New Consensus Algorithms for Multi-Agents Systems with Second-Order Dynamics

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Abstract— **This paper is concerned with consensus seeking in multi-agent systems with second-order dynamics under different interaction topologies. In the case of different graphs being used for the interaction topologies for the position and velocity information flows, the focus is placed upon developing a new consensus algorithm for multi-agent systems under some sufficient conditions on the fixed interaction topologies. In the case of the same graph being used for the interaction topologies for the position and velocity information flows, an in-depth study is made of the convergence of consensus of multi-agent systems under the fixed interaction topologies. A numerical example is provided to illustrate the effectiveness of the consensus algorithms.**

I. INTRODUCTION

The potential advantages of employing groups of autonomous agents have recently motivated the vast interest in the coordinated control of multi-agent systems. One of the critical problems is how to design a network "consensus algorithm"such that the group of agents can reach consensus on the shared information in the presence of limited or unreliable information exchanges and dynamically changing network topologies. The consensus problem for agents with single-integrator dynamics has recently been investigated from various perspectives (see, e.g., [3], [5], [9], [13]).

It is noticed that the literature mentioned above is mainly concerned with reaching consensus for agents with single-integrator dynamics. Taking into account that double-integrator dynamics can be used to model a broad class of complex processes, more and more attention has been paid recently to consensus related coordination problems for agents modeled by double-integrator dynamics, see e.g., [2], [6], [8], [14]. Nevertheless, there still exist some open, fundamental, and challenging consensus problems which have not yet been resolved. For example, it is usually assumed that the interaction topologies for the position information flow and velocity information flow are modeled by the same graph (see, e.g., all the literature mentioned above). However, there may exist the case that these two kinds of interaction topologies are different, which can be used to model more general classes of complex processes in practical applications. On the other hand, when the interaction topologies for the position and velocity information flows are modeled by the same directed graph, most of the existing literature (see [7], [8]) concerning the second-order consensus problem only provide some sufficient conditions for the agents under fixed interaction topology to reach consensus. One may ask whether there exists a necessary and sufficient condition on the interaction topology for the agents to reach consensus?

The purpose of this paper is twofold. First, we investigate under what conditions the agents can reach consensus when the interaction topologies for the position and velocity information flows are modeled by different digraphs. Next, when the interaction topologies for the velocity and position information flows are modeled by the same digraph (which will be called interaction topology for consistency), we seek necessary and/or sufficient conditions for the agents to reach (average) consensus under fixed interaction topology. Numerical results are presented to substantiate the theoretical findings.

II. PRELIMINARIES AND PROBLEM STATEMENT

A digraph (or directed graph) will be used to model interaction topology among agents. Let $G = (\mathcal{V}, \varepsilon, \mathcal{A})$ be a weighted digraph of order n with a finite nonempty set of nodes $\mathcal{V} = \{1, 2, \ldots, n\}$, a set of edges $\varepsilon \subset \mathcal{V} \times \mathcal{V}$, and a weighted *adjacency matrix* $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ with non-negative adjacency

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elements a_{ij}. An edge of G is denoted by (i,j). The adjacency elements associated with the edges are positive, i.e., $(j,i) \in \varepsilon \Leftrightarrow a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbors of node i is denoted by $N_i = \{j \in \mathcal{V} : (j,i) \in \varepsilon\}$. A digraph G is called balanced if and only if

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji},\ \forall i \in \mathcal{V}.$$

The Laplacian matrix $L = [l_{ij}] = \mathcal{L}(G)$ of a weighted digraph is defined by

$$l_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{n} a_{ik} & j = i \\ -a_{ij} & j \neq i \end{cases}$$

A digraph G is called *strongly connected* if between any pair of distinct nodes i, j in G, there is a directed path from node i to node j. A digraph G is called *weakly connected* if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. A directed graph has a spanning tree if there exists at least one node, called root node, having a directed path to all other nodes.

Given a nonnegative matrix $S = [s_{ij}] \in \mathbb{R}^{n\times n}$, the weighted digraph of S, denoted by $\Gamma(S)$, is the directed graph with node set $\mathcal{V} = \{1, 2, \ldots, n\}$ such that there is an edge in $\Gamma(S)$ from j to i if and only if $s_{ij} > 0$.

Let $I_n \in \mathbb{R}^{n\times n}$ denote the identity matrix and $\mathbf{1}_m \in \mathbb{R}^m$ be the column vector of all ones, where $m, n \in \mathbb{Z}_+$. Given any matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, let $\text{diag}(A)$ denote the diagonal matrix associated with A with the ith element equal to a_{ii}.

A matrix $M \in \mathbb{R}^{n\times n}$ is *nonnegative*, denoted as $M \geq 0$, if all its entries are nonnegative. A nonnegative matrix M is said to be *stochastic* if all its row sums are $+1$. Let $N \in \mathbb{R}^{n\times n}$. We write $M \geq N$ if $M - N \geq 0$. Given any nonnegative square matrices M and N, if $M \geq \gamma N$, where $\gamma > 0$, then $\Gamma(N)$ is a subgraph of $\Gamma(M)$.

Let $\prod_{i=1}^{k} M_i = M_k M_{k-1} \cdots M_1$ denote the left product of matrices $M_k, M_{k-1}, \cdots, M_1$. A stochastic matrix M is called *indecomposable and aperiodic* (SIA) if there exists a column vector $\mathbf{f} \in \mathbb{R}^n$ such that $\lim_{k\to\infty} M^k = \mathbf{1}_n \mathbf{f}^{\mathrm{T}}$.

Suppose that each agent is modeled by double-integrator dynamics

$$\dot{x}_i = v_i,\ \dot{v}_i = u_i,\ i \in \mathcal{V} = \{1, 2, \ldots, n\} \qquad (1)$$

where $x_i \in \mathbb{R}^m$ is the position state, $v_i \in \mathbb{R}^m$ is the speed state, and $u_i \in \mathbb{R}^m$ is the control input. For simplicity, we assume $m = 1$. However, all results still hold for any $m \in \mathbb{Z}_+$ by introducing the notation of Kronecker product, where $\mathbb{Z}_+$ denote the set of positive integers. In this paper, we mainly consider the following consensus algorithm which requires that both the relative position and velocity information can be measured by the neighboring agents:

$$\begin{aligned} u_i = &- \alpha v_i \\ &+ \sum\nolimits_{j\in N_i(t)} a_{ij}(t)\left[(x_j - x_i) + \gamma(v_j - v_i)\right] \end{aligned} \qquad (2)$$

where $\gamma > 0$ denotes the coupling strength of relative velocities between neighboring agents and $\alpha > 0$ denotes the absolute velocity damping gain. The weighting factor $a_{ij}(t) \geq 0$ if agent i can receive information from agent j at time t while $a_{ij}(t) = 0$ otherwise. Here, the weighting factors can be chosen from an infinite set. In particular, we assume that all the nonzero and hence positive weighting factors are both uniformly lower and upper bounded, i.e., $a_{ij}(t) \in [\underline{\alpha},\ \bar{\alpha}]$, where $0 < \underline{\alpha} < \bar{\alpha}$, if $j \in N_i(t)$. Note that $N_i(t)$ is variable when the interaction topology is dynamically changing. For the case that the interaction topologies for the position and velocity information flows, denoted as G_p and G_v respectively, are different, we propose the following consensus algorithm

$$\begin{aligned} u_i = &- \alpha v_i \\ &+ \sum_{j\in N_i^p} a_{ij}(x_j - x_i) + \gamma \sum_{j\in N_i^v} b_{ij}(v_j - v_i) \end{aligned} \qquad (3)$$

where $\alpha > 0, \gamma > 0$; N_i^p and N_i^v denote respectively the set of neighbors of agent i in G_p and G_v.

We say that the consensus is reached asymptotically for the group of agents if for any $x_i(0), v_i(0)$,

$$\lim_{t\to\infty} (x_i(t) - x_j(t)) = 0, \quad i \neq j$$

and

$$\lim_{t\to\infty} v_i(t) = 0.$$

In the next two sections, we will seek some new necessary and/or sufficient conditions for the agents to reach consensus by employing algorithms (3) and (2), respectively.

III. Setting with Position and Velocity Information Flows Modeled by Different Graphs

Let $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $\mathcal{B} = [b_{ij}] \in \mathbb{R}^{n\times n}$ be the adjacency matrices of the position information flow G_p and the velocity information flow G_v, respectively. We first perform the convergence analysis for the general consensus algorithm (3) in the setting that $\mathcal{A}$ and $\mathcal{B}$ are different.

Theorem 1: If the undirected graph G_p is connected and the digraph G_v is balanced, then for any $\alpha > 0$, $\gamma > 0$, consensus is reached by employing algorithm (3).

Proof: Let $x_{ij} = x_i - x_j$. Then, from (1) and (3) we have

$$\begin{cases} \dot{x}_{ij} = v_i - v_j \\ \dot{v}_i \ = -\alpha v_i \\ \qquad + \sum\limits_{j \in N_i^p} a_{ij} x_{ji} + \gamma \sum\limits_{j \in N_i^v} b_{ij}(v_j - v_i) \end{cases}. \quad (4)$$

Consider the following Lyapunov function candidate for (4)

$$\begin{aligned} V =& \frac{1}{2}\mathbf{x}^{\mathrm{T}} L \mathbf{x} + \frac{1}{2}\|\mathbf{v}\|^2 \\ =& \frac{1}{4}\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_{ij}^2 + \frac{1}{2}\sum_{i=1}^n v_i^2 \end{aligned}$$

where $\mathbf{x} = [x_1 \ \cdots \ x_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\mathbf{v} = [v_1 \ \cdots \ v_n]^{\mathrm{T}} \in \mathbb{R}^n$. From the fact that G_p, the undirected graph associated with $\mathcal{A}$, is connected, it follows that V is positive definite and radically unbounded with respect to $x_{ij}, \forall i \neq j$ and v_i. Differentiating V, we have

$$\begin{aligned} \dot{V} =& \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij}(v_i - v_j)x_{ij} + \sum_{i=1}^n v_i u_i \quad (5) \\ =& \sum_{i=1}^n \sum_{j=1}^n a_{ij} v_i x_{ij} \\ &+ \sum_{i=1}^n v_i[-\alpha v_i + \sum_{j=1}^n a_{ij} x_{ji} + \gamma \sum_{j=1}^n b_{ij}(v_j - v_i)] \quad (6) \\ =& -\alpha \sum_{i=1}^n v_i^2 + \gamma \sum_{i=1}^n \sum_{j=1}^n b_{ij} v_i (v_j - v_i) \quad (7) \end{aligned}$$

where (6) can be derived by replacing u_i with

$$-\alpha v_i + \sum_{j \in N_i} a_{ij} x_{ji} + \gamma \sum_{j \in N_i} b_{ij}(v_j - v_i)$$

and also using the fact that

$$-\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij} v_j x_{ij} = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n a_{ij} v_i x_{ij}$$

in (5). The latter argument is obtained by switching the dummy variables i and j and the order of the summation signs, and then employing the fact that $a_{ij} = a_{ji}$ and $x_{ij} = -x_{ji}$. Now consider the second term in (7). Let

$$\Delta = \frac{1}{2}\gamma \sum_{i=1}^n \sum_{j=1}^n b_{ij} v_j (v_j - v_i).$$

Using the similar manipulation techniques as that for the second term in (5), the second term in (7) then takes the following form

$$\begin{aligned} &\gamma \sum_{i=1}^n \sum_{j=1}^n b_{ij} v_i (v_j - v_i) \\ &= \frac{1}{2}\gamma \sum_{i=1}^n \sum_{j=1}^n b_{ij}\left(v_i - v_j\right)\left(v_j - v_i\right) + \Delta_1 \quad (8) \end{aligned}$$

where

$$\begin{aligned} \Delta_1 =& \frac{1}{2}\gamma \sum_{i=1}^n \sum_{j=1}^n b_{ji} v_j (v_i - v_j) + \Delta \\ =& \frac{1}{2}\gamma \sum_{i=1}^n \sum_{j=1}^n \left(b_{ji} - b_{ij}\right) v_j (v_i - v_j). \end{aligned}$$

Combining this with the fact that the digraph G_v associated with $\mathcal{B}$ is balanced yields

$$\begin{aligned} \Delta_1 =& -\frac{1}{2}\gamma \sum_{i=1}^n \sum_{j=1}^n \left(b_{ji} - b_{ij}\right) v_j^2 \\ &+ \frac{1}{2}\gamma \sum_{i=1}^n \sum_{j=1}^n \left(b_{ji} - b_{ij}\right) v_j v_i \\ =& -\frac{1}{2}\gamma \sum_{j=1}^n \left[\sum_{i=1}^n \left(b_{ji} - b_{ij}\right)\right] v_j^2 \\ =& \ 0 \end{aligned}$$

which, together with equalities (7) and (8), implies that

$$\dot{V} = -\alpha \sum_{i=1}^n v_i^2 - \frac{1}{2}\gamma \sum_{i=1}^n \sum_{j=1}^n b_{ij}\left(v_i - v_j\right)^2 \leq 0. \quad (9)$$

Let $S = \{(x_{ij}, v_i) : \dot{V} = 0\}$. From inequality (9), we know that $\dot{V} \equiv 0$ implies that $v_i \equiv 0$, and thus, $\dot{v}_i \equiv 0, \forall i \in \mathcal{V}$. It is worth noting that the conclusion that $v_i \equiv 0, \forall i \in \mathcal{V}$ is irrelevant to any kinds of connected properties of digraph G_v if only it is a balanced digraph. It then follows from (4) that

$$\sum_{j \in N_i} a_{ij} x_{ji} \equiv 0, \forall i \in \mathcal{V},$$

which also implies that $L_p \mathbf{x} \equiv 0$, where L_p is the Laplacian matrix of undirected graph G_p. Since G_p is connected, we have $\mathrm{rank}(L_p) = n - 1$ [1], which,

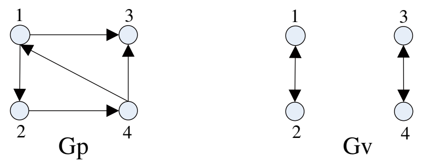

Fig. 1. Two graphs modeling the position and velocity information flow, respectively.

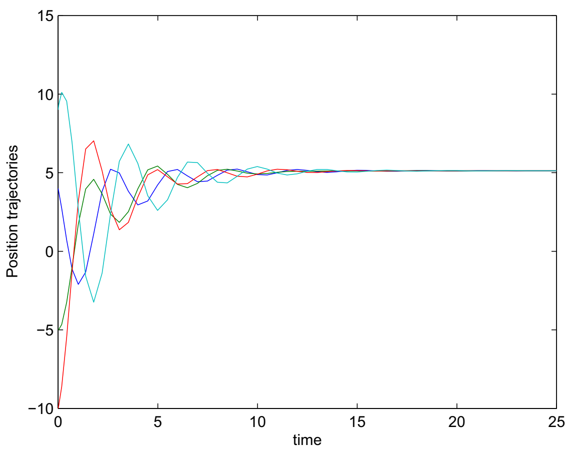

Fig. 2. Position trajectories for the four agents under algorithm (3).

in turn, implies $\mathbf{x} \in \text{span}\{\mathbf{1}_n\}$, i.e., $x_{ij} \equiv 0$, $\forall i \neq j$. Finally, it follows from the Lasalle's Invariance Principle [4] that $x_i(t) - x_j(t) \rightarrow 0$ and $v_i(t) \rightarrow 0$, $\forall i \neq j$, as $t \rightarrow \infty$, which completes the proof. ■

Example 1: The simulation is performed with four agents. The interaction topologies for the position and velocity information flows are modeled by G_p and G_v as shown in Figure 1, respectively. However, we further assume that G_p is undirected; that is, all the corresponding edges are bidirectional. Obviously, graph G_p is connected while G_v is a balanced digraph. For illustration, we choose α and γ as $\alpha = 0.4$ and $\gamma = 0.2$, respectively. Figure 2 and Figure 3 show that consensus can be reached by employing algorithm (3), which is in agreement with the result in Theorem 1.

IV. Setting with Position And Velocity for in Formation Flows Modeled by the Same Digraph

This section is mainly concerned with the consensus analysis for algorithm (2) under time-invariant and directed interaction topology in the setting with $G_p = G_v$.

The following lemma shows that a new digraph can

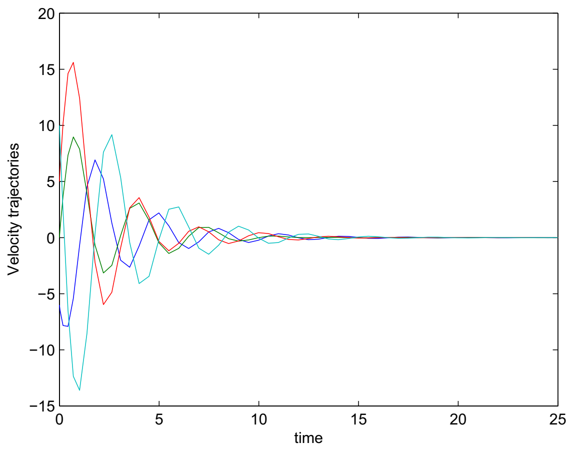

Fig. 3. Velocity trajectories for the four agents under algorithm (3).

be constructed with the same property of having a directed spanning tree as its original digraph. This serves to pave the way for specifying the intrinsic relations between the interaction topology and the associated digraph of the transformed system matrix, which will be shown through the proof of the main theorem to be presented.

Lemma 1: Let $G(\mathcal{V}, \varepsilon)$ be any given digraph. Assume that $G^{'}(\mathcal{V}^{'}, \varepsilon^{'})$ is a graph with n nodes and empty edge set, that is, $\mathcal{V}^{'} = \{n+1, n+2, \ldots, 2n\}$ and $\varepsilon^{'} = \varnothing$. Let $G^{*}(\mathcal{V}^{*}, \varepsilon^{*})$ be the digraph constructed from $G(\mathcal{V}, \varepsilon)$ and $G^{'}(\mathcal{V}^{'}, \varepsilon^{'})$ according to the following three rules.

- (A) $\mathcal{V}^{*} = \mathcal{V} \cup \mathcal{V}^{'} = \{1, \ldots, n, n+1, \ldots, 2n\}$;
- (B) There is no edge between node i and node j for any $i, j \in \mathcal{V}$, $i \neq j$. In addition, edge $(n+i, i) \in \varepsilon^{*}$ while edge $(i, n+i)$ might or might not exist in G^{*} for any $i \in \mathcal{V}$;
- (C) Edge $(n+j, n+i) \in \varepsilon^{*}$ if and only if edge $(j, i) \in \varepsilon$ for any $i, j \in \mathcal{V}$, $i \neq j$.

Then, digraph G has a directed spanning tree if and only if digraph G^{*} has a directed spanning tree.

Proof: Necessity: Denote by G_s a directed spanning tree of digraph G, and without loss of generality, assume that l is the root node of G_s. Define the following map from the edge set ε_s of G_s into the edge set ε^{*}:

$$\digamma : (p, q) \longmapsto (n+p, n+q).$$

Let

$$\bar{\varepsilon} = \bigcup\nolimits_{\forall (p,q) \in G_s} \{(n+p, n+q)\}.$$

It follows from Rule (C) that $\digamma$ is a one-to-one map from the edge set ε_s onto the edge set $\bar{\varepsilon}$. Let $\bar{G}$ be the digraph whose edge set is $\bar{\varepsilon}$ and node set consists of all the nodes which the edges in $\bar{\varepsilon}$ connect to. Rule (C) also implies that digraph $\bar{G}$ shares the same topological structure as that of G_s. Then it follows that $\bar{G}$ is a directed tree in digraph G^*, where $n+l$ must not be confined to be the only root node of $\bar{G}$. Clearly, the node set of $\bar{G}$ is just $\{n+1, n+2, \ldots, 2n\} = \mathcal{V}'$.

Now we can construct a directed spanning tree of digraph G^* based on the directed tree $\bar{G}$. From Rule (B), we know that $(n+i, i) \in \varepsilon^*, \forall i \in \mathcal{V}$. Thus, we can add n edges $(n+i, i)$, $i \in \mathcal{V}$ to the tree $\bar{G}$, which, in turn, creates a new tree G_s^* in digraph G^*. Notice, however, that the node set of G_s^* is exactly the node set $\mathcal{V}^*$, thus G_s^* is a directed spanning tree of G^* with root node $n+l$.

Sufficiency: Let G_s^* be a directed spanning tree of digraph G^*. Note that by the definition of G^*, digraph G can be obtained by contracting all the edges $(n+i, i)$ in digraph G^* for all $i \in \mathcal{V}$. Thus, the operation of edge contraction on G_s^* results in a directed spanning tree, say G_s, of digraph G. Here, we need to point out that if node l is the root of G_s^*, then node l must be the root of G_s; and if node $n+l$ is the root of G_s^*, node l must also be the root of G_s. ■

To move on further, we also need the following result, a summarized work of [5] and [9], which is regarding the consensus problem for agents modeled by single-integrator dynamics.

Lemma 2: Suppose that $\xi = [\xi_1 \ \ldots \ \xi_n]^{\mathrm{T}}$ with $\xi_i \in \mathbb{R}$. Then, consensus is reached asymptotically for system $\dot{\xi} = -L\xi$, i.e.,

$$\lim_{t\to\infty}(\xi_i(t) - \xi_j(t)) = 0, \ i, j \in \{1, 2, \ldots, n\}$$

if and only if G has a directed spanning tree, where G is the interaction topology for the group of agents and $L \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of digraph G. Moreover, if G has a directed spanning tree, then there exists a nonnegative column vector $\beta = [\beta_1 \ \ldots \ \beta_n]^{\mathrm{T}} \in \mathbb{R}^n$ satisfying $\beta^{\mathrm{T}} L = 0$ and $\beta^{\mathrm{T}} \mathbf{1}_n = 0$.

Theorem 2: Consider a directed network of agents with fixed interaction topology G. Assume that the velocity coupling strength γ satisfies $\gamma \geq \frac{1}{\alpha}$. Then, applying algorithm (2), consensus is reached, i.e., $x_i(t) \to x_j(t)$ and $v_i(t) \to 0$, $i, j \in \mathcal{V}$, as $t \to \infty$ if and only if G has a directed spanning tree.

Proof: To facilitate our analysis, we make the following transformation for the network dynamics, which is different from the one used in [7], [8] and [14]. Let $\mathbf{x} = [x_1 \ \cdots \ x_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\mathbf{y} = [y_1 \ \cdots \ y_n]^{\mathrm{T}} \in \mathbb{R}^n$, and $\xi = \begin{bmatrix} \mathbf{x}^{\mathrm{T}} & \mathbf{y}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{2n}$, where $y_i = \gamma v_i + x_i$. Applying algorithm (2) in equation (1), the network dynamics for all the agents can be written in matrix form as

$$\dot{\xi} = \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix} = \Xi(t) \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

where

$$\Xi(t) = \begin{bmatrix} -\frac{1}{\gamma} I_n & \frac{1}{\gamma} I_n \\ (\alpha - \frac{1}{\gamma}) I_n & (\frac{1}{\gamma} - \alpha) I_n - \gamma L(t) \end{bmatrix}$$

and $L(t)$ is the Laplacian matrix of the interaction topology at time t. When the interaction topology is fixed, the network dynamics is

$$\dot{\xi}(t) = \Xi \xi(t). \tag{10}$$

It follows from $L\mathbf{1}_n = 0$ that $\Xi \mathbf{1}_{2n} = 0$. On the other hand, since $\gamma \geq \frac{1}{\alpha}$, all the off-diagonal elements of matrix Ξ are nonnegative. Thus, matrix $-\Xi$ can be considered as the Laplacian matrix of a digraph G^* with $2n$ nodes, say, $1, \ldots, n, n+1, \ldots, 2n$, that is, $\mathcal{L}(G^*) = -\Xi$. Note that in this case, each node in G^* can be considered as an agent with single-integrator dynamics. And the network dynamics for these $2n$ agents is just (10). Combining this with equality $\mathbf{y} = \gamma \mathbf{v} + \mathbf{x}$, we know that the second-order consensus problem under investigation is equivalent to the first-order consensus problem for system (10). This paves the way for us to study the second-order consensus problem from a new viewpoint.

Denote matrices $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $\mathcal{A}^* = [b_{ij}] \in \mathbb{R}^{2n\times 2n}$ as the adjacency matrices of digraphs G and G^*, respectively. According to the definition of the Laplacian matrix, by some manipulations, we can get

$$\mathcal{A}^* = \begin{bmatrix} 0 & \frac{1}{\gamma} I_n \\ (\alpha - \frac{1}{\gamma}) I_n & \gamma \mathcal{A} \end{bmatrix}.$$

Sufficiency: Firstly, we will prove the argument that digraphs G and G^* share the same property of having a directed spanning tree; that is, G has a directed spanning tree if and only if G^* has a directed spanning tree. By checking the elements of matrix $\mathcal{A}^*$, we know that $b_{i,n+i} = \frac{1}{\gamma} > 0$ and $b_{n+i,i} = \alpha - \frac{1}{\gamma} \geq 0$, $\forall i \in \mathcal{V}$, which implies that there is an edge connecting node $n+i$ to node i while edge $(i, n+i)$ might or might not exist in digraph G^*. In addition, $b_{ij} = 0$ implies that there is no edge between node i and node j for any $i, j \in \mathcal{V}$, $i \neq j$. On the other hand, the equality

$$\begin{bmatrix} 0 & b_{n+1,n+2} & \cdots & b_{n+1,2n} \\ b_{n+2,n+1} & 0 & \cdots & b_{n+2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{2n,n+1} & b_{2n,n+2} & \cdots & 0 \end{bmatrix} = \gamma \mathcal{A} \quad (11)$$

implies that $a_{ij} = \frac{1}{\gamma} b_{n+i,n+j}$, which means that $a_{ij} > 0$ if and only if $b_{n+i,n+j} > 0$. That is to say, edge $(n+j, n+i)$ exists in digraph G^* if and only if edge (j, i) exists in digraph G for any $i, j \in \mathcal{V}$, $i \neq j$. Combining these arguments, we know that G^* is precisely the digraph constructed from digraph G and G' (with node set $\{n+1, n+2, \ldots, 2n\}$) according to the three rules defined in Lemma 1. This, together with Lemma 1, completes the proof for this argument.

Now consider the network of single integrator agents with fixed topology $G^* = (\mathcal{V}^*, \varepsilon^*, \mathcal{A}^*)$. According to Lemma 2, we can get $\lim_{t\to\infty}(x_i(t) - y_j(t)) = 0$, $\forall i, j \in \mathcal{V}$. Noticing that $y_i = \gamma v_i + x_i$, thus we have $\lim_{t\to\infty} v_i(t) = \frac{1}{\gamma}\lim_{t\to\infty}(y_i(t) - x_i(t)) = 0$, i.e., algorithm (2) asymptotically solves the consensus problem.

Necessity: If the interaction topology G does not have a directed spanning tree, then neither does digraph G^*. As pointed out above, $-\Xi$ can be considered as the Laplacian matrix of a digraph G^* with $2n$ nodes. Now consider system (10). Given that G^* does not have a spanning tree, it follows that there exist at least two subsystems of (10), among which there is no information exchange. Thus, consensus cannot be reached for system (10), which, in turn, implies that consensus cannot be reached by employing algorithm (2). This is a contradiction with the given condition that $x_i(t) \to x_j(t)$ and $v_i(t) \to 0$, $i, j \in \mathcal{V}$ as $t \to \infty$. ■

Remark 1: Theorem 4.3 in [8] presents a sufficient condition on the velocity coupling strength γ and interaction topology for the agents to reach consensus. However, the condition imposed on γ therein is complicated and the lower bound given for γ is difficult to determine, as it will vary with both the weighting factors and also the topological structure of the interaction topology. In contrast, the condition imposed on γ in this paper is easy to determine. Moreover, with the new condition, the convergence analysis for all the agents is irrelevant to the weighting factors of interaction topology which makes it easier to design the control input for each agent.

V. Conclusion

In this paper we have investigated the convergence of the consensus strategies for multiple agents with double-integrator dynamics under two kinds of different settings: the setting where the interaction topologies for the position and velocity information flows are modeled by different graphs and the setting where the interaction topologies for the position and velocity information flows being modeled by the same graph. In both settings, we have derived some sufficient (and necessary) conditions on the fixed interaction topologies for the agents to reach consensus.

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