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Abelian quasinormal subgroups of finite p-groups

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ABSTRACT

If $G = AX$ is a finite p-group, with A an abelian quasinormal subgroup and X a cyclic subgroup, then we find two composition series of G passing through A, all the members of which are quasinormal subgroups of G.

1. Introduction

A subgroup Q of a group G is said to be *quasinormal* (sometimes *permutable*) in G, and we write $Q\ qn\ G$, if $QH = HQ$ for all subgroups H of G. Equivalently $Q\ qn\ G$ if $QH = HQ$ for all *cyclic* subgroups H of G. Of course $QH = HQ$ if and only if $QH = \langle Q, H\rangle$. Normal subgroups are quasinormal, but the converse is false in general. In finite groups the difficulties associated with quasinormal subgroups are encountered in the p-groups. This is easy to explain. For, let G be a finite group with $Q\ qn\ G$ and suppose that Q is core-free. Then it is shown in [8] that Q lies in the hypercentre of G. In particular Q is nilpotent. Also, by [10, Lemma 6.2.16], each Sylow p-subgroup P of Q is quasinormal in G; and it is easy to see that each p'-element of G centralises P. Thus if S is a Sylow p-subgroup of G, then $P\ qn\ S$ and the complexities of the embedding of Q in G are reduced to those of P in S.

Very little is known about quasinormal subgroups of finite p-groups. Gross and Berger made a significant contribution in [6] and [1]; and several examples were constructed in [12] and [13] show-

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ing how complicated core-free quasinormal subgroups can be. How cyclic quasinormal subgroups are embedded in finite groups was discovered in [2–4]. Also in [14] it was shown that when Q is an *abelian* quasinormal subgroup of *any* group G (finite or infinite), then Q^n *qn* G, provided n is odd or divisible by 4. Apart from this, given Q *qn* G, virtually nothing is known about which subgroups of Q are also quasinormal in G, even when Q is abelian. It was proved in [5] that if Q has order p^2 (with p an odd prime), then there is always a quasinormal subgroup of G of order p lying in Q. But this subgroup is in no way canonical and its existence was established only by an exhaustive analysis of possible minimal counter-examples. This lack of knowledge becomes even more surprising when one considers the significance of quasinormal subgroups within finite p-groups. Indeed it is the quasinormal subgroups, not the normal subgroups, that are invariant under automorphisms and isomorphisms of the subgroup lattices.

Obvious questions spring to mind. In a finite p-group, is a minimal non-trivial quasinormal subgroup always of order p? Do maximal chains of quasinormal subgroups form a composition series? In fact we know that the answer to the latter question is in the negative, but that will appear elsewhere. Here we concentrate on *abelian* quasinormal subgroups A of a finite p-group G. Given the difficulty of dealing with the case where A has order p^2, progress is sure to be slow. Also since A *qn* G if and only if A *qn* AX for all cyclic subgroups X of G, we consider groups $G = AX$ with X cyclic. This is the hypothesis that Gross and Berger assumed in [6] and [1]. Also in [9] Nakamura showed that if Q *qn* $G = QX$, with X cyclic and G a finite p-group, then there is a quasinormal subgroup of G of order p lying in Q (in fact in $Z(Q)$). Thus we shall consider abelian quasinormal subgroups A of finite p-groups $G = AX$, where X is cyclic. We shall prove that there is a canonical series

$$1 = Q_0 \leqslant Q_1 \leqslant \cdots \leqslant Q_s = A,$$

with Q_i *qn* G and $|Q_{i+1} : Q_i| = p$, all i. If A is core-free, then there is a unique series of this type passing through the subgroups $\Omega_i(A)$, $i \geqslant 0$. Also, when p is odd, there is a series passing through the subgroups A^{p^i}, $i \geqslant 0$ (though not necessarily unique). We see this work as an essential first step before removing the hypothesis $G = AX$ with X cyclic.

Section 3 contains the proofs of our theorems and in Section 4 we construct some examples dealing with questions that arise naturally. Section 2 contains preliminary lemmas required for the theorems and examples.

Notation is standard. The centre of a group G is denoted by $Z(G)$. If H and K are subgroups of G, then $C_H(K)$ and $N_H(K)$ are the centraliser and normaliser, respectively, of K in H. The repeated commutator subgroup $[H, K, K, \ldots, K]$, with i K's, is written as $[H, {}_iK]$ and H_G, H^G are the (normal) core and closure, respectively, of H in G. A maximal proper subgroup H of G is denoted by $H <\cdot G$. If G is a p-group, then, for all $i \geqslant 0$, $\Omega_i(G) = \langle x \in G \mid x^{p^i} = 1\rangle$. For a cyclic group X, we shall write $X_i = \Omega_i(X)$. Finally C_n denotes a cyclic group of order n.

2. Preliminary lemmas

We begin with a key result.

Lemma 1. *Let $G = AX$ be a finite p-group, with A an elementary abelian subgroup and X a cyclic subgroup of order at least p^2. Suppose that $A \cap X = 1$ and $X_1 = \Omega_1(X) \lhd G$. Then with $X_2 = \Omega_2(X)$*

(i) *AX_1 is elementary abelian and $AX_1 \lhd AX_2$;*
(ii) *A qn G if and only if the fixed point of any regular X_2/X_1-submodule of AX_1 lies in A.*

Proof. Since $X_1 \leqslant Z(G)$, AX_1 is elementary abelian of index p in AX_2. Therefore $AX_1 \lhd AX_2$ and (i) follows.

For (ii), suppose that A *qn* G. Assume, for a contradiction, that there is a regular X_2/X_1-submodule W of AX_1 such that (in additive module notation)

$$W(1 - X_2)^{p-1} \nleqslant A.$$

Let $X_2 = \langle x_2 \rangle$. Then there is an element $a \in A$ such that $a(1-x_2)^{p-1} = a_1 x_1$, where $a_1 \in A$ and $\langle x_1 \rangle = X_1$. Replacing a by a power if necessary, we may also assume that $x_2^p = x_1$. Let $g = x_2^{-1}a$. Then

$$g^p = \left(x_2^{-1}a\right)^p = x_1^{-1}a\left(1-x_2^{-1}\right)^{p-1} = x_1^{-1}a(1-x_2)^{p-1} = a_1.$$

Now $AX_2 = A\langle g \rangle$. But $|A\langle g \rangle : A| = |\langle g \rangle : A \cap \langle g \rangle| = p$, while $|AX_2 : A| = |X_2| = p^2$, a contradiction.

Conversely, suppose that the fixed point of any regular X_2/X_1-submodule of AX_1 lies in A. We distinguish two possibilities.

Case 1. *Suppose that* $X = X_2$. Certainly $A\ qn\ AX_1$. Therefore let $g \in G \setminus AX_1$. Then $g = xa$, where $a \in A$ and $\langle x \rangle = X$. So $g^p = x_1 a(1-x)^{p-1}$, where $x_1 = x^p$. If $a(1-x)^{p-1} \neq 1$, then it must lie in A. Thus in any case $A\langle g^p \rangle = AX_1$ and $A\langle g \rangle = AX_1\langle g \rangle = G$. So $A\ qn\ G$.

Case 2. *Suppose that* $X > X_2$. Let $A \leqslant H \lessdot G$. Then by induction on $|X|$, we have

$$A\ qn\ H. \tag{1}$$

Now $A^G = AX_i \leqslant H$, for some $X_i \leqslant X$. Also the subgroups containing A form a chain. So A^G is generated by A and some conjugate of an element of A, i.e. an element of order p. Thus $|A^G : A| \leqslant p$, by (1), and then $A^G \leqslant K \lessdot H$. Let $g_1 \in G \setminus H$. Therefore $A^G\langle g_1^p \rangle = H$ and $g_1^p \notin K$. Hence $A\langle g_1^p \rangle = H$ (again by (1)) and so $A\langle g_1 \rangle = H\langle g_1 \rangle = G$. Then $A\ qn\ G$. □

As a consequence of this result, we find that A is always quasinormal in G in a certain situation.

Corollary 1. *Assume the hypotheses of Lemma* 1 *and in addition suppose that* $C_X(A) \geqslant X_2$. *Then* $A\ qn\ G$.

Proof. Clearly as X_2/X_1-module, AX_1 has no regular submodules. Therefore $A\ qn\ G$, by Lemma 1. □

We shall be considering situations where an elementary abelian subgroup A is quasinormal in the finite p-group $G = AX$, with X cyclic and $A \cap X = 1$. Then we always have

$$A^G = A \text{ or } AX_1, \tag{2}$$

as in Case 2 of Lemma 1(ii) above. In fact when p is odd, this does not require $A \cap X = 1$, by Lemma 2.1(iv) of [5]. But it does when $p = 2$. (See Example 1 in Section 4.)

We now give a simple criterion for deciding which subgroups of A in Corollary 1 are quasinormal in G.

Corollary 2. *Assume the hypotheses of Corollary* 1 *and let* $B \leqslant A$. *Then* $B\ qn\ G$ *if and only if* $BX = XB$.

Proof. The necessity of the condition is clear. Conversely, suppose that $BX = XB$. By Corollary 1, $A\ qn\ G$. So, by (2), AX_1 is an X-module. Let $A \leqslant M \lessdot G$. Then $M \cap BX = B(M \cap X)$ and therefore $B\ qn\ M$, by induction on $|X|$. Let $g \in G \setminus M$. Thus $g = xa$, where $\langle x \rangle = X$ and $a \in A$. It follows that with $|X| = p^{n+1}$ $(n \geqslant 1)$, we have

$$g^{p^n} = x^{p^n} a(1-x)^{p^n - 1} = x^{p^n},$$

because $|X/X_2| = p^{n-1}$. Hence

$$\Omega_1\left(\langle g \rangle\right) = X_1 = \Omega_1(X).$$

But $B^G = B^X \leqslant BX \cap AX_1 = BX_1$. So $BX_1 \lhd G$. Therefore $B\langle g \rangle = BX_1\langle g \rangle$ and $B\langle g \rangle$ is a subgroup. Thus $B\ qn\ G$. □

We shall see in Example 2 in Section 4 that when $A\ qn\ G$, we cannot remove the hypothesis $C_X(A) \geqslant X_2$ in Corollary 2.

Lemma 1 enables us to establish a criterion for deciding if an elementary abelian subgroup A is quasinormal in a finite p-group $G = AX$, where X is cyclic and AX_1 is an indecomposable X-module, for example when A is core-free in G.

Lemma 2. *Let $G = AX$ be a finite p-group, with A elementary abelian of rank $r-1$, X cyclic of order p^{n+1} ($n \geqslant 1$), $A \cap X = 1$, $X_1 = \Omega_1(X)$, $AX_1 = A \times X_1 = V \lhd G$, where V is an indecomposable X-module, and $C_X(A) = X_1$. Then $p^{n-1} < r \leqslant p^n$. Also $A\ qn\ G$ if and only if $r \leqslant p^{n-1}(p-1)$, in which case p is odd.*

Proof. Since V is indecomposable, $r \leqslant p^n$. Also if $r \leqslant p^{n-1}$, then $X_2 = \Omega_2(X)$ must centralise A, which is not the case. Thus $p^{n-1} < r$.

Suppose that $A\ qn\ G$. By Lemma 1(ii), the subgroup generated by the fixed points of the regular X_2/X_1-submodules of V lies in A. But X normalises this subgroup. Since $A_G = 1$, we deduce that there are no regular X_2/X_1-submodules of V. Let $X = \langle x \rangle$ and choose a basis $\{w_i \mid 1 \leqslant i \leqslant r\}$ of V such that (in additive notation)

$$w_i(x-1) = w_{i+1}, \quad 1 \leqslant i \leqslant r-1, \qquad w_r(x-1) = 1.$$

Then with $q = p^{n-1}$ and $y = x^q$, $w_i(y-1) = w_{i+q}$ and so

$$V = W_1 \times W_2 \times \cdots \times W_{p^{n-1}} \tag{3}$$

as a decomposition into indecomposable X_2/X_1-submodules (with W_i generated by w_i). Each W_i has rank at most $p-1$, since there are no regular submodules. Therefore $r \leqslant p^{n-1}(p-1)$, as required. (See Exercise 4 on page 227 of [10].)

Conversely, suppose that

$$r \leqslant p^{n-1}(p-1). \tag{4}$$

Suppose also that there is a regular X_2/X_1-submodule of V. Since the ranks of any two W_i's in (3) differ by at most 1, we must have

$$r \geqslant p + \big(p^{n-1}-1\big)(p-1) = p^{n-1}(p-1) + 1.$$

But this contradicts (4). Therefore V contains no regular X_2/X_1-submodules and $A\ qn\ G$ by Lemma 1(ii). □

Sufficient conditions for quasinormality in a more general situation can now be established.

Lemma 3. *Let $G = AX$ be a finite p-group with A elementary abelian, X cyclic of order p^{n+1}, $X_1 = \Omega_1(X) \lhd G$, $A \cap X = 1$ and $V = AX_1 \lhd G$. Suppose that all indecomposable X-submodules of V have rank at most $p^{n-1}(p-1)$. Then*

(i) *$A\ qn\ G$; and*
(ii) *$B \leqslant A$ and $BX = XB$ implies $B\ qn\ G$.*

Proof. (i) Clearly we may assume that $n \geqslant 1$ and $A_G = 1$. Then V has only 1 fixed point as X-module. Thus V is indecomposable and $A\ qn\ G$, by Corollary 1 and Lemma 2.

(ii) Let $A \leqslant M \lessdot G$. Since the rank hypothesis is satisfied for V as X^p-module, we may assume, by induction on $|X|$, that $B\ qn\ M$. Let $g \in G \setminus M$. Then $g = xa$, where $\langle x \rangle = X$ and $a \in A$. So (using additive module notation)

$$g^{p^n} = x^{p^n} a(1-x)^{p^n-1} = x^{p^n} = x_1,$$

where $\langle x_1 \rangle = X_1$. Thus $B\langle g \rangle = BX_1\langle g \rangle$. But $B^G = B^X \leqslant BX \cap V = BX_1$. Therefore $BX_1 \lhd G$. Hence $B\langle g \rangle$ is a subgroup and B *qn* G. □

It appears to be unknown whether a minimal (non-trivial) quasinormal subgroup of a finite p-group G has order p. However, if $G = AX$ with A abelian and quasinormal and X cyclic, then this is the case inside A, as we now show. (We use the term "lemma" here in the hope that it will help to establish a significantly more general result.)

Lemma 4. *Let $G = AX$ be a finite p-group with A an abelian quasinormal subgroup and X cyclic. Let B be a minimal* (*non-trivial*) *quasinormal subgroup of G lying in A. Then $|B| = p$ if* (i) *p is odd or* (ii) *B is elementary.*

Proof. Clearly we may assume that $B_G = 1$. So $B \cap X = 1$. Also if p is odd, then B^p *qn* G (by [14]) and so $B^p = 1$, i.e. B is elementary abelian. Thus $B^G = B^X = BX_1$, by (2). (Here $X_1 = \Omega_1(X)$.) Therefore there is a non-trivial central element of G of the form bx_1, where $b \in B$ and $x_1 \in X_1$. Hence $[b, X] = 1$ and so $b = 1$. Then $\langle x_1 \rangle = X_1 \leqslant Z(G)$.

Now $N_B(X)$ has order p. This is clear if p is odd. But if $p = 2$, having order 4 would involve a dihedral action and B would not be quasinormal. Thus let $N_B(X) = \langle b \rangle$, of order p, and suppose that B has rank $i+1$. Then $[B, {}_iX] \equiv \langle b \rangle \bmod X_1$. Let $A \leqslant M \lessdot G$. Thus $b \in Z(M)$. Let $g \in G \setminus M$. Then $B^G = BX_1 = B^{\langle g \rangle} = B\Omega_1(\langle g \rangle)$, an indecomposable $\langle g \rangle$-module. But $[B, {}_i\langle g \rangle] = [B, {}_iX]$. So b normalises $\langle g \rangle$. Therefore $\langle b \rangle$ *qn* G and $B = \langle b \rangle$ of order p. □

3. Main results

Theorem 1 of [14] proves that if A *qn* G, with A abelian, and if p is an odd prime, then A^p *qn* G. On the other hand, Example 2 of the same paper shows that when G is a finite p-group, then $\Omega_1(A)$ is *not* always quasinormal in G. In that example, $G = AX$ with X cyclic and $A_G \neq 1$. Our first main result shows that if $A_G = 1$ in this situation, then we do have $\Omega_i(A)$ *qn* G, for all $i \geqslant 1$, and for *all* primes p.

Theorem 1. *Let $G = AX$ be a finite p-group, with A an abelian quasinormal subgroup of G, $A_G = 1$ and X cyclic. Let $W_i = \Omega_i(A)$, $i \geqslant 0$. Then W_i qn G, for all i.*

Proof. We have $A \cap X = 1$. Let $|X| = p^{n+1}$ and let A have exponent p^r. So $r \leqslant n$, by [6]. (See also [10, Theorem 5.2.8].) Also, by the same reference,

$$\Omega_i(G) = W_iX_i \lhd G, \tag{5}$$

where $X_i = \Omega_i(X)$, for all i; and AX_i/W_iX_i is core-free in G/W_iX_i. Thus $W_iX_{i+1}/W_iX_i \lhd G/W_iX_i$. It follows, by induction on n, that

$$W_iX_1 \text{ } qn \text{ } G, \quad \text{all } i. \tag{6}$$

Since $\Omega_r(G) = AX_r \lhd G$ and $A_G = 1$, it follows that $A^G = AX_r$. Also $A \leqslant C_G(W_1) = C_G(W_1X_1) \lhd G$. So $[W_1, X_r] = 1$. Similarly $[W_2, X_r] \leqslant W_1X_1$ and therefore $[W_2, X_{r-1}] = 1$. Continuing in this way we obtain

$$[W_i, X_{r-i+1}] = 1, \tag{7}$$

for all $i \leqslant r$.

Let $g \in G$. We show that $W_i\langle g\rangle$ is a subgroup. By (7), we may assume that $g \notin AX_{r-i+1}$. We claim that some power of g has the form $w_i x_1$, where $w_i \in W_i$ and $\langle x_1\rangle = X_1$. For, if $A\langle g\rangle = AX_s$, then $A\langle g^p\rangle = AX_{s-1}$; and $s \geqslant 2$. Thus some power h of g belongs to $AX_{r-i+2} \setminus AX_{r-i+1}$. Then h belongs to $AX_{r+1} \leqslant W_{r+1}X_{r+1}$ if $i = 1$; and h belongs to $AX_r = W_r X_r$ if $i \geqslant 2$. Let $u = h^{p^{r-i+1}}$. Thus if $i = 1$, then $u \in W_1 X_1$; and if $i \geqslant 2$, then $u \in W_{i-1}X_{i-1}$. But $u \in AX_1 \setminus A$. Hence u has the form $w_1 x_1$ if $i = 1$ (with $w_1 \in W_1$ and $\langle x_1\rangle = X_1$); and the form $w_{i-1}x_1$ if $i \geqslant 2$ (with $w_{i-1} \in W_{i-1}$ and $\langle x_1\rangle = X_1$). Therefore our claim is true.

It follows that $W_i\langle g\rangle = W_i X_1\langle g\rangle$, which is a subgroup for all i, by (6). Thus W_i *qn* G. □

The next result shows that, with the same hypotheses, there is a chain of quasinormal subgroups of G between 1 and A, including the subgroups W_i, for all $i \geqslant 0$, such that each subgroup in the chain has index p in the one above.

Theorem 2. *Let $G = AX$ be a finite p-group, with A an abelian quasinormal subgroup, $A_G = 1$ and X cyclic. Let $W_i = \Omega_i(A)$, all $i \geqslant 0$. Then there exists a composition series of G passing through the W_i's in which every subgroup is quasinormal in G.*

Proof. Clearly $A \cap X = 1$. Also it is easy to establish the existence of the required composition series between A and G, using (5). Thus, by Theorem 1, it suffices to establish the existence of the series between W_i and W_{i+1}, for all i. Let A have exponent p^r and $|X| = p^{n+1}$. Then $r \leqslant n$, as in Theorem 1. Let $X_i = \Omega_i(X)$, $i \geqslant 0$.

Case (i). *Suppose that* $i \geqslant 1$. Again as in Theorem 1, we have, by induction on n, a composition series of G between $W_i X_1$ and $W_{i+1}X_1$ consisting of quasinormal subgroups of G. Let Q be one of these subgroups and let $Q_1 = A \cap Q$. Then the subgroups Q_1 form a composition series between W_i and W_{i+1}. We show that Q_1 *qn* G. Thus let $g \in G$. We prove that $Q_1\langle g\rangle$ is a subgroup. To see this, we may assume, by (7), that $g \notin AX_{r-i}$. Also, as in Theorem 1, we see that

$$\text{some power of } g \text{ has the form } w_i x_1,$$

where $w_i \in W_i$ and $\langle x_1\rangle = X_1$. It follows that $Q_1\langle g\rangle = Q_1 X_1\langle g\rangle = Q\langle g\rangle$, which is a subgroup. So Q_1 *qn* G in this case.

Case (ii). *Suppose that* $i = 0$. As X-module, $W_1 X_1$ $(= \Omega_1(G))$ is indecomposable (since $A_G = 1$) and so is uniserial by [7, Theorem VII 5.3]. Thus there is a unique chief series between X_1 and $W_1 X_1$. Let Q be a term of this series and let $Q_2 = A \cap Q \leqslant W_1$. Let $g \in G$. We show that $Q_2\langle g\rangle$ is a subgroup.

By (7), $[W_1, X_r] = 1$. Therefore we may assume that $g \notin AX_r$ $(= A^G)$. Then we claim that $\Omega_1(\langle g\rangle) = X_1$. For, some power h of g has the form ax_{r+1}, $a \in A$, $\langle x_{r+1}\rangle = X_{r+1}$. As we saw in the proof of Theorem 1, $AX_{r-1}/W_{r-1}X_{r-1}$ is quasinormal and core-free in $G/W_{r-1}X_{r-1}$, and $M = AX_r/W_{r-1}X_{r-1}$ is an indecomposable X/X_{r-1}-module. Since the fixed points of regular X_{r+1}/X_r-submodules of M all lie in $AX_{r-1}/W_{r-1}X_{r-1}$, by Lemma 1(ii), and since these elements generate a normal subgroup of $G/W_{r-1}X_{r-1}$, it follows that there are no regular X_{r+1}/X_r-submodules in M. Thus

$$h^p \equiv x_r \mod W_{r-1}X_{r-1},$$

where $x_r = x_{r+1}^p$. So $h^p \in W_{r-1}X_r$. Similarly $h^{p^2} \in W_{r-2}X_{r-1}$ and continuing we obtain $h^{p^r} \in X_1 \setminus \{1\}$. Therefore $\Omega_1(\langle g\rangle) = X_1$, as claimed.

Finally $Q_2\langle g\rangle = Q_2 X_1\langle g\rangle = Q\langle g\rangle$ is a subgroup. Thus Q_2 *qn* G. □

To end this section, we show that, when p is odd, the composition series of Theorem 2 is not unique in general.

Theorem 3. *Let p be an odd prime and let $G = AX$ be a finite p-group, with A an abelian quasinormal subgroup of G, $A_G = 1$ and X cyclic. Then there is a series*

$$1 = Q_0 \leqslant Q_1 \leqslant \cdots \leqslant Q_s = A, \tag{8}$$

with Q_i qn G, $|Q_{i+1} : Q_i| = p$, all i; and all powers A^{p^j} ($j \geqslant 1$) occur as Q_i's.

Proof. By Theorem 1 of [14], $B = A^p$ *qn* G. Form the products of B with the terms of the composition series of G between 1 and A passing through $W_i = \Omega_i(A)$, $i \geqslant 0$, given by Theorem 2. Then omitting repeats, we obtain a composition series of G between B and A consisting of quasinormal subgroups of G. Suppose that $H \geqslant K$ are adjacent terms of this composition series. So there exists $h \in H$ such that $H = K\langle h\rangle$ and $h^p \in K$. Therefore $H^p = K^p\langle h^p\rangle$ and $h^{p^2} \in K^p$. Thus $|H^p : K^p| = p$ or 1. Again by Theorem 1 of [14], H^{p^n} *qn* G for all $n \geqslant 1$. It follows that taking those powers of all H in the composition series between B and A and removing repetitions, we obtain the series (8). □

We shall show in Example 3 of the next section that in general there are more quasinormal subgroups of G lying in A than those described in Theorems 2 and 3.

4. Examples

We begin with an example of how the case $p = 2$ can differ from the case when p is an odd prime. In Lemma 2.1(iv) of [5], it is shown that if A is an elementary abelian quasinormal subgroup of a finite p-group G, then A^G always has exponent p if p is odd. Though A^G need not be abelian, as is shown by an example in the same paper. However, when $p = 2$, then A^G does not always have exponent 2 and is not always abelian.

Example 1. Let H be the group of order 2^7 generated by elements b_i, $0 \leqslant i \leqslant 3$, and x subject to the relations

$$b_i^2 = [b_i, b_j] = 1, \quad \text{all } i, j, \qquad x^8 = b_0, \qquad [b_i, x] = b_{i-1}, \quad 1 \leqslant i \leqslant 3.$$

Let $B = \langle b_i \mid 0 \leqslant i \leqslant 3\rangle$. Then $[B, X^4] = 1$ and B is an X/X^4-module. We define an automorphism α of H by

$$b_i \mapsto b_i, \quad \text{all } i, \qquad x \mapsto x^5 b_3.$$

Since $(x^5 b_3)^8 = (x b_3)^8 = x^8 b_3(1 - x)^7 = b_0$, we see that α preserves the relations of H. One checks easily that $[x^2, \alpha] = x^8 b_2$ and $[x^4, \alpha] = b_0$. Thus

$$\left[x, \alpha^2\right] = x^4 b_3 \left(x^4 b_3\right)^\alpha = x^4 b_3 x^4 b_0 b_3 = 1.$$

Therefore α is an automorphism of H of order 2.

Form the semi-direct product G of H by the cyclic group $\langle a\rangle$ of order 2, where the action of a on H is defined by α. Let $A = \langle a, B\rangle$. Then A is elementary abelian of rank 5. We have $B \lhd G$ and G/B is a modular group of order 16, i.e. all its subgroups are quasinormal. Thus A *qn* G. But $A^G = AX^4$ of exponent 4. Also A^G is not abelian.

Note that A here is not core-free in G. Indeed whenever A is a core-free elementary abelian quasinormal subgroup of a finite p-group G, then A^G is also elementary abelian, by [5, Lemma 2.1(ii)].

Let $G = AX$ be a finite p-group, with A elementary abelian and quasinormal in G and X cyclic. Suppose that $A_G = 1$. So $A \cap X = 1$ and $\Omega_1(X) \lhd G$. Let $B \leqslant A$ and suppose that $BX = XB$. Then

$B\ qn\ G$. For, if $C_X(A) \geqslant \Omega_2(X)$, then Corollary 2 applies; and if $C_X(A) = \Omega_1(X)$, then Lemmas 2 and 3 apply. In the light of this result, it is reasonable to ask if the hypothesis $A_G = 1$ is necessary here. In fact it is, even when $A \cap X = 1$, as we now show.

Example 2. Let p be an odd prime and let A be an elementary abelian p-group of rank $p+1$, with basis $a_1, a_2, \dots, a_p, b$. Let $X_1 = \langle x_1 \rangle$ be a cyclic group of order p and form the direct product $H = A \times X_1$. Now H has an automorphism ξ of order p defined by

$$a_i \mapsto a_i a_{i+1}, \quad 1 \leqslant i \leqslant p-1, \qquad a_p \mapsto a_p, \qquad b \mapsto b x_1^{-1}, \qquad x_1 \mapsto x_1.$$

Then by [11, 9.7.1(ii)], there is an extension G of H by a cyclic group of order p defined as follows:

$$G = \langle A, x \mid [a_i, x] = a_{i+1},\ 1 \leqslant i \leqslant p-1,\ [a_p, x] = 1,\ [x, b] = x^p \rangle.$$

Here $x^p = x_1$. Then $A_1 = \langle a_1, \dots, a_p \rangle \lhd G$ and G/A_1 has order p^3 with all its subgroups quasinormal. Therefore $A\ qn\ G$. Also with $B = \langle b \rangle$, $BX = XB$. But $(xa_1)^p = x^p a_1^{(1-x)^{p-1}} = x_1 a_p$. If B were quasinormal in G, then b would normalise $\langle xa_1 \rangle$, i.e. $[xa_1, b] \in \langle xa_1 \rangle^p = \langle x_1 a_p \rangle$. However, $[xa_1, b] = x_1$ and we would have a contradiction. Thus B is not quasinormal in G.

It is easy to construct a similar example when $p = 2$, by taking A of rank 5, X of order 8 and defining $[x, b] = x^4$.

Our final example shows that, in general, the subgroups given by Theorems 2 and 3 do not account for all the quasinormal subgroups of G lying in A.

Example 3. Let p be a prime $\geqslant 5$ (for convenience) and let M be the group of order p^5 defined by the presentation

$$\langle b, y, x \mid b^p = y^p = [b, y] = x^{p^3} = 1,\ [b, x] = y,\ [y, x] = x^{p^2} \rangle.$$

Let $W = \langle b, y \rangle \cong C_p \times C_p$ and $X = \langle x \rangle \cong C_{p^3}$. So $M = WX$ and $W \cap X = 1$. We construct an automorphism α of M as follows. Let $0 \leqslant i, j \leqslant p-1$ and define

$$\alpha : b \mapsto b, \qquad y \mapsto y, \qquad x \mapsto x^{1+p} b^i y^j.$$

One checks easily that $\alpha : x^p \mapsto x^{p+p^2}$ and $x^{p^2} \mapsto x^{p^2}$. Also α preserves the relations of M and therefore α defines an automorphism of M. For all $n \geqslant 1$,

$$\alpha^n : x \mapsto x^{(1+p)^n} \left(b^i y^j\right)^n.$$

So $|\alpha| = p^2$. Moreover $[x, \alpha^p] = x^{p^2} = [x, y^{-1}]$. Thus the action of α^p on M coincides with conjugation by y^{-1}. Then again by [11, 9.7.1(ii)], there is a group G of order p^6 which is an extension of M by a cyclic group of order p, generated by a, such that a acts on M according to α and $a^p = y^{-1}$. Therefore

$$G = \langle b, a, x \mid b^p = a^{p^2} = [b, a] = x^{p^3} = 1,\ [x, b] = a^p,\ [x, a] = x^p b^i a^{-jp} \rangle.$$

Let $A = \langle b, a \rangle$. We claim that

$$A\ qn\ G. \tag{9}$$

For, let $X_2 = \Omega_2(X)$. Since $\langle b, a^p\rangle \leqslant Z(AX_2)$, we have $A\ qn\ AX_2$. Let $g = x^\ell b^m a^n$, where $p \nmid \ell$. One checks that

$$g^p \equiv x^{\ell p} \mod \Omega_1(G)$$

and hence $g^{p^2} = x^{\ell p^2}$, i.e. $\Omega_1(\langle g\rangle) = X_1$ $(= \Omega_1(X))$. Thus $|A\langle g\rangle| = |A||\langle g\rangle| = p^6$ and $A\langle g\rangle = G$. So (9) is true.

Since $A_G = 1$, Theorems 2 and 3 apply to G and so

$$\langle a^p\rangle,\ \langle b, a^p\rangle \text{ and } A \text{ are quasinormal in } G.$$

Observe that there is no cyclic quasinormal subgroup of G of order p^2 here. Indeed if $i \neq 0$, then such a subgroup cannot exist. But suppose that $i = 0$. Then we claim that, for $0 \leqslant k \leqslant p-1$,

$$Q = \langle b^k a\rangle\ qn\ G. \tag{10}$$

For, since $Q^p = \langle a^p\rangle \leqslant Z(AX_2)$, factoring AX_2 by Q^p, we see from Lemma 3 that $Q\ qn\ AX_2$. Thus let $g = x^\ell b^m a^n$, $p \nmid \ell$. As above, $\Omega_1(\langle g\rangle) = X_1$, and so $Q\langle g\rangle = QX_1\langle g\rangle$. To see that this product is a subgroup, we may factor by X_1. Then $[x, a^p] = 1$ and so we may also factor by $\langle a^p\rangle$. But then applying Lemma 3 to

$$G/Q^pX_1 = \big(\langle b\rangle \times Q\big)X/Q^pX_1,$$

we see that Q/Q^pX_1 is quasinormal in this group, since $b^k a$ normalises X/Q^pX_1. Therefore $QX_1\langle g\rangle$ is a subgroup and (9) is true.

It follows that, when $i = 0$, there are p cyclic quasinormal subgroups of G of order p^2 lying in A, none of which appears in Theorems 2 and 3.

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