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On the Yang–Baxter equation for the six-vertex model

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Abstract

In this paper we review the theory of the Yang–Baxter equation related to the 6-vertex model and its higher spin generalizations. We employ a 3D approach to the problem. Starting with the 3D R-matrix, we consider a two-layer projection of the corresponding 3D lattice model. As a result, we obtain a new expression for the higher spin R-matrix associated with the affine quantum algebra $U_q(\widehat{sl(2)})$. In the simplest case of the spin $s = 1/2$ this R-matrix naturally reduces to the R-matrix of the 6-vertex model. Taking a special limit in our construction we also obtain new formulas for the Q-operators acting in the representation space of arbitrary (half-)integer spin. Remarkably, this construction can be naturally extended to any complex values of spin s. We also give all functional equations satisfied by the transfer-matrices and Q-operators.

1. Introduction

In this paper we analyze the properties of the six-vertex model in an external field and its higher spin generalizations based on a new 3D approach developed in [1–3]. This approach allows us to reveal new algebraic and analytic properties of the six-vertex model with arbitrary spin.

The theory of the six-vertex model goes back to the works of Lieb [4,5] who solved the famous two-dimensional ice model. These results were further extended by Sutherland [6] to

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the zero field six-vertex model and then generalized to the case of an arbitrary electric field by Yang, Sutherland et al. [7,8]. The main technique used was the Bethe ansatz [9]. However, in [10–13] Baxter introduced new analytic and algebraic methods which allowed him to solve the eight-vertex model in a zero field. The main ingredient of Baxter's approach is the theory of functional equations based on the concept of the Q-operator. This Q-operator satisfies the so-called TQ-relation which allows in principle to calculate eigenvalues of the transfer-matrix of the model. Starting with the work [14], the analytic Bethe ansatz [15] was developed where the TQ-relation (or an analogous equation) is used as a formal substitution to solve the transfer matrix functional equations.

The theory of functional relations allows us to determine eigenvalues of the higher transfer matrices associated with the so-called fusion procedure. This algebraic procedure provides a derivation of the functional relations for the higher transfer matrices based on decomposition properties of products of representations of the affine quantum groups. The notion of "higher" spin (or "fused") R-matrices was developed in [16] from the point of view of representation theory. These R-matrices for the six-vertex model acting in the tensor product of two highest weight modules were calculated in [17]. However, the formulas derived in [17] involve special projection operators and are not very convenient for practical calculations.

An alternative method for calculating the higher spin R-matrices was developed by Jimbo [18] (see also [19] for all simple Lie algebras). It is based on the spectral decomposition of the R-matrix and allows one to calculate the R-matrix in terms of spectral functions and quantum Clebsch–Gordan coefficients. For example, in the $U_q(sl(2))$ case, it results in a triple sum formula for the matrix elements of the "fused" R-matrices.

The main result of this paper is a new representation of the $U_q(sl(2))$ R-matrix $R_{I,J}(\lambda)$ in the tensor product of two highest weight representations with arbitrary weights I and J. It contains only one single summation and is expressed in terms of the basic hypergeometric series. The explicit formula reads

$$\begin{aligned}&\big[R_{I,J}(\lambda;\phi)\big]_{i,j}^{i',j'}\\&\quad=\delta_{i+j,i'+j'}\rho_{I,J}(\lambda)\phi^{2i}a_{ij}^{i'j'}(\lambda)\,{}_4\overline{\phi}_3\left(\begin{matrix}q^{-2i};q^{-2i'},\lambda^{-2}q^{J-I},\lambda^2q^{2+J-I}\\q^{-2I},q^{2(1+j-i')},q^{2(1+J-i-j)}\end{matrix}\,\middle|\,q^2,q^2\right)\end{aligned}\tag{1.1}$$

with

$$a_{ij}^{i'j'}(\lambda)=(-1)^i\frac{q^{i(i+j-2J-1)-jI+i'(I+j')}}{\lambda^{i+i'}(q^2;q^2)_i}\,\frac{(q^{-2J};q^2)_j(\lambda^{-2}q^{I-J};q^2)_{j-i'}}{(q^{-2J};q^2)_{j'}(\lambda^{-2}q^{-I-J};q^2)_{i+j}},\tag{1.2}$$

where $\rho_{I,J}(\lambda)$ is the normalization factor and we defined a *regularized* terminating basic hypergeometric series ${}_{r+1}\overline{\phi}_r$ as

$${}_{r+1}\overline{\phi}_r\big(q^{-n};\{a\}_r;\{b\}_r\big|q,z\big)=\sum_{k=0}^{n}z^k\frac{(q^{-n};q)_k}{(q;q)_k}\prod_{s=1}^{r}(a_s;q)_k\big(b_sq^k;q\big)_{n-k}.\tag{1.3}$$

It is easy to see that the hypergeometric series ${}_4\overline{\phi}_3$ entering (1.1) can be expressed in terms of the q-Racah polynomials [20]. The fact that q-Racah polynomials satisfy the Yang–Baxter equation with a spectral parameter is quite remarkable and should have some profound origins.

Another important property of the R-matrix (1.1) is that all its nonzero matrix elements can be made positive under the proper choice of the spectral parameter λ and the normalization factor $\rho_{I,J}(\lambda)$. This is explained in Section 4.

We notice that a similar formula with only one summation exists for the XXX spin chain in a holomorphic basis (see formula (2.17) in [21]). It would be interesting to understand its connection with (1.1).

As an application of the formula (1.1) we construct the Q-operators related to the $U_q(\widehat{sl(2)})$ algebra as special transfer matrices acting in the tensor product of arbitrary highest weight representations. The idea of the construction of the Q-operator in terms of some special transfer matrices belongs to Baxter [10]. It is a key element of his original solution of the 8-vertex model. For the simplest case of the six-vertex model the quantum space is built from 2-dimensional highest weight representations of the $U_q(sl(2))$ algebra at every site of the lattice.

The next step in a better understanding of the structure of the Q-operators related to the six-vertex model was achieved by Bazhanov and Stroganov [22]. They considered fundamental L-operators [23] intertwined by the R-matrix of the six-vertex model at the roots of unity $q^N = 1$. In this case, the highest weight representation of the $U_q(sl(2))$ algebra is replaced with a cyclic representation. Then all matrix elements of the Q-operator can be explicitly calculated as simple products involving only a two-spin interaction. Remarkably, these Q-operators coincide with the transfer matrix of the chiral Potts model [24–26].

A seemingly different method was developed by Pasquier and Gaudin [27] where they constructed the Q-operator for the Toda lattice in the form of an integral operator. Their Q-operator has a factorized kernel and its quasi-classical asymptotics gives a generating function for Backlund transformations in the corresponding classical system. It appears that this construction is naturally connected to a separation of variables (SoV) in quantum and classical integrable systems [28]. Later on this approach has been successfully applied to many other quantum lattice integrable systems and the general scheme of quantum SoV has been developed [29–32]. The integral Q-operator for the case of the XXX chain was first calculated in [33]. It is worth noting that taking the limit $N \to \infty$ [34] in the Bazhanov and Stroganov construction [22] one can recover the results of [27] and [33].

The main difference of the above approach from the original Baxter method is that the "quantum" representation space is infinite-dimensional. It has the structure of a tensor product of Verma modules with the basis chosen as multi-variable polynomials $p(x_1, \dots, x_M)$, where M is the size of the system. The Q-operators appear as integral operators with an explicit action on such a polynomial basis. A detailed construction can be found in [35,36] for the XXX case and its generalization to the XXZ case in [37,38]. The non-compact case and applications of the Q-operators to Liouville theory are discussed in [39–41]. It is worth mentioning that the representation of the Q-operator by an integral operator is known only for the XXX case [33]. The proper deformation of such integral operator for the case of the six-vertex model is still a challenging problem.

Another problem arises when spins take (half-) integer values. In this case the quantum space becomes reducible and the action of the Q-operator on the polynomial basis becomes singular. This difficulty can be overcome by expanding near the limit $2s \to \mathbb{Z}_+$ as shown in [35,36]. However, a removal of such a regularization is technically challenging and it is desirable to have an alternative approach which is free from this difficulty.

In 1997 Bazhanov, Lukyanov and Zamolodchikov (BLZ) suggested another method to derive the Q-operators related to the affine algebra $U_q(\widehat{sl(2)})$ [42,43]. Based on the universal R-matrix theory [44] they showed that the Q-operators can be constructed as special monodromy operators with the auxiliary space being an infinite-dimensional representation of the q-oscillator algebra. Although their original approach was developed in the context of quantum field theory, the results of [42,43] can be easily adjusted to the spin $s = 1/2$ XXZ chain [45]. However, the derivation of the local Q-operators from the universal R-matrix [46] quickly becomes unbearable for higher spins and has been completed only for the $s = 1/2$ case [45,47–49]. In principle, one can use the fusion procedure to derive the Q-operators in any highest weight representation with $2s \in \mathbb{Z}_+$, but this is also technically challenging.

The original motivation of this work was to understand a connection between the integral Q-operators with factorized kernels which appear in the case of infinite-dimensional representations (or the cyclic case $q^N = 1$) and the BLZ construction. As the first step, we need to calculate the local Q-operators acting in the tensor product of the q-oscillator algebra and the highest weight representation with the arbitrary weight $2s$.

It is known [42,43,50–52] that one can construct XXZ (or XXX) Q-operators by taking the infinite spin limit in the auxiliary space of the higher spin transfer-matrix. Our new formula (1.1) suits this purpose perfectly. We also notice here that a 3D approach we employ in this paper is useful for constructing universal R-matrices for higher rank algebras [53].

Taking the limit $I \to \infty$ in (1.1) we derive a generalization of the BLZ Q-operators acting in the tensor product of the highest weight Verma modules with the arbitrary weight $J \in \mathbb{C}$. The limit $J \to \mathbb{Z}_+$ is non-singular and gives the Q-operator in any finite-dimensional representation. The case $J = 1$ reduces to the previously known BLZ Q-operators.

The paper is organized as follows. In Section 2 we recall some basic facts about the $U_q(sl(2))$ and the q-oscillator algebras. We also give a definition of the terminating basic hypergeometric series and their regularized version, which we use in the paper. In Section 3 we define the 3D R-matrix and discuss its basic properties following [3]. In Section 4 we consider a two-layer projection and derive a formula for the matrix elements of the $U_q(sl(2))$ R-matrix $R_{I,J}(\lambda)$ acting in the tensor product of two highest weight representations with integer weights I and J. In Section 4 we discuss the properties of this R-matrix and show that for the case $I = 1$ it reduces to the standard $U_q(sl(2))$ L-operator acting in the $(J + 1)$-dimensional representation space. Then we transform the formula for the R-matrix from Section 2 to a remarkably simple formula (1.1) which contains only one summation and can be rewritten as a terminating balanced ${}_4\phi_3$ series. Using this representation we prove two important symmetry relations for the R-matrix $R_{I,J}(\lambda)$. We show that this construction can be generalized to the case of infinite-dimensional highest weight representations with $I, J \in \mathbb{C}$. In Section 5 we introduce two Q-operators $\mathbf{Q}_\pm(\lambda)$ acting in the tensor product of the highest weight modules with the weight I. Based on the factorization property for the transfer-matrix [42,43] we derive explicit expressions for the matrix elements of the local Q-operators for any values of I (including the infinite-dimensional case $I \in \mathbb{C}$). We also derive the standard TQ-relation and calculate the Wronskian of its two solutions $\mathbf{Q}_\pm(\lambda)$. In Section 6 we list a standard set of functional relations satisfied by higher-spin transfer matrices and Q-operators. Finally, in the Conclusion we summarize all results and outline further directions of research.

2. Conventions

First, let us recall some simple facts about the $U_q(sl(2))$ algebra. It is generated by three elements E, F and H with defining relations

$$q^H E q^{-H} = q^2 E, \qquad q^H F q^{-H} = q^{-2} F, \qquad [E, F] = \frac{[q^H]}{[q]} \tag{2.1}$$

and the following Casimir element

$$C = [q]^2 FE + \{q^{H+1}\} = [q]^2 EF + \{q^{H-1}\}, \tag{2.2}$$

where we used the following notations

$$[x] = x - x^{-1}, \qquad \{x\} = x + x^{-1}. \tag{2.3}$$

The Casimir element is normally parameterized by a complex number $J \in \mathbb{C}$

$$C = \{q^{J+1}\}. \tag{2.4}$$

For any $J \in \mathbb{C}$ one can introduce an infinite-dimensional Verma module V_J^+ with a basis v_j, $j \in \mathbb{Z}_+$. We define the infinite-dimensional representation π_J^+ of $U_q(sl(2))$ by the following action on the module V_J^+

$$Hv_j = (J - 2j)v_j, \qquad Ev_j = \frac{[q^j]}{[q]}v_{j-1}, \qquad Fv_j = \frac{[q^{J-j}]}{[q]}v_{j+1}. \tag{2.5}$$

When $J \in \mathbb{Z}_+$, the representation π_J^+ becomes reducible. The vectors v_j, $j > J$ span an irreducible submodule of V_J^+ isomorphic to V_{-J-2}^+ and one can introduce a finite-dimensional module V_J with the basis $\{v_0, \dots, v_J\}$ isomorphic to the quotient module V_J^+/V_{-J-2}^+. We denote the corresponding finite-dimensional representation as π_J.

Now let us consider the q-oscillator algebra

$$\mathsf{Osc}_q: \quad q^{\boldsymbol{N}}\boldsymbol{a}^{\pm} = q^{\pm 1}\boldsymbol{a}^{\pm}q^{\boldsymbol{N}} \qquad q\boldsymbol{a}^+\boldsymbol{a}^- - q^{-1}\boldsymbol{a}^-\boldsymbol{a}^+ = q - q^{-1}, \tag{2.6}$$

generated by three elements $\boldsymbol{N}$, $\boldsymbol{a}^+$ and $\boldsymbol{a}^-$ and impose an additional relation

$$q^{2\boldsymbol{N}} = (1 - \boldsymbol{a}^+\boldsymbol{a}^-) \equiv q^{-2}(1 - \boldsymbol{a}^-\boldsymbol{a}^+). \tag{2.7}$$

To make a link with the 3D R-matrix from the next section we shall introduce an infinite-dimensional Fock space $\mathcal{F}_q$, spanned by a set of vectors $|n\rangle$, $n = 0, 1, 2, \dots, \infty$, with the natural scalar product

$$\langle m|n\rangle = \delta_{m,n}, \qquad \boldsymbol{N}|n\rangle = n|n\rangle, \qquad \langle n|\boldsymbol{N} = \langle n|n. \tag{2.8}$$

The algebra (2.6) has two irreducible highest weight representations on the space $\mathcal{F}_q$ which we denoted as $\mathcal{F}_q^{\pm}$ in [3].

In this paper we shall only use one representation $\mathcal{F}_q^+$ with a slightly modified action comparing to [3]

$$\begin{aligned} &\boldsymbol{a}^-|0\rangle = 0, \qquad \boldsymbol{a}^+|n\rangle = |n+1\rangle, \qquad \boldsymbol{a}^-|n\rangle = (1 - q^{2n})|n-1\rangle, \\ &\langle 0|\boldsymbol{a}^+ = 0, \qquad \langle n|\boldsymbol{a}^+ = \langle n-1|, \qquad \langle n|\boldsymbol{a}^- = \langle n+1|(1 - q^{2+2n}), \end{aligned} \tag{2.9}$$

with $n = 0, 1, 2 \dots$.

Now we need to define the trace operation over the representations of π_j^+ and $\mathcal{F}_q$. Consider an operator $\mathbf{T}$ acting in the tensor product $V_J^+ \otimes W$, where W is some “quantum” representation space of the $U_q(sl(2))$ algebra. Then we define the trace $\mathbf{T}_J$ over V_J^+ of the operator $\mathbf{T}$ simply summing over all j

$$\mathbf{T}_J = \operatorname*{Tr}_{V_J^+}(\mathbf{T}). \tag{2.10}$$

The trace over the finite-dimensional representation π_J is defined in a similar way.

Now let us consider an operator $\mathbf{A}(\phi)$ acting in the tensor product $\mathcal{F}_q \otimes W$, where $\phi \in \mathbb{C}$ is a "horizontal" field. We define a normalized trace of $\mathbf{A}(\phi)$ on the space $\mathcal{F}_q$ by

$$\widehat{\mathop{\mathrm{Tr}}_{\mathcal{F}_q}}\big(\mathbf{A}(\phi)\big) = \frac{\mathrm{Tr}_{\mathcal{F}_q}(\mathbf{A}(\phi))}{\mathrm{Tr}_{\mathcal{F}_q}(\phi^{2\mathbf{N}} q^{-\mathbf{N}\otimes H})}, \tag{2.11}$$

where H is the generator of the $U_q(sl(2))$ algebra acting in the quantum space W. We always assume that the field variable ϕ is chosen in such a way that corresponding geometric series converge and then analytically continue to any values of ϕ. We also notice a relation between the field ϕ and the additive field h

$$\phi = q^h. \tag{2.12}$$

In this paper we prefer to use the exponential field ϕ.

In the last part of this section we remind a definition of the basic hypergeometric series [54] which we use in the next sections. We start with a q-Pochhammer symbol

$$(x;q)_n = \prod_{k=0}^{n-1}\big(1 - xq^k\big), \quad n \geqslant 0 \tag{2.13}$$

and

$$(x;q)_n = \frac{1}{(xq^n;q)_{-n}} = \frac{q^{n(n+1)/2}(-x/q)^n}{(q/x;q)_{-n}}, \quad n < 0. \tag{2.14}$$

In this paper we consider only terminating basic hypergeometric series ${}_{r+1}\phi_r$ which is defined by

$$\begin{aligned}{}_{r+1}\phi_r\big(q^{-n}, \{a\}_r; \{b\}_r \big| q, z\big) &\equiv {}_{r+1}\phi_r\left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} \middle| q, z\right) \\ &= \sum_{k=0}^{n} z^k \frac{(q^{-n};q)_k}{(q;q)_k} \prod_{s=1}^{r} \frac{(a_s;q)_k}{(b_s;q)_k}. \end{aligned} \tag{2.15}$$

The formula (2.15) is well defined for all $a_i, b_i \in \mathbb{C}$ except the case when some of the parameters b_i are equal to non-positive integer powers of q, i.e. $b_i = q^{-n}$, $n \in \mathbb{Z}_+$ for some i. To overcome this restriction we shall also introduce a *regularized* version of terminating basic hypergeometric series. Unlike usual hypergeometric functions ${}_{r+1}F_r$ there is no a commonly accepted definition for regularized basic hypergeometric series. So we find it convenient to define a regularized terminating basic hypergeometric series ${}_{r+1}\overline{\phi}_r$ as

$$\begin{aligned}{}_{r+1}\overline{\phi}_r\big(q^{-n}; \{a\}_r; \{b\}_r \big| q, z\big) &\equiv {}_{r+1}\overline{\phi}_r\left(\begin{matrix} q^{-n}; a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} \middle| q, z\right) \\ &= {}_{r+1}\phi_r\big(q^{-n}, \{a\}_r; \{b\}_r \big| q, z\big) \times \prod_{s=1}^{r}(b_s;q)_n \\ &= \sum_{k=0}^{n} z^k \frac{(q^{-n};q)_k}{(q;q)_k} \prod_{s=1}^{r}(a_s;q)_k \big(b_s q^k;q\big)_{n-k}. \end{aligned} \tag{2.16}$$

The formula (2.16) is obviously well defined for any $a_i, b_i \in \mathbb{C}$.

We also notice that the symmetry between q^{-n} and $a_1, \dots, a_r$ is broken and this is why we used an extra semicolon after the first argument of ${}_{r+1}\overline{\phi}_r$ in (2.16).

3. The 3D *R*-matrix

In [3] we defined the 3D R-matrix as the operator $\boldsymbol{R}$ acting in the tensor product of three Fock spaces $\mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q$. If we define states in $\mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q$ as $|n_1, n_2, n_3\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle$, then the operator $\boldsymbol{R}$ is completely determined by its matrix elements

$$R_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = \langle n_1, n_2, n_3| \boldsymbol{R} \big| n_1', n_2', n_3'\big\rangle, \quad n_i, n_i' = 0, 1, 2, \dots, \infty, \ i = 1, 2, 3, \tag{3.1}$$

where

$$R_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = \delta_{n_1+n_2,n_1'+n_2'} \delta_{n_2+n_3,n_2'+n_3'} \frac{q^{n_2(n_2+1)-(n_2-n_1')(n_2-n_3')}}{(q^2;q^2)_{n_2}} \\ \times Q_{n_2}\big(q^{-2n_1'}, q^{-2n_2'}, q^{-2n_3'}\big), \tag{3.2}$$

with $n_i, n_i' = 0, 1, 2, 3, \dots$ and we have introduced a set of (yet unknown) functions $Q_n(x, y, z)$ depending on the three variables $x = q^{-2n_1'}$, $y = q^{-2n_2'}$ and $z = q^{-2n_3'}$. We notice that the formula (3.2) contain two conservations laws

$$n_1 + n_2 = n_1' + n_2', \qquad n_2 + n_3 = n_2' + n_3', \tag{3.3}$$

which are similar to the conservation law of the 6-vertex model in two dimensions.

The specific q-dependent factor in (3.2) has been chosen to ensure that the functions $Q_n(x, y, z)$ are polynomials in x, y, z with coefficients which are themselves polynomials in the variable q. They are completely determined by initial conditions

$$Q_0(x, y, z) \equiv 1, \quad \forall x, y, z = 1, q^{-2}, q^{-4}, q^{-6}, \dots \tag{3.4}$$

and the following recurrence relation,

$$Q_{n+1}(x, y, z) = (x-1)(z-1) Q_n\big(xq^2, y, zq^2\big) + xz(y-1)q^{2n} Q_n\big(x, yq^2, z\big). \tag{3.5}$$

First two nontrivial polynomials read

$$Q_1(x, y, z) = 1 - (x + z) + xyz, \\ Q_2(x, y, z) = (1-x)\big(1 - xq^2\big)(1-z)\big(1 - zq^2\big) - x^2z^2q^4\big(1 - y^2\big) \\ - xzq^2\big(1 + q^2\big)(1-y)(1-x-z). \tag{3.6}$$

One can solve (3.5) with the initial condition (3.4) and derive the explicit formula valid for all values of n

$$Q_n(x, y, z) = (-x)^n q^{n(n-1)} {}_2\overline{\phi}_1\left(q^{-2n}, \frac{q^{2-2n}}{xy}; \frac{q^{2-2n}}{x}; q^2, yzq^{2n}\right) \tag{3.7}$$

where ${}_2\overline{\phi}_1$ is the regularized terminating basic hypergeometric series introduced in (2.16). This formula works for any values $x, y, z = 1, q^{-2}, q^{-4}, \dots$.

Using (3.7) one can rewrite (3.2) in a more transparent form convenient for further calculations

$$R_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = \delta_{n_1+n_2,n_1'+n_2'}\delta_{n_2+n_3,n_2'+n_3'}q^{n_2(n_2+1)-(n_2-n_1')(n_2-n_3')} \times \sum_{r=0}^{n_2} \frac{(q^{-2n_1'};q^2)_{n_2-r}}{(q^2;q^2)_{n_2-r}} \frac{(q^{2+2n_1};q^2)_r}{(q^2;q^2)_r} q^{-2r(n_3+n_1'+1)}. \tag{3.8}$$

As shown in [3] all nonzero matrix elements in (3.8) are positive for $0 < q < 1$. The R-matrix (3.8) possesses the following symmetries

$$R_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = R_{n_3,n_2,n_1}^{n_3',n_2',n_1'}, \qquad R_{n_2,n_1,n_3'}^{n_2',n_1',n_3} = q^{n_2-n_1-n_3^2+n_3'^2} \frac{(q^2;q^2)_{n_3}}{(q^2;q^2)_{n_3'}} R_{n_1,n_2,n_3}^{n_1',n_2',n_3'} \tag{3.9}$$

and solves the tetrahedron equation [3]

$$\boldsymbol{R}_{123}\boldsymbol{R}_{145}\boldsymbol{R}_{246}\boldsymbol{R}_{356} = \boldsymbol{R}_{356}\boldsymbol{R}_{246}\boldsymbol{R}_{145}\boldsymbol{R}_{123}. \tag{3.10}$$

It involves operators acting in six Fock spaces, where $\boldsymbol{R}_{ijk}$ acts non-trivially in the i-th, j-th and k-th spaces, but acts as the identity in other three spaces. In matrix form Eq. (3.10) reads

$$\sum_{\substack{n_1',n_2',n_3'\\ n_4',n_5',n_6'}} R_{n_1\,n_2\,n_3}^{n_1'\,n_2'\,n_3'} R_{n_1'\,n_4\,n_5}^{n_1''\,n_4'\,n_5'} R_{n_2'\,n_4'\,n_6}^{n_2''\,n_4''\,n_6'} R_{n_3'\,n_5'\,n_6'}^{n_3''\,n_5''\,n_6''} = \sum_{\substack{n_1',n_2',n_3'\\ n_4',n_5',n_6'}} R_{n_3\,n_5\,n_6}^{n_3'\,n_5'\,n_6'} R_{n_2\,n_4\,n_6'}^{n_2'\,n_4'\,n_6''} R_{n_1\,n_4'\,n_5'}^{n_1'\,n_4''\,n_5''} R_{n_1'\,n_2'\,n_3'}^{n_1''\,n_2''\,n_3''}. \tag{3.11}$$

Let $\lambda_i, \mu_i, i = 1, 2, \dots, 6$, be positive real numbers. Using the conservation laws it is easy to check that if $\boldsymbol{R}_{ijk}$ satisfies (3.10), then so does the "dressed" R-matrix

$$\boldsymbol{R}_{ijk}' = \left(\frac{\mu_k}{\lambda_i}\right)^{\boldsymbol{N}_j} \boldsymbol{R}_{ijk} \left(\frac{\lambda_j}{\lambda_k}\right)^{\boldsymbol{N}_i} \left(\frac{\mu_i}{\mu_j}\right)^{\boldsymbol{N}_k}, \tag{3.12}$$

where the indices (i, j, k) take four sets of values appearing in (3.10). Note that the twelve parameters λ_i, μ_i enter the four equations (3.12) only via eight independent ratios, so these equations define a solution of (3.10) containing eight continuous parameters. These new degrees of freedom allow to define a non-trivial family of commuting layer-to-layer transfer matrices.

In addition to (3.12) the tetrahedron equation is, obviously, invariant under diagonal similarity transformations

$$\boldsymbol{R}_{ijk}' = c_i^{\boldsymbol{N}_i} c_j^{\boldsymbol{N}_j} c_k^{\boldsymbol{N}_k} \boldsymbol{R}_{ijk} c_i^{-\boldsymbol{N}_i} c_j^{-\boldsymbol{N}_j} c_k^{-\boldsymbol{N}_k}, \tag{3.13}$$

where $c_1, c_2, \dots, c_6$ are arbitrary positive constants.

4. The 2-layer projection and a composite R-matrix

It is well known that any edge-spin model on the cubic lattice can be viewed as a two-dimensional model on the square lattice with an enlarged space of states for the edge spins (see [1] for additional explanations).

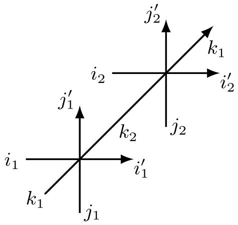

Fig. 1. A front-to-back line of the cubic lattice.

Here we are going to exploit only the simplest 2-layer case. Consider two vertices in the front-to-back direction as shown in Fig. 1 where we also assume the periodic boundary condition in the front-to-back direction.

For further convenience we associate indices $\{j_1, j_2\}$, $\{j_1', j_2'\}$ with the first direction and $\{i_1, i_2\}$, $\{i_1', i_2'\}$ with the second direction.

Let us define a composite R-matrix

$$\mathbb{S}_{\boldsymbol{i}\boldsymbol{j}}^{\boldsymbol{i}'\boldsymbol{j}'}(w) = \sum_{k_1,k_2} \boldsymbol{R}'^{\,j_1',i_1',k_2}_{\,j_1,i_1,k_1} \widetilde{\boldsymbol{R}}'^{\,j_2',i_2',k_1}_{\,j_2,i_2,k_2}. \tag{4.1}$$

The "dressed" R-matrices $\boldsymbol{R}'$ and $\widetilde{\boldsymbol{R}}'$ used in (4.1) are derived from $\boldsymbol{R}$ by combining both transformations (3.12)–(3.13) with different sets of fields in the first and second directions, i.e. $\{c_i, \lambda_i, \mu_i\}$ and $\{\widetilde{c}_i, \widetilde{\lambda}_i, \widetilde{\mu}_i\}$, $i = 1, 2$, but with the same fields $\{c_3, \lambda_3, \mu_3\}$ in the front-to-back direction. In the LHS of (4.1) we also introduced a new "spectral" parameter w which is a special combination of fields explicitly given below.

It follows from the conservation laws (3.3) that we can define two "global" conserved variables

$$I = i_1 + i_2 = i_1' + i_2', \qquad J = j_1 + j_2 = j_1' + j_2'. \tag{4.2}$$

Due to conservation laws (4.2) the R-matrix (4.1) acting in $(\mathcal{F}_q)^{\otimes 2} \otimes (\mathcal{F}_q)^{\otimes 2}$ decomposes into an infinite direct sum

$$\mathbb{S}(w) = \bigoplus_{I,J=0}^{\infty} \mathcal{R}_{I,J}(w) \tag{4.3}$$

of the $U_q(\widehat{sl(2)})$ R-matrices with weights I and J (or spins $I/2$ and $J/2$) [3]. So fixing the values I and J in (4.2) we can derive a general formula for matrix elements of such R-matrices.

Omitting some constant factors (depending on I and J and fields) one can derive after simple calculations

$$\big[\mathcal{R}_{I,J}(w)\big]_{i_1,j_1}^{i_1',j_1'} = \delta_{i_1+j_1,i_1'+j_1'} \phi_h^{i_1} \phi_v^{j_1} \psi_h^{i_1'-i_1} \psi_v^{j_1'-j_1} \sum_{k_1,k_2} w^{k_1} R_{j_1,i_1,k_1}^{j_1',i_1',k_2} R_{J-j_1,I-i_1,k_2}^{J-j_1',I-i_1',k_1}, \tag{4.4}$$

where

$$w = \frac{\mu_1\widetilde{\mu}_1}{\mu_2\widetilde{\mu}_2}, \qquad \phi_h = \frac{\widetilde{\lambda}_1}{\lambda_1}, \qquad \phi_v = \frac{\lambda_2}{\widetilde{\lambda}_2}, \qquad \psi_h = \frac{\mu_2\widetilde{c}_2}{\mu_1 c_2}, \qquad \psi_v = \frac{\widetilde{c}_1\lambda_2}{c_1\widetilde{\lambda}_2}. \tag{4.5}$$

The R-matrix (4.4) satisfies the Yang–Baxter equation

$$\sum_{i_1',i_2',i_3'} \big[\mathcal{R}_{I_1,I_2}(w)\big]_{i_1,i_2}^{i_1',i_2'} \big[\mathcal{R}'_{I_1,I_3}\big(ww'\big)\big]_{i_1',i_3}^{i_1'',i_3'} \big[\mathcal{R}''_{I_2,I_3}\big(w'\big)\big]_{i_2',i_3'}^{i_2'',i_3''}$$
$$= \sum_{i_1',i_2',i_3'} \big[\mathcal{R}''_{I_2,I_3}\big(w'\big)\big]_{i_2,i_3}^{i_2',i_3'} \big[\mathcal{R}'_{I_1,I_3}\big(ww'\big)\big]_{i_1,i_3'}^{i_1',i_3''} \big[\mathcal{R}_{I_1,I_2}(w)\big]_{i_1',i_2'}^{i_1'',i_2''}, \tag{4.6}$$

where $\mathcal{R}'$ and $\mathcal{R}''$ depend on different sets of fields $\{\phi_h', \phi_v', \psi_h', \psi_v'\}$ and $\{\phi_h'', \phi_v'', \psi_h'', \psi_v''\}$. These fields are not independent but satisfy the following constraints

$$\phi_v = \phi_v', \qquad \phi_h' = \phi_h'', \qquad \phi_v'' = \phi_h^{-1}, \qquad \psi_v'' = \frac{\psi_h \psi_v' \psi_h''}{\phi_h \psi_v \psi_h'}, \tag{4.7}$$

which easily follow from the conservation laws for the R-matrices (4.4) entering (4.6) similar to the 6-vertex model. Eq. (4.6) acts in the tensor product of three representation spaces with weights I_1, I_2, I_3 and is an immediate consequence of the Yang–Baxter equation for the q-oscillator R-matrix $\mathbb{S}(w)$ [3].

All ψ's fields are simple gauge transformations of the R-matrix and do not affect the spectrum of the transfer-matrix. Due to the conservation law in (4.4) the transfer-matrix splits into a tensor sum of blocks with equal sums of indices in the vertical direction. Similar to the 6-vertex model the vertical field ϕ_v will contribute the same factor in each block and can be set to 1 since it doesn't affect the spectrum. As a result we get the following Yang–Baxter equation

$$\mathcal{R}_{I_1,I_2}(w;1)\mathcal{R}_{I_1,I_3}\big(ww';\phi_h\big)\mathcal{R}_{I_2,I_3}\big(w';\phi_h\big)$$
$$= \mathcal{R}_{I_2,I_3}\big(w';\phi_h\big)\mathcal{R}_{I_1,I_3}\big(ww';\phi_h\big)\mathcal{R}_{I_1,I_2}(w;1) \tag{4.8}$$

where we explicitly showed a dependence on the horizontal field ϕ_h. As a consequence of (4.8) two transfer-matrices with the same ϕ_h will commute. From now on we shall assume that the R-matrix and the corresponding transfer-matrix depend on the horizontal field.

Some extra care should be taken while calculating the sum in (4.4). The summation goes over all non-negative k_1, k_2 satisfying the condition

$$i_1 + k_1 = i_1' + k_2. \tag{4.9}$$

For the case $i_1 \geqslant i_1'$ we can exclude the index k_2 and safely sum over k_1 from 0 to ∞. However, for $i_1 < i_1'$ the lower limit for k_1 should be $i_1' - i_1$. However, in this case all contributions to the sum in (4.4) from the values $0 \leqslant k_1 \leqslant i_1' - i_1 - 1$ are exactly zero. This happens because the second R-matrix in (4.4) becomes zero

$$R_{J-j_1,I-i_1,k_2}^{J-j_1',I-i_1',k_1} = 0 \tag{4.10}$$

for $i_1 < i_1'$ and $-(i_1' - i_1) \leqslant k_2 \leqslant -1$. So we can safely sum over k_1 from 0 to ∞ for all cases. The property (4.10) cannot be immediately seen from the definitions (3.2) and (3.7) and proved in Appendix A.

Now let us apply the second transformation from (3.9) to (4.4). After simple calculations we get the following symmetry of the matrix elements of the R-matrix

$$\big[\mathcal{R}_{J,I}\big(w;\{\phi\}\big)\big]_{j,i}^{j',i'} = q^{I-J} w^{i-i'} \big[\mathcal{R}_{I,J}\big(w;\big\{\phi^t\big\}\big)\big]_{i,j}^{i',j'}, \tag{4.11}$$

where we introduced two sets of fields $\{\phi\} \equiv \{\phi_h, \phi_v, \psi_h, \psi_v\}$ and $\{\phi^t\} \equiv \{\phi_v, \phi_h, \psi_v, \psi_h\}$.

We see that the R-matrix defined by (4.4) is not completely symmetric with respect to permutation of representation spaces with weights I and J. We shall repair it below by multiplying the R-matrix with the appropriate gauge and scalar factors.

Note that when we substitute the expression (3.8) for the R-matrix into (4.4) one can see that the sum over k_1 converges provided

$$w < q^{I+J}. \tag{4.12}$$

However, the resulting expression will be a rational function in w which can be analytically continued for any values of w.

Now let us remind that all nonzero elements of the 3D R-matrix (3.8) are positive for $0 < q < 1$. Then nonzero matrix elements of the composite R-matrix (4.4) are also positive provided that the field variables ϕ_h, ϕ_v, ψ_h and ψ_v are positive and condition (4.12) is satisfied. Indeed, in this case the LHS of (4.4) is given by a convergent series with positive terms.

For further convenience let us define new variables λ and ϕ and choose parameters in (4.4) as

$$w = \lambda^2, \qquad \phi_h = \phi^2, \qquad \phi_v = 1, \qquad \psi_h = 1, \qquad \psi_v = \lambda. \tag{4.13}$$

With such a choice of fields we define a properly normalized R-matrix by the following expression

$$\big[R_{I,J}(\lambda)\big]_{i,j}^{i',j'} = \sigma_{I,J}(\lambda) q^I \big[\mathcal{R}_{I,J}(\lambda)\big]_{i,j}^{i',j'} \tag{4.14}$$

where $\sigma_{I,J}(\lambda)$ is symmetric in I, J and defined by the following expression

$$\sigma_{I,J}(\lambda) = (-1)^{m(I,J)} q^{\frac{1}{2}IJ-\frac{1}{2}m(I,J)} \lambda^{-m(I,J)} \big(\lambda^2 q^{-I-J}; q^2\big)_{m(I,J)+1}, \tag{4.15}$$

where $m(i,j) = \min(i,j)$.

Finally, substituting (3.8) into (4.4) we arrive at the following explicit formula

$$\begin{aligned}
\big[R_{I,J}(\lambda;\phi)\big]_{i,j}^{i',j'} = {} & \delta_{i+j,i'+j'} (-1)^{m(I,J)} \phi^{2i-I} \lambda^{i-i'-m(I,J)} \\
& \times \frac{q^{i^2+(I-i)(J-j')-i'(i'-j)+2I+\frac{1}{2}IJ-\frac{1}{2}m(I,J)}}{(q^2;q^2)_i (q^2;q^2)_{I-i}} \\
& \times \big(\lambda^2 q^{-I-J}; q^2\big)_{m(I,J)+1} \sum_{k=0}^{i} \sum_{l=0}^{I-i} \frac{(-1)^{k+l} q^{2k(i'-j)-2l(J-I-j+i)}}{q^{k(k+1)+l(l+1)}(1-\lambda^2 q^{I-J-2k-2l})} \\
& \times \frac{(q^{-2i}, q^{2+2j}; q^2)_k (q^{-2j'}; q^2)_{i-k}}{(q^2;q^2)_k} \\
& \times \frac{(q^{-2(I-i)}, q^{2(1+J-j)}; q^2)_l (q^{-2(J-j')}; q^2)_{I-i-l}}{(q^2;q^2)_l}
\end{aligned} \tag{4.16}$$

with

$$0 \leqslant i, i' \leqslant I, \qquad 0 \leqslant j, j' \leqslant J. \tag{4.17}$$

One can show that with such a normalization the matrix elements of the R-matrix are the polynomials in λ and λ^{-1} of the degree $\leqslant m(I,J)$. We shall justify the choice of normalization (4.15) in the next section.

5. Properties of the higher spin R-matrix

In this section we shall analyze the formula (4.16) and derive a remarkably simple formula for the R-matrix $R_{I,J}(\lambda)$ in the form of a single sum.

First let us notice the formula (4.16) can be naturally extended to any values $J \in \mathbb{C}$. This corresponds to the case when the second space becomes an infinite-dimensional Verma module V_J^+ with indices j, j' running form 0 to ∞. In this case we will choose

$$m(I,J) = I, \quad I \in \mathbb{Z}_+, \ J \in \mathbb{C}. \tag{5.1}$$

As an example consider the case $I = 1$. The double sum in (4.16) contains only two nontrivial terms and we obtain

$$\left[R_{(1,J)}(\lambda;\phi)\right]_{i,j}^{i',j'} = \begin{pmatrix} \delta_{j,j'}\phi^{-1}[\lambda q^{\frac{1+J}{2}-j'}] & \delta_{j,j'+1}\phi^{-1}[q^{J-j'}]q^{j'-\frac{J-1}{2}} \\ \delta_{j+1,j'}\phi[q^{j'}]q^{\frac{J+1}{2}-j'} & \delta_{j,j'}\phi[\lambda q^{\frac{1-J}{2}+j'}] \end{pmatrix}_{i+1,i'+1}, \tag{5.2}$$

where $i, i' = 0, 1$.

As usual we can define the L-operator acting in the tensor product $\mathbb{C}^2 \otimes V_J^+$ as a two-by-two matrix with matrix elements coinciding with the matrix elements of $R_{1,J}(\lambda)$. To make a connection to the standard XXZ L-operator we introduce a rescaled spectral parameter $\mu = \lambda q^{1/2}$, set $\phi = 1$ and apply a simple similarity transformation $D = \mathrm{diag}(1, \lambda^{-1})$ in $\mathbb{C}^2$. Then we obtain

$$L(\mu) = \begin{pmatrix} \mu q^{H/2} - \mu^{-1}q^{-H/2} & \mu[q]Fq^{-H/2} \\ \mu^{-1}[q]q^{H/2}E & \mu q^{-H/2} - \mu^{-1}q^{H/2} \end{pmatrix} \tag{5.3}$$

where E, F and H are the generators of the quantum algebra $U_q(sl(2))$ with the action (2.5).

If $J \in \mathbb{Z}_+$, the module V_J^+ becomes reducible and first $J+1$ vectors $\{v_0, \dots, v_J\}$ form a basis of a $(J+1)$-dimensional representation space V_J.

Now let us assume that J is not a positive integer and perform a resummation in (4.16) by introducing a new variable $s = k + l$. Then we can rewrite the sum in (4.16) in the form of a pole expansion in λ^2 at the points $\lambda^2 = q^{J-I+2s}$, $s = 0, \dots, I$. Surprisingly the corresponding residues can be expressed in terms of terminating balanced ${}_4\phi_3$ series. Applying Sears' transformation for terminated balanced ${}_4\phi_3$ (B.1) from Appendix B we can rewrite (4.16) in the following form

$$\begin{aligned} &\left[R_{I,J}(\lambda;\phi)\right]_{i,j}^{i',j'} \\ &= \delta_{i+j,i'+j'}(-1)^{i+I}\phi^{2i-I}\lambda^{i-i'-I}q^{i(i+I-2J-1)+(I-i)(J-j)+i'(I+j')+I(3+J)/2} \\ &\quad \times \frac{(q^{-2J};q^2)_j}{(q^{-2J};q^2)_{j'}}\frac{(\lambda^2 q^{-I-J};q^2)_{I+1}}{(q^2;q^2)_i} \\ &\quad \times \sum_{s=0}^{I}\frac{(-1)^s}{1-\lambda^2 q^{I-J-2s}}\frac{q^{s(s-1)-2is}}{(q^2;q^2)_s(q^2;q^2)_{I-s}}c_{i,j}^{i',j'}(I,J;s) \end{aligned} \tag{5.4}$$

with coefficients $c_{i,j}^{i',j'}(I,J;s)$ given by

$$c_{i,j}^{i',j'}(I,J;s)=\frac{(q^{2(I-J-s)};q^2)_{j'-i}(q^{-2(s+J)};q^2)_I}{(q^{-2(s+J)};q^2)_{i+j}}\\ \times {}_4\bar{\phi}_3\left(\begin{matrix}q^{-2i};q^{-2i'},q^{-2s},q^{2(1+J-I+s)}\\ q^{-2I},q^{2(1+j'-i)},q^{2(1+J-i-j)}\end{matrix}\,\middle|\,q^2,q^2\right), \tag{5.5}$$

where we used our definition of regularized terminating hypergeometric series (2.16).

We note that the only problem for integer J comes from possible poles in (5.5) for $0\leqslant s< i+j-J$. However, one can show that for all such values of s there is exactly a matching zero coming from ${}_4\bar{\phi}_3$. So if we define coefficients $c_{i,j}^{i',j'}(I,J;s)$ for integer J as a limiting value from complex J, then the representation (5.4) for matrix elements works for any integer I and J provided that $I\leqslant J$.

Now let us notice the sum in (5.4) can be represented as a ratio

$$\frac{R_I(\lambda^2)}{(\lambda^2q^{-I-J};q^2)_{I+1}} \tag{5.6}$$

where $R_I(\lambda^2)$ is a polynomial of the degree I in λ^2. Such a polynomial can be reconstructed using a Lagrange interpolation formula. Applying this formula for any polynomial $P_n(x)$ of the degree n one can easily show that

$$x^{n+1}\left(x^{-1};q\right)_{n+1}\sum_{i=0}^{n}(-1)^i\frac{1}{x-q^i}\frac{q^{i(i+1)/2-ni}}{(q;q)_i(q;q)_{n-i}}P_n\left(q^i\right)=P_n(x). \tag{5.7}$$

This allows us to perform a summation over s in (5.4) and obtain the main result of this paper

$$\left[R_{I,J}(\lambda;\phi)\right]_{i,j}^{i',j'}=\delta_{i+j,i'+j'}\phi^{2i-I}a_{ij}^{i'j'}(\lambda)\\ \times {}_4\bar{\phi}_3\left(\begin{matrix}q^{-2i};q^{-2i'},\lambda^{-2}q^{J-I},\lambda^2q^{2+J-I}\\ q^{-2I},q^{2(1+j-i')},q^{2(1+J-i-j)}\end{matrix}\,\middle|\,q^2,q^2\right) \tag{5.8}$$

where

$$a_{ij}^{i'j'}(\lambda)=(-1)^i\frac{q^{i(i-J-1)+(I-i)(J-j)+i'(I+j')}}{\lambda^{i+i'-m(I,J)}q^{\frac{1}{2}IJ-\frac{1}{2}m(I,J)}}\frac{(q^{-2J};q^2)_j}{(q^{-2J};q^2)_{j'}}\\ \times\frac{(\lambda^{-2}q^{I-J};q^2)_{j-i'}(\lambda^{-2}q^{-I-J};q^2)_{m(I,J)}}{(q^2;q^2)_i(\lambda^{-2}q^{-I-J};q^2)_{i+j}}. \tag{5.9}$$

Let us make a few important remarks regarding (5.8). First, it is easy to see that the main ingredient of (5.8) comes from the coefficients $c_{i,j}^{i',j'}(I,J;s)$ with q^{2s} replaced by λ^2q^{I-J}. Assuming that λ is generic we no longer have poles coming from the pre-factor in (5.5). So (5.8) is well defined for integer values of J as well.

Second, we derived the formula (5.8) assuming that $m(I,J)=I$, i.e. $I\leqslant J$ for $I,J\in\mathbb{Z}_+$. However, as we already know from (4.11), the formula for the matrix elements should have a certain symmetry with respect to interchanging I and J. Namely, we should have

$$\mathcal{P}_{12}R_{I,J}(\lambda;1)\mathcal{P}_{12}=R_{J,I}(\lambda;1), \tag{5.10}$$

where $\mathcal{P}_{12}$ is the permutation operator. This symmetry of the R-matrix immediately follows from the symmetry transformation (B.3) for ${}_4\bar{\phi}_3$ basic hypergeometric series derived in Appendix B. It also allows us to restore all factors $m(I,J)$ correctly which were originally set to I.

The next comment concerns the normalization of the R-matrix $R_{I,J}(\lambda;\phi)$. One can easily derive from (5.8) that

$$\big[R_{I,J}(\lambda;\phi)\big]_{0,0}^{0,0}=\phi^{-I}q^{\frac{1}{2}IJ+\frac{1}{2}m(I,J)}\lambda^{m(I,J)}\big(\lambda^{-2}q^{-I-J};q^2\big)_{m(I,J)}. \tag{5.11}$$

We see that up to an overall normalization factor matrix elements of the R-matrix are rational functions in q^I and q^J and polynomials in λ and λ^{-1} of the degree determined by indices i, j, i', j'.

Now we can consider three different cases. If both $I, J \in \mathbb{Z}_+$, all matrix elements are polynomials in λ and λ^{-1} of the degree $d \leqslant m(I,J)$. This is also true in the case when only I (or J) is a positive integer and we set $m(I,J)=I$ (or $m(I,J)=J$).

However, the formula (5.8) works even in the case when the R-matrix acts in the tensor product of two infinite-dimensional Verma modules $V_I^+ \otimes V_J^+$, $I, J \in \mathbb{C}$. In this case we can choose a different normalization of the R-matrix, say,

$$\big[R_{I,J}(\lambda;1)\big]_{0,0}^{0,0}=1. \tag{5.12}$$

To confirm the last statement we could define the R-matrix $R_{I,J}(\lambda;1)$ in a zero field as a solution of the Yang–Baxter equation acting in $\mathbb{C}^2 \otimes V_I^+ \otimes V_J^+$, $I, J \in \mathbb{C}$

$$L_{1,I}(\mu)L_{1,J}(\lambda\mu)R_{I,J}(\lambda;1)=R_{I,J}(\lambda;1)L_{1,J}(\lambda\mu)L_{1,I}(\mu), \tag{5.13}$$

where the L-operators $L_{1,I}(\lambda)$ and $L_{1,J}(\mu)$ are defined as in (5.3). Substituting

$$\big[R_{I,J}(\lambda;1)\big]_{i,j}^{i',j'}=\delta_{i+j,i'+j'}S_{i,j}^{i',j'} \tag{5.14}$$

into (5.13) we obtain the system of three linearly independent recursions. These recurrence relations are given in Appendix C, (C.1)–(C.3). Up to a normalization $S_{0,0}^{0,0}$ they have a unique solution which coincides with (5.8). However, it is highly nontrivial to find a solution of (C.1)–(C.3) in terms of basic hypergeometric series. Our derivation of (5.8) is based on the 3D approach where it appears very naturally.

Now let us return to the case when $I, J \in \mathbb{Z}_+$. Then the R-matrix $R_{I,J}(\lambda;\phi)$ has another symmetry

$$\big[R_{I,J}(\lambda;\phi)\big]_{i,j}^{i',j'}=\big[R_{I,J}\big(\lambda;\phi^{-1}\big)\big]_{I-i,J-j}^{I-i',J-j'} \tag{5.15}$$

which will be used to define the second Q-operator in the next section.

Note that in the case $I=J=1$ (5.15) is equivalent to the invariance of the R-matrix under the conjugation by the operator $\mathcal{R}=\sigma_x\otimes\sigma_x$ and the transformation $\phi\to\phi^{-1}$. The proof of the relation (5.15) is reduced to applying the Sears' transformation (B.2) to (5.8).

Additionally the R-matrix possesses the following symmetry under a simultaneous transformation $\lambda\to\lambda^{-1}$ and $q\to q^{-1}$:

$$R_{I,J}\big(\lambda^{-1};\phi\big)\big|_{q\to q^{-1}}=(-1)^{m(I,J)}D_1\otimes D_2R_{I,J}(\lambda;\phi)D_1^{-1}\otimes D_2^{-1}, \tag{5.16}$$

where diagonal matrices D_1 and D_2 acting in V_I and V_J are

$$\begin{aligned}&[D_1]_{i,i'}=\delta_{i,i'}q^{-i(i-1)}, \quad i=0,\dots,I,\\ &[D_2]_{j,j'}=\delta_{j,j'}q^{-j(j-1)+j(J-I)}, \quad j=0,\dots,J.\end{aligned} \tag{5.17}$$

This symmetry can be proved by observing that the defining relations for the R-matrix (C.1)–(C.3) are invariant under combined transformations from (5.16).

In the case $I = J$ and $\lambda = \phi = 1$ the R-matrix reduces to the permutation operator

$$R_{I,I}(1;1) = q^{-\frac{1}{2}I(I+1)}\left(q^2;q^2\right)_I \mathcal{P}_{12} \tag{5.18}$$

which can be proved directly from the formula (5.8).

Finally, one can calculate the expansions of (5.8) near the points $\lambda = 0$ and $\lambda = \infty$. In the leading order we get

$$\left[R_{I,J}(\lambda;\phi)\right]_{i,j}^{i',j'} = \delta_{i,i'}\delta_{j,j'}\left(-\lambda q^{1/2}\right)^{-m(I,J)}\phi^{2i-I}q^{-\frac{1}{2}(I-2i)(J-2j)}\left(1+O(\lambda)\right) \quad \text{at } \lambda \to 0 \tag{5.19}$$

and

$$\left[R_{I,J}(\lambda;\phi)\right]_{i,j}^{i',j'} = \delta_{i,i'}\delta_{j,j'}\left(\lambda q^{1/2}\right)^{m(I,J)}\phi^{2i-I}q^{\frac{1}{2}(I-2i)(J-2j)}\left(1+O\left(\lambda^{-1}\right)\right) \quad \text{at } \lambda \to \infty. \tag{5.20}$$

We could calculate next order corrections in (5.20) and compare it with Kirillov and Reshetikhin expansion at $\lambda \to \infty$ in [17]. We leave this exercise to the reader.

6. Q-operators

The theory of the Q-operators related to the affine algebra $U_q(\widehat{sl(2)})$ has been developed in [42,43]. Two Q-operators $\mathbf{Q}_\pm(\lambda)$ appear as traces of special monodromy matrices over infinite-dimensional representations of the q-oscillator algebra introduced in Section 2. Similar to the usual transfer-matrices these monodromy matrices are derived from the tensor product of the local L-operators. In this section we construct these local L-operators acting in the $(I+1)$-dimensional highest weight module V_I. We also show that this construction can be naturally generalized to the case of infinite-dimensional Verma module V_I^+ with $I \in \mathbb{C}$.

Lets us first assume that $I \in \mathbb{Z}_+$. Then we can introduce the transfer matrix $\widehat{\mathbf{T}}_{J,I}(\lambda;\phi)$ associated with the infinite-dimensional Verma module, V_J^+, $J \in \mathbb{C}$ acting in the quantum space $W = \bigotimes_{i=1}^M V_I$ as

$$\widehat{\mathbf{T}}_{J,I}(\lambda;\phi) = \operatorname*{Tr}_{V_J^+}\big[\underbrace{R_{J,I}(\lambda;\phi)\otimes\cdots\otimes R_{J,I}(\lambda;\phi)}_{M \text{ times}}\big], \tag{6.1}$$

where the trace is defined as in (2.10).

We notice that usually the horizontal field is introduced via a global twist in the auxiliary space [42,43]. However, we prefer to use a local field and include it into the definition of the R-matrix. With such a definition the transfer-matrix still commutes with the shift operator along the periodic chain. We are going to exploit this fact in our construction of factorized Q-operators in the next publication.

Due to the conservation law in (5.8) the transfer matrix (6.1) has a block-diagonal form

$$\widehat{\mathbf{T}}_{J,I}(\lambda;\phi) = \bigoplus_{l=0}^{IM}\widehat{\mathbf{T}}_{J,I}^{(l)}(\lambda;\phi), \tag{6.2}$$

where for each block $\widehat{\mathbf{T}}_{J,I}^{(l)}(\lambda;\phi)$ the sum of in- and out-indices in the quantum space W is fixed to l, i.e.

$$\sum_{k=1}^M i_k = \sum_{k=1}^M i'_k = l. \tag{6.3}$$

Let us call the subspace in the quantum space W with a fixed l as the l-th sector.

The direct sum expansion (6.2) is also true for $I \in \mathbb{C}$, when the quantum space becomes infinite-dimensional, $W = \bigotimes_{i=1}^{M} V_I^+$. In this case the sum in (6.2) runs from zero to infinity, but all blocks with a fixed l are still finite-dimensional.

Using asymptotics (5.19)–(5.20) one can easily calculate asymptotics of the $\widehat{\mathbf{T}}_J^{(l)}(\lambda;\phi)$ in each block with a fixed l

$$\widehat{\mathbf{T}}_{J,I}^{(l)}(\lambda;\phi)\big|_{\lambda\to 0} = \left(-\lambda q^{1/2}\right)^{-IM}\phi^{-JM}\frac{q^{-\frac{1}{2}IJM+Jl}}{1-\phi^{2M}q^{IM-2l}}\left(\boldsymbol{I}_l + O(\lambda)\right), \tag{6.4}$$

$$\widehat{\mathbf{T}}_{J,I}^{(l)}(\lambda;\phi)\big|_{\lambda\to \infty} = \left(\lambda q^{1/2}\right)^{IM}\phi^{-JM}\frac{q^{\frac{1}{2}IJM-Jl}}{1-\phi^{2M}q^{2l-IM}}\left(\boldsymbol{I}_l + O\left(\lambda^{-1}\right)\right), \tag{6.5}$$

where $\boldsymbol{I}_l$ is the unit matrix of the dimension of the block.

When $J \in \mathbb{Z}_+$, the module V_J^+ becomes reducible as discussed in Section 2. The formula for the R-matrix (5.8) is analytic in q^J except the normalization factor which contain the function $m(I,J)$. The transfer matrix $\widehat{\mathbf{T}}_{J,I}(\lambda;\phi)$ splits into two terms

$$\widehat{\mathbf{T}}_{J,I}(\lambda;\phi) = \mathbf{T}_{J,I}(\lambda;\phi) + \widehat{\mathbf{T}}_{-J-2,I}(\lambda;\phi), \tag{6.6}$$

where $\mathbf{T}_{J,I}(\lambda;\phi)$ is the transfer matrix defined similar to (6.1) with the trace taken over the finite-dimensional representation π_J.

However, there is one subtlety related to our choice of normalization (5.11). When J is not integer, we assumed that $m(I,J) = I$ for $I \in \mathbb{Z}_+$ in our definition of the transfer-matrix (6.1). However, when J becomes a positive integer, we can expect in (6.6) some extra factor in front of $\mathbf{T}_{J,I}(\lambda;\phi)$ for the case $I > J$, since $m(I,J) = J$ in this case. Indeed, a detailed analysis shows that we need to slightly modify (6.6) as follows

$$h_{I-J}(\lambda)^M \mathbf{T}_{J,I}(\lambda;\phi) = \widehat{\mathbf{T}}_{J,I}(\lambda;\phi) - \widehat{\mathbf{T}}_{-J-2,I}(\lambda;\phi), \tag{6.7}$$

where

$$h_I(\lambda) = \prod_{k=0}^{I-1}\left[\lambda q^{\frac{I}{2}-k}\right] = \begin{cases} 1, & \text{for } I \leqslant 0, \\ (\lambda q^{1/2})^I(\lambda^{-2}q^{-I};q^2)_I, & \text{for } I > 0. \end{cases} \tag{6.8}$$

Let us illustrate Eq. (6.7) with the case $I = M = 1$. Using formula (5.8) we obtain after simple calculations

$$\widehat{\mathbf{T}}_{J,1}(\lambda;\phi) = \begin{pmatrix} \frac{\lambda\phi^{-J}q^{\frac{1+J}{2}}}{1-\phi^2/q} - \frac{\lambda^{-1}\phi^{-J}q^{-\frac{1+J}{2}}}{1-\phi^2 q} & 0 \\ 0 & \frac{\lambda\phi^{-J}q^{\frac{1-J}{2}}}{1-\phi^2 q} - \frac{\lambda^{-1}\phi^{-J}q^{\frac{J-1}{2}}}{1-\phi^2/q} \end{pmatrix} \tag{6.9}$$

Then for any integer $J \geqslant 1$ we get from (6.7)

$$\mathbf{T}_{J,1}(\lambda;\phi) = \begin{pmatrix} \phi^{-J}\sum_{k=0}^{J}\phi^{2k}[\lambda q^{\frac{1+J}{2}-k}] & 0 \\ 0 & \phi^{-J}\sum_{k=0}^{J}\phi^{2k}[\lambda q^{\frac{1-J}{2}+k}] \end{pmatrix} \tag{6.10}$$

which can be verified by direct calculations using the explicit formula (5.8) for the R-matrix $R_{J,I}(\lambda;\phi)$.

We also notice the following normalization of the transfer-matrix for $J=0$ and $I \in \mathbb{Z}_+$

$$\mathbf{T}_{0,I}(\lambda;\phi) = \boldsymbol{I}, \tag{6.11}$$

which is an immediate consequence of (5.8).

Now we turn to the construction of the Q-operators for any highest weight representation in the quantum space. The main algebraic properties of the Q-operators are encoded into the *fundamental fusion relation* discovered in [42,43]

$$\mathbf{Wr}(\phi)\widehat{\mathbf{T}}_{J,I}(\lambda;\phi) = \mathbf{Q}_+^{(I)}\big(\lambda q^{-\frac{J+1}{2}}\big)\mathbf{Q}_-^{(I)}\big(\lambda q^{\frac{J+1}{2}}\big), \tag{6.12}$$

where the Wronskian $\mathbf{Wr}(\phi)$ does not depend on the spectral parameter λ. The Wronskian is the diagonal operator and its eigenvalues are the same in every l-th sector. We notice that our factorization relation (6.12) is slightly different from the one used in [42,43] due to the fact that their λ is, in fact, our λ^{-1}.

The relation (6.12) has been derived in [42,43] irrespective of the choice of the quantum space using the universal R-matrix approach [44]. In principle, using the explicit expression of the $U_q(\widehat{sl(2)})$ universal R-matrix [46] one can construct the Q-operators for any highest weight representation. However, calculations for higher spins quickly become unbearable. In our approach we derive the Q-operators for an arbitrary weight I based on the explicit construction of the R-matrix (5.8).

Although the eigenvalues of the transfer-matrix are polynomials in λ and λ^{-1}, the eigenvalues of the Q-operators are not. In the twisted case $\phi \neq 1$ they are polynomials multiplied by simple exponential factors depending on the horizontal field. Namely, set

$$\lambda = e^{iu}, \qquad \phi = q^h \tag{6.13}$$

and define two operators $\mathbf{A}_\pm^{(I)}(\lambda)$ as

$$\mathbf{Q}_\pm^{(I)}(\lambda) = e^{\pm iuhM}\mathbf{A}_\pm^{(I)}(\lambda) = \lambda^{\pm hM}\mathbf{A}_\pm^{(I)}(\lambda), \tag{6.14}$$

where M is the size of the system. Then the eigenvalues of the operators $\mathbf{A}_\pm^{(I)}(\lambda)$ will be polynomials in λ and λ^{-1} (see, for example, [55] for detailed explanations).

Then we can rewrite the relation (6.12) as

$$\phi^{(J+1)M}\mathbf{Wr}(\phi)\widehat{\mathbf{T}}_{J,I}(\lambda;\phi) = \mathbf{A}_+^{(I)}\big(\lambda q^{-\frac{J+1}{2}}\big)\mathbf{A}_-^{(I)}\big(\lambda q^{\frac{J+1}{2}}\big). \tag{6.15}$$

Let us now make the following substitution into (6.15)

$$\lambda = \mu q^{-\frac{J+1}{2}} \tag{6.16}$$

and consider the limit $J \to \infty$ with μ being fixed. In the RHS of (6.15) we shall get the operator $\mathbf{A}_-^{(I)}(\mu)$ pre-multiplied by a constant matrix $\mathbf{A}_+^{(I)}(\infty)$. We can always absorb this matrix into $\mathbf{A}_-^{(I)}(\mu)$ by redefining it, so we assume that

$$\mathbf{A}_+^{(I)}(\infty) \sim \boldsymbol{I} \tag{6.17}$$

and check a consistency of (6.17) later. Now taking the limit $J \to \infty$ and substituting (6.16) in (5.8) we define the local L-operator $A_-^{(I)}(\mu)$

$$\big[A_-^{(I)}(\mu)\big]_{n,i}^{n',i'} = \lim_{J\to\infty}\big\{\big[R_{J,I}\big(\mu q^{-\frac{J+1}{2}};\phi\big)\big]_{n,i}^{n',i'}U_{i,i'}^{n}(I,J)\big\} \tag{6.18}$$

where

$$U^n_{i,i'}(I,J)=\phi^J(-1)^{II-i}\lambda^{-i'}q^{(i-i')(j+j')+J(3i'-i)/2+(i'-i)/2}. \tag{6.19}$$

Let us notice that we introduced some additional factor (6.19) into (6.18). It is needed to compensate divergent contributions in J and simplify the formula for the L-operator. It is easy to prove the following identity

$$\sum_{k=1}^{M}(i_k-i'_k)(n_k+n'_k)=\left[\sum_{k=1}^{M}(i_k-i'_k)\right]^2 \tag{6.20}$$

provided that $i_k+n_k=i'_k+n'_k$, $k=1,\dots,M$. Therefore, the factor $U^n_{i,i'}(I,J)$ can only contribute a constant to each block with a fixed l (see (6.3)).

After simple calculations we obtain the following result

$$\begin{aligned}\big[A^{(I)}_-(\lambda)\big]^{n',i'}_{n,i}&=\delta_{i+n,i'+n'}\phi^{2n}\lambda^{i-I}q^{iI+ii'+n(I-i-i')}\\&\quad\times\frac{(\lambda^2q^{1-I+2(i'-n)};q^2)_{I-i-i'}}{(q^2;q^2)_i}\,{}_3\overline{\phi}_2\left(\begin{matrix}q^{-2i};q^{-2i'},\lambda^2q^{1-I}\\q^{-2I},q^{2(1+n-i')}\end{matrix}\,\middle|\,q^2,q^2\right).\end{aligned} \tag{6.21}$$

The formula (6.21) defines a local L-operator $A^{(I)}_-(\lambda)$ acting in the tensor product $\mathcal{F}_q\otimes V_I$. To define the corresponding global Q-operator we take a tensor product of M copies of (6.19), take a normalized trace (2.11) over the Fock space $\mathcal{F}_q$ and multiply by the exponential factor from (6.14)

$$\mathbf{Q}^{(I)}_-(\lambda)=\lambda^{-hM}\,\widehat{\underset{\mathcal{F}_q}{\mathrm{Tr}}}\big\{\underbrace{A^{(I)}_-(\lambda)\otimes\cdots\otimes A^{(I)}_-(\lambda)}_{M\text{ times}}\big\}. \tag{6.22}$$

We could use the same strategy and define the second Q-operator $\mathbf{Q}^{(I)}_+(\lambda)$ by taking a different limit in (6.15). However, we prefer to use a different approach. Let us remind that the operators $\mathbf{Q}^{(I)}_\pm(\lambda)$ are two linearly independent solutions of the TQ-relation (which we prove later) with the transfer matrix

$$\mathbf{T}_{1,I}(\lambda;\phi)=\underset{V_1}{\mathrm{Tr}}\big[\underbrace{R_{1,I}(\lambda;\phi)\otimes\cdots\otimes R_{1,I}(\lambda;\phi)}_{M\text{ times}}\big]. \tag{6.23}$$

This transfer matrix has the following symmetry

$$\mathbf{T}_{1,I}(\lambda;\phi)^{i'_1,\dots,i'_M}_{i_1,\dots,i_M}=\mathbf{T}_{1,I}\big(\lambda;\phi^{-1}\big)^{I-i'_1,\dots,I-i'_M}_{I-i_1,\dots,I-i_M}, \tag{6.24}$$

which is a consequence of the symmetry (5.15) of the R-matrix.

It follows that we can define the second L-operator

$$\big[A^{(I)}_+(\lambda)\big]^{n',i'}_{n,i}=\big[A^{(I)}_-(\lambda)\big]^{n',I-i'}_{n,I-i}\big|_{\phi\to\phi^{-1}} \tag{6.25}$$

and the second Q-operator $\mathbf{Q}^{(I)}_+(\lambda)$ by

$$\mathbf{Q}^{(I)}_+(\lambda)=\lambda^{hM}\,\widehat{\underset{\mathcal{F}_q}{\mathrm{Tr}}}\big\{\underbrace{A^{(I)}_+(\lambda)\otimes\cdots\otimes A^{(I)}_+(\lambda)}_{M\text{ times}}\big\}. \tag{6.26}$$

Due to the symmetry (6.24) $\mathbf{Q}_{+}^{(I)}(\lambda)$ will satisfy the same TQ-relation. Since TQ-relation has only two linear independent solutions and $\mathbf{Q}_{\pm}^{(I)}(\lambda)$ cannot mix,[1] we have constructed the second Q-operator.

In principle, (6.25) completely determines matrix elements of the L-operator $A_{+}^{(I)}(\lambda)$. However, it has a big disadvantage, since it can be applied only for integer values of I. In fact, using the transformation (B.4) for hypergeometric series ${}_3\bar{\phi}_2$ one can transform (6.25) to the following neat form

$$\begin{aligned}\big[A_{+}^{(I)}(\lambda)\big]_{n,i}^{n',i'} &= \delta_{i+n',i'+n}\phi^{-2n}(-1)^{i+i'}\lambda^{-i}q^{i(i+1)-i'(i'+1)+i'(I+i)+n(I-i-i')}\\ &\quad\times\frac{(q^2;q^2)_{n'}}{(q^2;q^2)_n(q^2;q^2)_i}\,{}_3\bar{\phi}_2\left(\begin{matrix}q^{-2i};q^{-2i'},\lambda^2q^{1-I}\\ q^{-2I},q^{2(1+n-i)}\end{matrix}\,\middle|\,q^2,q^2\right).\end{aligned}\tag{6.27}$$

Comparing (6.21) and (6.27) we observe that the L-operator $A_{+}^{(I)}(\lambda)$ coincides with $A_{-}^{(I)}(\lambda)^{t_2}$ up to a simple diagonal transformation, transformation $\phi\to\phi^{-1}$ and equivalence transformations in both auxiliary and quantum space.

Let us comment on the obtained results. First, it is clear that both Q-operators $\mathbf{Q}_{\pm}^{(I)}(\lambda)$ have the same block-diagonal form as the transfer-matrix (6.23) due to the presence of the delta-functions in (6.21) and (6.27). Second, $\mathbf{Q}_{\pm}^{(I)}(\lambda)$ commute with the transfer-matrix and this is the consequence of the Yang–Baxter equation for the R-matrix (5.8).

Finally, it is clear that the operator $\mathbf{A}_{+}^{(I)}(\lambda)$ is well defined even for $I\in\mathbb{C}$, since the dependence on I is analytic in (6.27). The matrix elements of this operator are always polynomials in λ, λ^{-1} as well as its eigenvalues in any finite-dimensional block with fixed l.

It is well known that for non-integer I the eigenvalues of the second Q-operator $\mathbf{A}_{-}^{(I)}(\lambda)$ are not polynomials. Having the explicit form (6.21) we can clarify this in details. The only non-analytic term in I in (6.21) is the Pochhammer symbol in the numerator which can be transformed as follows

$$\begin{aligned}&\big(\lambda^2q^{1-I+2(i'-n)};q^2\big)_{I-i-i'}\\ &\quad=\big(-\lambda^2q^{i'-i-2n}\big)^{I-i-i'}\big(\lambda^{-2}q^{1-I};q^2\big)_I\frac{(\lambda^{-2}q^{1+I};q^2)_{n-i'}}{(\lambda^{-2}q^{1-I};q^2)_{n+i}}.\end{aligned}\tag{6.28}$$

So we see that for $I\in\mathbb{C}$ matrix elements of the operator $\mathbf{A}_{-}^{(I)}(\lambda)$ contain a meromorphic function

$$\big(\lambda^{-2}q^{1-I};q^2\big)_I=\frac{(\lambda^{-2}q^{1-I};q^2)_\infty}{(\lambda^{-2}q^{1+I};q^2)_\infty}\tag{6.29}$$

which doesn't depend on matrix indices. The rest of the formula (6.21) is the rational function of q^I and can be analytically continued to complex values of $I\in\mathbb{C}$. As we have seen before in the case of the transfer-matrix, both Q-operators for $I\in\mathbb{C}$ can be decomposed into an infinite direct sum of finite-dimensional matrices with a fixed number of in- and out-spins. Therefore, formulas (6.21), (6.27) allow to construct the Q-operators even in the case when the quantum space is the tensor product of Verma modules V_I^+ with $I\in\mathbb{C}$.

Now let us consider some simple examples. First, take $I=1$. Then we can write the operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ as 2-by-2 matrix operators acting in the auxiliary space of the q-oscillator algebra (2.6). In the representation of the q-oscillator algebra defined by (2.9) we obtain

[1] The operators $\mathbf{Q}_{\pm}^{(I)}(\lambda)$ satisfy different quasi-periodicity conditions under the shift $u\to u+\pi$ due to (6.14).

$$A_+^{(1)}(\lambda) = \phi^{-2N}\begin{pmatrix} q^N & -q^{-1}\boldsymbol{a}^- \\ \lambda^{-1}\boldsymbol{a}^+ & [\lambda^{-1}q^N] \end{pmatrix}, \tag{6.30}$$

$$A_-^{(1)}(\lambda) = \phi^{2N}\begin{pmatrix} [\lambda^{-1}q^N] & \lambda^{-1}\boldsymbol{a}^+ \\ -q^{-1}\boldsymbol{a}^- & q^N \end{pmatrix}, \tag{6.31}$$

where $[x]$ is defined in (2.3).

Similarly, one can calculate from (6.27) the matrix elements of $A_+^{(I)}(\lambda)$ at $I = 2$ and obtain

$$A_+^{(2)}(\lambda) = \phi^{-2N}\begin{pmatrix} q^{2N} & -\boldsymbol{a}^- q^{N-1} & q^{-2}(\boldsymbol{a}^-)^2 \\ \lambda^{-1}\{q\}\boldsymbol{a}^+ q^N & \lambda^{-1}(\{q\}q^{2N} - q^{-1}) - \lambda & q^{-\frac{5}{2}}\{q\}\boldsymbol{a}^-[\lambda q^{-N+\frac{1}{2}}] \\ \lambda^{-2}(\boldsymbol{a}^+)^2 & -q^{\frac{1}{2}}\lambda^{-1}\boldsymbol{a}^+[\lambda q^{-N-\frac{1}{2}}] & [\lambda q^{-N-\frac{1}{2}}][\lambda q^{-N+\frac{1}{2}}] \end{pmatrix}. \tag{6.32}$$

The second L-operator $A_-^{(2)}(\lambda)$ is simply obtained from (6.32) by reflection along rows and columns and changing $\phi \to \phi^{-1}$.

Now let us calculate the Wronskian $\mathbf{Wr}(\phi)$ in (6.15). We do it by considering the limit $\lambda \to \infty$ and restricting (6.15) to the l-th sector where the Wronskian is proportional to the identity matrix.

We start with calculating the limit of $\mathbf{A}_+^{(I)}(\lambda)$ at $\lambda \to \infty$. It is easy to see from (6.27) that matrix elements of $A_+^{(I)}(\lambda)$ at $\lambda \to \infty$ behave like

$$\left[A_+^{(I)}(\lambda)\right]_{n,i}^{n',i'} \sim \begin{cases} \lambda^{2i'-i}(1 + O(\lambda^{-2})), & \text{for } i > i' \\ \lambda^{i}(1 + O(\lambda^{-2})), & \text{for } i \leqslant i'. \end{cases} \tag{6.33}$$

Therefore, only the matrix elements with the indices $i_k \leqslant i'_k$, $k = 1, \ldots, M$ will contribute to the leading order in λ. It follows from (6.3) that we need to consider only diagonal matrix elements of $\mathbf{A}_+^{(I)}(\lambda)$. Evaluating (6.27) at $i' = i$ and taking the normalized trace (2.11) over the auxiliary space $\mathcal{F}_q$ in the l-th sector we get

$$\mathbf{A}_+^{(I)}(\lambda)\big|_{\lambda\to\infty} = -(-\lambda)^l \phi^{2M} q^{2l-IM}\big(\boldsymbol{I} + O\big(\lambda^{-2}\big)\big), \tag{6.34}$$

where $\boldsymbol{I}$ is the identity matrix and the trace in the denominator of the RHS of (2.11) is equal to

$$\operatorname*{Tr}_{\mathcal{F}_q}\big(\phi^{2N} q^{-N\otimes H}\big) = \frac{1}{1 - \phi^{2M} q^{2l-IM}}. \tag{6.35}$$

The asymptotics (6.34) justifies our assumption (6.17) made to calculate the operator $\mathbf{A}_-^{(I)}(\lambda)$.

Taking the limit $\lambda \to \infty$ in (6.21) we obtain by the similar arguments the asymptotics of the second Q-operator $\mathbf{A}_-^{(I)}(\lambda)$

$$\mathbf{A}_-^{(I)}(\lambda)\big|_{\lambda\to\infty} = (-\lambda)^{IM-l}\big(\boldsymbol{I} + O\big(\lambda^{-2}\big)\big). \tag{6.36}$$

Substituting (6.5), (6.34), (6.36) into (6.15) we obtain the expression for the Wronskian in the l-th sector

$$\mathbf{Wr}^{(l)}(\phi) = -(-1)^{IM}\phi^M q^{l-IM}\big(1 - \phi^{2M} q^{2l-IM}\big)\boldsymbol{I}. \tag{6.37}$$

As expected the Wronskian does not depend on the spin J in the auxiliary space V_J^+. We also notice that a normalization factor in (6.37) depends on the particular choice of a λ-dependent normalization of the L-operators $A_\pm^{(I)}(\lambda)$.

7. Functional relations

In the previous section we used two fundamental functional relations (6.7) and (6.12) which relate transfer matrices $\widehat{\mathbf{T}}^{(l)}_{J,I}(\lambda;\phi)$, $\mathbf{T}^{(l)}_{J,I}(\lambda;\phi)$ and Q-operators $\mathbf{Q}^{(I)}_{\pm}(\lambda)$. Once they derived, no further algebraic work is required. All other functional relations are a consequence of these two. In this section we shall assume that $I\in\mathbb{Z}_+$ and use operators $\mathbf{A}^{(I)}_{\pm}(\lambda)$ instead of $\mathbf{Q}^{(I)}_{\pm}(\lambda)$ since they are related by a simple transformation (6.14).

Since all the above operators commute, functional equations can be rewritten in terms of its eigenvalues. Let $\widehat{\mathcal{T}}^{(l)}_{J,I}(\lambda;\phi)$, $\mathcal{T}^{(l)}_{J,I}(\lambda;\phi)$ and $\mathcal{A}^{(I)}_{\pm}(\lambda)$ be the eigenvalues of the corresponding operators.

In the l-th sector of the quantum space we have

$$\mathcal{A}^{(I)}_{+}(\lambda)=\rho_+\prod_{k=1}^{l}[\lambda/\lambda^+_k],\qquad \mathcal{A}^{(I)}_{-}(\lambda)=\rho_-\prod_{k=1}^{IM-l}[\lambda/\lambda^-_k]. \tag{7.1}$$

Let us start with the Wronskian relation between the eigenvalues of the two Q-operators. Setting $J=0$ in (6.7) we obtain

$$\phi^{-M}\mathcal{A}^{(I)}_{-}(\lambda q^{1/2})\mathcal{A}^{(I)}_{+}(\lambda q^{-1/2})-\phi^{M}\mathcal{A}^{(I)}_{+}(\lambda q^{1/2})\mathcal{A}^{(I)}_{-}(\lambda q^{-1/2})=\mathcal{W}r(\phi)h_I(\lambda)^M, \tag{7.2}$$

where $\mathcal{W}r(\phi)$ denotes the eigenvalues of the Wronskian $\mathbf{Wr}(\phi)$ and $h_I(\lambda)$ is defined in (6.8).

The functional equation (7.2) completely determines both polynomials $\mathcal{A}^{(I)}_{\pm}$ up to normalization factors $\rho_\pm$. Indeed, substituting into (7.2) $\lambda=\lambda^+_k q^{\pm 1/2}$ and $\lambda=\lambda^-_k q^{\pm 1/2}$ we obtain

$$\begin{aligned}
&-\phi^{M}\mathcal{A}^{(I)}_{+}(q\lambda^+_k)\mathcal{A}^{(I)}_{-}(\lambda^+_k)=\mathcal{W}r(\phi)h_I(\lambda^+_k q^{1/2})^M,\\
&\phi^{-M}\mathcal{A}^{(I)}_{+}(\lambda^+_k/q)\mathcal{A}^{(I)}_{-}(\lambda^+_k)=\mathcal{W}r(\phi)h_I(\lambda^+_k q^{-1/2})^M
\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}
&-\phi^{M}\mathcal{A}^{(I)}_{-}(\lambda^-_k/q)\mathcal{A}^{(I)}_{+}(\lambda^-_k)=\mathcal{W}r(\phi)h_I(\lambda^-_k q^{-1/2})^M,\\
&\phi^{-M}\mathcal{A}^{(I)}_{-}(q\lambda^+_k)\mathcal{A}^{(I)}_{+}(\lambda^-_k)=\mathcal{W}r(\phi)h_I(\lambda^-_k q^{+1/2})^M.
\end{aligned} \tag{7.4}$$

From (7.3), (7.4) we get two sets of Bethe ansatz equations

$$\phi^{\pm 2M}\frac{\mathcal{A}^{(I)}_{\pm}(q\lambda^\pm_k)}{\mathcal{A}^{(I)}_{\pm}(q^{-1}\lambda^\pm_k)}=-\left(\frac{h_I(\lambda^\pm_k q^{1/2})}{h_I(\lambda^\pm_k q^{-1/2})}\right)^M. \tag{7.5}$$

Of course, our derivation of the Bethe ansatz equations assumes that ϕ is not equal to a special value when $\mathcal{W}r(\phi)=0$. We will not discuss here this and further subtleties like the root of unity case $q^N=1$ (see [55] for further discussions and [49] for the case $q=1$).

Combining (6.7) and (6.12) for an arbitrary J we obtain

$$\begin{aligned}
&h_{I-J}(\lambda)^M\mathcal{W}r(\phi)\mathcal{T}_{J,I}(\lambda;\phi)\\
&\quad=\phi^{-(J+1)M}\mathcal{A}^{(I)}_{+}(\lambda q^{-\frac{J+1}{2}})\mathcal{A}^{(I)}_{-}(\lambda q^{\frac{J+1}{2}})-\phi^{(J+1)M}\mathcal{A}^{(I)}_{+}(\lambda q^{\frac{J+1}{2}})\mathcal{A}^{(I)}_{-}(\lambda q^{-\frac{J+1}{2}})
\end{aligned} \tag{7.6}$$

In particular, for $J=1$ we have

$$\mathcal{T}_{1,I}(\lambda;\phi)=\frac{\phi^{-2M}\mathcal{A}^{(I)}_{+}(\lambda q^{-1})\mathcal{A}^{(I)}_{-}(\lambda q)-\phi^{2M}\mathcal{A}^{(I)}_{+}(\lambda q)\mathcal{A}^{(I)}_{-}(\lambda q^{-1})}{h_{I-1}(\lambda)^M\mathcal{W}r(\phi)} \tag{7.7}$$

Multiplying (7.7) by $\mathcal{A}_{\pm}^{(I)}(\lambda)$ and using (7.2) we immediately arrive at the following equation

$$\mathcal{T}_{1,I}(\lambda;\phi)\mathcal{A}_{\pm}^{(I)}(\lambda)=\phi^{\pm M}\big[\lambda q^{\frac{1-I}{2}}\big]^M\mathcal{A}_{\pm}^{(I)}(q\lambda)+\phi^{\mp M}\big[\lambda q^{\frac{1+I}{2}}\big]^M\mathcal{A}_{\pm}^{(I)}\big(q^{-1}\lambda\big). \tag{7.8}$$

We can rewrite (7.8) back in matrix form in terms of the original operators $\mathbf{Q}_{\pm}^{(I)}(\lambda)$. Using (6.14) we obtain the famous Baxter's TQ-relation

$$\mathbf{T}_{1,I}(\lambda;\phi)\mathbf{Q}_{\pm}^{(I)}(\lambda)=\big[\lambda q^{\frac{1-I}{2}}\big]^M\mathbf{Q}_{\pm}^{(I)}(q\lambda)+\big[\lambda q^{\frac{1+I}{2}}\big]^M\mathbf{Q}_{\pm}^{(I)}\big(q^{-1}\lambda\big). \tag{7.9}$$

We just proved that the operators $\mathbf{Q}_{\pm}^{(I)}(\lambda)$ are two solutions of (7.9). Their linear independence has been proved earlier.

The TQ-relation (7.8) has a natural extension for any positive $J>1$. Combining (7.6) with the Wronskian relation (7.2) one can show that

$$\mathcal{T}_{J,I}(\lambda;\phi)=\frac{\mathcal{A}_{\pm}^{(I)}(\lambda q^{-\frac{J+1}{2}})\mathcal{A}_{\pm}^{(I)}(\lambda q^{\frac{J+1}{2}})}{h_{I-J}(\lambda)^M}\sum_{k=0}^{J}\frac{\phi^{\pm M(J-2k)}h_I(\lambda q^{k-\frac{J}{2}})^M}{\mathcal{A}_{\pm}^{(I)}(\lambda q^{k-\frac{J+1}{2}})\mathcal{A}_{\pm}^{(I)}(\lambda q^{k-\frac{J-1}{2}})}. \tag{7.10}$$

Note that the formula (7.10) allows to express $\mathcal{T}_{J,I}(\lambda)$ in terms of the eigenvalues of only one Q-operator. So it can be more convenient in cases when there is a problem to find the second linearly independent Q-operator. In particular, (7.9) is useful in the limit $\phi\to 1$ when two operators $\mathbf{A}_{\pm}^{(I)}(\lambda)$ can become linearly dependent and we can't apply (7.6) due to the Wronskian being zero.

To conclude this section we shall give standard fusion relations satisfied by the eigenvalues of the higher-spin transfer-matrices. For higher spin representations of the XXZ spin chain they first appeared in [17]. Since our normalization of the R-matrix is different, the scalar functions in our formulas are modified comparing to [17]. We have

$$\begin{aligned}\mathcal{T}_{1,I}(\lambda;\phi)\mathcal{T}_{J,I}\big(\lambda q^{\frac{J+1}{2}}\big)&=f_{IJ}^{-}\big(\lambda q^{\frac{1-I}{2}}\big)^M\mathcal{T}_{J-1,I}\big(\lambda q^{\frac{J}{2}+1}\big)\\&\quad+f_{IJ}^{+}\big(\lambda q^{\frac{1+I}{2}}\big)^M\mathcal{T}_{J+1,I}\big(\lambda q^{\frac{J}{2}}\big),\end{aligned} \tag{7.11}$$

where

$$f_{IJ}^{+}(\lambda)=\begin{cases}1, & \text{for } I>J,\\ [\lambda], & \text{for } I\leqslant J;\end{cases}\qquad f_{IJ}^{-}(\lambda)=\begin{cases}[\lambda][\lambda q^{I+1}], & \text{for } I\geqslant J,\\ [\lambda], & \text{for } I<J\end{cases} \tag{7.12}$$

and

$$\begin{aligned}\mathcal{T}_{1,I}(\lambda)\mathcal{T}_{J,I}\big(\lambda q^{-\frac{J+1}{2}}\big)&=g_{IJ}^{-}\big(\lambda q^{\frac{I-1}{2}}\big)^M\mathcal{T}_{J-1,I}\big(\lambda q^{-\frac{J}{2}-1}\big)\\&\quad+g_{IJ}^{+}\big(\lambda q^{-\frac{1+I}{2}}\big)^M\mathcal{T}_{J+1,I}\big(\lambda q^{-\frac{J}{2}}\big),\end{aligned} \tag{7.13}$$

$$g_{IJ}^{+}(\lambda)=\begin{cases}1, & \text{for } I>J,\\ [q\lambda], & \text{for } I\leqslant J;\end{cases}\qquad g_{IJ}^{-}(\lambda)=\begin{cases}[q\lambda][\lambda q^{-I}], & \text{for } I\geqslant J,\\ [q\lambda], & \text{for } I<J.\end{cases} \tag{7.14}$$

We dropped a dependence on ϕ in the functional relations (7.12), (7.14) for brevity. The proof of these relations is similar. We substitute the explicit expressions for the eigenvalues of the transfer-matrices (7.6) and (7.7) into (7.12), (7.14) and using the Wronskian relation (7.2) reduce them to identity. All calculations are slightly tedious but straightforward.

We also notice that all functional relations considered above can be generalized to the case of complex $I\in\mathbb{C}$. The quantum space will be infinite-dimensional $W=\bigotimes_{i=1}^M V_I^+$ and decompose into the direct sum of finite-dimensional blocks. All functional relations will still be satisfied in any such block with properly modified scalar functions. We will leave this as the exercise for the reader.

8. Conclusion

In this paper we derived a new formula for the $U_q(sl(2))$ R-matrix acting in the tensor product of two highest weight modules with arbitrary weights I and J. When $I = 1$, this R-matrix reduces to the standard XXZ L-operator. The formula for the matrix elements contains only one summation and is expressed in terms of the basic hypergeometric series ${}_4\phi_3$. Taking the limit $I \to \infty$ we generalized the Bazhanov, Lukyanov and Zamolodchikov construction of Q-operators to the XXZ spin chain with arbitrary spin. This includes the infinite-dimensional case, when each L-operator acts in the infinite-dimensional Verma module with a complex weight $J \in \mathbb{C}$. These Q-operators are represented as special transfer-matrices with an auxiliary space being the infinite-dimensional representation of the q-oscillator algebra. What is remarkable is that this construction is non-singular in the limit $J \to \mathbb{Z}_+$.

However, as explained in the Introduction there is an alternative construction of "factorized" Q-operators [37,38] based on the factorization property of the $U_q(sl(2))$ L-operator. This approach works well for the infinite-dimensional representations (or cyclic case $q^N = 1$), but its restriction to a finite-dimensional case requires a regularization.

The natural question now is how these two constructions of the Q-operators are related to each other. Since both Q-operators commute with the transfer-matrix, there should be a transformation between them which becomes singular in the limit of integer weights.

Another interesting challenge is to construct the XXZ Q-operator as the integral operator with a factorized kernel for the case of infinite-dimensional representations. This would allow to calculate the action of the Q-operator on the proper (polynomial) basis directly and compare it with the results of [38]. We are going to address these problems in our next publication.

Let us remind that a 3D approach of [1–3] works for the $U_q(\widehat{sl(n)})$ case with $n \geqslant 2$. So it would be interesting to obtain a generalization of the result (1.1) for higher ranks.

Finally, we notice that the formula for the R-matrix looks similar to the expression of quantum $U_q(sl(2))$ $6j$-symbols [56] in terms of the q-Racah polynomials [20] (see also [57]). A better understanding of this connection and its possible generalization to the elliptic case deserves a separate study.

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Appendix A

We start with the property (4.10) which we reformulate as

$$R_{n_1,n_2,n_3}^{n_1',n_2',n_3'} = 0, \quad \text{when } n_2 > n_2', \ -(n_2 - n_2') \leqslant n_3 \leqslant -1. \tag{A.1}$$

Using (3.2) and (3.7) we find that the matrix element in (A.1) is proportional to

$${}_2\phi_1\big(q^{-2n_2}, x; xq^{-2n_2'}; q^2, z\big), \tag{A.2}$$

where $x = q^{2+2n_1}$ and $z = q^{-2n_3}$. Using Heine's transformation of ${}_2\overline{\phi}_1$ (see (III.3) in [54]) we obtain

$$
\begin{aligned}
{}_2\overline{\phi}_1\bigl(q^{-2n_2}, x; xq^{-2n_2'}\big|q^2, z\bigr) = {} & \frac{(q^2/z; q^2)_{n_2-n_2'}(-z/q^2)^{n_2-n_2'}}{q^{(n_2-n_2')(n_2-n_2'+1)}} \\
& \times {}_2\overline{\phi}_1\bigl(q^{-2n_2'}, xq^{2(n_2-n_2')}; xq^{-2n_2}; q^2, zq^{2(n_2'-n_2)}\bigr). \qquad \text{(A.3)}
\end{aligned}
$$

The factor $(q^2/z)_{n_2-n_2'}$ in the right hand side of (A.3) is equal to zero for $z = q^{2k}$, $k = 1, \dots, n_2 - n_2'$ which exactly corresponds to the range for the index n_3 from (A.1).

Appendix B

Here we prove the identity between terminating balanced ${}_4\phi_3$ series which we used to prove (5.10). Let us start with the second Sears' transformation [54]

$$
\begin{aligned}
& {}_4\phi_3\left(\begin{matrix} q^{-m}, a, b, c \\ d, e, f \end{matrix}\,\middle|\, q, q\right) \\
& \quad = \frac{(a, ef/ab, ef/ac; q)_m}{(e, f, ef/abc; q)_m}\, {}_4\phi_3\left(\begin{matrix} q^{-m}, q^{1-m}/d, e/a, f/a, \\ q^{1-m}/a, ef/ab, ef/ac \end{matrix}\,\middle|\, q, q\right) \qquad \text{(B.1)}
\end{aligned}
$$

where $def = abcq^{1-m}$, $m \in \mathbb{Z}_+$.

One can rewrite (B.1) in terms of regularized terminating series (2.16) as follows

$$
{}_4\overline{\phi}_3\left(\begin{matrix} q^{-m}; a, b, c \\ d, e, f \end{matrix}\,\middle|\, q, q\right) = q^{m(m-1)}(ad)^m\, {}_4\overline{\phi}_3\left(\begin{matrix} q^{-m}; q^{1-m}/d, e/a, f/a, \\ q^{1-m}/a, ef/ab, ef/ac \end{matrix}\,\middle|\, q, q\right). \qquad \text{(B.2)}
$$

Now let us choose $d = q^{1-m+n}$, where m and n are two positive integers and apply the formula (B.2) first with respect to the index m and then with respect to the index n. After simple transformations we get the following result

$$
\begin{aligned}
& {}_4\overline{\phi}_3\left(\begin{matrix} q^{-m}; a, b, c \\ q^{1-m+n}, e, f \end{matrix}\,\middle|\, q, q\right) \\
& \quad = \frac{(-1)^{m+n}(ab)^n(a, b, c; q)_{m-n}}{q^{n+\frac{(m-n)(m-n-1)}{2}}}\, {}_4\overline{\phi}_3\left(\begin{matrix} q^{-n}; \frac{q}{a}, \frac{q}{b}, cq^{m-n} \\ q^{1-n+m}, \frac{qe}{ab}, \frac{qf}{ab} \end{matrix}\,\middle|\, q, q\right), \qquad \text{(B.3)}
\end{aligned}
$$

where $abc = efq^n$.

We also need another useful identity which allows to relate matrix elements of two local L-operators $A_+^{(J)}(\lambda)$ and $A_-^{(J)}(\lambda)$. It reads

$$
\begin{aligned}
& (-1)^{j+j'}\frac{(q^2; q^2)_{J-j}(\lambda^2 q^{1-J-2(n-j')}; q^2)_{J-j-j'}}{(q^{2+2n}; q^2)_{j-j'}(q^2; q^2)_j}\, {}_3\overline{\phi}_2\left(\begin{matrix} q^{-2j}; q^{-2j'}, \lambda^2 q^{1-J} \\ q^{-2J}, q^{2(1+n-j')} \end{matrix}\,\middle|\, q^2, q^2\right) \\
& \quad = q^{j(j-1)-j'(j'-1)+2J(J-2j-n)+2n(j+j')}\, {}_3\overline{\phi}_2\left(\begin{matrix} q^{-2(J-j)}; q^{-2(J-j')}, \lambda^2 q^{1-J} \\ q^{-2J}, q^{2(1+n-J+j)} \end{matrix}\,\middle|\, q^2, q^2\right) \\
& \qquad \text{(B.4)}
\end{aligned}
$$

provided that $0 \leqslant j, j' \leqslant J$, $J \in \mathbb{Z}_+$ and $n, n + j - j' \geqslant 0$. This formula can be proved by using transformations (III.9–III.13) in [54] for ${}_3\overline{\phi}_2$ series. We omit the details.

Appendix C

In this appendix we derive the recurrence relations which completely determine matrix elements $[R_{I,J}(\lambda;1)]_{i,j}^{i'j'}$. Eq. (5.13) can be represented as a two-by-two matrix equation and leads to four difference equations for matrix elements. These four equations split into six equations if we decouple them with respect to the spectral parameter μ. Further algebra shows that only three of them are linearly independent provided that the indices satisfy the condition $i+j=i'+j'$. Using the notation (5.15) we obtain

$$\begin{aligned}&\big(1-\lambda^2 q^{2(1+i+j)-I-J}\big)\big(1-q^{2+2i'}\big)S_{i,j}^{i',j'}\\&\quad-\lambda q^{i'-j}\big(1-q^{2(1+i+i'-I)}\big)\big(1-q^{2+2j}\big)S_{i,j+1}^{i'+1,j'}\\&\quad-q^{3j-J-j'}\big(1-q^{2+2i}\big)\big(1-\lambda^2 q^{2(1+i'-j)+J-I}\big)S_{i+1,j}^{i'+1,j'}=0,\end{aligned}\tag{C.1}$$

$$\begin{aligned}&\lambda q^{i-j-2i'+J-2}\big(1-q^{2(2+j+j'-J)}\big)\big(1-q^{2+2i'}\big)S_{i,j+1}^{i',j'+1}\\&\quad-\big(1-\lambda^2 q^{2i-2j'+J-I}\big)\big(1-q^{2+2j'}\big)S_{i,j+1}^{i'+1,j'}\\&\quad+q^{3i-i'-I-1}\big(1-\lambda^2 q^{I+J-2(1+i+j)}\big)\big(1-q^{4+2j}\big)S_{i,j+2}^{i'+1,j'+1}=0,\end{aligned}\tag{C.2}$$

$$\begin{aligned}&\lambda q^{3+3i-j-2I+J}\big(1-q^{2I-2i}\big)\big(1-q^{2(J-j-j')}\big)S_{i,j}^{i',j'}\\&\quad+\big(1-q^{2(1+J-j)}\big)\big(1-\lambda^2 q^{2(1+i-j')+J-I}\big)S_{i+1,j-1}^{i',j'}\\&\quad-q^{3+3i-i'-I}\big(1-q^{2J-2j'}\big)\big(1-\lambda^2 q^{I+J-2i-2j}\big)S_{i+1,j}^{i',j'+1}=0.\end{aligned}\tag{C.3}$$

We can exclude shifts in two spins in the system (C.1)–(C.3) and derive a second order recurrence relation in one spin variable. This recurrence relation is very similar to a recursion satisfied by q-Racah polynomials. It has a unique solution which truncates for negative values of indices and is given by (5.8).

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