Modeling high-frequency volatility with three-state FIGARCH models

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ABSTRACT

Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity (FIGARCH) models have enjoyed considerable popularity over the past decade because of their ability to capture the features of volatility clustering and long-memory persistence. However, in the presence of structural changes, it is well known that the estimate of long memory will be spurious. Consequently, two modeling approaches are developed to incorporate structural changes into the FIGARCH framework. One approach is to model the intercept in the conditional variance equation via a certain function of time. Based on this approach, the Adaptive-FIGARCH (A-FIGARCH) and Time-Varying FIGARCH (TV-FIGARCH) models are proposed. The second approach is to model the time-series in separate stages. In the first stage, a certain algorithm is applied to detect the change points. The FIGARCH model is fitted to the time-series in the next stage, with the intercept (and other parameters) being allowed to vary between change points. An example of a recently developed algorithm for detecting change points is the Nonparametric Change Point Model (NPCPM), which can be readily applied to the standard FIGARCH framework (NPCPM-FIGARCH). In this paper, we adopt the second approach but use the Markov Regime-Switching (MRS) model to detect the change points and identify three economic states depending on the scale of volatility. This new 2-stage Three-State FIGARCH (3S-FIGARCH) framework is compared with other FIGARCH-type models via Monte-Carlo simulations and high-frequency datasets. From the comparison, we find that the 3S-FIGARCH model can largely improve the fit and potentially lead to a more reliable estimator of the long-memory parameter.

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1. Introduction

Persistence of a time series describes how fast the effect of current shock will die away. It has been extensively observed and studied in various fields of economics and finance in the past few decades (Aggarwal et al., 1999; Fan et al., 2008; Granger and Hyung, 2004; Jensen, 2000; Narayan and Narayan, 2007; Narayan and Narayan, 2011). The analysis of persistence can help researchers understand how the time series evolves and improve the forecasting quality (Franses and van Dijk, 1996; Ho et al., 2013; Liu, 2000; Narayan and Sharma, 2014; Westerlund and Narayan, 2012). In particular, the long-memory persistence describes the property that the effects of shocks last far longer than the usual autoregressive moving average (ARMA) process (Baillie and Morana, 2009; Baillie et al., 1996; Belkhouja and Boutahary, 2011; Bollerslev and Mikkelsen, 1996; Diebold and Inoue, 2001). A widely accepted definition of long memory is $\text{var}(S_T) = O(T^{2d+1})$, where $S_T = \sum_{t=1}^T y_t$, $\{y_t\}$ is a sequence of financial series and T is the number of observations (Diebold and Inoue, 2001).

Then d is the long-memory parameter, and a positive value suggests the existence of long memory. Among recent finance studies, there is growing evidence suggesting that the long-memory persistence significantly exists in the volatility of financial return series (Barunk and Dvoráková, 2015; Caporale and Gil-Alana, 2013; Li, 2012).

In particular, the time-varying volatility of financial returns has been a considerable field of research since the introduction of the GARCH model. To incorporate long-memory persistence within this framework, the Fractional Integrated GARCH (FIGARCH) model is then proposed (Baillie et al., 1996; Bollerslev and Mikkelsen, 1996). The FIGARCH model has thus received considerable interest because of its ability to capture the long-memory persistence in the volatility (Baillie and Morana, 2009; Belkhouja and Boutahary, 2011; Ho et al., 2013). Despite this ability, it has the same main weakness of the original GARCH model, which is the assumption that the conditional volatility has only one regime over the entire period. However, many studies demonstrate that structural changes are common in financial datasets (Beltratti and Morana, 2006; Engle and Rangel, 2008). Further, Diebold and Inoue (2001) argue that the existence of structural changes or stochastic regime switching is not only related to long memory but also easily confused with it. This finding is supported by many empirical studies, where spurious long-memory is found when structural changes are present (Granger and Hyung, 2004; Mikosch and Starica, 2004; Yalama and Celik, 2013). As a result, many researchers have suggested

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that structural changes should be incorporated into long-memory models to properly fit financial return volatility (Baillie and Morana, 2009; Beine et al., 2001; Belkhouja and Boutahary, 2011; Martens et al., 2004; Morana and Beltratti, 2004). For instance, the Adaptive FIGARCH (A-FIGARCH) model developed by Baillie and Morana (2009) and the Time-Varying FIGARCH (TV-FIGARCH) model developed by Belkhouja and Boutahary (2011) allow the intercept in the conditional variance equation to be time-varying. Essentially, this is achieved by modeling the intercept via parametric functions.

Among the existing literature, another approach to incorporate the structural changes in the GARCH-type framework is to fit the model in stages (Aggarwal et al., 1999; Malik et al., 2005; Ross, 2013). First, the return series is fitted by a certain algorithm to detect the abrupt change points. The intercept (and other parameters) in the conditional variance equation is then allowed to be different for each period between the change points. In the first stage, the most widely employed method is the Iterated Cumulative Sum of Squares (ICSS) algorithm proposed by Incan and Tiao (1994). However, as pointed out by Ross (2013), the original ICSS only works for Gaussian distribution. To overcome this problem, Ross (2013) develops the Nonparametric Change Point Model (NPCPM) algorithm, which employs the Mood test Mood (1954) to detect the change points. The NPCPM-GARCH model is then proposed, which can effectively work for both Gaussian and non-Gaussian data. This approach can be straightforwardly extended to the FIGARCH framework. A potential problem of the NPCPM algorithm is that it identifies the change points without considering the economic states. For example, sample periods with different structures are detected based on the change points only, but they will not be combined and studied subsequently according to their economic similarity. This may lead to a model that lacks parsimony because economic similarity is neglected. Besides, instead of assuming that the volatility series will switch back and forth between different regimes with some probability, the NPCPM algorithm assumes that the switch to a different regime is permanent. In addition, as NPCPM requires the return series to be independent, Ross (2013) suggests that it should be applied to the standardized residuals from the (FI)GARCH model. This might cause some problems such as the lack of economic interpretation of the detected change points and loss of information.

In this paper, we propose a two-stage Three-State FIGARCH (3S-FIGARCH) model, which also incorporates the structural changes by modeling the FIGARCH process in stages. In the first stage, the MRS framework proposed by Hamilton (1989) is employed to detect change points directly. The MRS model assumes that there are two economic states (low- and high-volatility states) in the financial return series. Also, the series can switch between the states over time, and the state process is a stationary, irreducible Markov process. We further use the three-state classification proposed by Wilfling and Maennig (2001) and Wilfling (2009) to classify the underlying state process: calm (extremely low volatility), turbulent (extremely high volatility) and intermediate (others). In the second stage, parameters of the FIGARCH process are allowed to be different for each state. As there are only three possible values for each parameter, our model should be more parsimonious than the NPCPM-FIGARCH framework. Moreover, the MRS model does not require the financial return series to be originally independent, so that the detected states (change points) are more reliable. Finally, the MRS model takes the economic information (low and high volatility) of the return series into consideration, and the detected states can have meaningful economic interpretation. To demonstrate the usefulness of the model, we firstly conduct a series of simulation studies. It is shown that the 3S-FIGARCH model outperforms the other structure-changing specifications (A-, TV- and NPCPM-FIGARCH) in all cases.

We also compare their performance via empirical studies on four world stock indexes. They are hourly data collected from 1 January 2001 to 31 December 2012, including: (1) the NASDAQ, which consists of over 3000 stocks listed on the NASDAQ stock market; (2) the DAX, which consists of 30 large Germany companies; (3) the Nikkei, which

consists of 225 Japanese companies; and (4) the ASX, which consists of 50 large Australian companies. Assumptions of Gaussian, Student's *t* and General Error (GED) distributions are modeled individually for each model and stock index.¹ The results suggest that models with non-Gaussian distribution assumptions outperform models with Gaussian distribution assumptions. More importantly, the 3S-FIGARCH specification generally gives a better fit to the data when measured using standard model selection criteria. It also provides a potentially more reliable estimate of the long-memory parameter. Thus, our 3S-FIGARCH framework could be a widely useful tool for modeling the long-memory persistence of high-frequency financial volatility in other contexts.

The remainder of this paper proceeds as follows. Section 2 describes the existing and structure-changing FIGARCH models, including FIGARCH, A-FIGARCH, TV-FIGARCH and NPCPM-FIGARCH, as well as the likelihood functions for Gaussian, Student's *t* and GED distribution assumptions. Section 3 explains the 3S-FIGARCH model proposed in this paper and compares its performance with other FIGARCH-type models via a series of simulation studies. We discuss the empirical results in Section 4. Section 5 concludes the paper.

2. The original and existing structure-changing FIGARCH models

2.1. The original FIGARCH model

The FIGARCH model proposed by Baillie et al. (1996) is extended from the family of GARCH models. In addition to the features of incorporating volatility clustering and providing good in-sample estimates (Franses and van Dijk, 1996; French et al., 1987), FIGARCH is particularly designed to model the long-memory persistence of financial volatility.

The original FIGARCH(1, *d*, 1) model is specified as follows:

$$\begin{aligned} r_t &= \mu + \varepsilon_t \text{ and } \varepsilon_t = \eta_t \sqrt{h_t} \\ b(L)h_t &= \omega + [b(L) - \phi(L)(1-L)^d] \varepsilon_t^2 \\ b(L) &= 1 - b_1 L \text{ and } \phi(L) = 1 - \phi_1 L \end{aligned} \quad (1)$$

where ε_t is the error at time *t*, h_t is the conditional volatility of ε_t at time *t*, η_t is an identical and independent sequence following a specific distribution, *L* is the lag operator, $(1-L)^d$ is the fractional differencing operator as defined by Hosking (1981) and *d* is the long-memory parameter. We have a stationary long-memory process for volatility when $0 < d < 1$. If $d = 1$, the process has a unit root and thus a permanent shock effect, which is equivalent to the IGARCH model. If $d = 0$, the process reduces to an ordinary GARCH process without long-memory persistence (Baillie et al., 1996).

2.2. A-FIGARCH model

To control for the effects of structural changes, Baillie and Morana (2009) suggest that the intercept of the conditional variance equation should be time dependent. Based on Andersen and Bollerslev's (1997) flexible functional form, Baillie and Morana (2009) propose the A-FIGARCH model, and its conditional variance equation is shown below:

$$\begin{aligned} b(L)h_t &= \omega + [b(L) - \phi(L)(1-L)^d] \varepsilon_t^2 + \omega_t \\ \text{and } \omega_t &= \sum_{j=1}^k [\gamma_j \sin(2\pi jt/T) + \delta_j \cos(2\pi jt/T)] \end{aligned} \quad (2)$$

¹ Although the FIGARCH model is originally based on Gaussian distribution (Baillie et al., 1996), significant evidence suggests that the financial return series is rarely Gaussian but typically leptokurtic and exhibits heavy-tail behavior (Bollerslev, 1987; Ho et al., 2013; Lin and Fei, 2013; Susmel and Engle, 1994). Student's *t*-distribution and GED are two widely used alternatives in finance study (Chkili et al., 2012; Fan et al., 2008; Ho et al., 2013; Zhu and Galbraith, 2011) and are employed in this paper to be compared with the Gaussian distribution.

where T is the number of observations. Moreover, they argue that adequate approximations can be achieved with very parsimonious specifications of $k = 1$ or 2 .

2.3. TV-FIGARCH model

Belkhouja and Boutahary (2011) provide an alternative way to model the variation in the intercept and propose the TV-FIGARCH model.²

The conditional variance equation of their model is constructed as:

$$b(L)h_t = \omega + \left[b(L) - \phi(L)(1-L)^d \right] \varepsilon_t^2 + \sum_{r=1}^R \omega_r F_r(s_t, \gamma_r, c_r) \quad (3)$$

where $s_t = t/T$ and $F_r(s_t, \gamma_r, c_r) = (1 + \exp\{-\gamma_r(s_t - c_r)\})^{-1}$.

In addition, γ_r controls the degree of smoothness and must be positive, while c_r is the threshold parameter with the constraint $c_1 \leq c_2 \leq \dots \leq c_R$. $s_t = t/T$ is the transition variable. When $\gamma_r \rightarrow \infty$, the switch from one state to another is abrupt; that is, a smooth change approaches a structural change at the threshold parameter c_r .

2.4. NPCPM-FIGARCH model

Another approach to incorporate the structural changes is to fit the GARCH family models in stages (Aggarwal et al., 1999; Malik et al., 2005; Ross, 2013). First, the return series is fitted by a certain algorithm to detect the abrupt change points in volatility. Then, the intercept (and other parameters) in the conditional variance equation is allowed to be different for each period between the change points. Based on this idea, Ross (2013) develops the NPCPM algorithm,³ which employs the Mood test (Mood, 1954) to detect the change points, and thereby proposes the NPCPM-GARCH model. Compared with the widely employed ICSS-GARCH, this model can effectively work for both Gaussian and non-Gaussian data.

We extend this approach to the FIGARCH framework, and the conditional equation of the NPCPM-FIGARCH model is:

$$b(L)h_t = \omega_t + \left[b(L) - \phi(L)(1-L)^d \right] \varepsilon_t^2 \quad (4)$$

where ω_t is equal to some constant k_0 until the first change point, then switches to k_1 until the next change point, and so on.⁴

2.5. Distributions of innovation and parameter estimation

Originally, the FIGARCH model is developed with the assumption that η_t in Eq. (1) follows a Gaussian distribution. Since financial data are rarely Gaussian, using alternatives to the Gaussian distribution may lead to more efficient estimations. To compare the performance of models with different distributions, we further employ two widely

used alternative distributions, the Student's t and GED. Their specific density functions are listed below:

$$\begin{aligned} \text{Normal} : f(\varepsilon_t | \theta, \Omega_{t-1}) &= \frac{1}{\sqrt{2\pi}h_t} e^{-\varepsilon_t^2/(2h_t)} \\ \text{Student-}t : f(\varepsilon_t | \theta, \Omega_{t-1}) &= \frac{\Gamma[(v+1)/2]}{\Gamma(v/2)\sqrt{\pi(v-2)}h_t} \left[1 + \frac{\varepsilon_t^2}{(v-2)h_t} \right]^{-(v+1)/2} \\ \text{GED} : f(\varepsilon_t | \theta, \Omega_{t-1}) &= \frac{v e^{-1/2|\varepsilon_t|(\lambda\sqrt{h_t})}}{\lambda 2^{(v+1)/v}\Gamma(1/v)} \text{ and } \lambda = \left[\frac{2^{(-2/v)}\Gamma(1/v)}{\Gamma(3/v)} \right]^{1/2} \end{aligned} \quad (5)$$

where Ω_{t-1} is the information set at time $t-1$, θ is the vector of parameters, ε_t and h_t are from the FIGARCH models in Sections 2.1–2.4, and v is the degree of freedom for Student's t or GED distributions. Finally, the log-likelihood function corresponding to Eq. (5) is:

$$L(\theta | \varepsilon) = \sum_{t=2}^T \ln f(\varepsilon_t | \theta, \Omega_{t-1}) \text{ where } \varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_T)', \quad (6)$$

and the Maximum Likelihood Estimator (MLE) $\hat{\theta}$ is obtained by maximizing Eq. (6).

3. The 3S-FIGARCH model

3.1. Step 1: MRS model with Student's t -innovation (MRS- t)

As discussed in the introduction, a potential problem of existing algorithms for detecting change points (such as ICSS and NPCPM) is that they fail to take economic information into consideration. Therefore, we propose a new approach that employs the MRS model to identify the economic states of financial return series.

The MRS model is proposed by Hamilton (1989). Let $\{s_t\}$ be a stationary, irreducible Markov process with discrete state space $\{1, 2\}$ and transition matrix $P = [p_{jk}]$, where $p_{jk} = P(s_{t+1} = k | s_t = j)$ is the transition probability of moving from state j to state k ($j, k \in \{1, 2\}$). Then, an MRS- t model is specified as follows:

$$r_t = \mu + \varepsilon_{s_t,t} \text{ and } \varepsilon_{s_t,t} = \sigma_{s_t,t} \eta_t \text{ where } \eta_t \stackrel{iid}{\sim} t(0, 1, v). \quad (7)$$

$\varepsilon_{s_t,t}$ is the error at time t in state s_t , η_t follows a Student's t -distribution, with 0 mean, unit standard deviation and v degrees of freedom.⁵ $\sigma_{s_t,t}$ is the standard deviation of $\varepsilon_{s_t,t}$ at time t in state s_t . In this paper, we impose the constraint that $\sigma_1 < \sigma_2$ so that states 1 and 2 will indicate the low-volatility and high-volatility economic states, respectively, for the financial return series.

We estimate the parameters of the MRS- t model using the maximum likelihood estimation (MLE). The conditional density of $\varepsilon_{s_t,t}$ is given as:

$$\begin{aligned} \Omega_{t-1} &= \{\varepsilon_{s_{t-1},t-1}, \varepsilon_{s_{t-2},t-2}, \dots, \varepsilon_{s_{1,1}}\} \\ \theta &= (\mu, p_{11}, p_{22}, v, \sigma_1, \sigma_2)' \\ f(\varepsilon_{s_t,t} | s_t = j, \theta, \Omega_{t-1}) &= \frac{\Gamma[(v+1)/2]}{\Gamma(v/2)\sqrt{\pi(v-2)}\sigma_j^2} \left[1 + \frac{\varepsilon_{j,t}^2}{(v-2)\sigma_j^2} \right]^{-\frac{v+1}{2}} \end{aligned} \quad (8)$$

where $\Gamma(\cdot)$ is the Gamma function and $f(\varepsilon_{s_t,t} | \theta, \Omega_{t-1})$ is the conditional density of $\varepsilon_{s_t,t}$. This stems from the fact that at time t , $\varepsilon_{s_t,t}$ follows a

² Since the unconditional variance is not defined for the FIGARCH process (Conrad and Haug, 2006), the approach adopted by Amado and Teräsvirta (2014) to model the unconditional variance of GARCH model cannot be directly employed. However, TV-FIGARCH essentially uses the same transition function as in Amado and Teräsvirta (2014) to model the intercept of the FIGARCH process, so it could lead to similar result.

³ For details of the NPCPM algorithm, see Section 3 of Ross (2013).

⁴ We consider the case that also allows b and ϕ in the conditional variance equation to be time-varying. The results are robust in the simulation and empirical studies. They are not reported here due to space constraints but are available from the authors upon request.

⁵ We assume that the innovations follow the Student's t -distribution rather than the Gaussian distribution because the former can accommodate the feature of excess kurtosis (Bollerslev, 1987). In addition, as noted by Klaassen (2002), Ardia (2009) and Haas (2009), if regimes are not Gaussian but leptokurtic, the use of within-regime normality can seriously affect the identification of the regime process. The reason can be found in Haas and Paoletta (2012), who further argue that the Quasi MLE (QMLE) based on Gaussian components does not provide a consistent estimator of the MRS model.

Student's t -distribution with mean 0, variance σ_{s_t} and degrees of freedom v given time $t - 1$.

Plugging the filtered probability in state j at time $t - 1$, $\rho_{j,t-1} = P(s_{t-1} = j | \theta, \Omega_{t-1})$, into Eq. (8) and integrating out the state variable s_{t-1} , the density function in Eq. (8) becomes:

$$f(\varepsilon_{s_t,t} | \theta, \Omega_{t-1}) = \sum_{j=1}^2 \sum_{k=1}^2 \left(p_{jk} \rho_{j,t-1} f(\varepsilon_{s_t,t} | s_t = j, \theta, \Omega_{t-1}) \right). \quad (9)$$

$\rho_{j,t-1}$ can be obtained by an integrative algorithm given in Hamilton (1989). The log-likelihood function corresponding to Eq. (9) is shown below.

$$L(\theta | \varepsilon) = \sum_{t=2}^T \ln f(\varepsilon_{s_t,t} | \theta, \Omega_{t-1}) \text{ where } \varepsilon = (\varepsilon_{s_t,1}, \varepsilon_{s_t,2}, \dots, \varepsilon_{s_t,T})'. \quad (10)$$

The MLE $\hat{\theta}$ can be obtained by maximizing Eq. (10).

To identify which economic state the financial return series lies in at time t , we extract the smoothing probability of the low-volatility state as follows (Hamilton, 1989):

$$P(s_t = 1 | \theta, \Omega_T) = \rho_{1,t} \left[\frac{p_{11} P(s_{t+1} = 1 | \theta, \Omega_T)}{P(s_{t+1} = 1 | \theta, \Omega_t)} + \frac{p_{12} P(s_{t+1} = 2 | \theta, \Omega_T)}{P(s_{t+1} = 2 | \theta, \Omega_t)} \right]. \quad (11)$$

Using the fact that $P(s_T = 1 | \theta, \Omega_T) = \rho_{1,T}$, the smoothing probability series $P(s_t = 1 | \theta, \Omega_T)$ can be generated by iterating Eq. (11) backwards from T to 1.

As suggested by Hamilton (1989), a widely recognized rule is that if $P(s_t = 1 | \theta, \Omega_T)$ is less than 0.5, r_t is assumed to lie in the high-volatility state; otherwise, r_t lies in the low-volatility state. In addition, as argued by Wilfling and Maennig (2001) and Wilfling (2009), r_t lies in an extremely high volatility state when $P(s_t = 1 | \theta, \Omega_T)$ is close to 0, in an extremely low volatility state when $P(s_t = 1 | \theta, \Omega_T)$ is close to 1 and in an intermediate state otherwise. Following this idea, we further redefine three economic states. If $P(s_t = 1 | \theta, \Omega_T) < P_1$, r_t lies in the turbulent state. If $P(s_t = 1 | \theta, \Omega_T) > P_2$, r_t lies in the calm state. Otherwise, r_t lies in the intermediate state.

3.2. Step 2: FIGARCH process modeling

After identifying the three economic states, we can fit the 3S-FIGARCH model with a conditional variance equation:

$$b(L)h_t = \omega_{s_t} + [b(L) - \phi(L)(1-L)^d] \varepsilon_t^2 \quad (12)$$

where ω_{s_t} is the intercept at time t in state s_t . As there are only three economic states in this framework, ω_{s_t} will have three possible values that satisfy the condition $\omega_1 < \omega_2 < \omega_3$ (1, 2 and 3 indicate the calm, intermediate and turbulent states, respectively). Further, to make the model more flexible, we can allow ϕ and b in Eq. (12) to vary with time⁶:

$$b_{s_t}(L)h_t = \omega_{s_t} + [b_{s_t}(L) - \phi_{s_t}(L)(1-L)^d] \varepsilon_t^2. \quad (13)$$

The model with the conditional variance equation described in Eq. (12) will be called the 3S-FIGARCH model. The one with the conditional variance equation described in Eq. (13) will be indicated as a Three-State-Varying FIGARCH (3S-V-FIGARCH) model. Both models can be estimated via MLE for different distributions of innovations as described in Section 2.5.

Since FIGARCH model belongs to the family of $ARCH(\infty)$, its asymptotic properties can be derived under standard conditions (Baillie and Morana, 2009). In particular, our 3S-(V-)FIGARCH model has a two-

stage framework, and its second stage is essentially a FIGARCH model with time-varying intercept (and short memory terms) in the conditional variance equation. Also, the break points of the time-varying coefficients are completely determined by the estimated smoothing probability series at the first stage. Hence, the 3S-(V-)FIGARCH, together with A-FIGARCH, TV-FIGARCH and NPCPM-FIGARCH, belongs to the class of $ARCH(\infty)$ processes with time-varying coefficients. The asymptotic properties of MLE for this class of processes have been comprehensively studied in Dahlhaus et al. (2006). Further, as pointed out by Baillie and Morana (2009), although a FIGARCH model with time-varying coefficients is non-strictly stationary and non-ergodic, the results of Dahlhaus et al. (2006) still hold in this non-standard framework. Therefore, just like related structure-changing FIGARCH models such as A-FIGARCH and TV-FIGARCH, under fairly general conditions, the asymptotic distribution of MLE of the 3S-(V-)FIGARCH model is

$$\sqrt{T}(\hat{\theta} - \theta_0) \overset{d}{\rightarrow} N(0, B(\theta_0)^{-1}) \quad (14)$$

where $\hat{\theta}$ is the MLE vector, θ_0 is the true parameter vector, A stands for asymptotic convergence and $B(\theta_0)$ is the Hessian matrix.

Finally, we discuss the parameter constancy of the 3S-(V-)FIGARCH model. As mentioned above, the time-varying coefficients of the 3S-(V-)FIGARCH model are completely decided by a pre-estimated state series $\hat{s}_t$. Thus, the 3S-FIGARCH nests the original FIGARCH model, given the same distributional assumption about the innovations.⁷ Hence, the constancy of ω can be tested via the Likelihood Ratio Test (LRT). More specifically, the test statistic $2(L^{3S} - L^F)$ follows a chi-square distribution with 2 degrees of freedom, where L^F (L^{3S}) is the log likelihood of the (3S-)FIGARCH model. A significant rejection indicates that ω of the FIGARCH model is time-varying. Similarly, since 3S-V-FIGARCH nests the 3S-FIGARCH model given the same distributional assumption about the innovations,⁸ the constancy of $\phi(L)$ and $b(L)$ can also be tested via the LRT. For the (1,d,1) specification, $2(L^{3SV} - L^{3S})$ follows a chi-square distribution with 4 degrees of freedom, where L^{3SV} is the log likelihood of the 3S-V-FIGARCH model. A significant rejection indicates that ϕ_1 and/or b_1 of the 3S-FIGARCH is/are time-varying.

3.3. Comparison with existing structure-changing FIGARCH models

In this section, we perform a series of simulation studies to compare the 3S-(V-)FIGARCH model with the A-FIGARCH, TV-FIGARCH and NPCPM-FIGARCH models.

Following the design of simulation conducted by Belkhouja and Boutahary (2011), we choose three sets of FIGARCH(1,d,1) parameters: $d = 0.25$, $\phi_1 = 0.60$ and $b_1 = 0.20$; $d = 0.50$, $\phi_1 = 0.30$ and $b_1 = 0.30$; and $d = 0.75$, $\phi_1 = 0.20$ and $b_1 = 0.60$. For each set, 3000, 4000 and 5000 simulated observations are generated, respectively. Thus, there are 9 combinations of different sets of parameters and different sample sizes. 300 replicates of the FIGARCH process with 2 change points are simulated for each combination. That is, in Eq. (1) we let $\omega = \omega_t$, where $\omega_t = 0.1, 1$ and 1 for $0 \leq t \leq T/3$, $T/3 < t < 2T/3$ and $2T/3 \leq t \leq T$, respectively. Further, since financial return series is rarely Gaussian, the innovation of this simulated FIGARCH process is assumed to follow a Student's t -distribution with 3 degrees of freedom.⁹

To avoid the starting bias, 10,000 observations are generated for each simulation, and then only the last 3000, 4000 or 5000 observations are used. Moreover, to avoid simulation bias, 500 such replicates are produced for each combination, while the first 200 are discarded.

⁷ If $\omega_1 = \omega_2 = \omega_3$, 3S-FIGARCH reduces to the original FIGARCH case.

⁸ For example, since both models depend on the same pre-estimated $\hat{s}_t$, 3S-V-FIGARCH(1,d,1) reduces to the 3S-FIGARCH(1,d,1) if $\phi_1^1 = \phi_1^2 = \phi_1^3$ and $b_1^1 = b_1^2 = b_1^3$. Superscripts 1, 2 and 3 stand for the parameters of state 1, 2 and 3, respectively.

⁹ We also conduct the simulation studies with CED assumption which lead to consistent results. They are not reported here but are available upon request.

⁶ Since long memory is a long-term characteristic, d should not change for finite-sample data. Otherwise, it could lead to problematic interpretation.

Each simulation is fitted into the original FIGARCH and structure-changing FIGARCH models, including A-FIGARCH, TV-FIGARCH, NPCPM-FIGARCH, 3S-FIGARCH and 3S-V-FIGARCH. The Student's *t*-distribution is used in all models. As to the value of *K* in the A-FIGARCH model, there is no suggested selection criteria provided by Baillie and Morana (2009), so we set it to be 3, which is the number of unique values in the ω_t . In terms of the *R* in the TV-FIGARCH model, as done by Belkhouja and Boutahary (2011), we test the following null hypotheses:

$$\begin{aligned} H_{05} : \omega_5 &= 0 \\ H_{04} : \omega_4 &= 0 | \omega_5 = 0 \\ H_{03} : \omega_3 &= 0 | \omega_4 = \omega_5 = 0 \\ H_{04} : \omega_2 &= 0 | \omega_3 = \omega_4 = \omega_5 = 0 \\ H_{05} : \omega_1 &= 0 | \omega_2 = \omega_3 = \omega_4 = \omega_5 = 0 \end{aligned}$$

and the selected *R* corresponds to the lowest *p*-value among all the *p*-values of the rejected null hypotheses.

All models are fitted with a (1,*d*,1) specification. To compare the model evaluations, we generate the logarithm of likelihood, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) for each estimation. We then differentiate the logarithm of likelihood, AIC and BIC of the FIGARCH model from those of each structure-changing FIGARCH model, respectively. The results for these different models are summarized in Table 1.

In Table 1, all structure-changing FIGARCH models can generally improve the logarithm of likelihood. For AIC and BIC, it appears that the A-FIGARCH model is not always better than the FIGARCH model, while the NPCPM-FIGARCH model can lead to consistent improvements which are greater than those of TV-FIGARCH. Compared with the three

existing structure-changing FIGARCH models, our 3S-FIGARCH model can generate the greatest overall improvement in all cases. In addition, compared with the 3S-FIGARCH model, the 3S-V-FIGARCH model further increases the logarithm of likelihood and decreases AIC in all cases. However, in relation to BIC, which favors parsimony more than AIC, the 3S-FIGARCH model generally outperforms the 3S-V-FIGARCH model.

To compare the estimates, the generated *d* of all models is summarized in Table 2. Considering the FIGARCH model, the estimated *d* is positively biased when the true value is 0.25, which is consistent with Diebold and Inoue (2001). Further, the bias is considerably small and becomes negative when *d* = 0.50 and 0.75, respectively. Also, with the increase of *T*, the biases are reduced in most cases. Despite the fact that A-FIGARCH is preferred to FIGARCH by various model evaluation criteria, the biases of the A-FIGARCH are mostly worse than those of the FIGARCH. TV-FIGARCH can lead to reduced biases when *d* = 0.25, but it fails to reduce them when *d* = 0.75. NPCPM-FIGARCH model can further reduce the biases when *d* = 0.25 and 0.75. On the other hand, 3S-FIGARCH model produces the smallest biases in all cases when *d* = 0.25 and 0.75. Those results are robust when the 3S-V-FIGARCH model is employed. When *d* = 0.50, the biases of the original FIGARCH are quite small, and all the other models except A-FIGARCH lead to similar results. Finally, with the increase of *T*, the SEs in the 3S- and 3S-V-FIGARCH models all consistently decrease. This further supports the asymptotic properties discussed in Section 3.2.

In conclusion, our proposed 3S-FIGARCH framework outperforms the existing A-FIGARCH, TV-FIGARCH and NPCPM-FIGARCH models with respect to various model evaluation criteria and long-memory parameter estimates. Additionally, with the existence of structural breaks,

Table 1
Simulations of structure-changing FIGARCH models.

<i>d</i>	ϕ_1	b_1	<i>T</i>	<i>Mean_A</i>	<i>SE_A</i>	<i>Mean_{TV}</i>	<i>SE_{TV}</i>	<i>Mean_{NP}</i>	<i>SE_{NP}</i>	<i>Mean_{3S}</i>	<i>SE_{3S}</i>	<i>Mean_{3SV}</i>	<i>SE_{3SV}</i>
<i>Panel A: Improved Log Lik.</i>													
.25	0.60	0.20	3000	40.9586	10.7460	20.6337	11.7315	38.8496	18.5423	47.8546	10.6417	50.9873	10.8228
.50	0.30	0.30		29.1742	8.2127	17.3762	10.3008	21.2690	15.8411	48.1731	10.1790	51.7079	10.5719
.75	0.20	0.60		26.9632	8.0193	14.5317	8.4341	16.6577	15.9594	47.2837	9.4596	70.2853	14.1332
.25	0.60	0.20	4000	35.6871	10.8743	16.6189	9.0436	44.9559	16.7244	51.9440	11.4659	55.1612	11.6439
.50	0.30	0.30		25.7710	9.5372	15.7058	8.9374	26.6204	16.2615	56.2669	11.9454	60.1811	11.8929
.75	0.20	0.60		20.6948	7.9211	12.6904	6.6800	21.3041	16.1237	55.7406	11.5111	87.5747	16.6881
.25	0.60	0.20	5000	28.1230	10.9813	16.1330	10.5166	48.4087	16.2421	52.6061	12.4222	55.8955	13.8080
.50	0.30	0.30		19.3635	9.6293	13.6800	8.3819	30.4816	14.2754	62.0995	13.8854	66.2810	13.9548
.75	0.20	0.60		14.6471	6.8343	12.8009	6.4939	24.8629	15.1396	62.2889	12.5842	100.4606	18.0951
<i>Panel B: Improved AIC</i>													
.25	0.60	0.20	3000	−69.9173	21.4919	−35.2673	23.4631	−73.6658	36.2628	−91.7092	21.2834	−89.9746	21.6456
.50	0.30	0.30		−46.3484	16.4254	−28.7523	20.6017	−39.3779	30.4466	−92.3462	20.3579	−91.4157	21.1439
.75	0.20	0.60		−41.9264	16.0387	−23.0633	16.8682	−30.5153	30.4028	−90.5674	18.9192	−128.5706	28.2664
.25	0.60	0.20	4000	−59.3741	21.7486	−27.2378	18.0873	−85.4718	32.8383	−99.8880	22.9318	−98.3223	23.2878
.50	0.30	0.30		−39.5420	19.0745	−25.4115	17.8748	−49.6208	31.2863	−108.5337	23.8907	−108.3622	23.7858
.75	0.20	0.60		−29.3895	15.8422	−19.3807	13.3600	−39.3149	30.6458	−107.4812	23.0223	−163.1494	33.3762
.25	0.60	0.20	5000	−44.2459	21.9627	−26.2660	21.0332	−92.1974	31.9694	−101.2121	24.8445	−99.7911	27.6160
.50	0.30	0.30		−26.7271	19.2585	−21.3600	16.7638	−56.8299	27.5056	−120.1990	27.7709	−120.5620	27.9096
.75	0.20	0.60		−17.2943	13.6687	−19.6018	12.9879	−45.7390	28.9090	−120.5777	25.1684	−188.9212	36.1902
<i>Panel C: Improved BIC</i>													
.25	0.60	0.20	3000	−33.8791	21.4919	−17.2482	23.4631	−61.5530	34.0471	−79.6964	21.2834	−53.9364	21.6456
.50	0.30	0.30		−10.3102	16.4254	−10.7332	20.6017	−29.8879	27.0257	−80.3334	20.3579	−55.3775	21.1439
.75	0.20	0.60		−5.8882	16.0387	−5.0442	16.8682	−22.1064	26.2763	−78.5547	18.9192	−92.5324	28.2664
.25	0.60	0.20	4000	−21.6099	21.7486	−8.3557	18.0873	−71.4990	31.2924	−87.2999	22.9318	−60.5580	23.2878
.50	0.30	0.30		−1.7777	19.0745	−6.5294	17.8748	−38.2285	27.7487	−95.9456	23.8907	−70.5979	23.7858
.75	0.20	0.60		8.3748	15.8422	−0.4986	13.3600	−28.9507	26.0672	−94.8931	23.0223	−125.3851	33.3762
.25	0.60	0.20	5000	−5.1428	21.9627	−6.7144	21.0332	−77.1426	30.7970	−88.1778	24.8445	−60.6879	27.6160
.50	0.30	0.30		12.3761	19.2585	−1.8085	16.7638	−43.3610	24.6439	−107.1646	27.7709	−81.4588	27.9096
.75	0.20	0.60		21.8089	13.6687	−0.0502	12.9879	−32.7481	25.1911	−107.5433	25.1684	−149.8180	36.1902

Note: this table presents the model evaluations of simulations fitted into A-FIGARCH, TV-FIGARCH, NPCPM-FIGARCH, 3S-FIGARCH and 3S-V-FIGARCH models. *LogLik.* is the logarithm of likelihood. *AIC* and *BIC* are Akaike Information Criterion and Bayesian Information Criterion, respectively. *SE* is the standard error. Subscripts *A*, *TV*, *NP*, *3S* and *3SV* stand for the statistic for the A-FIGARCH, TV-FIGARCH, NPCPM-FIGARCH, 3S-FIGARCH and 3S-V-FIGARCH models, respectively. All the figures presented in this table are the difference between the model evaluation statistics of FIGARCH model and those of the corresponding structure-changing FIGARCH model. There are 300 replicates for each set of simulations.

Table 2
Parameter estimates of FIGARCH-type models.

d	ϕ_1	b_1	T	$Bias_F$	SE_F	$Bias_A$	SE_A	$Bias_{TV}$	SE_{TV}	$Bias_{NP}$	SE_{NP}	$Bias_{3S}$	SE_{3S}	$Bias_{3SV}$	SE_{3SV}
0.25	0.60	0.20	3000	0.1233	0.0934	0.1805	0.1727	0.1085	0.1002	0.0647	0.1472	0.0345	0.1954	0.0382	0.1988
.50	0.30	0.30		0.0448	0.1129	0.1281	0.1338	0.0271	0.1145	0.0371	0.1177	0.0366	0.1930	0.0418	0.2075
.75	0.20	0.60		−0.0636	0.0996	0.0758	0.1093	−0.0751	0.1006	−0.0388	0.1109	0.0277	0.1108	0.0242	0.1301
.25	0.60	0.20	4000	0.0928	0.0770	0.1317	0.1408	0.0767	0.0707	0.0311	0.1241	0.0297	0.1925	0.0231	0.1936
.50	0.30	0.30		0.0189	0.1062	0.0836	0.1221	0.0036	0.0937	0.0221	0.1082	0.0118	0.2064	0.0233	0.2202
.75	0.20	0.60		−0.0652	0.0854	0.0243	0.0928	−0.0701	0.0844	−0.0331	0.0913	0.0159	0.0869	0.0069	0.1091
.25	0.60	0.20	5000	0.0818	0.0728	0.1172	0.0952	0.0647	0.0529	0.0273	0.1068	0.0232	0.1685	0.0234	0.1758
.50	0.30	0.30		−0.0003	0.0931	0.0456	0.0998	−0.0111	0.0831	0.0166	0.0935	0.0069	0.1697	0.0155	0.1806
.75	0.20	0.60		−0.0718	0.0808	−0.0107	0.0903	−0.0744	0.0800	−0.0270	0.0849	0.0054	0.0807	−0.0115	0.1048

Note: this table presents the estimated d of simulations fitted into the FIGARCH, A-FIGARCH, TV-FIGARCH, NPCPM-FIGARCH, 3S-FIGARCH and 3S-V-FIGARCH models. Subscript F stands for the statistic of the FIGARCH model. $Bias$ is the difference between the estimated d and the true value. For explanations of other variables, please see Table 1.

our simulation study suggests that positive (negative) bias of d is generated when the true value is smaller (greater) than 0.50. When $d = 0.50$, the bias is basically negligible.

4. Empirical results

We apply all the FIGARCH-type models to four world stock indexes. They are: (1) the NASDAQ, which consists of over 3000 stocks listed on the NASDAQ stock market; (2) the DAX, which consists of 30 large Germany companies; (3) the Nikkei, which consists of 225 Japanese companies; and (4) the ASX, which consists of 50 large Australian companies. The hourly closing prices for each index over the period 1 January 2001 to 31 December 2012 are obtained from Thomson Reuters Tick History (TRTH) database, which contains microsecond-time-stamped tick data dating back to January 1996. The database covers 35 million over-the-counter (OTC) and exchange-traded instruments worldwide that are provided by the Securities Industry Research Center of Australasia (SIRCA). The four series are plotted in Fig. 1.

For each index, the return in the percentage series is defined as the logarithm of the hourly closing price differences times 100; that is,

$r_t = 100 \times \log(S_t/S_{t-1})$, where S_t is the hourly closing price at time t . A group of summary statistics for each index is presented in Table 3. It can be seen that all indexes have a mean close to 0. The standard deviations of NASDAQ, DAX and Nikkei are quite similar and slightly over 0.5. ASX has a smaller standard deviation at around 0.4. NASDAQ is positively skewed, while the others are negatively skewed. For kurtosis, it is clear that none are close to 0, indicating a non-Gaussian distribution. For further confirmation, p-values from both the Kolmogorov–Smirnov and Jarque–Bera normality tests are close to 0, suggesting the rejection of the null hypothesis that return is normally distributed. Finally, the Ljung–Box test indicates that all stock indexes have significant autocorrelation in the squared return, suggesting a time-varying volatility.

4.1. Structural change test

To fit the structure-changing FIGARCH models, we need to firstly examine if there is any structural change in the volatility. If structural change exists, the locations of where changes (change points) occur need to be detected for the NPCPM- and 3S-(V-)FIGARCH models. Hence, we can use the Mood test to verify the existence of structural

Fig. 1. Hourly closing prices for the stock indexes. Note: this figure presents the hourly closing prices for the stock indexes NASDAQ, DAX, Nikkei and ASX from 1/1/2001 to 31/12/2012.

Table 3
Summary statistics for stock indexes.

	NASDAQ	DAX	Nikkei	ASX
Mean	0.0009	0.0006	−0.0014	0.0017
S.D.	0.5309	0.5253	0.5737	0.3988
Skew	0.2223	−0.3482	−0.6378	−0.2992
Kurt.	14.1575	20.8320	33.6778	27.4681
K.S.	0.0000	0.0000	0.0000	0.0000
J.B.	0.0000	0.0000	0.0000	0.0000
Q_{10}^2	0.0000	0.0000	0.0000	0.0000

Note: this table presents the summary statistics for hourly NASDAQ, DAX, Nikkei and ASX indexes ranging from 1/1/2001 to 31/12/2012. Mean is the mean, S.D. is the standard deviation, Skew is the skewness, Kurt. is the kurtosis, K.S. is the p-value of Kolmogorov–Smirnov normality test, J.B. is the p-value of Jarque–Bera normality test and Q_{10}^2 is the p-value of the Ljung–Box test for r_t^2 at lags 10.

change. As noted by Ross (2013), the Mood test only works for independent series. Therefore, we firstly fit the original FIGARCH model for the stock indexes to extract the conditional variance series h_t . We then apply the NPCPM algorithm to the standardized residuals $\varepsilon_t/\sqrt{h_t}$ to detect the change points, if any.¹⁰ If there are no change points, the Mood test will suggest that there are no structural changes in the volatility. Our results suggest that at least 1 change point is detected for each index. Hence, it is significant evidence indicating that during our sample period, all of the four indexes do have structural changes in their volatility. The discovered change points, along with the return series of each index, are plotted in Fig. 2.

As described in Section 3.1, the MRS-t model is fitted for each index to extract $P(s_t = 1|\theta, \Omega_T)$ series. Without loss of generality, we firstly set $P_1 = 1/3$ and $P_2 = 2/3$, so that the series will have the same probability to lie in each of the three states. The smoothing probability series $P(s_t = 1|\theta, \Omega_T)$, along with threshold probabilities P_1 and P_2 , are plotted in Fig. 3. Generally, all indexes lie in the turbulent state during the 2008 Global Financial Crisis (GFC) period. From 2001 to 2003, NASDAQ and Nikkei remained in the turbulent state, which could be due to the burst of the information technology (IT) bubble at the beginning of the twenty-first century. Apparently, the Australian stocks were not seriously affect by the burst of the bubble, and ASX remained in the calm state during this period. Between 2003 and 2008, NASDAQ, DAX and ASX tend to lie in the calm state, while the state of Nikkei is unstable. After 2010 (the end of the 2008 GFC), all indexes appear to switch back to, and remain in the calm state most of the time. In conclusion, the identification of economic states from the MRS-t model for all indexes is apparently consistent with major macroeconomic events.

4.2. Model performance comparison

After detecting the change points for all stock indexes, the NPCPM-FIGARCH and 3S-FIGARCH models can be fitted, along with the original FIGARCH, A-FIGARCH and TV-FIGARCH models. For the A-FIGARCH model, K is set equal to the number of change points detected by the NPCPM algorithm plus 1, indicating the number of subsamples with shifted volatility. As for the TV-FIGARCH model, we perform the five tests to select the order of R as mentioned in Section 3.3. It is interesting to notice that the order of 1 is preferred in all the indexes. All models are fitted with a $(1, d, 1)$ specification. The logarithm of likelihood, AIC and BIC under different distributional assumptions are presented in Table 4. In addition, as argued by Diebold and Inoue (2001), long memory can be related to structural changes and is easily confused with it. Therefore, if structural changes are present and can be correctly modeled, we expect that the estimate of the long-memory parameter d will be smaller. As a result, estimates of d are also reported in Table 4 for comparison.

¹⁰ We use the cpm package in R to apply the NPCPM algorithm. In particular, to minimize the false identification of change points, the ARLO parameter in cpm is set to 50,000. We thank Ross (2013) for making the package available.

For the original FIGARCH models, the AIC and BIC indicate that models with non-Gaussian innovation distributions outperform the Gaussian models for all stock indexes. This result is consistent with the fact that Gaussian distribution assumptions are rejected in all cases. More specifically, FIGARCH models with GED work best for NASDAQ and Nikkei, while those using Student's t are preferred for DAX and ASX. Turning to the long-memory parameter d , all estimates are significant, suggesting that long-memory persistence exists in the conditional volatility of the four stock indexes from 1 January 2001 to 31 December 2012.

In terms of the A-FIGARCH, TV-FIGARCH and NPCPM-FIGARCH models, there are some common conclusions for all the indexes. First, by comparing AIC and BIC, all three models generally outperform the original FIGARCH model in most cases. This suggests that incorporating the structural changes can lead to better model performance. Considering each distributional assumption, TV-FIGARCH models generally outperform A-FIGARCH models for NASDAQ, DAX and Nikkei, when BICs are compared. For ASX, A-FIGARCH models have smaller AIC and BIC than TV-FIGARCH models. More importantly, NPCPM-FIGARCH models outperform both A-FIGARCH and TV-FIGARCH models in all cases. This result indicates that allowing the intercept to vary according to the detected change points has better performance than modeling it via time-dependent functions. In relation to the estimates of d , A-FIGARCH, TV-FIGARCH and NPCPM-FIGARCH models tend to generate similar results. More specifically, estimates of the A-FIGARCH tend to be relatively smaller, except for NASDAQ and ASX when the Student- t distribution is adopted. However, compared with the estimates of d from the original FIGARCH model, the differences are fairly small.

In contrast, the 3S-FIGARCH model achieves the smallest AIC and BIC and leads to the smallest estimates of d . This is consistent across different distributions and stock indexes. In terms of AIC and BIC, the improvements of 3S-FIGARCH models over NPCPM-FIGARCH models are much greater than those of NPCPM-FIGARCH models over A-FIGARCH or TV-FIGARCH models. For example, for NASDAQ with GED assumption, the difference between the BIC of the 3S-FIGARCH and NPCPM-FIGARCH models is more than 200. The difference between the BIC of the NPCPM-FIGARCH models and the A-FIGARCH or TV-FIGARCH models is less than 100. In addition, the 3S-FIGARCH framework (number of parameters is 8) is more parsimonious compared with the NPCPM-FIGARCH framework (number of parameters is 16). As a result, using the information of $P(s_t = 1|\theta, \Omega_T)$ to identify the economic states can lead to a much better model performance. Additionally, it is worth noticing that using log likelihoods of the 3S-FIGARCH and FIGARCH models, the LRT suggests that ω is significantly time-varying in all indexes with all distributional assumptions. This is consistent with the result of NPCPM that the structural changes exist in all cases. More importantly, the 3S-FIGARCH model produces the smallest estimate of d . Compared with the estimate from the original FIGARCH model, the difference can be up to more than 50%. This further confirms that using the information of $P(s_t = 1|\theta, \Omega_T)$ to identify the economic states and estimate the long-memory parameter is potentially more effective and reliable.

As discussed in Section 3.3, short-memory persistence parameters b and ϕ in the conditional variance equation can also be allowed to vary. Thus, we fit the 3S-V-FIGARCH models with three different distributional assumptions for all stock indexes and present the results in Table 5. Comparing the log likelihoods of the 3S-V-FIGARCH with those of the 3S-FIGARCH, the LRT suggests that the null hypothesis that b and ϕ are constant is rejected in all indexes with different distributional assumptions. Hence, in addition to the ω , short memory terms are also time-varying. Overall, estimates of d are consistent with those of 3S-FIGARCH models.

4.3. Further analysis and robustness check

As described in Section 3.1, we use threshold probabilities P_1 and P_2 to identify the economic state. Since the calm and turbulent states

Fig. 2. Hourly return and change points discovered for stock indexes. Note: this figure presents the hourly return and change points discovered for stock indexes NASDAQ, DAX, Nikkei and ASX from 1/1/2001 to 31/12/2012. The vertical red dash line indicates the change points in hourly return discovered using the NPCPM-FIGARCH model. There are 7, 11, 5 and 1 change points detected for NASDAQ, DAX, Nikkei and ASX, respectively.

respectively indicate extremely low and high volatility states, we may expect that the probability of lying in the extreme volatility state is smaller than that of lying in the intermediate state. Therefore, we refit

both the 3S-FIGARCH and 3S-V-FIGARCH models for all stock indexes with three different sets of P_1 and P_2 . The results are presented in Table 6.

Fig. 3. Smoothing probability of calm state for the stock indexes. Note: this figure presents the smoothing probability of calm state for the stock indexes NASDAQ, DAX, Nikkei and ASX. The probability series are all generated from MRS-t models. The horizontal red dash line indicates the bounds ($P_1 = 1/3$ and $P_2 = 2/3$) to identify calm, intermediate and turbulent states based on smoothing probability using the 3S(-V)-FIGARCH models.

Table 4

Summary output of various FIGARCH models for stock indexes.

Models	NASDAQ				DAX				Nikkei				ASX			
	<i>d</i>	<i>log. lik</i>	<i>AIC</i>	<i>BIC</i>	<i>d</i>	<i>log. lik</i>	<i>AIC</i>	<i>BIC</i>	<i>d</i>	<i>log. lik</i>	<i>AIC</i>	<i>BIC</i>	<i>d</i>	<i>log. lik</i>	<i>AIC</i>	<i>BIC</i>
<i>Panel A: Normal</i>																
FIGARCH	0.3841*	−13,823	27,657	27,697	0.3887*	−15,701	31,411	31,451	0.3088*	−14,609	29,228	29,268	0.3021*	−5969	11,948	11,988
A-FIGARCH	0.3363*	−13,776	27,594	27,763	0.3019*	−15,607	31,272	31,511	0.3034*	−14,552	29,137	29,272	0.2592*	−5870	11,759	11,830
TV-FIGARCH	0.3509*	−13,788	27,592	27,656	0.3751*	−15,684	31,384	31,447	0.2865*	−14,608	29,231	29,295	0.2848*	−5917	11,851	11,915
NP-FIGARCH	0.3545*	−13,777	27,578	27,674	0.3318*	−15,631	31,294	31,422	0.3081*	−14,557	29,134	29,214	0.2882*	−5938	11,888	11,936
S-FIGARCH	0.2162*	−13,623	27,259	27,315	0.2034*	−15,383	30,780	30,836	0.2512*	−14,426	28,866	28,921	0.1600*	−5662	11,338	11,394
<i>Panel B: Student-t</i>																
FIGARCH	0.5374*	−11,456	22,924	22,971	0.4234*	−12,560	25,132	25,180	0.5582*	−9773	19,559	19,607	0.3695*	−880	1772	1819
A-FIGARCH	0.9081*	−11,337	22,717	22,895	0.4109*	−12,395	24,851	25,098	0.4103*	−9626	19,288	19,430	0.4173*	−818	1656	1735
TV-FIGARCH	0.5078*	−11,434	22,885	22,957	0.4236*	−12,525	25,068	25,140	0.5325*	−9703	19,424	19,495	0.3764*	−841	1701	1773
NP-FIGARCH	0.4581*	−11,387	22,800	22,903	0.3898*	−12,417	24,867	25,003	0.4974*	−9584	19,189	19,277	0.3701*	−842	1697	1753
S-FIGARCH	0.3458*	−11,257	22,529	22,593	0.2862*	−12,301	24,618	24,682	0.4822*	−9450	18,916	18,980	0.3361*	−753	1522	1586
<i>Panel C: GED</i>																
FIGARCH	0.4405*	−10,861	21,733	21,781	0.3747*	−12,566	25,143	25,191	0.3162*	−9066	18,143	18,191	0.2415*	−1041	2095	2142
A-FIGARCH	0.3666*	−10,820	21,684	21,861	0.2799*	−12,488	25,036	25,283	0.2246*	−8989	18,015	18,157	0.2097*	−1022	2063	2143
TV-FIGARCH	0.4108*	−10,841	21,699	21,771	0.3594*	−12,549	25,117	25,188	0.2598*	−9017	18,052	18,123	0.2229*	−1027	2072	2143
NP-FIGARCH	0.3802*	−10,819	21,664	21,767	0.3034*	−12,492	25,018	25,153	0.2386*	−8983	17,988	18,075	0.2229*	−1026	2066	2121
S-FIGARCH	0.2126*	−10,743	21,503	21,566	0.2096*	−12,381	24,777	24,841	0.1623*	−8927	17,871	17,934	0.1459*	−966	1947	2011

Note: this table presents the summary output for FIGARCH, A-FIGARCH, TV-FIGARCH, NPCPM-FIGARCH and 3S-FIGARCH fitted with Normal, Student's t and GED distributions. The data are hourly NASDAQ, DAX, Nikkei and ASX indexes ranging from 1/1/2001 to 31/12/2012. For explanations of other variables, please see Table 1. 3S-FIGARCH model includes two stages. Firstly, an MRS-t model is fitted for the index data. The estimated smooth probability $P(S_t = 1|T)$ is then collected. Secondly, we estimate a FIGARCH model with two thresholds probabilities 1/3 and 2/3, where the intercept is defined as ω_0^1 when $P(S_t|T) \geq 2/3$, ω_0^2 when $1/3 < P(S_t|T) < 2/3$ and ω_0^3 when $P(S_t|T) \leq 1/3$.

* Denote significance at the 5% level.

In terms of model performance, reducing the probability to lie in the extreme volatility state generally leads to better results. This is consistent across different distributions, models and stock indexes. For example, for NASDAQ, 3S-FIGARCH models with GED produce BIC values of 21,555, 21,534 and 21,522 for the threshold probabilities $P_1 = 0.25$ and $P_2 = 0.75$, $P_1 = 0.15$ and $P_2 = 0.85$, and $P_1 = 0.05$ and $P_2 = 0.95$, respectively. Recall that for $P_1 = 1/3$ and $P_2 = 2/3$, the corresponding BIC value is 21,566. The similar pattern also roughly applies in the estimates of d . For example, for NASDAQ, 3S-FIGARCH models with Student's t generate estimates of d of 0.3390, 0.3212 and 0.3039 for the three sets of threshold probabilities as described above. The corresponding estimate of d is 0.3458 for $P_1 = 1/3$ and $P_2 = 2/3$.

In conclusion, for the 3S-FIGARCH and 3S-V-FIGARCH models, changing the probability to lie in the extreme volatility state leads to consistent model performance and estimates of d . More specifically, by reducing that probability, model performance generally tends to be better, while estimates of d are roughly inclined to be smaller.

5. Conclusion

This paper proposes a two-stage 3S-FIGARCH model as a new approach to detect the structural changes and incorporate them into

the FIGARCH framework. By using simulation studies and the hourly returns of four major stock indexes (NASDAQ, DAX, Nikkei and ASX) from 1 January 2001 to 31 December 2012, we demonstrate that the 3S-FIGARCH model outperforms the original and existing structure-changing FIGARCH frameworks including A-, TV- and NPCPM-FIGARCH models. In addition, Diebold and Inoue (2001) argue that long memory can be related to structural changes and is easily confused with it. As evidenced by our empirical studies, the estimated d from the 3S-FIGARCH framework is generally smaller than those from other FIGARCH-type models. Therefore, this finding further suggests that our proposed approach can lead to potentially more reliable estimate of the long-memory parameter. Finally, we show that the above conclusions also hold when short-memory parameters in the conditional variance equations are allowed to be time-varying and when threshold probabilities to identify the economic states are changed.

Further, our findings can be extended to other contexts in economics and finance, where long memory is the main focus. For instance, our model can be used to enhance the accuracy of dynamic hedging strategies and derivatives pricing models, as high-frequency asset volatility is a key input in these strategies and models (Chronopoulou and Viens, 2012; Hyung et al., 2008; Stentoft, 2005). As noted by Hyung et al. (2008), the FIGARCH specification dominates various short- and long-

Table 5

Summary output of 3S-V-FIGARCH models for stock indexes.

	NASDAQ				DAX			
	<i>d</i>	<i>log. lik</i>	<i>AIC</i>	<i>BIC</i>	<i>d</i>	<i>log. lik</i>	<i>AIC</i>	<i>BIC</i>
Normal	0.2154*	−13,567	27,157	27,244	0.2047*	−15,322	30,666	30,754
Student-t	0.3353*	−11,240	22,503	22,599	0.2861*	−12,282	24,588	24,684
GED	0.1983*	−10,724	21,473	21,568	0.2102*	−12,359	24,741	24,837
	Nikkei				ASX			
	<i>d</i>	<i>log. lik</i>	<i>AIC</i>	<i>BIC</i>	<i>d</i>	<i>log. lik</i>	<i>AIC</i>	<i>BIC</i>
Normal	0.2454*	−14,388	28,798	28,886	0.1705*	−5626	11,274	11,361
Student-t	0.4776*	−9432	18,887	18,983	0.3469*	−749	1522	1618
GED	0.1509*	−8913	17,849	17,945	0.1544*	−958	1939	2035

Note: this table presents the summary output for the 3S-V-FIGARCH models fitted with Normal, Student's t and GED distributions. For explanations of other variables, please see Table 1 and Table 4.

* Denote significance at the 5% level.

Table 6

Summary output of 3S-FIGARCH and 3S-V-FIGARCH models for stock indexes.

Models	NASDAQ				DAX				Nikkei				ASX			
	d	log. lik	AIC	BIC	d	log. lik	AIC	BIC	d	log. lik	AIC	BIC	d	log. lik	AIC	BIC
<i>Panel A: $P_1 = 0.25$ and $P_2 = 0.75$</i>																
S-Normal	0.2125*	−13,616	27,246	27,302	0.1948*	−15,354	30,721	30,776	0.2487*	−14,410	28,835	28,890	0.1579*	−5664	11,342	11,398
S-Student-t	0.3390*	−11,248	22,513	22,576	0.2817*	−12,295	24,606	24,669	0.5639*	−9492	19,000	19,064	0.3303*	−743	1501	1565
S-GED	0.2056*	−10,738	21,492	21,555	0.2037*	−12,372	24,760	24,823	0.1571*	−8923	17,862	17,926	0.1411*	−960	1936	2000
S-V-Normal	0.2068*	−13,561	27,143	27,231	0.1939*	−15,292	30,606	30,693	0.2461*	−14,374	28,770	28,857	0.1680*	−5629	11,280	11,367
S-V-Student-t	0.3288*	−11,236	22,495	22,591	0.2832*	−12,277	24,579	24,674	0.5537*	−9469	18,961	19,056	0.3415*	−740	1503	1598
S-V-GED	0.1976*	−10,722	21,467	21,562	0.2047*	−12,350	24,724	24,819	0.1456*	−8908	17,841	17,936	0.1487*	−954	1931	2026
<i>Panel B: $P_1 = 0.15$ and $P_2 = 0.85$</i>																
S-Normal	0.2023*	−13,600	27,214	27,270	0.1710*	−15,278	30,570	30,625	0.2361*	−14,391	28,796	28,851	0.1475*	−5644	11,302	11,358
S-Student-t	0.3212*	−11,232	22,481	22,544	0.2576*	−12,259	24,535	24,598	0.4519*	−9422	18,860	18,923	0.3231*	−734	1483	1547
S-GED	0.1904*	−10,727	21,471	21,534	0.1827*	−12,339	24,693	24,756	0.1439*	−8914	17,843	17,907	0.1333*	−953	1922	1986
S-V-Normal	0.2067*	−13,557	27,136	27,223	0.1653*	−15,210	30,441	30,528	0.2303*	−14,341	28,705	28,792	0.1577*	−5610	11,242	11,330
S-V-Student-t	0.2972*	−11,218	22,460	22,555	0.2544*	−12,242	24,509	24,604	0.4490*	−9390	18,804	18,899	0.3368*	−730	1484	1579
S-V-GED	0.1820*	−10,707	21,438	21,533	0.1804*	−12,315	24,654	24,749	0.1293*	−8894	17,811	17,906	0.1413*	−946	1916	2011
<i>Panel C: $P_1 = 0.05$ and $P_2 = 0.95$</i>																
S-Normal	0.2046*	−13,600	27,214	27,270	0.1724*	−15,243	30,499	30,555	0.3010*	−14,382	28,778	28,834	0.1435*	−5501	11,016	11,071
S-Student-t	0.3039*	−11,225	22,466	22,529	0.2493*	−12,241	24,497	24,561	0.4375*	−9423	18,862	18,926	0.3128*	−710	1436	1500
S-GED	0.1879*	−10,721	21,459	21,522	0.1783*	−12,317	24,650	24,713	0.1382*	−8910	17,836	17,899	0.1293*	−922	1859	1923
S-V-Normal	0.1872*	−13,535	27,092	27,179	0.1677*	−15,166	30,353	30,441	0.2951*	−14,369	28,760	28,847	0.1439*	−5489	11,000	11,087
S-V-Student-t	0.2891*	−11,211	22,447	22,542	0.2466*	−12,220	24,465	24,560	0.4318*	−9393	18,811	18,906	0.3157*	−707	1438	1533
S-V-GED	0.1763*	−10,701	21,426	21,521	0.1761*	−12,290	24,603	24,698	0.1218*	−8891	17,805	17,900	0.1308*	−918	1861	1956

Note: this table presents the summary output for 3S-FIGARCH and 3S-V-FIGARCH models fitted with Normal, Student's t and GED distributions. The threshold probabilities are set to $P_1 = 0.25$ and $P_2 = 0.75$, $P_1 = 0.15$ and $P_2 = 0.85$, and $P_1 = 0.05$ and $P_2 = 0.95$, respectively. 3S-Normal, 3S-Student-t and 3S-GED are 3S-FIGARCH models with Normal, Student's t and GED distributions, respectively. 3S-V-Normal, 3S-V-Student-t and 3S-V-GED are 3S-V-FIGARCH models with Normal, Student's t and GED distributions, respectively. For explanations of other variables, please see Tables 1 and 4.

* Denote significance at the 5% level.

memory volatility models in terms of its out-of-sample forecasting performance for forecast horizons of 10 days and beyond. The long-memory characteristic has important implications for volatility forecasting and option pricing. Option pricing in a stochastic volatility setting requires a risk premium for the unhedgeable volatility risk. Fractionally integrated series lead to volatility forecasts that are larger than those from short-memory models and these forecasts usually translate into higher option prices. This finding could be an explanation for the better pricing performance of the FIGARCH model (Ho et al., 2013). Stentoft (2005) further notes that incorporating FIGARCH features in option pricing models can potentially help explain some empirically well-documented systematic pricing errors. In addition, out-of-sample performance shows that FIGARCH effects are important when pricing options on individual stocks and can lead to improvements over the constant volatility model. Therefore, when structural changes are likely to exist, applying our framework in place of the FIGARCH model may potentially lead to more reliable estimates and conclusions.

As a future direction to extend this research, a suitable approach to choose the threshold probabilities may further improve our model. To the best of our knowledge, there is no well-accepted method of finding the proper thresholds in the Markov-Switching-type models. For a two-state case, Hamilton (1989) suggests that 0.5 should be used as a rule of thumb. This simple choice is then widely used in relevant empirical studies in finance. Hence, a similar approach is adopted in this paper for the three-state case. Future research should focus on how threshold probabilities can be determined.

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