two adjacent angular estimates. By selecting $n_{int} = 10$, the average\(^1\) resolving power of the relative localisation improves from 2.98° to 0.29° and from 9.71° to 2.14° for angular measurements (for best and worst conditions mentioned earlier) and from $11.0 \times 10^{-3}$ m to $3.2 \times 10^{-3}$ m for distance measurements\(^2\). The increase in resolution reduces quantization effects which otherwise affects the output of the relative localisation system. This directly contributes to the higher precision of estimates shown by the experimental results presented later in chapter 7.

### 4.4 Relative localisation of an AUV

Previous sections focused on how acoustic source localisation concepts can be utilised to produce an angle and a distance to an acoustic source. The sources of uncertainty of estimates were identified and analysed and a scheme for improving the resolution and precision of the estimates was also presented. In the relative localisation system being presented in this thesis, an acoustic

---

\(1\). Using the three differences between each of the four points selected to calculate the average best and worst errors.

\(2\). Under ideal synchronisation conditions with zero time-jitter.
4.4 Relative localisation of an AUV

Sending event consists of two pings emitted by the sender AUV, first from the bow end and the next from the stern end. Observer AUVs which have the sender within their sensing range would estimate the angle and the distance to each of the two sources upon receiving the acoustic pings, resulting in two sub-azimuth and four sub-range estimates. These are then used to derive the compound localisation estimates of azimuth $\theta$, range $r$ and heading $\alpha$ of the sender relative to each of their body-fixed coordinate frames of the observers. Figure 4.11 illustrates these compound estimates when considering one observer AUV ($R_1$) and a sender AUV ($R_2$). The measuring conventions, the derivation and uncertainty analyses of these quantities are presented in the subsequent sections.

4.4.1 Measuring conventions and ranges

Figure 4.11 illustrates AUV labelled $R_1$ (observer) measuring the azimuth, range and heading of AUV $R_2$ (sender). These relative measurements are based upon a polar coordinate system fixed on $R_1$. The pole $O_1$ is at the center of the AUV coinciding with the center of buoyancy and the polar axis runs across the pole along the center line of the vehicle, pointing towards the bow end.

**Azimuth**

The azimuth $\theta$ is the angle between the line $O_1O_2$ and the polar axis on $R_1$. This quantity, with a range of $-180^\circ < \theta \leq 180^\circ$ is measured as positive clockwise and as negative counter-clockwise from the polar axis. For example, if $R_2$ was directly ahead of $R_1$, the azimuth would be $0^\circ$ and if $R_2$ was in parallel alongside $R_1$ either on port side or starboard side would result in an azimuth of $90^\circ$ or $-90^\circ$ respectively.

**Range**

The range $r$ is a positive scalar quantity which gives the Euclidean distance between the poles $O_1$ and $O_2$, of the coordinate systems fixed on the observer ($R_1$) and sender ($R_2$). Even though the subsequent formula used to derive the range allows this quantity to be zero, in practise it is strictly greater than zero due to the physical size of the AUVs as can be seen from the diagram in figure 4.11.

**Heading**

The heading $\alpha$ is the relative rotation angle between the polar axes of $R_1$ and $R_2$, measured with respect to the observer, $R_1$. As depicted in figure 4.11, it can be seen as translating the coordinate system of $R_1$ on to $R_2$, such that the two poles coincide and then measure the angle between the two polar axes. Similar to the azimuth, the heading has a range of $-180^\circ < \alpha \leq 180^\circ$ and is measured positive and negative for clock-wise and counter-clock-wise rotations respectively. That is,
4.4 Relative localisation of an AUV

for whatever azimuth if the two AUVs are travelling (or pointing) in the same direction (same heading), then the relative heading of $R_2$ measured by $R_1$ would be $0^\circ$.

The relative behaviour of the two AUVs $R_1$ and $R_2$ can be interpreted using the azimuth and heading as follows; if the azimuth and heading of $R_2$ measured by $R_1$ is the same ($\alpha = \theta$), then $R_2$ is pointed away from $R_1$ (heading directly away, if in motion i.e. the range $r$ is increasing) and if the two quantities are separated by $180^\circ$ ($\alpha = -\theta$), then $R_2$ is pointed at $R_1$ (heading directly towards, if in motion i.e. range $r$ is decreasing).

4.4.2 Transducer placement

Chapter 4 described and explained how TDOAs and TOFs of acoustically transmitted MLS signals are used to measure angles and distances to a sound source. The following sections will describe how these measurements are incorporated into the relative localisation system to produce azimuth, range and heading estimates. Figure 4.12 shows the mounting configuration of hydrophones and projectors on a hull of a Serafina Mk II class AUV (Serafina website, 2009) and how it corresponds with the sender and receiver positions $P_1$, $P_2$ and $H_1$, $H_2$ mentioned in chapter 4. The hydrophone spacing $d$ is set to 0.3m and projector spacing $l$ is set to 0.5m, based on the physical size of the AUV hull. The pole of the coordinate frame $O$ lies at the intersection of two lines $H_1H_2$ and $P_1P_2$ and the polar axis is along the center line of the AUV which goes through the projectors $P_1$ and $P_2$ and points towards the bow end of the vehicle. This configuration of hydrophone and projector placement is further illustrated in chapter 6 which explains the experimental setup.
4.4 Relative localisation of an AUV

4.4.3 Geometric description

Figure 4.13 shows a geometrical abstraction of two vehicles, the sender with its two projectors $P_1$ and $P_2$ and the observer with its two hydrophones $H_1$ and $H_2$. $O_1$ and $O_2$ represent the poles of the body fixed polar coordinate frames of the observer and the sender respectively. The azimuth, range and heading of the sender, relative to the observer is given by $\theta$, $r$ and $\alpha$. As explained in the previous chapter, emission of two MLS signal pings constitutes a sending event. These pings are emitted from the projectors $P_1$ and $P_2$, first from the bow end ($P_1$) then the stern end ($P_2$) of the sender AUV. Each of these signals are received by two hydrophones $H_1$ and $H_2$ mounted on port and starboard sides of the observer AUV.

Azimuth

According to the diagram in figure 4.13 and the definition given in section 4.4, the azimuth $\theta$ is the angle between the polar axis going through $O_1$ and the line $O_1O_2$. However, according to the source localisation scheme given in chapter 4, the angles are measured to the actual sound sources, and with regard to the diagram the sources are at $P_1$ and $P_2$. The angle measurement
using TDOA gives $\beta_1$ and $\beta_2$ corresponding to the angles which the lines $P_1O_1$ and $P_2O_2$ makes with $H_1H_2$ using (4.3) as follows:

$$\beta_i = \tan^{-1}\left(\frac{\sqrt{d^2 - \delta_i^2}}{\delta_i}\right), \ i = 1, 2 \quad (4.30)$$

where $d$ is the base distance between the hydrophones $H_1$ and $H_2$. $\delta_1$ and $\delta_2$ are the acoustic path length differences calculated according to (4.12), based on the two TDOAs corresponding to the two MLS signals emitted from $P_1$ and $P_2$. According to the measurement convention given in section 4.4, the two angles returned by (4.30) need to be transformed to $\theta_1$ and $\theta_2$ which are measured against the polar axis instead of the line $H_1H_2$. This transformation is given by:

$$\theta_i = [90^\circ - \beta_i]_{adj}, \ i = 1, 2 \quad (4.31)$$

where the adjustment function $[\ ]_{adj}$ is defined as:

$$[x]_{adj} = \begin{cases} x, & -180^\circ < x \leq 180^\circ \\ \text{sgn}(x)((|x| - 360^\circ), & x \leq -180^\circ, x > 180^\circ \\ \end{cases} \quad (4.32)$$

which return the angles conforming to the measuring convention.

This adjustment is required since (4.3) produces $\beta_i$ with the range $-180^\circ < \beta_i \leq 180^\circ$ when implemented with the $\tan^{-1}(x, y)$ function which considers the quadrant of the complex value $x + iy$ where $y/x = \tan\beta$. However, in the experimental implementation of the relative localisation system, the range of (4.3) was limited to $0^\circ \leq \beta_i \leq 180^\circ$ due to the directivity of the hydrophones used as explained in section 4.1.1.

The two angular measurements $\theta_1$ and $\theta_2$ obtained from (4.31) are combined to produce the azimuth estimate $\theta$ according to the geometry of the diagram in figure 4.13 as follows:

$$\theta = \frac{\theta_1 + \theta_2}{2} \quad (4.33)$$

**Range**

According to figure 4.13 the range $r_i$ which is the Euclidean distance between the poles of the coordinate frames fixed to the sender and observer vehicles, is given as the length of line $O_iO_j$.

However, as with the azimuth measurement, the actual measurements are the distances to the sound sources $P_1$ and $P_2$ from each of the receivers $H_1$ and $H_2$. If $P_1H_1$, $P_1H_2$, $P_2H_1$, and $P_2H_2$ are denoted as $r_{11}$, $r_{12}$, $r_{21}$, and $r_{22}$ respectively, the distances $r_1 (P_1O_1)$ and $r_2 (P_2O_2)$ can be calculated using (4.18) based on the TOF measurements given in chapter 4 as follows:

$$r_i = \sqrt{\frac{r_{11}^2 + r_{12}^2}{2} - \left(\frac{d}{2}\right)^2}, \ i = 1, 2 \quad (4.34)$$
4.4 **Relative localisation of an AUV**

where $d$ is the distance between hydrophones on the observer vehicle. Once $r_1$ and $r_2$ are calculated, the range $r$ can be calculated using the formula:

$$r = \sqrt{\frac{r_1^2 + r_2^2 - \left(\frac{l}{2}\right)^2}{2}}$$  \hspace{1cm} (4.35)

where $l$ is the separation between the projectors on the sender vehicle.

**Heading**

The relative heading $\alpha$ of the sender vehicle as seen by the observer vehicle can be estimated with the aid of the component measurements $\theta_1$, $\theta_2$, $r_1$ and $r_2$ used earlier to derive the azimuth and the range. The range adjusted heading can be expressed as:

$$\alpha = \tan^{-1}\left(\frac{r_1 \sin \theta_1 - r_2 \sin \theta_2}{r_1 \cos \theta_1 - r_2 \cos \theta_2}\right)_{adj}$$  \hspace{1cm} (4.36)

where the $\theta_i$ values are given by (4.31) with the range $-180^\circ < \theta_i < 180^\circ$ and $r_i$ values are given by (4.34). The adjustment function is the same as given in (4.32). While $r_1$, $r_2 \geq 0$, by inspecting figure 4.13 it is also clear that they cannot both be zero at the same time given the constraint $l > d$ which holds true in practise.

As with the azimuth calculation given previously in the current section, the returned $\alpha$ value has a range of $-180^\circ < \alpha < 180^\circ$ when implemented with the $\tan^{-1}(x, y)$ function which considers the quadrant of the complex value $x + iy$ where $y/x = \tan \alpha$.

**4.4.4 Reverse hyperbolic localisation**

The azimuth estimation presented in the previous sections is based on hyperbolic localisation (TDOA measurement) schemes while the range estimation is based on spherical localisation (TOF measurement) schemes. Overall, the source localisation presented can be viewed as a hybrid approach. The heading estimation relies on the azimuth and range as shown by (4.36). However, the range estimation and the subsequent heading estimation requires implicit synchronisation between the sender and observer as with traditional TOF based spherical localisation schemes. This synchronisation provided by the underlying communication scheduling system enables the relative localisation system to measure TOF as described in section 4.2.

With the two sound sources at $P_1$, $P_2$ and the two receivers at $H_1$, $H_2$, a novel reverse hyperbolic scheme was devised to calculate the range and heading which does not require a TOF measurement, consequently eliminating the dependence on sender-observer synchronisation.
4.4 Relative localisation of an AUV

Reverse azimuth estimation

The acoustic path length differences $\delta_1$ ($P_1H_1 - P_1H_2$) and $\delta_2$ ($P_2H_1 - P_2H_2$) mentioned in section were based on the two hyperbolae centred on $H_1$ and $H_2$ corresponding to the two acoustic sources at $P_1$ and $P_2$. By considering two more path length differences denoted by $\eta_1$ and $\eta_2$ which are equivalent to the quantities $P_1H_1 - P_2H_1$ and $P_1H_2 - P_2H_2$, two additional hyperbolae centred on $P_1$ and $P_2$ can be realised. The diagram in figure 4.14 depicts the two hyperbolae centred on $H_1$, $H_2$ in blue and the two hyperbolae centred on $P_1$, $P_2$ in red.

The observer vehicle receives four acoustic signal channels for each sending event. They comprise of two channels received by the hydrophones $H_1$, $H_2$ with $P_1$ as the source, denoted by $x_{11}(t)$, $x_{12}(t)$ and the two channels received with $P_2$ as the source denoted by $x_{21}(t)$, $x_{22}(t)$. The values for $\delta_1$ and $\delta_2$ were obtained from (4.12) after performing cross-correlation as described in section 4.1.2 on the channel pairs $x_{11}(t)$, $x_{12}(t)$ and $x_{21}(t)$, $x_{22}(t)$ respectively. Similarly, cross-correlating channel pairs $x_{11}(t)$, $x_{21}(t)$ and $x_{12}(t)$, $x_{22}(t)$ would give the corresponding sample-domain delays which can be converted to the required path length differences $\eta_1$ and $\eta_2$ using the following modified version of (4.12):

$$\eta_i = \frac{\tau'_i f_s}{\nu}, \ i = 1, 2$$

(4.37)

where $\nu$ is the speed of sound underwater, $f_s$ the sampling frequency of the analogue to digital converter and $\tau'_0, \tau'_0$ the sample-domain delays that maximise the respective cross-correlation.
functions. Just as $\beta_1$ and $\beta_2$ defined the angles $P_1O_1$ and $P_2O_1$ made with $H_1H_2$, two new angles $\varphi_1$ and $\varphi_2$ can be defined as the angles $H_1O_2$ and $H_2O_2$ makes with $P_1P_2$. These two angles can be obtained from a slightly modified version of (4.30) using the path length differences $\eta_1$ and $\eta_2$ as follows:

$$\varphi_i = \text{sgn}^*(\theta_2 - \theta_1) \tan^{-1}\left(\frac{\sqrt{l^2 - \eta_i^2}}{\eta_i}\right), \quad i = 1, 2$$

(4.38)

where $l$ is the separation between the two projectors $P_1$, $P_2$ and the modified sign function is defined as:

$$\text{sgn}^*(y) = \begin{cases} \frac{y}{|y|} & |y| > 0 \\ 1 & y = 0 \end{cases}$$

(4.39)

The angles returned from (4.38) has the range $-180^\circ < \varphi_1, \varphi_2 \leq 180^\circ$ and can be combined to produce the angle $\varphi$ as:

$$\varphi = \frac{\varphi_1 + \varphi_2}{2}$$

(4.40)

This angle $\varphi$ can be identified as the reverse azimuth of the sender vehicle or the azimuth of the observer relative to the sender vehicle. The geometry of these quantities along with the measuring convention is shown in figure 4.15.

**Heading and range estimates based on reverse azimuth**

According to the geometry shown in figure 4.15, an alternate relative heading ($\alpha'$) of the sender with respect to the observer vehicle can be given as:

$$\alpha' = [\varphi - \theta]_{adj}$$

(4.41)

The adjustment function is the same as given in (4.32). The azimuth $\theta$ is as given by (4.33).

A new calculation for range based on the reverse azimuth $\varphi$ and $\alpha'$ derived above yields $r'$ as follows:

$$r' = \left| \frac{d \cos(\varphi_1 + \alpha')}{2 \sin(\theta - (\varphi_1 + \alpha'))} \right|$$

(4.42)
where $d$ is the base distance between the hydrophones $H_1$, $H_2$. The other variables have the ranges $-180^\circ < \varphi_1, \alpha', \theta \leq 180^\circ$. As explained earlier in section 4.4.1, even though (4.42) allows for $r'$ to be zero, in practise it is always strictly greater than zero.

As these alternate calculations are based on two hyperbolas, each with foci at $P_1$ and $P_2$, considering their asymptotes which pass through $H_1$ and $H_2$, substituting these points in the polar coordinate equations for $O_2H_1$ and $O_2H_2$ and adjusting for the measuring conventions, the separate range components can be calculated using the following formula:

$$
 r'_i = \left| r' \cos(\theta_1 - \theta) \pm \sgn^*(\alpha') \sgn^*(\theta) \sqrt{r'^2 \cos(\theta_1 - \theta)^2 - \left(r^2 - \left(\frac{l}{2}\right)^2\right)} \right|, 
 i = 1, 2 \quad (4.43)
$$

where the '+' sign yield $r'_1$ and the '−' sign yield $r'_2$. The variables have the following ranges; $r', l > 0, -180^\circ < \theta_1, \alpha', \theta \leq 180^\circ$ and the modified sign function is as defined in (4.39). The sign functions are used in this formula to resolve the ambiguity introduced by the existence of two asymptotes per hyperbola mentioned earlier.

Figure 4.15: The geometric description of the new angles $\varphi_1$, $\varphi_2$, $\varphi$ and their relationship to azimuth $\theta$ and alternate values for range $r'$ and heading $\alpha$. 
Properties of alternate heading and range estimates

The alternate heading and range estimates obtained above do not require sender-observer synchronisation or any knowledge of the sending time of the acoustic signal. This is due to the calculations being purely based on TDOA measurements. This form of independence from the underlying scheduling system provides an additional level of redundancy and increases the robustness of the relative localisation system against synchronisation failures and timing drifts. This also facilitates a more fault tolerant outlier rejection scheme with two independent estimates for the range and heading.

However, in its current form, the independence from sender-observer synchronisation comes at a cost. The errors associated with the alternate range are greater than those of the direct estimation. On the other hand, the errors associated with the alternate heading are lower than its direct counterpart and these errors are not dependant on the range as with the case of the direct heading estimation. The errors and associated resolution of the direct and alternate estimates are discussed in the following sections.

4.4.5 Estimation errors and resolution

The previous chapter presented the uncertainties associated with the angle and distance estimates in (4.26) and (4.25). These angle and distance estimates were further developed earlier in this chapter to form the components of the pose vector; azimuth, range, heading and the alternate versions of range and heading.
Azimuth

By applying the transformation given in (4.31), the angular uncertainties given by (4.25) in the previous chapter can be expressed as:

\[ \Delta \theta_i = \pm \frac{v \Delta \tau}{f_i d \cos \theta_i}, \quad i = 1, 2 \]  

(4.44)

By applying the general error propagation formula to (4.33) and substituting for the sub-azimuths errors given above, the error for the main azimuth estimate is given as:

\[ \Delta \theta = \pm \frac{v \Delta \tau}{2 f \Delta \theta} \left( \frac{1}{\cos^2 \theta_1} + \frac{1}{\cos^2 \theta_2} \right) \]  

(4.45)

Typical values for \( v, f \) and \( d \) are substituted while using \( \Delta \tau = 0.05 \), which corresponds to a sub-sample segmentation of \( n_{\text{seg}} = 10 \) as explained in section 4.3.2, to plot the absolute variation of \( \Delta \theta \) as \( \theta_1 \) and \( \theta_2 \) spans the range \(-90^\circ \rightarrow 90^\circ\). This plot is shown in figure 4.16. The error associated with azimuth is independent of the distance between the sender and the observer. The minimum errors are achieved around \( 0^\circ \) for both the sub-azimuths and the cross-sections of this plot retains the shape described by the initial plot given in figure 4.6 with an offset of \( -90^\circ \) along the x-axis due to (4.31). The resolution of the azimuth estimation follows a similar shape as the surface of the plot in figure 4.16 multiplied by a scaling factor of 2.0. This suggests that the estimation can resolve changes in azimuth of less than \( 1.0^\circ \) for most of

![Figure 4.17: Variation of the absolute range error \( |\Delta r| \) as the range \( r \) increases.](image-url)
the $\theta_1$ and $\theta_2$ combinations, but performance trails off as either of the sub-azimuths approach ±90°.

**Range**

The uncertainty for distance given by (4.26) in the previous chapter can be extended to both sub-ranges $r_1$ and $r_2$ as:

$$
\Delta r_i = \pm \frac{\nu}{2fr_i} \sqrt{2(\Delta r^2 + \Delta r_{L}^2)(r_i^2 + (\frac{d}{2})^2)}, \; i = 1, 2
$$

(4.46)

By applying the general error propagation formula to (4.35) and substituting $\Delta r_1$ and $\Delta r_2$ from (4.46) the estimation error for range can be derived by:

$$
\Delta r = \pm \frac{\nu}{4fr} \sqrt{(\Delta r^2 + \Delta r_{L}^2)(4r^2 + l^2 + d^2)}
$$

(4.47)

Despite the fact that $r$ is used in the above equation, as the plot in figure 4.17 suggests, the absolute error in range $\Delta r$ tends to be constant and independent of $r$ for ranges satisfying $r > l$. As with the error in azimuth, this plot is obtained by substituting typical values to the parameters $\nu, f, d$ and $l$ in (4.47), using $\Delta r_{L} = 9.6$ and $\Delta r = 0.05$ corresponding to $n_{lat} = 10$. The initial spike shown in the plot is for ranges where $r < d$ which does not occur due to physical constraints of the system.

Accordingly the range estimates produced by the relative localisation system would have an upper error bound of ±0.3m as suggested by the constant estimation error depicted in the plot. However, the experimental results presented in chapter 7 shows that the nominal errors are much

![Figure 4.18: Variation of absolute error in heading $|\Delta \alpha|$; a) as sub-azimuths are varied while $r = 2.0m$, b) as azimuth and range are varied.](image-url)
4.4 Relative localisation of an AUV

lower than this bound and the resolving power approaches $3.2 \times 10^{-3} \text{ m}$, which is the value predicted in section 4.3.2 for zero-jitter conditions.

Heading

Unlike the azimuth and range which are independently estimated, the heading is a compound estimation based on both the sub-azimuths and sub-ranges as shown in (4.36). As a consequence, applying the general error propagation formula results in an error for the heading estimate which is related to all four quantities; $\theta_1$, $\theta_2$, $r_1$ and $r_2$ as:

$$
\Delta \alpha = \frac{\pm v \Delta \tau}{f l^2} \frac{\sin^2(\theta_1 - \theta_2)}{\cos^2 \theta_1 + \cos^2 \theta_2} + \frac{1}{d^2} \left[ \frac{\cos \phi_1 + \cos \phi_2}{\sin^2 \phi_1 + \sin^2 \phi_2} \right]
$$

The plots in figures 4.18 and 4.19 show the variation of $|\Delta \alpha|$ as two quantities are changed while the others are kept constant. By observing plots shown in figure 4.18.b and figure 4.19.b, it is apparent that the error in heading and subsequently the resolution of the heading estimate is very sensitive to the range, i.e. the distance between the sender and the observer. This predicts that the performance of the heading estimation would deteriorate with increasing range and would be unable to resolve heading changes of $\pm 45^\circ$ for ranges beyond 5.0 m.

Alternate heading

The estimation error associated with the alternate calculation for heading given by (4.41) is

$$
\Delta \alpha' = \frac{\pm v \Delta \tau}{2f l^2} \left( \frac{1}{\cos^2 \theta_1} + \frac{1}{\cos^2 \theta_2} \right) + \frac{1}{d^2} \left( \frac{1}{\sin^2 \phi_1} + \frac{1}{\sin^2 \phi_2} \right)
$$
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where $\varphi_1$ and $\varphi_2$ are the reverse sub-azimuths introduced in section . Figure 4.20 shows how $|\Delta \alpha'|$ varies with the pose vector components, $\theta$, $r$ and $\alpha$, considering two at a time. It can be seen from plots b) and c) in figure 4.20 that the errors increase whenever $|\theta + \alpha| \approx 0^\circ, 180^\circ$. By inspecting the formula in (4.41), this condition corresponds to the reverse azimuth $\varphi$ approaching $0^\circ$ or $180^\circ$. Since the reverse azimuth errors behave similar to the azimuth errors explained earlier section 4.3(before applying the adjustments given in (4.31)), the magnitude of the errors rapidly increases as $\varphi \rightarrow 0^\circ, 180^\circ$. However, these errors do not display the deterioration with increasing range as displayed by the direct heading estimation errors explained in the previous section. In fact, apart from the areas affected by the aforementioned condition related to the reverse azimuth, the alternate heading estimation errors remain invariant with increasing range. Furthermore, precisely knowing the behaviour of the error when $|\theta + \alpha| \approx 0^\circ, 180^\circ$ and being able to separately estimate $\theta$ and $\alpha$ (although with lower precision) allows the relative localisation system to gracefully handle this situation.

Figure 4.20: Variation of $|\Delta \alpha'|$ a) as $r$ and $\theta$ are varied with $\alpha = 90^\circ$, b) as $\alpha$ and $r$ are varied with $\theta = 0^\circ$ and c) as $\alpha$ and $\theta$ varied with $r = 2.0m$. 
4.4 Relative localisation of an AUV

Estimation error associated with the alternate range given by (4.42) can be stated as follows:

\[
\Delta r' = \pm \frac{v r' \Delta \tau}{2 f d} \sqrt{\frac{X}{Y}} \left( \frac{d + 2 r' \sin \left( (\theta_1 + \theta_2)/2 \right)}{\sin(2(\varphi_1 + \alpha'))} \right)^2 + \frac{Y(4 r'^2 - d^2 \cos^2 (\varphi_1 + \alpha'))}{4 d^2 \cos^2 (\varphi_1 + \alpha')} \tag{4.50}
\]

where \( X \) and \( Y \) are:

\[
X = \frac{5}{\sin^2 \varphi_1} + \frac{1}{\sin^2 \varphi_2}, \quad Y = \frac{1}{\cos^2 \theta_1} + \frac{1}{\cos^2 \theta_2} \tag{4.51}
\]

As with the alternate heading, the absolute errors associated with the estimation of the alternate range are plotted in figure 4.21 as they vary with \( \theta \), \( r \) and \( \alpha \), considering two at a time while keeping the third quantity constant. Since the alternate range is coupled with the alternate heading, errors associated with these two quantities show similarity in behaviour. Therefore the condition \( |\theta + \alpha| \approx 90^\circ \) causes the errors to increase. However, due to the structure of (4.50), the
4.5 Discussion

This chapter described the methodology and techniques used in the measurement and estimation process which leads to the derivation of the compound estimates for azimuth, range and heading. In addition, an alternative scheme to estimate heading and short ranges without the need for sender-observer synchronisation was also presented. These quantities are used in assembling the pose vector which is described in chapter 5. Formulae of relationships between the component quantities and the uncertainty of the compound quantities were also derived. They give a statistical basis to analyse the behaviour of the system in the presence of random errors and provides theoretical bounds to the precision of the estimates. By inspecting these and the associated plots, it was shown that the uncertainties of azimuth, range and alternative heading estimates are independent of distance between the sender and the observers. However, as the sender position approaches the angular and radial sensing limits of the hydrophones on the observers, the resulting loss of SNR leads to deterioration of the estimation accuracy. This effect will be discussed along with strategies to handle such situations in chapter 5 with experimental evaluation presented in chapter 7. The uncertainty analysis reveals that the precision of compound estimates for direct heading and alternative range are severely affected as the sender-observer distance increases. The behaviour of these estimates are experimentally evaluated in chapter 7.

The following chapter explains the way in which the techniques and methodologies presented throughout this as well as the previous chapter are combined in developing the distributed relative localisation system. Furthermore, issues such as interference handling, outlier handling and computational complexity will also be addressed.