Solutions of Yule-Walker equations for singular AR processes

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A study is presented on solutions of the Yule-Walker equations for singular AR processes that are stationary outputs of a given AR system. If the Yule-Walker equations admit more than one solution and the order of the AR system is no less than two, the solution set includes solutions which define unstable AR systems. The solution set also includes one solution, the minimal norm solution, which defines an AR system whose characteristic polynomial has either only stable zeros (implying that only one stationary output exists for this system and it is linearly regular) or has stable zeros as well as zeros of unit modulus, (implying that stationary solutions of this system are a sum of a linearly regular process and a linearly singular process). The numbers of stable and unit circle zeros of the characteristic polynomial of the defined AR system can be characterized in terms of the ranks of certain matrices, and the characteristic polynomial of the AR system defined by the minimal norm solution has the least number of unit circle zeros and the most number of stable zeros over all possible solutions.

Keywords: Singular AR process; minimal norm solution; stationary process.

JEL classifications: C32; C67.

1. INTRODUCTION

In this article we consider multivariable AR systems of the form

$$z_t = e_1 z_{t-1} + \dots + e_p z_{t-p} + v_t, \quad (1)$$

where $e_j \in \mathbb{R}^{r \times r}$, $j = 1, \dots, p$ and v_t is a white-noise $r \times 1$ vector. We are only interested in stationary solutions of (1). In addition, we assume that v_t is uncorrelated with the past z_s , $s < t$. In that case, v_t is the innovation of z_t . Our emphasis is on the case where $\Sigma_v = \mathbb{E}[v_t v_t^T]$ is singular.

According to the Wold decomposition (e.g. Hannan and Deistler (1988)) a stationary process z_t can be decomposed into a sum of a linearly regular part z_t^r and linearly singular part z_t^s which are mutually orthogonal, i.e.

$$z_t = z_t^r + z_t^s. \quad (2)$$

For any stationary solution z_t of (1), if its corresponding covariance matrices are denoted as $\gamma_j = \mathbb{E}[z_t z_{t-j}^T]$, $j = 0, 1, 2, \dots$, then its associated Yule-Walker equations are of the form

$$[e_1 \dots e_p] \Gamma_p = [\gamma_1 \dots \gamma_p] \quad (3)$$

$$\Sigma_v = \gamma_0 - [e_1 \dots e_p][\gamma_1 \dots \gamma_p]^T, \quad (4)$$

where

$$\Gamma_p = \begin{bmatrix} \gamma_0 & \dots & \dots & \gamma_{p-1} \\ \vdots & \gamma_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \gamma_{p-1}^T & \dots & \dots & \gamma_0 \end{bmatrix}.$$

Here we are interested in the case where Σ_v is singular and we will call the corresponding AR system *singular*. Singular AR systems (1) can be represented as

$$e(z)z_t = v_t = f e_t, \quad (5)$$

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where z is used for the backward shift operator as well as a complex variable, $e(z) = I - e_1 z - \dots - e_p z^p$, and $f \in \mathbb{R}^{r \times q}$ where $\Sigma_v = ff^T$ and $rk(\Sigma_v) = q < r$. In this article, we call $\det(e(z))$ the *characteristic polynomial* of an AR system of the form (5). A particular example arises when $f = 0$ and thus $\Sigma_v = 0$. The AR system is then degenerate, the solutions of which are simply a linearly singular process.

In the standard case where the innovation covariance matrix Σ_v is non-singular, it is well known that a stationary solution exists for (1) if and only if the roots of $\det(e(z))$ lie outside the unit circle. In that case the stationary solution is unique, as are the covariance matrices γ_j , and these matrices and the matrices e_j and Σ_v are linked by the Yule-Walker equations given by (3) and (4). Moreover, the matrix Γ_p is non-singular. Further, given the matrix sequence of γ_j such that the matrix Γ_p is non-singular, the matrices e_j can be uniquely determined using the Yule-Walker equations.

However, when Σ_v is singular, the simple conclusions mentioned in the previous paragraph no longer hold in general and things can become rather complicated. To appreciate this, we consider a simple example.¹

EXAMPLE 1: Consider the following AR system.

$$(I - ez)z_t = f\varepsilon_t, \quad (6)$$

where e is a 2×2 non-singular matrix with two distinct eigenvalues, f is a 2×1 vector, and ε_t is scalar.

Since e is a 2×2 non-singular matrix with two distinct eigenvalues, there exists a non-singular matrix M and a diagonal matrix δ such that $\delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} = MeM^{-1}$.

Define $\omega_t = [\omega_{1t} \ \omega_{2t}]^T = Mz_t$ and $g = [g_1 \ g_2]^T = Mf$, then it follows from (6) that

$$(I - \delta z)\omega_t = g\varepsilon_t \quad (7)$$

which can be rewritten as

$$\begin{aligned} (1 - \delta_1 z)\omega_{1t} &= g_1 \varepsilon_t, \\ (1 - \delta_2 z)\omega_{2t} &= g_2 \varepsilon_t. \end{aligned} \quad (8)$$

The following observations can be made on (8):

- If $|\delta_1| < 1$ and $|\delta_2| < 1$, then there exists only one stationary solution for (6), which is linearly regular.
- If $|\delta_1| < 1$ and $|\delta_2| > 1$, then stationary solutions for (6) exist only if $g_2 = 0$. When $g_2 = 0$, the solution is unique: $\omega_{1t} = (1 - \delta_1 z)^{-1} g_1 \varepsilon_t$, $\omega_{2t} = 0$, which is linearly regular.
- If $|\delta_1| < 1$ and $|\delta_2| = 1$, then stationary solutions for (6) exist only if $g_2 = 0$. When $g_2 = 0$, the stationary solutions are: $\omega_{1t} = (1 - \delta_1 z)^{-1} g_1 \varepsilon_t$, $\omega_{2t} = \alpha$, where α is any stochastic variable uncorrelated with ε_s for all s . The only linearly regular solution in this stationary solution set corresponds to a stochastic variable α that equals to zero almost everywhere. For the linearly regular solution, it is obvious that

$$\Gamma_1 = \gamma_0 = \mathbb{E}[z_t z_t^T] = M^{-1} \mathbb{E}[\omega_t \omega_t^T] M^{-T} = M^{-1} \begin{bmatrix} \mathbb{E}[\omega_{1t}^2] & 0 \\ 0 & 0 \end{bmatrix} M^{-T},$$

which implies that Γ_1 is singular. For a solution which additionally has a singular component, the above matrix Γ_1 is replaced by

$$M^{-1} \begin{bmatrix} \mathbb{E}[\omega_{1t}^2] & 0 \\ 0 & \mathbb{E}[\alpha^2] \end{bmatrix} M^{-T}.$$

This illustrates that the covariance appearing in the Yule-Walker equations actually depends on the particular admissible solution for (6) [or (1)].

The last two observations imply that unlike regular AR systems, (i) singular AR systems may admit stationary solutions even if the roots of $\det(e(z))$ lie inside or on the unit circle; (ii) singular AR systems can have multiple stationary solutions; and (iii) the matrix Γ_p corresponding to a stationary solution can be singular.

Our interest in singular AR systems was triggered by the analysis of Generalized Dynamic Factor Models (see e.g. Forni *et al.* (2000), Forni and Lippi (2001), Anderson and Deistler (2009), and Deistler *et al.* (2010)). In a certain sense these systems are more complicated compared to the regular AR case. This in particular applies to the problem of identifiability see Anderson and Deistler (2008), Anderson *et al.* (2009). The elements of the equivalence class of left coprime $(e(z), f)$ are related by left multiplication by unimodular matrices, see Hannan and Deistler (1988).

A detailed analysis of singular AR systems has been given in Inouye (1983) which is the main reference for our article.

2. SOLUTIONS OF YULE-WALKER EQUATIONS

For AR systems (1), if $rk(\Sigma_v) > 0$ holds, a stationary solution (if it exists) has a linearly regular part

$$z_t^r = e^{-1}(z)f\varepsilon_t \quad (9)$$

¹We are grateful to a reviewer for suggesting this example.

and the corresponding transfer function $e^{-1}(z)f$ has no poles inside and on the unit circle. Note that this is not the same as asserting that the characteristic polynomial $\det(e(z))$ has no zeros inside or on the unit circle, because $(e(z), f)$ is not necessarily left coprime. However, there always exists a left coprime pair $(\bar{e}(z), f)$ with the same transfer function such that $\det(\bar{e}(z))$ has all zeros outside the unit circle. The linearly singular part of the stationary solution is a solution of the homogeneous equation

$$e(z)z_t^s = 0 \quad (10)$$

and is obtained by a suitable choice of initial values and corresponds to the zeros of $\det(e(z))$ on $|z| = 1$. Note that for singular AR systems the matrix Γ_{p+1} must be singular and Γ_p may be singular (for a more detailed discussion see Deistler *et al.* (2010)). If Γ_p is non-singular, the Yule-Walker equations have a unique solution. If Γ_p is singular then the Yule-Walker equations are still always solvable and the solution set for each row of $[e_1 \dots e_p]$ is an affine subspace obtained by a particular solution plus an arbitrary vector in the kernel of Γ_p .

The (row-wise) minimum norm solution in the solution set is given by

$$[\hat{e}_1 \dots \hat{e}_p] = [\gamma_1 \dots \gamma_p] \Gamma_p^\#, \quad (11)$$

where $\Gamma_p^\#$ is the Moore-Penrose pseudo-inverse of Γ_p , which is defined as

$$\Gamma_p^\# = O_p^T \begin{bmatrix} \Lambda_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} O_p, \quad (12)$$

where $O_p^T O_p = I$, and Λ_1 is diagonal and non-singular and consists of all nonzero eigenvalues of Γ_p .

Making use of the minimal norm solution, it can be shown that all solutions of the Yule-Walker equations given by (3) and (4) can be written in the following form

$$[e_1 \dots e_p] = [\hat{e}_1 \dots \hat{e}_p] + [0 \ \Phi] O_p \quad (13)$$

with Φ being any matrix with r rows and $pr - rk(\Gamma_p)$ columns. Note also that Σ_v in (4) is the same for all solutions $[e_1 \dots e_p]$.

3. MAIN RESULTS

Consider the system

$$\hat{e}(z)z_t = v_t = f \varepsilon_t \quad (14)$$

defined by the minimal norm solution (11) and by $ff^T = \Sigma_v$.

The AR system given by (1) or (5) is said to be *stable* if its characteristic polynomial satisfies $\det(e(z)) \neq 0$, $|z| \leq 1$. It is said to be *unstable* if its characteristic polynomial has a zero inside the unit circle. Thus, in this sense, if the characteristic polynomial has zeros outside and on the unit circle, the AR system is neither stable nor unstable.

As has been shown in Deistler *et al.* (2010), the solution set of the Yule-Walker equations contains a solution $[e_1 \dots e_p]$ that corresponds to a stable AR system (1) if and only if the minimum norm solution (11) defines a stable AR system (For non-singular Γ_p , this is of course trivial; it is quite nontrivial when Γ_p is singular). Much of this paper explores the consequences of this condition not being satisfied.

It is straightforward to transform (1) into an AR(1) system of the form

$$x_t = E x_{t-1} + G \varepsilon_t, \quad (15)$$

where $x_t = [z_t^T \dots z_{t-p+1}^T]^T$, and

$$E = \begin{bmatrix} e_1 & e_2 & \dots & e_{p-1} & e_p \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix}, \quad G = \begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (16)$$

Then for stationary z_t , $\Gamma_p = \mathbb{E} x_t x_t^T$ and the following Lyapunov equation (e.g. Gajic and Qureshi (1995)) holds

$$\Gamma_p - E \Gamma_p E^T = Q, \quad (17)$$

$$\text{where } Q = \begin{bmatrix} \Sigma_v & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & & \\ 0 & & & 0 \end{bmatrix}.$$

For the minimal norm solution $[\hat{e}_1 \dots \hat{e}_p]$, define $\hat{E}$ analogously as E ; there holds

$$\Gamma_p - \hat{E} \Gamma_p \hat{E}^T = Q. \quad (18)$$

The matrix pair $[E, G]$ is said to be *reachable* if $[\lambda I - E \ G]$ is of full row rank for all $\lambda \in \mathbb{C}$, or equivalently the matrix $[G \ EG \ \dots \ E^{p-1}G]$ is of full row rank. Given any eigenvalue λ_0 of E , if $[\lambda_0 I - E \ G]$ is of full row rank, then λ_0 is said to be a *reachable mode*. Otherwise, it is called an *unreachable mode* (e.g. Kailath (1980) or Gajic and Qureshi (1995)). One can derive the following results, cast in the language of linear system theory.

LEMMA 1. Assume that z_t is a stationary solution of (1), define $x_t = [z_t^T \ \dots \ z_{t-p+1}^T]^T$, let E, G be defined as in (16) and consider (17) with $\Sigma_v = \bar{f}\bar{f}^T$. Then

- [i] If the pair $[E, G]$ is reachable, then Γ_p is positive definite.
- [ii] If $|\lambda_i(E)| < 1$ and Γ_p is positive definite, then the pair $[E, G]$ is reachable.
- [iii] If Γ_p is positive definite and reachability does not hold, then E has one or more eigenvalues on the unit circle corresponding to unreachable modes.
- [iv] Suppose (17) for fixed E and Σ_v has a nonnegative solution $\bar{\Gamma}_p$. Define $\Gamma_{\min} = \sum_{j=0}^{\infty} E^j G G^T (E^T)^j$. Then $\Gamma_{\min}$ exists and satisfies (17), and any nonnegative solution Γ_p of (17) satisfies $\Gamma_p \geq \Gamma_{\min}$.

PROOF. [i] Note that $x_t = [z_t^T \ \dots \ z_{t-p+1}^T]^T$ is stationary as z_t is stationary by assumption. It is shown in Section 2 that $z_t = z_t^r + z_t^s$, where z_t^r and z_t^s are the linearly regular part and the linearly singular part respectively. Accordingly, we have $x_t = x_t^r + x_t^s$, where $x_t^r = [(z_t^r)^T \ \dots \ (z_{t-p+1}^r)^T]^T$ and $x_t^s = [(z_t^s)^T \ \dots \ (z_{t-p+1}^s)^T]^T$ and are the linearly regular part and the linearly singular part of x_t respectively. Since the linearly regular part x_t^r of the stationary process x_t , corresponds to the transfer function $[I - EZ]^{-1}G$, this transfer function can only have poles outside the unit circle. We have $x_t^r = [GEGE^2G \ \dots][e_t^T \ e_{t-1}^T \ e_{t-2}^T \ \dots]^T$ and thus by reachability

$$\Gamma_p^r = [GEGE^2G \ \dots][GEGE^2G \ \dots]^T \quad (19)$$

has full rank. Therefore the same holds for Γ_p as it is the sum of Γ_p^r and Γ_p^s , the latter being the covariance of x_t^s .

[ii] Under the assumption $|\lambda_i(E)| < 1$, Γ_p is equal to its linearly regular part Γ_p^r . Thus the result follows immediately from (19).

[iii] If reachability does not hold, then Γ_p is singular and thus $\Gamma_p - \Gamma_p^r \neq 0$ and corresponds to the linearly singular component which corresponds to eigenvalues on the unit circle.

[iv] Define

$$\Gamma_{(i)} = \sum_{j=0}^i E^j \text{diag} [\Sigma, 0, \dots, 0] (E^T)^j, \quad (20)$$

and $Q = \text{diag} [\Sigma, 0, \dots, 0]$.

We claim that $\bar{\Gamma}_p \geq \Gamma_{(i+1)} \geq \Gamma_{(i)} \forall i$. Observe that $\bar{\Gamma}_p = E\bar{\Gamma}_p E^T + Q \geq Q = \Gamma_0$. Assume for an induction that $\bar{\Gamma}_p \geq \Gamma_{(i)}$. Then

$$\bar{\Gamma}_p = Q + E\bar{\Gamma}_p E^T \geq Q + E\Gamma_{(i)} E^T = \Gamma_{(i+1)} \quad (21)$$

Hence $\Gamma_{\min}$ is well defined as $\lim_{i \rightarrow \infty} \Gamma_{(i)}$, and it satisfies the Lyapunov equation (17), as well as $\bar{\Gamma}_p \geq \Gamma_{\min}$. Furthermore, since the above arguments can be carried over to any nonnegative solution Γ_p of the Lyapunov equation (17), we have $\Gamma_p \geq \Gamma_{\min}$. This completes the proof of conclusion [iv].

The main results are presented in the following four theorems, where the first theorem deals with the case that Γ_p is non-singular and the remaining three theorems deal with the case that Γ_p is singular.

THEOREM 1. Consider a stationary solution of the AR system (1) and suppose that the corresponding Γ_p is non-singular. Let $[e_1 \ \dots \ e_p]$ denote the unique solution of the Yule-Walker equations (3) and (4) and let E be the corresponding block companion matrix, f be such that $\bar{f}\bar{f}^T = \Sigma_v$, and G be defined by (16). Then the associated AR system (14) is stable if and only if $\text{rk}([G \ EG \ \dots \ E^{p-1}G]) = \text{rk}(\Gamma_p)$. When this condition fails, all zeros of the characteristic polynomial $e(z)$ lie outside or on the unit circle, with the number of unit circle zeros equal to $\text{rk}(\Gamma_p) - \text{rk}([G \ EG \ \dots \ E^{p-1}G])$.

THEOREM 2. Suppose that the AR system (1) has one or more stationary solutions, and let Γ_p be defined using one of these solutions. Suppose that Γ_p is singular and $p > 1$. Then there always exists at least one solution $[e_1 \ \dots \ e_p]$ of the Yule-Walker equations (3) and (4) such that the AR system (1) is unstable.

THEOREM 3. Consider a stationary solution of the AR system (1) and suppose that the corresponding Γ_p is singular. Let $\hat{e}_i$, $i = 1, \dots, p$ denote the minimal norm solution of the Yule-Walker equations (3) and (4) and let $\hat{E}$ be the corresponding block companion matrix, $\hat{f}$ be such that $\hat{f}\hat{f}^T = \Sigma_v$ and G be defined as in (16). Then the minimal norm solution defines a stable AR system (14) if and only if $\text{rk}([G \ \hat{E}G \ \dots \ \hat{E}^{p-1}G]) = \text{rk}(\Gamma_p)$.

THEOREM 4. Consider a stationary solution of the AR system (1) and suppose that the corresponding Γ_p is singular. Let $\hat{e}_i$, $i = 1, \dots, p$ denote the minimal norm solution of the Yule-Walker equations (3) and (4) and let $\hat{E}$ be the corresponding companion matrix. If the AR system (14) is not stable, then the characteristic polynomial of the minimum norm solution has the following properties

- [i] It can not have zeros inside the unit circle and must have zeros both outside and on the unit circle.
- [ii] The number of unit circle zeros is equal to $\text{rk}(\Gamma_p) - \text{rk}([G \hat{E} G \dots \hat{E}^{p-1} G])$.
- [iii] It has the least number of unit circle zeros among all solutions of the Yule Walker equations.
- [iv] It has the largest number of stable zeros among all solutions of the Yule Walker equations.

The proofs of these theorems are placed in the appendix.

So as to gain further insights into Theorem 4, a simple example is provided below to show that the minimal norm solution of the Yule-Walker equations may be not stable when Γ_p is singular.

EXAMPLE 2: Consider the following AR system.

$$(I - e_1 z - e_2 z^2)z_t = f \varepsilon_t, \quad (22)$$

where $z_t = [z_{1t} \ z_{2t}]^T$, ε_t is scalar and $\mathbb{E}[\varepsilon_t^2] = 1$,

$$e_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} -0.6 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The system equation (22) can be rewritten as

$$\begin{aligned} (1 - 0.5z + 0.6z^2)z_{1t} &= \varepsilon_t, \\ (1 - z)z_{2t} &= 0. \end{aligned} \quad (23)$$

It is obvious that the stationary solutions of (22) are: $z_{1t} = (1 - 0.5z + 0.6z^2)^{-1} \varepsilon_t$, $z_{2t} = \alpha$, where α is any stochastic variable uncorrelated with ε_s for all s .

Now take z_{2t} as any stochastic variable α , which is uncorrelated with ε_s for all s and whose variance is 1. Then this stationary solution of (23) leads to

$$\Gamma_p = \begin{bmatrix} 1.7316 & 0 & 0.5411 & 0 \\ 0 & 1 & 0 & 1 \\ 0.5411 & 0 & 1.7316 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad [\gamma_1 \ \gamma_2] = \begin{bmatrix} 0.5411 & 0 & -0.7684 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \quad (24)$$

Obviously Γ_p is singular as it has two rows that are the same. The minimal norm solution is computed as

$$[\hat{e}_1 \ \hat{e}_2] = [\gamma_1 \ \gamma_2] \Gamma_p^\# = \begin{bmatrix} 0.5000 & 0 & -0.6000 & 0 \\ 0 & 0.5000 & 0 & 0.5000 \end{bmatrix}. \quad (25)$$

It is easy to check that $\det(\hat{e}(z)) = (1 - 0.5z + 0.6z^2)(1 - 0.5z - 0.5z^2)$. The roots of $\det(\hat{e}(z))$ are $0.4167 + 1.2219i$, $0.4167 - 1.2219i$, -2 , 1 , where the first three roots are outside the unit circle and the last one is on the unit circle. Since $\det(\hat{e}(z))$ has one root on the unit circle, the minimal norm solution of the Yule-Walker equations is indeed *not* stable.

Based on the above example, a remark can be made as follows: It has been shown that when Γ_p is singular and the minimum norm solution is not stable, the minimum norm solution is the best possible solution in the sense of Theorem 4. In other words, it is impossible to find a stable characteristic polynomial by solving the Yule-Walker equations associated with Γ_p . Recall also that when the pair $(\hat{e}(z), f)$ is not coprime, there is a coprime pair $(\bar{e}(z), f)$ with the same transfer function and such that $\det(\bar{e}(z))$ is stable. This however would correspond to different covariances γ_p (which are associated with the linearly regular part z_t^r of z_t).

4. CONCLUSIONS

We have obtained a number of interesting properties of solutions of the Yule-Walker equations of a singular AR system with a stationary output process. In case that the Yule-Walker equations have one unique solution, the associated AR system has a characteristic polynomial whose zeros are confined to the exterior of the unit circle, or the unit circle itself. The number of unit circle zeros is easily characterized by a rank test. In case that the Yule-Walker equations admit multiple solutions, it has been proved first that there must exist a solution that defines an AR system that is unstable if the maximal lag of the AR system is larger than or equal to two. Then, sufficient and necessary conditions have been derived for the minimal norm solution to define a stable AR system, or equivalently, for there to exist any solution of the Yule-Walker equations defining a stable AR system. Next, when the minimal norm solution cannot define a stable AR system, it has also been proved that it will always define an AR system whose characteristic

polynomial cannot have unstable zeros but has both stable and unit circle zeros. Finally, the number of unit circle zeros of the characteristic polynomial is characterized by a rank test that is the same as the rank test for the unique solution case, and the characteristic polynomial of the AR system defined by the minimal norm solution is shown to have the most number of stable zeros and the least number of unit circle zeros.

APPENDIX: PROOFS OF THEOREMS

PROOF OF THEOREM 1. If the associated AR system (14) is stable, then the linearly singular part of the stationary solution is zero. Thus, $\Gamma_p = \Gamma_p^r$. Now, by the assumption that Γ_p is non-singular, it follows from (19) that $rk([G \ EG \ \dots \ E^{p-1}G]) = rk(\Gamma_p)$. Conversely, the condition $rk([G \ EG \ \dots \ E^{p-1}G]) = rk(\Gamma_p)$ says that the pair $[E, G]$ is reachable, which is equivalent to saying that $(\lambda I - E, G)$ is left coprime, and thus, since the transfer function is stable, $e(z)$ has to be stable.

If $rk([G \ EG \ \dots \ E^{p-1}G]) \neq rk(\Gamma_p)$, then $rk([G \ EG \ \dots \ E^{p-1}G]) < rk(\Gamma_p)$ and the pair $[E, G]$ is not reachable. Then it is a well known fact from linear systems theory (see Kailath (1980)) that there exists a non-singular matrix $\bar{T}$ such that

$$\bar{T}E\bar{T}^{-1} = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} \\ 0 & \bar{E}_{22} \end{bmatrix}, \quad \bar{T}G = \begin{bmatrix} g_1 \\ 0 \end{bmatrix}, \quad (26)$$

where the pair $[\bar{E}_{11}, g_1]$ is reachable, every eigenvalue of $\bar{E}_{22}$ is an unreachable mode of the pair $[E, G]$, and the number of rows (columns) of $\bar{E}_{11}$ is equal to $rk([G \ EG \ \dots \ E^{p-1}G])$.

As the pair $[\bar{E}_{11}, g_1]$ is reachable and the corresponding transfer function $[I - \bar{E}_{11}z^{-1}g_1]$ has its poles outside the unit circle (as e_t is persistently exciting and the output is stationary), all eigenvalues of $\bar{E}_{11}$ must be of modulus smaller than 1.

Let λ be an eigenvalue of $\bar{E}_{22}$. Since λ is an unreachable mode of the pair $[E, G]$, then its associated eigenvector x satisfies $x^*E = \lambda x^*$, $x^*G = 0$. Using (18) it is immediate that $(1 - |\lambda|^2)x^*\Gamma_p x = 0$. Since Γ_p is non-singular and thus positive definite, it follows that $|\lambda| = 1$. This proves that all eigenvalues of $\bar{E}_{22}$ are on the unit circle.

Note that E and Γ_p has the same number of rows and columns and the number of rows (columns) of $\bar{E}_{11}$ is equal to $rk([G \ EG \ \dots \ E^{p-1}G])$, it follows that the number of rows (columns) of $\bar{E}_{22}$ is equal to $rk(\Gamma_p) - rk([G \ EG \ \dots \ E^{p-1}G])$. This fact together with the facts that all eigenvalues of $\bar{E}_{11}$ are inside the unit circle and all eigenvalues of $\bar{E}_{22}$ are on the unit circle implies that all zeros of the characteristic polynomial $e(z)$ lie outside or on the unit circle, with the number of unit circle zeros equal to $rk(\Gamma_p) - rk([G \ EG \ \dots \ E^{p-1}G])$. $\square$

PROOF OF THEOREM 2. Define $\hat{F} = O_p \hat{E} O_p^T$ and $\bar{F} = O_p \bar{E} O_p^T$, where $O_p^T O_p = I$ and $O_p^T \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} O_p = \Gamma_p$. It has been established in the proof Theorem 5 in Deistler et al. (2010) that

$$\hat{F} = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ 0 & \hat{F}_{22} \end{bmatrix}, \quad (27)$$

where all eigenvalues of $\hat{F}_{22}$ are within the unit circle (no matter where the eigenvalues of $\hat{F}_{11}$ lie), and

$$\bar{F} = \begin{bmatrix} \bar{F}_{11} & \bar{F}_{12} \\ 0 & \bar{F}_{22} \end{bmatrix}, \quad (28)$$

where $\bar{F}_{11} = \hat{F}_{11}$. Note that $\hat{F}_{22}$ and $\bar{F}_{22}$ have the same number of rows and columns as the number of zero eigenvalues of Γ_p .

Partition O_p according to $\hat{F}$ as

$$O_p = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix}. \quad (29)$$

Using (13) and noting that $\hat{F}_{22}$ and Φ have the same number of columns, it is easy to show that

$$\bar{F} = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} + O_{11}\Phi \\ 0 & \hat{F}_{22} + O_{21}\Phi \end{bmatrix}. \quad (30)$$

We claim that $O_{21} \neq 0$. To obtain a contradiction, assume that $O_{21} = 0$; then it follows from $O_p^T O_p = I$ that $O_{12} = 0$.

This implies that $O_p = \begin{bmatrix} O_{11} & 0 \\ 0 & O_{22} \end{bmatrix}$. Then $\begin{bmatrix} O_{11}^T \Lambda_1 O_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \gamma_0 & \cdots & \cdots & \gamma_{p-1} \\ \vdots & \gamma_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \gamma_{p-1}^T & \cdots & \cdots & \gamma_0 \end{bmatrix}$. If $pr - rk(\Gamma_p) \geq r$, then the bottom right

zero block of Γ_p has at least r rows and columns and thus $\gamma_0 = 0$ by looking at the matrix γ_0 sitting at the bottom right corner of Γ_p , which is a contradiction to the definition of the AR system (1). If $pr - rk(\Gamma_p) < r$ and $p > 1$, the bottom right zero block of Γ_p has

strictly less than r rows and columns and thus γ_0 sitting at the right corner of Γ_p must have the form $\gamma_0 = \begin{bmatrix} \gamma_{01} & 0 \\ 0 & 0 \end{bmatrix}$, which is not positive definite. However, since $O_{11}^T \Lambda_1 O_{11}$ is positive definite and its dimension is equal to $rk(\Gamma_p)$ (which satisfies that $rk(\Gamma_p) > (p-1)r \geq r$), γ_0 , which sits at the top left corner of Γ_p must sit at the top left corner of $O_{11}^T \Lambda_1 O_{11}$, and thus must be positive definite, another contradiction results. This proves the claim.

Since $O_{21} \neq 0$, then the matrix pair $\hat{F}_{22}, O_{21}$ has at least one reachable mode. Therefore, there always exists a matrix Φ such that $\hat{F}_{22} + O_{21}\Phi$ will have at least one eigenvalue that lies outside the unit circle. This completes the proof.

PROOF OF THEOREM 3. If the minimal norm solution defines a stable AR system (14), then as in the proof of Theorem 1, it can be shown that $rk([G \hat{E}G \cdots \hat{E}^{p-1}G]) = rk(\Gamma_p)$.

Conversely, suppose that $rk([G \hat{E}G \cdots \hat{E}^{p-1}G]) = rk(\Gamma_p)$, it is left to show that the minimal norm solution defines a stable AR system (14). As mentioned in the proof of Theorem 2, it has been established in the proof Theorem 5 in Deistler *et al.* (2010) that

$$\hat{F} = O_p \hat{E} O_p^T = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ 0 & \hat{F}_{22} \end{bmatrix}, \quad (31)$$

where all eigenvalues of $\hat{F}_{22}$ are within the unit circle (no matter where the eigenvalues of $\hat{F}_{11}$ lie).

It has also been established in the proof of Theorem 5 in Deistler *et al.* (2010) that the matrix $\hat{F}$ satisfies:

$$\begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} - \hat{F} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \hat{F}^T = O_p G G^T O_p^T = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad (32)$$

where $rk(\Lambda_1) = rk(\Gamma_p)$ and the first and last matrices in the above equation are partitioned the same way as $\hat{F}$.

Let $\hat{G} = O_p G$; it is easy to show that $\hat{G}^T = [\hat{G}_1^T \ 0]$, $\Sigma_1 = \hat{G}_1 \hat{G}_1^T$.

Since

$$rk([G \hat{E}G \cdots \hat{E}^{p-1}G]) = rk(O_p [G \hat{E}G \cdots \hat{E}^{p-1}G]) = rk([\hat{G}_1 \ \hat{F}_{11} \hat{G}_1 \cdots \hat{F}_{11}^{p-1} \hat{G}_1]),$$

there holds

$$rk([\hat{G}_1 \ \hat{F}_{11} \hat{G}_1 \cdots \hat{F}_{11}^{p-1} \hat{G}_1]) = rk(\Gamma_p) = rk(\Lambda_1).$$

This equation implies that the pair $[\hat{F}_{11}, \hat{G}_1]$ is reachable. Then it follows from $\Lambda_1 - \hat{F}_{11} \Lambda_1 \hat{F}_{11}^T = \hat{G}_1 \hat{G}_1^T$ that all eigenvalues of $\hat{F}_{11}$ are within the unit circle. $\square$

PROOF OF THEOREM 4. As in the proof of Theorem 3, we have

$$\hat{F} = O_p \hat{E} O_p^T = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ 0 & \hat{F}_{22} \end{bmatrix} \quad (33)$$

and

$$\begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} - \hat{F} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \hat{F}^T = \begin{bmatrix} \hat{G}_1 \\ 0 \end{bmatrix} \begin{bmatrix} \hat{G}_1^T \\ 0 \end{bmatrix}^T, \quad (34)$$

where all eigenvalues of $\hat{F}_{22}$ are within the unit circle and Λ_1 is positive definite, $rk(\Lambda_1) = rk(\Gamma_p)$ and the first and last matrices in the above equation are partitioned the same way as $\hat{F}$. Since Γ_p is singular, $\hat{F}_{22}$ always exists and has at least one eigenvalue inside the unit circle. This proves that the characteristic polynomial of the AR system defined by the minimal norm solution will always have a stable zero.

If the pair $[\hat{F}_{11}, \hat{G}_1]$ is reachable, all eigenvalues of $\hat{F}_{11}$ must lie within the unit circle. This implies that $\hat{E}$ can only have eigenvalues inside the unit circle, which contradicts the assumption that the minimal norm solution does not define a stable AR system.

Hence, the pair $[\hat{F}_{11}, \hat{G}_1]$ is not reachable. Then there exists a non-singular matrix $\bar{T}$ such that

$$\bar{T} \hat{F}_{11} \bar{T}^{-1} = \begin{bmatrix} \hat{F}_{11,1} & \hat{F}_{11,2} \\ 0 & \hat{F}_{11,3} \end{bmatrix}, \bar{T} \hat{G}_1 = \begin{bmatrix} g_{1,T} \\ 0 \end{bmatrix}, \quad (35)$$

where $[\hat{F}_{11,1}, g_{1,T}]$ is reachable. Thus, as the corresponding transfer function $[I - \hat{F}_{11,1}z]^{-1}g_{1,T}$ has its poles outside the unit circle, all eigenvalues of $\hat{F}_{11,1}$ must be of modulus smaller than 1.

Again, partition $\bar{T} \Lambda_1 \bar{T}^T$ conformally as

$$\bar{T} \Lambda_1 \bar{T}^T = \begin{bmatrix} \bar{\Lambda}_{11} & \bar{\Lambda}_{12} \\ \bar{\Lambda}_{12}^T & \bar{\Lambda}_{22} \end{bmatrix}, \quad (36)$$

and define $\bar{\Sigma}_{ij}$ as the associated submatrix of $\bar{T} \hat{G}_1 \hat{G}_1^T \bar{T}^T$.

Noting that $\bar{T}\hat{G}_1\hat{G}_1^T\bar{T}^T = \begin{bmatrix} \bar{\Sigma}_{11} & 0 \\ 0 & 0 \end{bmatrix}$. It follows from $\Lambda_1 - \hat{F}_{11}\Lambda_1\hat{F}_{11}^T = \hat{G}_1\hat{G}_1^T$ that

$$\begin{bmatrix} \bar{\Lambda}_{11} & \bar{\Lambda}_{12} \\ \bar{\Lambda}_{12}^T & \bar{\Lambda}_{22} \end{bmatrix} - \begin{bmatrix} \hat{F}_{11,1} & \hat{F}_{11,2} \\ 0 & \hat{F}_{11,3} \end{bmatrix} \begin{bmatrix} \bar{\Lambda}_{11} & \bar{\Lambda}_{12} \\ \bar{\Lambda}_{12}^T & \bar{\Lambda}_{22} \end{bmatrix} \begin{bmatrix} \hat{F}_{11,1} & \hat{F}_{11,2} \\ 0 & \hat{F}_{11,3} \end{bmatrix}^T = \begin{bmatrix} \bar{\Sigma}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad (37)$$

which implies

$$\bar{\Lambda}_{22} - \hat{F}_{11,3}\bar{\Lambda}_{22}\hat{F}_{11,3}^T = 0.$$

Let x be an eigenvector of $\hat{F}_{11,3}$ corresponding to eigenvalue λ . It is immediate that $(1 - |\lambda|^2)x^*\bar{\Lambda}_{22}x = 0$. Since $\bar{\Lambda}_{22}$ is positive definite, it follows that we must have $|\lambda| = 1$. Since the eigenvalues of $\hat{E}$ are the union of the eigenvalues of $\hat{F}_{11,1}$, $\hat{F}_{11,3}$ and $\hat{F}_{22}$, conclusion [i] is proved.

It has been shown that the number of eigenvalues of $\hat{E}$ that lie on the unit circle is equal to number of columns (or rows) of the matrix $\hat{F}_{11,3}$, which is equal to $rk(\Lambda_1) - rk([g_{1,T} \hat{F}_{11,1}g_{1,T} \cdots \hat{F}_{11,1}^{p-1}g_{1,T}])$. conclusion [ii] follows from the fact that $rk(\Lambda_1) - rk([g_{1,T} \hat{F}_{11,1}g_{1,T} \cdots \hat{F}_{11,1}^{p-1}g_{1,T}]) = rk(\Gamma_p) - rk([G \hat{E}G \cdots \hat{E}^{p-1}G])$.

Using (30) and the facts that $\bar{F}_{11} = \hat{F}_{11}$ and the eigenvalues of $\hat{F}_{22}$ are all within the unit circle, conclusions [iii] and [iv] are immediate. $\square$

Acknowledgements

Support by the ARC Discovery Project Grant DP1092571, by the FWF (Austrian Science Fund) under contracts P17378 and P20833/N18 and by the Oesterreichische Forschungsgemeinschaft is gratefully acknowledged. The authors are indebted to Per Enqvist at KTH, Stockholm for drawing their attention to the article by Inouye (1983).

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