Joint 48th IEEE Conference on Decision and Control and
28th Chinese Control Conference
Shanghai, P.R. China, December 16-18, 2009

A Network Synthesis Theorem for Linear Dynamical Quantum Stochastic Systems

Hendra I. Nurdin, Matthew R. James and Andrew C. Doherty

Abstract— **In this paper we introduce a network synthesis theorem for linear dynamical quantum stochastic systems that are encountered in linear quantum optics and in phenomenological models of quantum linear circuits. In particular, such a theorem will be important in the physical realization of coherent/fully quantum linear stochastic controllers for quantum control. We show how general linear dynamical quantum stochastic systems can be systematically constructed by assembling an appropriate interconnection of one degree of freedom open quantum harmonic oscillators and, in the quantum optics setting, provide an explicit illustrative example of the systematic synthesis of a two degrees of freedom open quantum harmonic oscillator.**

I. Introduction

In recent years there has been increased interest in exploiting quantum mechanical systems as a basis for new quantum technologies, giving birth to the field of quantum information science. To develop quantum technologies, it has been recognized that quantum control systems will be very important for tasks such as manipulating a quantum mechanical system to perform a desired function or to protect it from external disturbances [1].

In particular, more recently there have been theoretical and experimental investigations of fully quantum and mixed quantum-classical linear controllers that are able to manipulate quantum signals [2]–[7]. The line of research in [3]–[6] has raised the important open problem of how one would systematically build or implement a general, arbitrarily complex, linear quantum controller, at least approximately, from basic quantum devices, such as quantum optical devices. This problem can be viewed as a quantum analogue of the synthesis problem of classical electrical networks that asks the question of how to build arbitrarily complex linear electrical circuits from elementary passive and active electrical components such as resistors, capacitors, inductors, voltage and current sources. Therefore the synthesis problem is not only relevant for the construction of linear quantum stochastic controllers, but also fundamental to the development of linear quantum circuit theory that arises, for example, in quantum optics and when working with phenomenological models of quantum RLC circuits such as described in [8].

The main result of this paper is a new synthesis theorem that allows systematic construction of arbitrarily complex linear quantum stochastic systems based around a cascade connection of one degree of freedom open quantum harmonic oscillators as basic building blocks. An explicit yet simple example that illustrates the application of the theorem to the synthesis of a two degrees of freedom open quantum harmonic oscillator is provided.

H. I. Nurdin and M. R. James are with the Department of Information Engineering, Australian National University, Canberra, ACT 0200, Australia. Hendra.Nurdin,Matthew.James@anu.edu.au. Research supported by the Australian Research Council

A. C. Doherty is with the School of Physical Sciences, University of Queensland, Queensland 4072, Australia. doherty@physics.uq.edu.au

II. Linear Quantum Stochastic Systems

In previous works [3], [9] quantum linear stochastic systems are essentially considered as open quantum harmonic oscillators. Here we shall consider a more general class of linear quantum stochastic systems consisting of the cascade of a static passive optical network with an open quantum harmonic oscillator. However, in this paper we restrict our attention to synthesis of linear systems with purely quantum dynamics, whereas the earlier work [3] considers a more general scenario where a mixture of both quantum and classical dynamics are allowed (via the concept of an augmentation of a quantum linear stochastic system). The class of mixed classical and quantum controllers will be considered in a separate work. To this end, let us first recall the definition of an open quantum harmonic oscillator (for further details, see [3], [4], [9]).

In this paper we shall use the following notations: $i = \sqrt{-1}$, * will denote the adjoint of a linear operator as well as the conjugate of a complex number, if $A = [a_{jk}]$ is a matrix of linear operators or complex numbers then $A^{\#} = [a^*_{jk}]$, and $A^{\dagger}$ is defined as $A^{\dagger} = (A^{\#})^T$, where ^{T} denotes matrix transposition. We also define, $\Re\{A\} = (A + A^{\#})/2$ and $\Im\{A\} = (A - A^{\#})/2i$ and denote the identity matrix by I whenever its size can be inferred from context and use $I_{n\times n}$ to denote an $n \times n$ identity matrix.

Let $q_1, p_1, q_2, p_2, \ldots, q_n, p_n$ be the canonical position and momentum operators, satisfying the canonical commutation relations $[q_j, p_k] = 2i\delta_{jk}$, $[q_j, q_k] = 0$, $[p_j, p_k] = 0$, of a *quantum harmonic oscillator* with a quadratic Hamiltonian $H = \frac{1}{2}x_0^T R x_0$ ($x_0 = (q_1, p_1, \ldots, q_n, p_n)^T$), where $R = R^T \in \mathbb{R}^{n\times n}$. The integer n will be referred to as the *degrees of freedom* of the oscillator. Let $A_1(t), A_2(t), \ldots, A_m(t)$ be bosonic vacuum quantum noise fields satisfying the quantum Ito multiplication rules [10], [11]: $dA_j dA_k^* = \delta_{jk}dt$ and $dA_j dA_k = 0$ $\forall j, k = 1, \ldots, m$, where δ_{jk} is the Kronecker delta which takes on the value 0 unless $j = k$ in which case it takes on the value 1. Formally speaking, the fields $A_1(t), A_2(t), \ldots, A_m(t)$ are, respectively, integrated versions of singular quantum white noise processes $\eta_1(t), \eta_2(t), \ldots, \eta_m(t)$ satisfying the singular commutation relations : $[\eta_j(t), \eta_k(t')^*] = \delta_{jk}\delta(t - t')$, where $\delta(t)$ denotes the Dirac delta function, $\forall j, k$ and $\forall t, t' \geq 0$. An *open* *quantum harmonic oscillator*, or simply an *open oscillator*, is defined as a quantum harmonic oscillator coupled to $A(t)$ via the formal time-varying *idealized* interaction Hamiltonian [12, Chapter 11]

$$H_{Int}(t) = i(L^T\eta(t)^* - L^\dagger\eta(t)),$$

where L is a *linear* coupling operator given by $L = Kx_0$ with $K \in \mathbb{C}^{m\times n}$ and $\eta(t) = (\eta_1(t), \eta_2(t), \ldots, \eta_m(t))^T$. Although the Hamiltonian is formal since the $\eta_j(t)$'s are singular quantum white noise processes, it can be given a rigorous interpretation in terms of Markov limits (e.g., [13], [12, Chapter 11]). The evolution of the open oscillator is then governed by the unitary process $\{U(t)\}_{t\geq 0}$ satisfying the quantum stochastic differential equation (QSDE) [3], [9], [12]:

$$dU(t) = (-iHdt + L^T dA(t)^\# - L^\dagger dA(t) - \frac{1}{2}L^\dagger L dt)U(t);\ U(0) = I. \quad (1)$$

The time evolved canonical operators are given by $x(t) = U(t)^* x_0 U(t)$ and satisfy the QSDE:

$$dx(t) = 2\Theta(R + \Im\{K^\dagger K\})x(t)dt + 2i\Theta[\ -K^\dagger \quad K^T\ ]\begin{bmatrix} dA(t) \\ dA(t)^\# \end{bmatrix}, \quad x(0) = x_0, \quad (2)$$

where Θ is a canonical commutation matrix of the form $\Theta = \mathrm{diag}(J, J, \ldots, J)$ with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

while the output bosonic fields $Y(t) = (Y_1(t), \ldots, Y_n(t))^T$ that results from interaction of $A(t)$ with the harmonic oscillator are given by $Y(t) = U(t)^* A(t) U(t)$ and satisfy the QSDE:

$$dY(t) = Kx(t)dt + dA(t). \quad (3)$$

Note that the dynamics of $x(t)$ and $Y(t)$ are linear.

The input $A(t)$ of an open oscillator can first be passed through a static passive linear optical network (see, e.g., [14], [15] for details) without affecting the linearity of the overall system dynamics, this is shown in Fig. 1. Such an operation effects the transformation $A(t) \mapsto \tilde{A}(t) = SA(t)$, where $S \in \mathbb{C}^{m\times m}$ is a complex unitary matrix (i.e., $S^\dagger S = SS^\dagger = I$). Thus $\tilde{A}(t)$ will be a new set of vacuum noise fields satisfying the same Ito rule as $A(t)$.

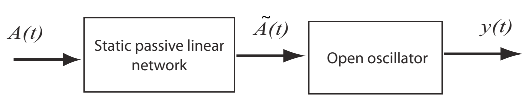

Fig. 1. A generalized open oscillator

Letting $S = [S_{ij}]_{i,j=1,\ldots,m}$, it can be shown by straightforward calculations using the quantum Ito stochastic calculus that the cascade is equivalent (in the sense that it produces the same dynamics for $x(t)$ and $y(t)$) to a linear quantum system whose dynamics is governed by a unitary process $\{\tilde{U}(t)\}_{t\geq 0}$ satisfying the QSDE (for a general treatment, see [16]):

$$d\tilde{U}(t) = (\sum_{j,k=1}^{m}(S_{jk} - \delta_{jk})d\Lambda_{jk}(t) - iHdt + L^T dA(t)^\# - L^\dagger dA(t) - \frac{1}{2}L^\dagger L dt)\tilde{U}(t);\ \tilde{U}(0) = I, \quad (4)$$

where $\Lambda_{jk}(t)$ $(j, k = 1, \ldots, m)$ are fundamental processes, called the gauge processes, satisfying the quantum Ito rules:

$$d\Lambda_{jk}d\Lambda_{j'k'} = \delta_{lk'}d\Lambda_{kl'}dt;\ dA_j(t)dA_{kl}(t) = \delta_{jk}dA_l(t).$$

For convenience, in the remainder of the paper we shall refer to the cascade of a static passive linear optical network with an open oscillator as a *generalized open oscillator*.

Let G be a generalized open oscillator that evolves according to the QSDE (4) with given parameters S, $L = Kx_0$ and $H = \frac{1}{2}x_0^T R x_0$. For compactness, we shall use a shorthand notation of [16] and denote such a generalized open oscillator by $G = (S, L, H)$. In the next section we briefly recall the concatenation and series product developed in [16] that allows one to systematically obtain the parameters of a generalized open oscillator built up from an interconnection of generalized open oscillators of one degree of freedom.

III. The Concatenation and Series Product of Generalized Open Oscillators and Reducible Quantum Networks

In this section we will recall the formalisms of concatenation product, series product and reducible network developed in [16] for the manipulation of networks of generalized open oscillators.

Let $G_1 = (S_1, K_1 x_{1,0}, \frac{1}{2}x_{1,0}^T R_1 x_{1,0})$ and $G_2 = (S_2, K_2 x_{2,0}, \frac{1}{2}x_{2,0}^T R_2 x_{2,0})$ be two generalized open oscillators, where $x_{k,0} = x_k(0)$. The concatenation product $G_1 \boxplus G_2$ of G_1 and G_2 is defined as $G_1 \boxplus G_2 = (S_{1\boxplus 2}, (K_1 x_{1,0}, K_2 x_{2,0})^T, \frac{1}{2}x_{1,0}^T R_1 x_{1,0} + \frac{1}{2}x_{2,0}^T R_2 x_{2,0})$, where

$$S_{1\boxplus 2} = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}.$$

It is important to note here that the possibility that $x_{1,0} = x_{2,0}$ or that some components of $x_{1,0}$ coincide with those of $x_{2,0}$ are allowed. If G_1 and G_2 are independent oscillators (i.e., the components of $x_{1,0}$ commute with those of $x_{2,0}$) then the concatenation can be interpreted simply as the "stacking" or grouping of the variables of two non-interacting generalized open oscillators to form a larger generalized open oscillator.

It is also possible to feed the output of a system G_1 to the input of system G_2, with the proviso that G_1 and G_2 have the same number of input and output channels. This operation of cascading or loading of G_2 onto G_1 is represented by the series product $G_2 \triangleleft G_1$ defined by:

$$G_2 \triangleleft G_1 = (S_2S_1, K_2x_{2,0} + S_2K_1x_{1,0}, \frac{1}{2}x_{1,0}^T R_1 x_{1,0} + \frac{1}{2}x_{2,0}^T R_2 x_{2,0} + \frac{1}{2i}x_{2,0}^T(K_2^\dagger S_2 K_1 - K_2^T S_2^\# K_1^\#)x_{1,0}).$$

Note that $G_2 \triangleleft G_1$ is again a generalized open oscillator with a scattering matrix, coupling operator and Hamiltonian as given by the above formula.

With concatenation and series products having been defined we now come to the important notion of a *reducible network* of open quantum systems [16]. This type of network consists of a set of l generalized open oscillators $\mathcal{G} = \{G_k = (S_k, K_k x_{k,0}, \frac{1}{2}x_{k,0}^T R_k x_{k,0});\ k = 1,\ldots,l\}$ having the same number of input and output fields, along with the specification of a direct interaction Hamiltonian $H^d = \sum_j \sum_{k=j+1} x_j^T R_{jk} x_k$ ($R_{jk} \in \mathbb{R}^{2\times 2}$) and a list $\mathcal{S} = \{G_j \triangleleft G_k\}$ of series connections among generalized open oscillators G_j and G_k, $j \neq k$, with the condition that each input and each output (relative to the decomposition $G = \boxplus_{j=1}^{l} G_j$) has at most one connection, i.e., lists of connections such as $\{G_2 \triangleleft G_1, G_3 \triangleleft G_2, G_1 \triangleleft G_3\}$ are disallowed.

Let $\mathcal{C}$ denote the set of maximal-length chains drawn from the list of series connections $\mathcal{S}$ (here a (series) chain is a system of the form $G_{j_p} \triangleleft G_{k_p} \triangleleft \cdots \triangleleft G_{j_1} \triangleleft G_{k_1}$ with $p \leq l/2$, $G_{j_\cdot} \triangleleft G_{k_\cdot} \in \mathcal{S}$, and $(j_r, k_r) \neq (j_{r'}, k_{r'})$ whenever $r \neq r'$), and let $\mathcal{U}$ denote the set of components in $\mathcal{G}$ not involved in any series connection. Then a reducible network $\mathcal{N}$ can be expressed (up to a reordering of the input and output fields, since the concatenation product is not commutative) as $\mathcal{N} = (\boxplus_{C_k \in \mathcal{C}} C_k) \boxplus (\boxplus_{G_l \in \mathcal{U}} G_l) \boxplus (I, 0, H^d)$. Therefore, such a reducible network $\mathcal{N}$ again forms a generalized open oscillator and is denoted by $\mathcal{N} = \{\{G_k\}_{k=1,\ldots,l}, H^d, \mathcal{S}\}$. Note that if $\mathcal{N}_0$ is a reducible network defined as $\mathcal{N}_0 = \{\{G_k\}_{k=1,\ldots,l}, 0, \mathcal{S}\} = (S_0, L_0, H_0)$ then $\mathcal{N}$, which is $\mathcal{N}_0$ equipped with the direct interaction Hamiltonian H^d, is simply given by $\mathcal{N} = (S_0, L_0, H_0 + H^d)$.

IV. Main Synthesis Theorem

The main result of this paper is the following synthesis theorem:

Theorem 4.1: Let G be an n-degrees of freedom generalized open oscillator with Hamiltonian matrix $R \in \mathbb{R}^{2n\times 2n}$, coupling matrix $K \in \mathbb{C}^{m\times 2n}$ and unitary scattering matrix $S \in \mathbb{C}^{m\times m}$. Let R be written in terms of blocks of 2×2 matrices as $R = [R_{jk}]_{j,k=1,\ldots,n}$, where the R_{jk}'s are real 2×2 matrix satisfying $R_{kj} = R_{jk}^T$ for all j, k, and let K be written as

$$K = [\begin{array}{cccc} K_1 & K_2 & \ldots & K_n \end{array}],$$

where for each j, $K_j \in \mathbb{C}^{m\times 2}$. For $j = 1,\ldots,n$, let $G_j = (S_j, \tilde{K}_j x_j, \frac{1}{2}x_j^T R_{jj} x_j)$ be independent one degree of freedom generalized open oscillators with canonical operators $x_j = (q_i, p_i)^T$, m output fields, Hamiltonian matrix R_{jj}, coupling matrix $\tilde{K}_j$ and scattering matrix S_j. Also, define $S_{k\twoheadleftarrow j}$ for $j \leq k+1$ as $S_{k\twoheadleftarrow j} = S_k S_{k-1}\cdots S_j$ for $j < k$, $S_{k\twoheadleftarrow k} = S_k$ and $S_{k\twoheadleftarrow k+1} = I_{m\times m}$, and let H^d be a direct interaction Hamiltonian given by

$$H^d = \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} x_k^T \left(R_{jk}^T - \frac{1}{2i}(\tilde{K}_k^\dagger S_{k\twoheadleftarrow j+1}\tilde{K}_j - \tilde{K}_k^T S_{k\twoheadleftarrow j+1}^\# \tilde{K}_j^\#)\right) x_j \quad (5)$$

If $S_1,\ldots,S_n$ satisfies $S_n S_{n-1}\cdots S_1 = S$ and $\tilde{K}_k$ satisfies $\tilde{K}_k = S_{n\twoheadleftarrow k+1}^\dagger K_k$ for $k = 1,\ldots,n$ then the reducible network of harmonic oscillators $\mathcal{N}$ given by $\mathcal{N} = \{\{G_1,\ldots,G_n\}, H^d, \{G_2 \triangleleft G_1, G_3 \triangleleft G_2, \ldots, G_n \triangleleft G_{n-1}\}\}$ is equivalent to G. That is, G can be synthesized via a series connection $G_n \triangleleft \ldots \triangleleft G_2 \triangleleft G_1$ of n one degree of freedom generalized open oscillators along with a suitable bilinear direct interaction Hamiltonian involving the canonical operators of these oscillators. In particular, if $S = I_{m\times m}$ (no scattering) then S_k can be chosen to be $S_k = I_{m\times m}$ and $\tilde{K}_k$ can be chosen to be $\tilde{K}_k = K_k$ for $k = 1,\ldots,n$.

For a proof of the theorem, see [17]. Therefore, according to the theorem, synthesis of an arbitrary n-degrees of freedom linear quantum stochastic system is in principle possible if the following two requirements can be met:

1) Arbitrary one degree of freedom open oscillators $G = (I, L, H)$ with m input and output fields can be synthesized. In particular, it follows from this that one degree of freedom generalized open oscillators $G' = (S, L, H)$ can be synthesized as $G' = (I, L, H) \triangleleft (S, 0, 0)$.
2) The bilinear interaction Hamiltonian H^d as given by (5) can be synthesized.

Due to space limitation, in this paper we will not completely address the problem of systematically building arbitrary one degree of freedom open oscillators and implementing arbitrary bilinear interaction Hamiltonians between them, the details can be found in [17]. However, in order to give a better feel for Theorem 4.1, in the next section we shall develop a simple illustrative example of the synthesis of a two degrees of freedom open oscillator from two one degree of freedom open oscillators, in the setting of quantum optics.

V. Illustrative Synthesis Example

Consider a two degrees of freedom open oscillator G coupled to a single external bosonic noise field $A(t)$ given by $G = (I_{4\times 4}, Kx, \frac{1}{2}x^T \mathrm{diag}(R_1, R_2)x)$ with $x = (q_1, p_1, q_2, p_2)^T$, $K = [\begin{array}{cccc} 3/2 & 1/2i & 1 & i \end{array}]$, $R_1 = \left[\begin{array}{cc} 2 & 0.5 \\ 0.5 & 3 \end{array}\right]$ and $R_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$.

Let G_1 and G_2 be two independent one degree of freedom open oscillators given by $G_1 = (I_{2\times 2}, K_1 x_1, \frac{1}{2}x_1^T R_{11} x)$ and $G_2 = (I_{2\times 2}, K_2 x_2, \frac{1}{2}x_2^T R_2 x_2)$ with $x_1 = (q_1, p_1)^T$, $x_2 = (q_2, p_2)^T$, $K_1 = [\begin{array}{cc} 3/2 & i/2 \end{array}]$ and $K_2 = [\begin{array}{cc} 1 & i \end{array}]$. Since the scattering matrix for G is an identity matrix, it follows from Theorem 4.1 that G may be constructed as a reducible network given by $G = \{\{G_1, G_2\}, H_{12}^d, G_2 \triangleleft G_1\}$ with the direct interaction Hamiltonian H_{12}^d between G_1 and G_2 given by (cf. (5)):

$$\begin{aligned} H_{12}^d &= -\frac{1}{2i} x_2^T (K_2^\dagger K_1 - K_2^T K_1^\#) x_1 \\ &= \frac{1}{2} x_2^T \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} x_1. \end{aligned}$$

This network is depicted in Fig. 2.

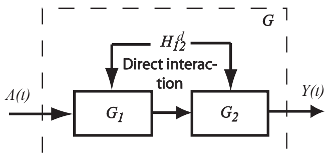

Fig. 2. Cascade connection of G_1 and G_2 with indirect interaction H_{12}^d that realizes G

In the following we shall consider how to build G_1 and G_2 and how H_{12}^d can be implemented to synthesize the overall system G.

A. Synthesis of G_1 and G_2

Let us now consider the synthesis of $G_1 = (I_{2\times 2}, K_1 x_1, \frac{1}{2} x_1^T R_1 x_1)$. We first note that a general quadratic Hamiltonian of the form $H_1 = \frac{1}{2} x_1^T R_1 x_1$ can be constructed in the same fashion as a Degenerate Parametric Amplifier (DPA) based around an optical cavity inserted with an optical nonlinear crystal that is pumped by a classical pump beam, following the treatment in [12, Section 10.2]. Such a system realizes a Hamiltonian of the form: $H_1 = \omega_{cav} a_1^* a_1 + i\frac{1}{2}(\epsilon e^{-i\omega_p t}(a_1^*)^2 - \epsilon^* e^{-i\omega_p t} a_1^2)$ [12, eq. 10.2.1], where $a_1 = \frac{q_1 + ip_1}{2}$ is the cavity annihilation operator (or cavity mode), ω_{cav} is the cavity mode (angular) frequency, ω_p is the pump beam frequency, and ϵ is a complex number representing the pump beam effective intensity. Let Δ be the *detuning frequency* between the cavity mode and reference frequency ω_r given by $\Delta = \omega_{cav} - \omega_r$. By taking a reference frequency $\omega_r = \omega_p/2$ and making a transformation to a *rotating frame* with respect to ω_r via the substitution $a_1 \to a_1 e^{i\omega_r t}$, H_1 can be reformulated in this frame in the form: $H_1 = \Delta a_1^* a_1 + i\frac{1}{2}(\epsilon (a_1^*)^2 - \epsilon^* a_1^2)$, and be written compactly as $H_1 = \frac{1}{2} x_1^T R_1' x_1 + c$ with

$$R_1' = \frac{1}{2} \begin{bmatrix} \Delta + \frac{i}{2}(\epsilon - \epsilon^*) & \frac{1}{2}(\epsilon + \epsilon^*) \\ \frac{1}{2}(\epsilon + \epsilon^*) & \Delta - \frac{i}{2}(\epsilon - \epsilon^*) \end{bmatrix},$$

and c is some complex constant. Since c has no effect on the dynamics of the system observables, we may simply discard it. Therefore, it is clear from this that any choice of a real symmetric coupling matrix R_1' can be realized by appropriately choosing the parameters Δ and ϵ. In particular, $R_1 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 3 \end{bmatrix}$ is realized by choosing $\Delta = 5$ and $\epsilon = 0.5 + 0.5i$.

The coupling operator $L_1 = K_1 x_1$ can be realized by a combination of two squeezers and a mirror (see the Appendix for a more detailed discussion). Here an external vacuum noise field $A_1(t)$ is passed through a squeezer Q that implements the transformation $A_1(t) \mapsto Z_1(t) = \frac{1}{\sqrt{3}} \Re\{A(t)\} + i\sqrt{3}\Im\{A(t)\}$. $Z_1(t)$ is a new noise field (it satisfies the canonical commutation relation $[dZ_1(t), dZ_1(t)^*] = dt$) which is a squeezed version of $A_1(t)$ in which the real quadrature of $A_1(t)$ is squeezed by an amount of $\frac{1}{\sqrt{3}} < 1$. Then $Z_1(t)$ is allowed to interact with the cavity mode a_1 via a partially transmitting mirror with a coupling coefficient γ of $\gamma = 3$. To obtain the output $Y_1(t)$ of G_1, the light reflected from this mirror, say $Z_{1,out}(t)$, is then passed through another squeezer Q^{-1}, the inverse of Q, that implements the transformation $Z_{1,out}(t) \mapsto Y_1(t) = \sqrt{3}\Re\{Z_{1,out}(t)\} + i\frac{1}{\sqrt{3}}\Im\{Z_{1,out}(t)\}$, i.e., Q^{-1} squeezes the imaginary quadrature of $Z_{1,out}$ by $\frac{1}{\sqrt{3}}$. Overall, the open oscillator G_1 with Hamiltonian H_1 and coupling operator L_1 can be implemented around a ring cavity structure as shown in Fig. 3.

Remark 5.1: In the figures, black rectangles will be used to denote mirrors which are fully reflecting at the cavity frequencies and fully transmitting at the pump frequency, while white rectangles lines will denote partially transmitting mirrors at the cavity frequencies.

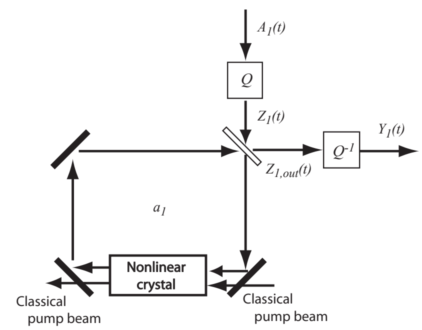

Fig. 3. Realization of G_1

The open oscillator G_2 can be implemented in a similar way to G_1. Taking the same reference frequency ω_r as before, in the rotating frame with respect to ω_r the Hamiltonian $H_2 = \frac{1}{2} x_2^T R_2 x_2$ can be implemented in the same way as H_1 by the choice $\Delta = 1$ and $\epsilon = 0$. Since $\epsilon = 0$, this means no nonlinear optical crystal and pump beam are required to implement R_2, but it suffices to have a cavity that is detuned from ω_r by an amount $\Delta = 1$. The coupling operator $L_2 = q_2 + ip_2 = 2a_2$, where a_2 is the annihilation operator/cavity mode of cavity, is standard (e.g., see [3]) and can be implemented via one partially transmitting mirror with coupling coefficient κ of $\kappa = 4$, on which an external vacuum noise field $A_2(t)$ interacts with the cavity mode a_2 to produce an outgoing field $Y_2(t)$. The implementation of G_2 is shown in Fig. 4.

B. Synthesis of H_{12}^d

We now consider the implementation of the direct interaction Hamiltonian H_{12}^d given by $H_{12}^d = \frac{1}{2} x_2^T \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} x_1$. To proceed, we first note that H_{12}^d may be reexpressed in terms of the cavity modes a_1 and a_2 as $H_{12}^d = -2i(a_1^* a_2 -$

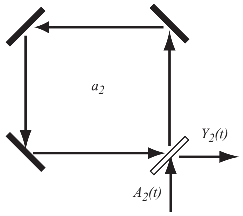

Fig. 4. Realization of G_2

$a_1a_2^*) + i(a_1^*a_2^* - a_1a_2)$. Define $H_{12,1}^d = -2i(a_1^*a_2 - a_1a_2^*)$ and $H_{12,2}^d = i(a_1^*a_2^* - a_1a_2)$ so that $H_{12}^d = H_{12,1}^d + H_{12,2}^d$.

The first part $H_{12,1}^d = -2i(a_1^*a_2 - a_1a_2^*)$ can be simply implemented as a beam splitter with a rotation/mixing angle of -2 (e.g., see [15, Section 3.3]) as shown in Figure 5.

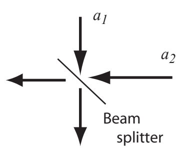

Fig. 5. Beam splitter the direct interaction Hamiltonian $H_{12,1}^d$

On the other hand, the second part $H_{12,2}^d = i(a_1^*a_2^* - a_1a_2)$ can be implemented by having the two modes a_k and a_l interact in a suitable $\chi^{(2)}$ nonlinear crystal using a classical pump beam of frequency $\omega_p = 2\omega_r$ and effective intensity $\epsilon = 1$ in an amplification process as depicted in Figure 6.

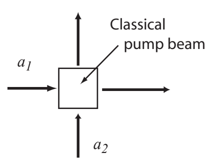

Fig. 6. Implementation of direct interaction Hamiltonian $H_{12,2}^d$

Remark 5.2: Although experimental realization of bilinear interaction Hamiltonians as proposed here may be challenging for systems with more than a few degrees of freedom with current technology, it is in principle possible and may become easier to implement with the development of new methods and technologies in experimental quantum optics. Moreover, alternative architectures for implementing bilinear interaction Hamiltonians are possible and one such architecture is investigated in [18].

Remark 5.3: It is assumed here that the equations for the dynamics of generalized open operators are given with respect to a common rotating frame of frequency ω_r, including the transformation of all bosonic noises $A_i(t)$ according to $A_i(t) \mapsto A_i(t)e^{i\omega_r(t)}$, and that classical pumps employed for interaction between different cavity modes in a nonlinear crystal are all of frequency $\omega_p = 2\omega_r$. This is a natural setting in quantum optics where a rotating frame is essential for obtaining linear time invariant QSDE models for active devices that require an external source of quanta. In a control setting, this means both the quantum plant and the controller equations have been expressed in the same rotating frame.

C. *Complete realization of* $G = \{\{G_1, G_2\}, H_{12}^d, G_2 \triangleleft G_1\}$

The overall two degrees of freedom open oscillator G can now be realized by (i) positioning the arms of the two (ring) cavities of G_1 and G_2 to allow their internal light beams to "overlap" at two points where a beam splitter and a non-linear crystal are placed to implement $H_{12,1}^d$ and $H_{12,2}^d$, respectively, and (ii) passing the output $Y_1(t)$ of G_1 as input to G_2. This implementation is shown in Fig. 7.

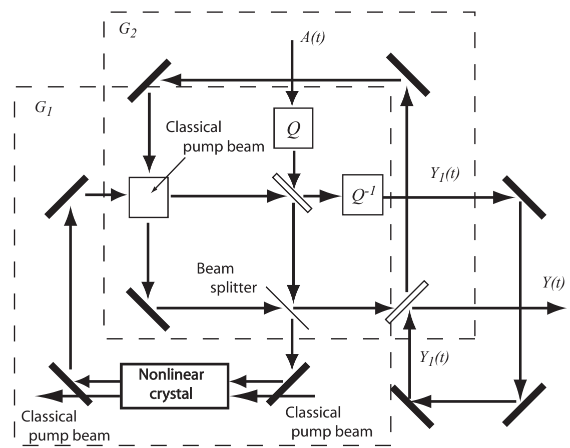

Fig. 7. Realization of G

VI. Conclusions

In this paper we have developed a network theory for synthesizing arbitrarily complex linear dynamical quantum stochastic systems from one degree of freedom open oscillators in a systematic way. We also provide an explicit synthesis example where a two degrees of freedom open oscillator was constructed from a network of two one degree of freedom open oscillators, in the setting of quantum optics. A complete theory that addresses how to build arbitrary one degree of freedom open oscillators and arbitrary direct interaction between them is presented in [17].

Together with advances in experimental physics and the availability of high quality basic quantum devices, it is hoped the results of this paper will assist in the systematic realization of coherent linear quantum stochastic controllers and linear photonic circuits in the laboratory for applications in quantum control and quantum information science.

Appendix
Realization of the linear coupling $L_1 = \frac{3}{2}q_1 + \frac{i}{2}p_1$

The purpose of this Appendix is discuss in more detail the realization of the linear coupling $L_1 = \frac{3}{2}q_1 + \frac{i}{2}p_1$ for the open oscillator G_1 of Section V. To this end, first let $\alpha = 3/2$ and $\beta = i/2$. Then in terms of $a_1 = \frac{q_1+ip_1}{2}$ and $a_1^* = \frac{q_1-ip_1}{2}$, L_1 can be expressed as $L_1 = \tilde{\alpha}a_1 + \tilde{\beta}a_1^*$ with $\tilde{\alpha} = \alpha - i\beta = 2$ and $\tilde{\beta} = \alpha + i\beta = 1$. Define $\gamma = |\tilde{\alpha}|^2 - |\tilde{\beta}|^2 = 3$ and consider the interaction Hamiltonian H_{Int} corresponding to L_1:

$$\begin{aligned} H_{Int}(t) &= i(L_1\eta_1(t)^* - L_1^*\eta_1(t)) \\ &= i((\tilde{\alpha}a_1 + \tilde{\beta}a_1^*)\eta_1(t)^* - \\ &\quad (\tilde{\alpha}^*a_1^* + \tilde{\beta}^*a_1)\eta_1(t)), \end{aligned}$$

where η_1 is a vacuum quantum white noise field interacting with G_1 via L_1. Let us rewrite this Hamiltonian as follows:

$$\begin{aligned} H_{Int}(t) &= i(a_1(\tilde{\alpha}\eta_1(t)^* - \tilde{\beta}^*\eta_1(t)) - \\ &\quad a_1^*(\tilde{\alpha}^*\eta_1(t) - \tilde{\beta}\eta_1(t)^*)) \\ &= i\sqrt{\gamma}(a_1\xi_1(t)^* - a_1^*\xi_1(t)), \end{aligned}$$

where $\xi_1(t) = \frac{1}{\sqrt{\gamma}}(\tilde{\alpha}^*\eta_1(t) - \tilde{\beta}\eta_1(t)^*) = \frac{1}{\sqrt{3}}(2\eta_1(t) - \eta_1(t)^*)$. Let $A_1(t) = \int_0^t \eta_1(s)ds$ and $Z_1(t) = \int_0^t \xi_1(s)ds$. Then we have that

$$\begin{bmatrix} dZ_1(t) \\ dZ_1(t)^* \end{bmatrix} = Q \begin{bmatrix} dA_1(t) \\ dA_1(t)^* \end{bmatrix}; \quad Q = \begin{bmatrix} \frac{\tilde{\alpha}^*}{\sqrt{\gamma}} & -\frac{\tilde{\beta}}{\sqrt{\gamma}} \\ -\frac{\tilde{\beta}^*}{\sqrt{\gamma}} & \frac{\tilde{\alpha}}{\sqrt{\gamma}} \end{bmatrix}.$$

Since $|\tilde{\alpha}/\sqrt{\gamma}|^2 - |\tilde{\beta}/\sqrt{\gamma}|^2 = 1$ it follows that Q is a quasi-unitary linear transformation [15, Section 3.1] that preserves the field commutation relations. Hence $Z_1(t)$ is a new field satisfying $[dZ_1(t), dZ_1(t)^*] = dt$ and the Ito rules:

$$\begin{bmatrix} dZ_1(t) \\ dZ_1(t)^* \end{bmatrix} \begin{bmatrix} dZ_1(t) & dZ_1(t)^* \end{bmatrix} = Q \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Q^T dt.$$

In particular, since $Z_1(t) = \frac{1}{\sqrt{3}}\Re\{A_1(t)\} + i\sqrt{3}\Im\{A_1(t)\}$, it is a real/amplitude quadrature squeezed version of $A_1(t)$ (i.e., the quadrature $\Re\{A_1(t)\}$ is squeezed by $1/\sqrt{3}$ while $\Im\{A_1(t)\}$ is amplified by $\sqrt{3}$) and is obtained by passing $A_1(t)$ through a corresponding squeezer which we denote by Q.

The main idea here is that instead of considering an oscillator mode a_1 interacting with $A_1(t)$, we consider the same oscillator interacting with the new field $Z_1(t)$ via the interaction $H_{Int}(t) = i\sqrt{\gamma}(a_1\xi_1(t)^* - a_1^*\xi_1(t))$. This interaction can be implemented in one arm of a ring cavity with a fully reflecting mirror M and a partially transmitting mirror M' with coupling coefficient γ, with $Z_1(t)$ incident on M'. After the interaction, an output field $Z_{1,out}(t)$ is reflected by M' given by

$$\begin{aligned} Z_{1,out}(t) &= U(t)^*Z_1(t)U(t) \\ &= \frac{\tilde{\alpha}^*}{\sqrt{\gamma}}U(t)^*A_1(t)U(t) - \frac{\tilde{\beta}}{\sqrt{\gamma}}U(t)^*A_1(t)^*U(t). \end{aligned}$$

However, the actual output that is of interest is the output $Y_1(t) = U(t)^*A_1(t)U(t)$ when the oscillator interacts directly with the field $A_1(t)$. To recover $Y_1(t)$ from $Z_{1,out}(t)$, first notice that since Q is a quasi-unitary transformation it has an inverse Q^{-1} which is again quasi-unitary (as quasi-unitary matrices form a group). Hence $Y_1(t)$ can be recovered from $Z_{1,out}(t)$ by exploiting the following relation that follows directly from the fact that $(Z_1(t), Z_1(t)^*)^T = Q(A_1(t), A_1(t)^*)^T$:

$$\begin{bmatrix} Y_1(t) \\ Y_1(t)^* \end{bmatrix} = Q^{-1} \begin{bmatrix} Z_{1,out}(t) \\ Z_{1,out}(t)^* \end{bmatrix}.$$

Q^{-1} as the inverse of Q is an imaginary/phase quadrature squeezing operation (i.e., it squeezes $\Im\{Z_{1,out}(t)\}$ by $1/\sqrt{3}$, while $\Re\{Z_{1,out}(t)\}$ is amplified by $\sqrt{3}$), and $Y_1(t)$ is the output of passing $Z_{1,out}(t)$ through a phase quadrature squeezer that realizes Q^{-1}. Overall, this yields the implementation of L_1 as depicted in Fig. 3.

REFERENCES

[1] J. P. Dowling and G. J. Milburn, "Quantum technology: The second quantum revolution," *Philosophical Transactions: Mathematical, Physical and Engineering Sciences*, vol. 361, no. 1809, pp. 1655–1674, August 2003.

[2] M. Yanagisawa and H. Kimura, "Transfer function approach to quantum control-part II: Control concepts and applications," *IEEE Trans. Automatic Control*, vol. 48, no. 12, pp. 2121–2132, 2003.

[3] M. R. James, H. I. Nurdin, and I. R. Petersen, "H^∞ control of linear quantum stochastic systems," *IEEE Trans. Automat. Contr.*, vol. 53, no. 8, pp. 1787–1803, 2008.

[4] H. I. Nurdin, "Topics in classical and quantum linear stochastic systems," Ph.D. dissertation, The Australian National University, April 2007.

[5] H. I. Nurdin, M. R. James, and I. R. Petersen, "Quantum LQG control with quantum mechanical controllers," in *Proceedings of the 17th IFAC World Congress (Seoul, South Korea, July 6-11, 2008)*, 2008, pp. 9922–9927.

[6] ——, "Coherent quantum LQG control," *Automatica J. IFAC*, vol. 45, pp. 1837–1846, 2009. [Online]. Available: http://arxiv.org/pdf/0711.2551v2 (preprint)

[7] H. Mabuchi, "Coherent-feedback quantum control with a dynamic compensator," *Phys. Rev. A*, vol. 78, p. 032323, 2008.

[8] B. Yurke and J. S. Denker, "Quantum network theory," *Phys. Rev. A*, vol. 29, no. 3, pp. 1419–1437, 1984.

[9] S. C. Edwards and V. P. Belavkin, "Optimal quantum filtering and quantum feedback control," August 2005. [Online]. Available: http://arxiv.org/pdf/quant-ph/0506018

[10] R. L. Hudson and K. R. Parthasarathy, "Quantum Ito's formula and stochastic evolution," *Commun. Math. Phys.*, vol. 93, pp. 301–323, 1984.

[11] K. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*. Berlin: Birkhauser, 1992.

[12] C. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics*, 2nd ed., ser. Springer Series in Synergetics. Springer, 2000.

[13] J. Gough, "Quantum Stratonovich calculus and the quantum Wong-Zakai theorem," *J. Math. Phys.*, vol. 47, no. 113509, 2006.

[14] U. Leonhardt and A. Neumaier, "Explicit effective Hamiltonians for linear quantum-optical networks," *J. Opt. B*, vol. 6, pp. L1–L4, 2004.

[15] U. Leonhardt, "Quantum physics of simple optical instruments," *Rep. Prog. Phys.*, vol. 66, pp. 1207–1249, 2003.

[16] J. Gough and M. R. James, "The series product and its application to quantum feedforward and feedback networks," 2009, IEEE Transactions on Automatic Control, to appear. [Online]. Available: http://arxiv.org/abs/0707.0048v3

[17] H. I. Nurdin, M. R. James, and A. C. Doherty, "Network synthesis of linear dynamical quantum stochastic systems," 2009, to appear in SIAM Journal on Control and Optimization. [Online]. Available: http://arxiv.org/abs/0806.4448v1 (preprint)

[18] H. I. Nurdin, "Synthesis of linear quantum stochastic systems via quantum feedback networks," conditionally accepted for publication, IEEE Transactions on Automatic Control. [Online]. Available: http://arxiv.org/abs/0905.0802 (preprint)