Generalised finite radon transform for $N \times N$ images [☆]

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Abstract

This paper extends the domain of the finite radon transform (FRT) to apply to square arrays of arbitrary size. The FRT is a discrete formalism of the Radon transform that assumes the image is periodic over the finite array $\mathbb{Z}_N^2$ and requires only arithmetic operations for both forward and exact inverse transformation. The FRT is useful in image processing applications such as tomographic reconstruction [I. Svalbe, D. van der Spek, Reconstruction of tomographic images using analog projections and the digital Radon transform, *Linear Algebra and Its Applications* 339 (15) (2001) 125–145.], image representation [M. Do, M. Vetterli, Finite ridgelet transform for image representation, *IEEE Transactions on Image Processing* 12(1).] image convolution [D. Lun, T. Chan, T. Hsung, D. Feng, Y. Chan, Efficient blind image restoration using discrete periodic Radon transform, *IEEE Transactions on Image Processing* 13(2) (2004) 188–200.], image watermarking and encryption [A. Kingston, I. Svalbe, Projective transforms on periodic discrete image arrays, to appear in P. Hawkes (Ed), *Advances in Imaging and Electron Physics* (2006).], and robust data transmission [A. Kingston, I. Svalbe, Geometric shape effects in redundant keys used to encrypt data transformed by finite discrete Radon projections, In: *Proc. 8th Int. Conf. on Digital Image Computing: Techniques and Applications* (2005).].

The original definition by Matúš and Flusser in 1993 [F. Matúš, J. Flusser, Image representation via a finite Radon transform, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 15(10) (1993) 996–1006.] was restricted to apply only to square arrays of prime size, $p \times p$. Hsung, Lun and Siu developed an FRT that also applied to dyadic square arrays, $2^n \times 2^n$, called the discrete periodic radon transform (DPRT) [T. Hsung, D. Lun, W. Siu, The discrete periodic Radon transform, *IEEE Transactions on Signal Processing* 44(10) (1996) 2651–2657.]. Kingston further extended this to define an FRT that applies to *prime*-adic arrays, $p^n \times p^n$.

This paper defines a generalised FRT that applies to square arrays of arbitrary size, $N \times N$ for $N \in \mathbb{N}$. The Fourier slice theorem and convolution property (two important properties of the classical Radon transform) are established for this FRT. The original image can be reconstructed exactly from the FRT projections using Fourier inversion and back-projection. New methods are identified to correct for the over-representation of pixels due to the compositeness of N . A remarkable result is established that enables 2D sampling patterns to be corrected by an angle invariant 1D filter prior to back-projection.

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1. Introduction

The classical radon transform (RT) projects a continuous 2D function f to a set of 1D functions

[☆] Part of this work was originally presented at DGCI-05 in Poitiers [1]. Here the proofs are more rigorous and the section on reconstruction from projections has been extended, in particular to include the result obtained using filtered back projection with a digital form of the Ram Lak filter providing exact reconstruction applicable to discrete $N \times N$ arrays.

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$r = \{r_\theta | 0 \leq \theta < \pi\}$ called projections. An element of the projection, $r_\theta(\rho)$, is found as the integral of f over the line $y \sin \theta + x \cos \theta = \rho$. A projection, r_θ , is the set of all parallel line integrals $r_\theta = \{r_\theta(\rho) | \rho \in \mathbb{R}\}$. Johan Radon first demonstrated in 1917 that the original function could be recovered from this projection mapping [9]. The existence of the inverse transform is one of the most important properties of the RT, enabling nondestructive tomographic imaging of an object from its projected images. This property is utilised in areas ranging from astronomy and seismology to medical imaging [9].

Other important properties of the RT include the Fourier slice theorem and the convolution property. The Fourier slice theorem relates the RT to the Fourier transform (FT), as the 1D FT of a projection, $\hat{r}_\theta$, represents the spatial frequencies along a slice at angle θ through the 2D FT of the original function, $\hat{f}$, i.e. $\hat{r}_\theta(\omega) = \hat{f}(\omega \cos \theta, \omega \sin \theta)$ [10]. The convolution property as a consequence enables the 2D convolution of functions to be performed on the RT as a set of 1D convolutions over the projections.

The classic RT described above maps a point in the image, (j, k) , to a sinusoid in the RT as $\rho = \sqrt{j^2 + k^2} \sin(\tan^{-1}(j/k) + \theta)$, giving rise to the name sinogram for this representation of the RT. Another mapping used by seismologists (known as the τ - p transform) and described by Edholm and Herman in 1987 [11], maps the image to a linogram where line integrals are based on gradient, m and intercept t , according to $x + my = t$. In this representation the point (j, k) is mapped to a line—hence the term linogram—and back-projection is also performed using the same projection operator. It is common to perform the linogram mapping in two stages, *essentially* horizontal lines: $sx + y = t$ for $-1 \leq s < 1$ and *essentially* vertical lines: $x + my = t$ for $-1 \leq m < 1$, to ensure $|\tan \theta| \leq 1$.

To reconstruct exactly a continuous function from projections, continuous projection data is required at all angles in $[0, \pi)$. In practice however, only a finite set of projections are obtained, each with a finite resolution set by the detector aperture, thus only a discrete approximation to the original function can be reconstructed. As well as tomographic reconstruction, the RT is useful for many image processing applications such as locating linear features, detecting and isolating motion, denoising, blur deconvolution and feature representation. Since, all these applications of the RT are in essence discrete, a discrete formalism of the RT that minimises the need for interpolation in projection and reconstruction would be ideal.

The RT has no 1D analogue, therefore the definition of a discrete RT is nontrivial and many formalisms have been proposed [12–14]. The finite radon transform (FRT) [7] is a very simple but powerful definition requiring only arithmetic operations for the transform and exact inversion with no interpolation. It is *almost* orthogonal with a minimum angle between basis functions of $\cos^{-1}(1/p)$ approaching $\pi/2$ with increasing p [15]. The FRT applies to square image data, $p \times p$ where p is prime, and assumes the image is periodic with p in both the x and y directions (as does the 2D discrete Fourier transform (DFT)) by defining the $p \times p$ array as the finite group $\mathbb{Z}_p^2$ under addition. Both the Fourier slice theorem and convolution property hold for the FRT [7]. Since, p is prime there are only $p + 1$ distinct line directions (thus $p + 1$ projections) corresponding to the $p + 1$ unique subgroups. Every discrete line samples p points in the image, has p translates, and intersects any other distinct line only once (or never if parallel). All these attributes match those of lines in continuous space.

The FRT is a discrete form of a linogram with lines mapped according to gradient and intercept (i.e. points in

the image are mapped to discrete lines in the FRT) and the back-projection operator is a symmetry of the projection operator. Matúš and Flusser showed that p of the projections, $R_m = \{R_m(t) | t \in \mathbb{Z}_p\}$ for $m \in \mathbb{Z}_p$ can be defined as the sum of pixels centred on lines $x \equiv my + t \pmod{p}$ and the remaining projection, $R_0^\perp = \{R_0^\perp(t) | t \in \mathbb{Z}_p\}$, can be found as a *perpendicular* line sum over $y \equiv 0x + t \pmod{p}$, i.e.

$$R_m(t) = \sum_{y=0}^{p-1} I(\langle my + t \rangle_p, y)$$

$$\text{for } 0 \leq m < p, R_0^\perp(t) = \sum_{x=0}^{p-1} I(x, t), \quad (1)$$

where $\langle x \rangle_\eta$ denotes $x \pmod{\eta}$. An example of these discrete lines for $p = 7$ is presented in Fig. 1a and b.

The FRT has found application in tomographic reconstruction [2,16,17], image compression [7,15], image denoising [18], image representation [3], blind blur deconvolution [4], watermarking [19,5], encrypted and robust data transmission [5,6]. An adaptive FRT for a sliding and zooming window was developed in [20] to facilitate the location of small linear features and for object tracking. The major restriction of the FRT in these applications is that it requires square prime sized arrays.

The FRT formalism was extended by Hsung, et al., in 1996 to apply to dyadic square arrays, $2^n \times 2^n$ for any positive integer n [8]. This version, termed the discrete periodic Radon transform (DPRT), makes the transform more conducive to image processing, which commonly utilises square dyadic images for computational efficiency. The DPRT has *essentially* horizontal projections $R_m(t)$, as defined above, for $0 \leq m < N = 2^n$. However, since the array size is not prime, the image size N now has factors other than $N \equiv 0 \pmod{N}$. Additional *perpendicular* (or *essentially* vertical) projections are required as $R_s^\perp(t)$ for $0 \leq s < N/2$, which are discrete line sums along $y \equiv 2sx + t \pmod{N}$. An example of these discrete lines for $N = 8$ is

Fig. 1. Examples of discrete lines on $p \times p$ arrays. Shaded points lie on the centre of the wrapped lines, the values of these pixels are summed to give the value of one element in the transform. (a) Pixels on $x \equiv 2y + 3 \pmod{7}$ sum to give $R_2(3)$. Since, the array size is prime, only one element on each row and column is sampled by a discrete wrapped line. (b) Pixels on $y \equiv 0x + 3 \pmod{7}$ sum to give $R_0^\perp(3)$.

presented in Fig. 2a and b. Including all these discrete projections represents $I(x, y)$ exactly in the transform space. The DPRT is then defined as

$$R_m(t) = \sum_{y=0}^{N-1} I(\langle my + t \rangle_N, y) \quad \text{for } 0 \leq m < N,$$

$$R_s^\perp(t) = \sum_{x=0}^{N-1} I(x, \langle 2sx + t \rangle_N) \quad \text{for } 0 \leq s < \frac{N}{2}. \quad (2)$$

The DPRT is a redundant representation of $I(x, y)$ as it has $N(1 + 1/2)$ projections of length N . A non-redundant form of the DPRT with orthogonal bases was presented by Lun et al. in 2003 [21], it is termed the orthogonal DPRT (ODPRT).

The formalism for the FRT applied to arrays of size $p^n \times p^n$ is a natural extension of the DPRT and was developed in [22]. Projections are defined as for the DPRT, with p replacing two in (2). This is a redundant transform but requires only $N(1 + 1/p)$ projections of length N . A non-redundant form of this FRT, with orthogonal bases, was also presented in [22].

The above FRT definitions are all restricted to apply to square arrays with dimensions based on a single prime, i.e. prime, dyadic or *prime-adic*. Should the FRT be required of an array of arbitrary size $M \times N$, the image must be padded with zeroes to a square array, $N \times N$, with a minimum size being the smallest $N = p^n \geq \sup\{M, N\}$. Padding arrays adds redundant information and unwanted computational complexity. The motivation for this work is to remove these restrictions and this paper extends the above FRT definitions to apply to square arrays of any composite size $N = p_1^{n_1} p_2^{n_2} p_3^{n_3} \dots$, where p_i is prime and n_i is any positive integer giving the prime decomposition of N . This definition is referred to in this paper as the generalised FRT. An FRT for non-square arrays is the subject of ongoing research.

Section 2 establishes the minimal set of projections that are required for an exact and invertible FRT for $N \times N$

arrays and provides a method to obtain that set. This leads to the definition for the generalised FRT in Section 3. A discrete form of the Fourier slice theorem that applies to this FRT and the related convolution property is also derived in this section, along with a method to perform a fast FRT. A means to determine the level of redundancy in the sampling patterns that result on composite arrays is presented in Section 4. Knowledge of this over-sampling pattern is required for the inverse transform; three distinct inversion processes for exact image reconstruction from FRT projections are then presented (i)-Fourier inversion (Section 4.2), (ii)-multi-resolutional corrected back-projection (Section 4.3) and (iii)-filtered back-projection (Section 4.4).

2. Projection sets for invertible mappings

This section begins by reviewing the properties of the projection set required for the FRT and shows how this set is extended to apply to arrays of powers of two in the DPRT (or dyadic FRT) and prime powers in the DRT over $p^n \times p^n$ images (or *prime-adic* FRT). A method is then formulated to determine the required basis that matches these conditions for an FRT over arrays of composite size.

2.1. Projection set for $\mathbb{Z}_p^2$

The set of pixels sampled by each discrete line used for integration in the FRT produces a 2D lattice with basis vectors $(m, 1)$ and $(p, 0)$ for lines in R_m , denoted $\mathcal{L}\{(m, 1), (p, 0)\}$, and $(1, 0)$ and $(0, p)$ for lines in $R_0^\perp$, denoted $\mathcal{L}\{(1, 0), (0, p)\}$. These lattices can be generated by $(0, p)$ and/or $(p, 0)$, along with any other vector in the lattice $(m, 1), (2m, 2), \dots, ((p-1)m, p-1)$, for lines in R_m and $(1, 0), (2, 0), \dots, (p-1, 0)$, for lines in $R_0^\perp$, reduced modulo p . These vectors are the same for all of the parallel lines, translated by t , that make up a projection. Therefore each vector specifies the orientation for an entire projection. Let Θ_p be the set of these vectors that define the projection set for a $p \times p$ array. Θ_p contains $(m, 1)$ for $0 \leq m < p$ and $(1, 0)$ (or any set of basis vectors that generate the equivalent lattices). To demonstrate this principle, the lattice generated by the discrete line $x \equiv 5y + 3 \pmod{7}$ in Fig. 3a has been depicted in Fig. 3b.

It is useful to store the FRT basis vectors as the shortest vector in each lattice $\mathcal{L}\{(m, 1), (p, 0)\}$ as this gives the direction of the lines with maximum point density. The reduced basis vectors for the example in Fig. 3 are shown as the black vectors in Fig. 3c, these are the shortest vectors which generate the equivalent lattice to $\mathcal{L}\{(5, 1), (7, 0)\}$ shown as grey vectors. Note that to determine the reduced basis vectors for the lattice, the infinite lattice $\mathcal{L}\{(m, 1), (p, 0)\}$ is required, as opposed to the periodic lattice $\mathcal{L}_p\{(m, 1)\}$. For the example given in Fig. 3, the shortest vector in the lattice is $(-2, 1)$, which becomes $(5, 1)$ in $\mathcal{L}_7$ and is no longer the shortest vector (in $\mathcal{L}_7$ the reduced basis vectors are $\{(1, 3), (3, 2)\}$).

Fig. 2. Examples of discrete lines on $2^n \times 2^n$ arrays. Shaded points lie on the centre of the wrapped lines, the values or these pixels are summed to give the value of one element in the transform. (a) Pixels on the *essentially* horizontal line $x \equiv 2y + 3 \pmod{8}$ sum to give $R_2(3)$. (b) Pixels on the *essentially* vertical line $y \equiv 1x + 3 \pmod{8}$ sum to give $R_1^\perp(3)$, i.e. $s = 3$. Since the array size is composite and the slope m or $2s$ may have a common factor with 2^n , each row and column is no longer necessarily uniquely sampled.

Fig. 3. An FRT discrete line sum $x \equiv my + t \pmod{p}$ with $p = 7$, $m = 5$ and $t = 3$. (a) Shaded points are those summed in the periodic discrete line. (b) $\{\mathcal{L}(5,1), (7,0)\}$ generates the sampling pattern of this periodic discrete line. (c) The lattice basis $\{(5, 1), (7, 0)\}$ shaded grey can be reduced to vectors of minimum length (shown in black) which still generate the same lattice ($\{(-2, 1), (1, 3)\}$ in this case).

The points of each of the $p + 1$ distinct lattices correspond to all the unique cyclic subgroups of order p in the group $\mathbb{Z}_p^2$ under addition. Each subgroup contains the identity $(0, 0)$ and, since p is prime, each subgroup contains $p - 1$ unique elements. Let $\Upsilon(N)$ represent the number of discrete lines in the projection set for the FRT of an $N \times N$ array, i.e. $\Upsilon(p)$ is the cardinality of Θ_p . It can be seen $\Upsilon(p) = p + 1 = p(1 + 1/p)$ where p is prime. $\Upsilon(N)$ also gives the number of unique cyclic subgroups of order N in the group $\mathbb{Z}_N^2$ under addition.

2.2. Projection set for $\mathbb{Z}_{2^n}^2$ and $\mathbb{Z}_{p^n}^2$

For the dyadic FRT over $\mathbb{Z}_{2^n}^2$, the array size has factors of 2^j . Any two distinct lines will intersect at any of $2^{n-1}, 2^{n-2}, \dots, 2, 1$ locations or not at all if parallel. Therefore the set of elements included in lines generated by the projection set defined for the FRT does not include all of $\mathbb{Z}_{2^n}^2$ and additional *perpendicular* projections based on discrete lines $y \equiv 2sx + t \pmod{2^n}$ are necessary. Here Θ_{2^n} contains $(m, 1)$ for $0 \leq m < 2^n$ and $(1, 2s)$ for $0 \leq s < 2^{n-1}$. There are $N + N/2$ cyclic subgroups of order N in the group $\mathbb{Z}_N^2$ under addition when $N = 2^n$. There are also cyclic subgroups of order less than N , however it is unnecessary to include discrete lines resulting from these, as each one is entirely contained within a cyclic subgroup of order N . Therefore $\Upsilon(2^n) = 2^n + 2^{n-1} = 2^n(1 + 1/2)$. Similarly, for the *prime*-adic FRT over $\mathbb{Z}_{p^n}^2$, Θ_{p^n} contains $(m, 1)$ for $0 \leq m < p^n$ and $(1, ps)$ for $0 \leq s < p^{n-1}$ with $\Upsilon(p^n) = p^n + p^{n-1} = p^n(1 + 1/p)$.

2.3. Projection set for $\mathbb{Z}_N^2$

Define the discrete lines for the FRT over square arrays of composite size as $ax \equiv by + t \pmod{N}$. Θ_N then contains a number of vectors (b, a) . How many of these are there to ensure the transform is invertible and how are they found?

A function $f(p)$ is said to be multiplicative if $\gcd(p, q) = 1$ implies that $f(pq) = f(p)f(q)$ [23]. It can be shown that $\Upsilon(N)$ is multiplicative and is therefore found as

$$\Upsilon(N) = N \prod_{p|N} (1 + 1/p). \quad (3)$$

where $p|N$ denotes some prime, p , that divides N .

Suppose that $\gcd(p, q) = 1$ and denote Θ_p to be the complete projection set for $\mathbb{Z}_p^2, \{(b_p^j, a_p^j) | j \in [0, \Upsilon(p) - 1]\}$. Also denote Θ_q to be the complete projection set for $\mathbb{Z}_q^2, \{(b_q^k, a_q^k) | k \in [0, \Upsilon(q) - 1]\}$. Any point $(x_p, y_p) \in \mathbb{Z}_p^2$ can be represented as $\eta_p(b_p^j, a_p^j)$ for some $\eta_p \in \mathbb{Z}_p$ and any point $(x_q, y_q) \in \mathbb{Z}_q^2$ can be represented as $\eta_q(b_q^k, a_q^k)$ for some $\eta_q \in \mathbb{Z}_q$.

The set $\{q(x_p, y_p) + p(x_q, y_q) \pmod{pq} | (x_p, y_p) \in \mathbb{Z}_p^2, (x_q, y_q) \in \mathbb{Z}_q^2\}$ gives $\mathbb{Z}_{pq}^2$. There are $p^2 q^2$ points generated in this set. It is required to establish that each point in this set is unique. Assume the converse is true, that the point $(x_{pq}, y_{pq}) \in \mathbb{Z}_{pq}^2$ generated as $(x_{pq}, y_{pq}) \equiv q(x_p^1, y_p^1) + p(x_q^1, y_q^1) \pmod{pq}$ can also be generated as $q(x_p^2, y_p^2) + p(x_q^2, y_q^2) \pmod{pq}$; then

$$q(x_p^1, y_p^1) + p(x_q^1, y_q^1) \equiv q(x_p^2, y_p^2) + p(x_q^2, y_q^2) \pmod{pq}.$$

Using the property, $jk \equiv jl \pmod{q} \Rightarrow k \equiv l \pmod{q/\gcd(j, q)}$, this becomes the congruence

$$q(x_p^1, y_p^1) + p(x_q^1, y_q^1) \equiv q(x_p^2, y_p^2) + p(x_q^2, y_q^2) \pmod{\frac{pq}{\gcd(q, pq)}},$$

and so

$$(x_p^1, y_p^1) \equiv (x_p^2, y_p^2) \pmod{p}.$$

All points contributing to (x_{pq}, y_{pq}) from $\mathbb{Z}_p^2$ are unique. Similarly $(x_q^1, y_q^1) \equiv (x_q^2, y_q^2) \pmod{q}$, so all the points generated are incongruent $\pmod{pq}$ and together form the complete set $\mathbb{Z}_{pq}^2$.

The projection set $\Theta_{pq} = \{(b_{pq}^l, a_{pq}^l) | l \in [0, \Upsilon(pq))\}$ is required such that any point in $\mathbb{Z}_{pq}^2$ can be represented as $\eta_{pq}(b_{pq}^l, a_{pq}^l)$ for some $\eta_{pq} \in \mathbb{Z}_{pq}$ and $(b_{pq}^l, a_{pq}^l) \in \Theta_{pq}$. From the above theorem,

$$(x_{pq}, y_{pq}) \equiv \begin{cases} q(x_p, y_p) + p(x_q, y_q) \\ q(\eta_p(b_p^j, a_p^j)) + p(\eta_q(b_q^k, a_q^k)) \end{cases} \pmod{pq}. \quad (4)$$

It is possible to find some $\eta_{pq} \in \mathbb{Z}_{pq}$, which satisfies $q\eta_{pq} \equiv q\eta_p \pmod{pq}$ and also satisfies $p\eta_{pq} \equiv p\eta_q \pmod{pq}$ giving

$$\eta_{pq} \equiv (q\eta_p - p\eta_q)\bar{q} - \bar{p} \pmod{pq}, \quad (5)$$

where $\bar{x} \pmod{\eta}$ denotes the multiplicative inverse of x modulo η . Substitute this into (4) to obtain

$$(x_{pq}, y_{pq}) \equiv \begin{cases} \eta_{pq}(q(b_p^j, a_p^j) + p(b_q^k, a_q^k)) & (\text{mod } pq) \\ \eta_{pq}(b_{pq}^l, a_{pq}^l) & (\text{mod } pq) \end{cases} \quad (6)$$

Here $(b_{pq}^l, a_{pq}^l) \equiv q(b_p^j, a_p^j) + p(b_q^k, a_q^k) \pmod{pq}$. This shows the complete projection set for $\mathbb{Z}_{pq}^2$ can be generated from those of its co-prime factors as $\Theta_{pq} \equiv \{q\Theta_p + p\Theta_q\} \pmod{pq}$; it has been proven above that all these vectors are unique. This set contains $Y(pq) = Y(p)Y(q)$ vectors, showing the function $Y(N)$ is indeed multiplicative. Below is an example demonstrating how to obtain the projection set for $N = pq = 6$ from the sets for the two primes, $p = 2$ and $q = 3$:

$$\begin{aligned} \text{Given } \begin{cases} \Theta_2 = \{(1,0), (0,1), (1,1)\}. \\ \Theta_3 = \{(1,0), (0,1), (1,1), (2,1)\}. \end{cases} \\ + \quad (1,0) \ (0,1) \ (1,1) \ (2,1) \times 2 \\ (1,0) \ (5,0) \ (3,2) \ (5,2) \ (1,2) \\ \Theta_6 = \{2\Theta_3 + 3\Theta_2\} = (0,1) \ (2,3) \ (0,5) \ (2,5) \ (4,5) \\ (1,1) \ (5,3) \ (3,5) \ (5,5) \ (1,5) \\ \times 3 \end{aligned}$$

This property is the foundation for generating the basis vectors, which specify the projections included in the FRT of an image defined on a square array of arbitrary composite size, $N \times N$.

3. Generalised FRT formalism

3.1. Definition of the generalised FRT

Denote each element of the FRT as $R_{b,a}(t)$, defined as the sum of all pixels in $I(x, y)$ such that $ax \equiv by + t \pmod{N}$. For the prime, dyadic and prime-adic FRT definitions $R_m(t) = R_{m,1}(t)$ and $R_s^\perp(t) = R_{1,2p}(t)$. The generalised FRT over square arrays of composite size, $N \times N$, can be defined as

$$R_{b,a}(t) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} I(x,y) \delta\langle ax - by - t \rangle_N \quad \text{for } (b,a) \in \Theta_N. \quad (7)$$

where $\delta\langle x \rangle_\eta$ is 1 when $x \equiv 0 \pmod{\eta}$ and 0 otherwise and Θ_N is the projection set required for $N \times N$ arrays as defined in Section 2.

3.2. Properties of the generalised FRT

A discrete form of the Fourier slice theorem can be demonstrated for the generalised FRT. Denote $\hat{R}_{b,a}(u)$ as the 1D Discrete Fourier Transform (DFT) of $R_{b,a}(t)$ for $(b, a) \in \Theta_N$, then

$$\begin{aligned} \hat{R}_{b,a}(u) &= \sum_{t=0}^{N-1} R_{b,a}(t) e^{-i2\pi ut/N} \\ &= \sum_{t=0}^{N-1} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} I(x,y) e^{-i2\pi ut/N} \delta\langle ax - by - t \rangle_N \\ &= \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} I(x,y) e^{-i2\pi(axu - byu)/N} = \hat{I}(au, -bu), \end{aligned} \quad (8)$$

where $\hat{I}(u,v)$ is the 2D DFT of $I(x, y)$. The DFT of the discrete projection with basis (b, a) , which is based on discrete line sums over $ax \equiv bx + t$ for some $t \in N$, is a wrapped discrete line through the origin of Fourier space, $-bu \equiv av \pmod{N}$, perpendicular to the discrete line of projection (as the product of these gradients is -1).

The FRT of an image, $I(x, y)$, can be obtained in $O(NY(N) \log N)$ operations by utilising this property. $\hat{I}(u,v)$ can be obtained in $O(N^2 \log N)$ operations. Each of the $Y(N)$ projections of length N can be obtained as the inverse 1D DFT of the corresponding discrete slices in $O(N \log N)$ operations.

A discrete form of the convolution property is also conserved in the generalised FRT as a result of the discrete Fourier slice theorem (8). Assume the function $F(x, y)$ on an $N \times N$ array is to be determined from the $N \times N$ function $G(x, y)$ and $L \times M$ function $H(x, y)$, $0 < L, M \leq N$, by the 2D cyclic convolution

$$F(x,y) = \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{M-1} G(\langle x - \alpha \rangle_N, \langle y - \beta \rangle_N) H(\alpha, \beta). \quad (9)$$

The same result can be obtained by convolution operations on each 1D projection in the FRT. Denote the generalised FRT of $\eta(x, y)$ as $R_{b,a}^n(t)$. The value of $F(x, y)$ can be found through the DRT of $G(x, y)$ and $H(x, y)$ as

$$\begin{aligned} R_{b,a}^F(t) &= \sum_{y=0}^{N-1} F(x,y) \delta\langle ax - by - t \rangle_N \\ &= \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{M-1} G(\langle x - \alpha \rangle_N, \langle y - \beta \rangle_N) H(\alpha, \beta) \delta\langle ax - by - t \rangle_N \\ &= \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{M-1} \sum_{q=0}^{N-1} \sum_{r=0}^{N-1} G(q,r) \delta\langle aq - br + ax - b\beta - t \rangle_N H(\alpha, \beta) \\ &= \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{M-1} R_{b,a}^G(\langle t - a\alpha + b\beta \rangle_N) H(\alpha, \beta) \\ &= \sum_{k=0}^{N-1} R_{b,a}^G(k) \sum_{\alpha=0}^{L-1} \sum_{\beta=0}^{M-1} H(\alpha, \beta) \delta\langle a\alpha - b\beta - t + k \rangle_N \\ &= \sum_{k=0}^{N-1} R_{b,a}^G(\langle t - k \rangle_N) R_{b,a}^H(k). \end{aligned} \quad (10)$$

The convolution of 2D images G and H can be performed in FRT space as a set of 1D circular convolutions of discrete projections, $R_{b,a}^G$ and $R_{b,a}^H$. This reduces the computational complexity for 2D applications in filtering or

matching, as outlined for the prime and dyadic FRT (or DPRT) [8]. It is expected that this property will extend to 3D and nD and this remains a topic for further research.

4. Generalised FRT inversion

The inverse transform (or image reconstruction from projections) for the FRT can be achieved either via Fourier inversion or back-projection. To recover the value of the original function at pixel (x, y) , the discrete line sums in each of the $Y(N)$ projections that contain this pixel are summed. This incorporates the value of the pixel (x, y) $Y(N)$ times and all other pixels at least once (depending on the degree of compositeness of N). It is first required to determine the degree with which some pixels are over-represented; this is the subject of the next section and the following section establishes three distinct methods to correct for that over-representation, allowing the exact value of I for each pixel (x, y) to be recovered.

4.1. Sampling function, $v_N(x, y)$

Let $v_N(x, y)$ represent the number of times a pixel (x, y) is sampled by all discrete lines or subgroups generated by vectors in Θ_N . This *over-sampling* may be found by considering the set of all discrete lines that pass through the image origin, i.e. $R_{b,a}(0)$. The function, $v_N(x, y)$, shows the over-representation each pixel will have after back-projection when reconstructing the origin. From the discrete Fourier slice theorem (8), v_N defined about the origin of the image also gives the over-representation of spatial frequencies in $I_{(u,v)}$, due to the symmetry: $v_N(v, -u) = v_N(u, v)$. This over-representation must be accounted for when reconstructing the original function. Since the array is congruent (mod N), the over-representation of each pixel (x, y) after back-projection to reconstruct pixel (j, k) can be found as $v_N(\langle x-i \rangle_N, \langle y-j \rangle_N)$, so it is sufficient to investigate reconstructing the origin only.

For the FRT case, where $N = p$, a prime, $v_p(x, y) = 1$ for all pixels except the origin where $v_p(0, 0) = Y(N) = p + 1$. An example of this (for $p = 7$) is shown in Fig. 4a. This can be written as

$$v_p(x, y) = d \prod_{p|d} (1 + 1/p) \quad \text{where } d = \gcd(x, y, p). \quad (11)$$

For the dyadic FRT case, where $N = 2^n$, it was shown in [8] that $v_{2^n}(x, y) = \gcd(x, y, 2^n)$ for all pixels, except the origin, where $v_{2^n}(0, 0) = Y(N) = 2^n(1 + 1/2)$. An example of this for $N = 2^3$ is shown in Fig. 4b. This can be written as

$$v_{2^n}(x, y) = d \prod_{p|d} \begin{cases} 1 + 1/2, & \text{if } d = 2^n, \\ 1 & \text{otherwise,} \end{cases} \quad \text{where } d = \gcd(x, y, 2^n). \quad (12)$$

Similarly, for the *prime-adic* FRT case, where $N = p^n$, it was shown in [22] that

$$v_{p^n}(x, y) = d \prod_{p|d} \begin{cases} 1 + 1/p, & \text{if } d = p^n, \\ 1 & \text{otherwise,} \end{cases} \quad \text{where } d = \gcd(x, y, p^n). \quad (13)$$

An example of this for $N = 3^2$ is shown in Fig. 4c.

Suppose that $\gcd(m, n) = 1$ and $\{(a_m^j, b_m^j) | j \in [0, v_m(x, y))\}$ is the set of solutions $a_m^j x - b_m^j y \equiv 0 \pmod{m}$ and $\{(a_n^k, b_n^k) | k \in [0, v_n(x, y))\}$ is the set of solutions $a_n^k x - b_n^k y \equiv 0 \pmod{n}$. Then $\{(b_{pq}^l, a_{pq}^l) = (n(a_m^j, b_m^j + m(b_n^k, a_n^k)) | j \in [0, v_m(x, y)), k \in [0, v_n(x, y)), l = jk\}$ gives the solution set (mod pq) and there are $v_m(x, y)v_n(x, y)$ discrete lines. The proof that each new discrete line is unique is identical to that given for $Y(N)$ in Section 2. All of the new solutions so defined are incongruent (mod pq) and form the complete set of $Y(N)N$ solutions. Therefore, we can say $v_N(x, y)$ is multiplicative and define

$$v_N(x, y) = d \prod_{p|d} \begin{cases} 1 + 1/p, & \text{if } \gcd(p, N/d) = 1, \\ 1, & \text{otherwise,} \end{cases} \quad \text{where } d = \gcd(x, y, N). \quad (14)$$

The following example demonstrates how to find $v_6(2, 4)$ from $v_3(2, 4)$ and $v_2(2, 4)$.

(a)	(b)	(c)
8 1 1 1 1 1 1	12 1 2 1 4 1 2 1	12 1 1 3 1 1 3 1 1
1 1 1 1 1 1 1	1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1	2 1 2 1 2 1 2 1	1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1	1 1 1 1 1 1 1 1	3 1 1 3 1 1 3 1 1
1 1 1 1 1 1 1	4 1 2 1 4 1 2 1	1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1	1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1	2 1 2 1 2 1 2 1	3 1 1 3 1 1 3 1 1
	1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1
		1 1 1 1 1 1 1 1 1

Fig. 4. Examples of the sampling function (with origin in the top left corner) for (a) prime FRT, $v_7(x, y)$ (b) dyadic FRT (or DPRT), $v_8(x, y)$ *prime-adic* FRT, $v_9(x, y)$.

$$\text{Given } \begin{cases} v_2(2,4) = v_2(0,0) = 3 & \text{as } (1,0), (0,1), (1,1). \\ v_3(2,4) = v_3(2,1) = 1 & \text{as } (2,1). \end{cases}$$

$$v_6(2,4) = 3 \text{ is found as } (2,1) \begin{array}{c|ccc} + & (1,0) & (0,1) & (1,1) & \times 3 \\ \hline \times 2 & (1,2) & (4,5) & (1,5) & \end{array}$$

$$\text{so the lines } \begin{cases} 2x \equiv 1y \pmod{6} \\ 5x \equiv 4y \pmod{6} \\ 5x \equiv y \pmod{6} \end{cases} \text{ all include pixel } (j, k).$$

A graphical representation of these discrete lines is presented in Fig. 5. The lines in this figure have been drawn in the direction of the shortest vector in each lattice. For example, in the 3×3 case, the lattice generated by $(2, 1)$, i.e. $\mathcal{L}\{(2,1), (p,0)\}$, has been drawn with basis $(-1, 1)$ which generates the same lattice, but with maximum point density along the selected line direction.

4.2. Fourier inversion

The reconstruction of an image from its set of discrete periodic projections can be performed in the frequency domain by taking the 1D DFT of each projection (in $O(NY(N) \log N)$ operations) and mapping it onto the 2D DFT of the image, using the discrete Fourier slice theorem (8). This will over-represent some spatial frequencies according to $v_N(u, v)$. Dividing the value at each spatial frequency by $v_N(u, v)$ and applying the inverse 2D DFT to this data in $O(N^2 \log N)$ operations recovers the original image exactly.

It was shown in [24] for square prime sized arrays that Fourier inversion has a large constant of proportionality and that discrete back-projection of $O(N^3)$ is actually faster for array sizes less than $O(600)$. The greater efficiency of Fourier inversion with increasing array size implies its implementation speed overtakes that of back-projection with increasing N and is consistently faster for array sizes greater than $O(1800)$. The speed of Fourier inversion depends on how well N can be divided to perform a fast Fourier transform (FFT). Prime array sizes give the worst case scenario and show that, for smaller array sizes, it may

Fig. 5. Depiction of the discrete lines presented in the example for $v_6(2, 4)$. The numbers indicate the number of discrete lines that sample each pixel. Note in the 6×6 case, the reduced basis vector $(2, 1)$ generates the same lattice as $(4, 5)$ and the reduced basis vector $(-1, 1)$ generates the same lattice as $(1, 5)$.

often be beneficial to perform image reconstruction via discrete back-projection. The remainder of this section details two distinct methods to achieve this: multi-resolutional corrected back-projection and filtered back-projection.

4.3. Multi-resolutional correction for $v_N(x, y)$ in back-projection

For the FRT, $v_p(x, y)$ is $Y(p) = p + 1$ at the origin and 1 at all other (x, y) . The sum of all discrete line sums containing a specific pixel (j, k) , [i.e. $R_{1,0}(j)$ and $R_{m,1}\langle i-m_j \rangle_p$ for $0 \leq m < p$] gives the sum of the entire image, I_{sum} , with the value of I at (j, k) included an additional p times, (see example in Fig. 4a). The over-representation can be corrected for by subtracting the sum of the image, I_{sum} and dividing the result by p . The image is recovered from its FRT as

$$I(x, y) = \frac{1}{p} \left(\sum_{m=0}^{p-1} R_{m,1}\langle x - ym \rangle_p + R_{1,0}(y) - I_{\text{sum}} \right), \quad (15)$$

where I_{sum} can be obtained as the sum of all discrete line sums from any one projection, i.e. $I_{\text{sum}} = \sum_{t=0}^{p-1} R_{b,a}(t)$ for any $(b, a) \in \Theta_p$, as all projections sample the entire image once.

For the prime-adic (and hence dyadic) FRT, $v_p(x, y)$ is $Y(p^n)$ at the origin and $\gcd(x, y, p^n)$ at all other (x, y) . This gives an over-representation at each resolution $p \times p, p^2 \times p^2, \dots, p^{n-1} \times p^{n-1}$ as depicted in Fig. 4b and c. This can be corrected for using a multi-resolutional scheme. To correct at each resolution $p^k \times p^k$ within a $p^n \times p^n$ array, an important property must be introduced.

Firstly notice that $\Theta_{p^j} \subset \Theta_{p^n}$ for $j < n$. Since the image is assumed to be periodic, the infinite lattice produced by taking a line generated by $(b, a) \in \Theta_{p^j}$, for all discrete lines with intercept $t \equiv 0 \pmod{p^j}$, (i.e. $t = 0, p^j, 2p^j, \dots$), gives the same infinite lattice as that produced by $(b, a) \in \Theta_{p^j}$ in the $p^j \times p^j$ array, i.e.

$$\mathcal{L}\{(m, 1), (p^j, 0), (p^n, 0)\} \equiv \mathcal{L}\{(m, 1), (p^j, 0)\} \quad \text{and} \\ \mathcal{L}\{(1, ps), (0, p^j), (0, p^n)\} \equiv \mathcal{L}\{(1, ps), (0, p^j)\},$$

for any $(m, 1), (1, ps) \in \Theta_{p^j}$.

Therefore taking all the line sums from Θ_{p^j} that include the origin and all other parallel line sums with $t \equiv 0 \pmod{p^j}$

in $\mathbb{Z}_{p^n}^2$ (i.e. $\sum_{k=0}^{n-j-1} R_{b,a}(0 + kp^j)$ for all (b, a) in Θ_{p^j}) yields the sampling pattern for $\mathbb{Z}_{p^j}^2$, i.e. $v_{p^j}(x, y)$ replicated $(\text{mod } p^j)$ in the x and y directions. Examples of this for $p^j = 2^1$ and $p^j = 2^2$ within $\mathbb{Z}_{2^3}^2$ are depicted in Fig. 6b and d.

These replicating patterns can be used to correct for $v_{p^n}(x, y)$ at resolution $p^j \times p^j$. The initial back-projected value, $I'(x, y)$, can be corrected through a multi-resolutional process reducing the over-representation from v_{p^n} to zero over all $\mathbb{Z}_{p^n}^2 \setminus \{(x, y)\}$; each step, ξ_{p^j} , corrects for the over-representation described above at resolution $p^j \times p^j$. The inversion process is given as

Fig. 6. (a) $v_2(x, y)$ (b) All discrete lines in Θ_2 on the 8×8 array, with intercept 0 (mod 2) yields $v_2(x, y)$ replicated (mod 2) (c) $v_4(x, y)$ (d) All discrete lines in Θ_4 on the 8×8 array, with intercept 0 (mod 4) yields $v_4(x, y)$ replicated (mod 4).

$$I(x, y) = \frac{1}{N} \left(\zeta_{p^n}(x, y) - \sum_{j=1}^{n-1} \frac{(p-1)p^j}{N} \zeta_{p^j}(x, y) - \frac{p}{N} I_{\text{sum}} \right), \quad (16)$$

where

$$\zeta_{p^j} = \left[\sum_{m=0}^{p^j-1} \sum_{k=0}^{\frac{N}{p^j}-1} R_{m,1}(\langle x - my \rangle_{p^j} + kp^j) + \sum_{s=0}^{p^{j-1}-1} \sum_{j=0}^{\frac{N}{p^j}-1} R_{1,s}(\langle y - psx \rangle_{p^j} + kp^j) \right].$$

The correction at each resolution $p^j \times p^j$ corresponds to all factors, F , of N , denoted $F|N$. Eq. (16) can be rewritten as

$$I(x, y) = \frac{1}{N} \left(\zeta_{N=p^n}(x, y) + \sum_{F=p^j|N} A_F \zeta_j(x, y) + A_{1=p^0} I_{\text{sum}} \right), \quad (17)$$

where the fraction of contribution required at each resolution, $F = p^j$, is given as

$$A_{F=p^j} = \frac{p^j}{N} \prod_{p^i|N} \begin{cases} -p, & \text{if } \gcd(p, p^i) = 1, (\text{i.e. } p^i = 1), \\ 1 - p, & \text{otherwise.} \end{cases} \quad (18)$$

The inversion process for composite N is a natural extension of that for $N = p^n$. It is also undertaken by correcting for the over-representation at each resolution after

back projection of pixel (x, y) , which is defined as $1/N \sum_{(a,b) \in \Theta_N} R_{b,a} \langle ay - bx \rangle_N$. As for the over-sampling function

$v_N(x, y)$ defined in the previous section, the resolutions that require correction correspond to all the factors, F , of N ($F|N$). For the example shown in Fig. 7a, where $N = 3^2 5 = 45$, the over-sampling must be corrected for at scales 15×15 , 9×9 , 5×5 and 3×3 . Subtracting a fraction of I_{sum} corrects at resolution 1×1 , leaving only a multiple (N) of $I(x, y)$.

The sampling achieved by taking the projection set for an $F \times F$ array, for some $F|N$, for all discrete line sums with translates $t \pmod F$ yields the sampling pattern $v_F(x, y)$ replicated (mod F) in the x and y directions. An example of this is depicted in Fig. 7b for $N = 45$ and $F = 9$. This $F \times F$ back-projection sampling pattern, achieved as

$$\zeta_F^N(x, y) = \sum_{(b,a) \in \Theta_F} \sum_{k=0}^{N/F-1} R_{b,a} \langle ay - bx + kF \rangle_N, \quad (19)$$

can be used to correct for the over-representation from back-projection at the resolution of $F \times F$ (9×9 in the example) by adding a fraction, A_F^N , of this back-projection, $\zeta_F^N(x, y)$. This technique is applied to correct the back-projection at each resolution, $F|N$.

Since the over-sampling of pixel (x, y) after back-projection of the origin, $v_N(x, y)$, is a multiplicative function, the scaling factors, A_F^N , for correction after back-projection at

Fig. 7. Top left corner of (a) $v_{45}(x, y)$ (b) All discrete lines in Θ_9 on the 45×45 array, with intercept 0 (mod 9), i.e. $\zeta_9^{45}(x, y)$, yields $v_9(x, y)$ (as shown in Fig. 4c) replicated (mod 9). To fully correct (a), $\zeta_9^{45}(x, y)$, $\zeta_5^{45}(x, y)$, $\zeta_3^{45}(x, y)$, and I_{sum} are required.

each resolution $F \times F$ of the multi-resolutional back-projections, $\xi_F^N(x, y)$ are also multiplicative. Given $\gcd(m, n) = 1$, the scaling factor required for resolution $F \times F$ in a $p \times q$ array for $F|pq$ is found as

$$A_F^{pq} \equiv A_f^m A_{f/f}^n \quad \text{for some } f \in \mathbb{Z}_m \text{ that divides } F. \quad (20)$$

For example, in the 45×45 array, the back-projection, $\xi_9^{45}(0, 0)$, which gives the sampling shown in Fig. 7b, corrects for the over-representation after back-projection at resolution 9×9 , by adding it $A_9^{45} = A_9^9 A_1^5 = 1 \cdot -1 = -1$ times. The value for A_F^N can therefore be found as

$$A_F = \frac{F}{N} \prod_{p|N/F} \begin{cases} -p, & \text{if } \gcd(p, F) = 1, \\ 1 - p, & \text{otherwise.} \end{cases} \quad (21)$$

Note that in (15), $I_{\text{sum}} = \sum_{t=0}^{p-1} R_{b,a}(t)$ for any $(b, a) \in \Theta_p$. If the projection set for a 1×1 array is defined as $\Theta_1 = \{(0, 1)\}$, then I_{sum} can be found as $\xi_1^N(x, y)$ for any $(x, y) \in \mathbb{Z}_N^2$. Therefore the entire inversion process can be written as

$$I(x, y) = \frac{1}{N} \sum_{F|N} A_F^N \xi_F^N(x, y). \quad (22)$$

The result of the correction process at each resolution for the inversion of the $Y(175) \times 175$ FRT of a 175×175 Lena image is depicted in Fig. 8.

4.4. Filtered back-projection

The original image can be reconstructed exactly from back-projection without a multi-resolutional correction if the projections are *filtered* prior to back-projection. This is

similar to filtered back-projection from acquired projections for the continuous RT, where the projections are convolved with a form of the Ram-Lak filter (a low pass filtered ramp function in frequency domain). The 2D IDFT of $\hat{I}(u, v)$,

$$I(x, y) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \hat{I}(u, v) \exp[i2\pi(ux + vy)/N], \quad (23)$$

can be rewritten as a sum over all spatial frequencies $\hat{I}(aw, -bw)$ for $w \in [0, N]$ for each discrete line or subgroup generated by $(b, a) \in \Theta_N$. Dividing by the over-sampling function to account for over-representation of some frequencies, (23) becomes

$$I(x, y) = \frac{1}{N^2} \sum_{(b,a) \in \Theta_N} \sum_{w=0}^{N-1} \frac{1}{v_N(aw, -bw)} \hat{I}(aw, -bw) \exp[i2\pi(ax - by)w/N]. \quad (24)$$

By definition, $\gcd(a, b) = 1$, therefore $\gcd(aw, -bw, N) = \gcd(w, N)$ and the 2D over-sampling function, $v_N(aw, -bw)$ can be reduced to a 1D function according to

$$v_N(w) = d \prod_{p|d} \begin{cases} 1 + 1/p, & \text{if } \gcd(p, N/d) = 1, \\ 1, & \text{otherwise,} \end{cases} \quad (25)$$

where $d = \gcd(w, N)$.

This shows the remarkable fact that all discrete lines with basis (b, a) , such that $\gcd(a, b) = 1$, through the origin of $\hat{I}(u, v)$, have an identical over-sampling function. An example of this function for $N = 60$ is presented in Fig. 9.

From the discrete Fourier Slice theorem (8), $\hat{R}_{b,a}(w) = \hat{I}(aw, -bw)$, so (24) becomes

Fig. 8. Depiction of the correction made by each resolution of the reconstruction of the $Y(175) = 240 \times 175$ FRT of a 175×175 Lena image. (a) Uncorrected back-projection. (b) Corrected at resolution 35×35 . (c) Corrected at resolution 25×25 . (d) Corrected at resolution 7×7 . (e) Corrected at resolution 5×5 and (f) subtracting $A_1 I_{\text{sum}}$ completes all required corrections and exactly reproduces the original image.

Fig. 9. A plot of the 1D over-sampling function $v_N(w)$ for $N = 2^2 \times 3 \times 5 = 60$. The reciprocal of this function gives the filter required to correct for the over-representation in the frequency domain.

$$\begin{aligned} I(x, y) &= \frac{1}{N^2} \sum_{(b, a) \in \Theta_N} \sum_{w=0}^{N-1} \frac{1}{v_N(w)} \hat{R}_{b,a}(w) \exp[i2\pi(ax - by)w/N] \\ &= \frac{1}{N^2} \sum_{(b, a) \in \Theta_N} \hat{R}_{b,a}(ax - by). \end{aligned} \quad (26)$$

This is simply a back-projection of filtered projections, $\hat{R}_{b,a}(t)$. These filtered projections are found as

$$\begin{aligned} \hat{R}_{b,a}(\eta) &= \sum_{w=0}^{N-1} \frac{1}{v_N(w)} \hat{R}_{b,a}(w) \exp[i2\pi\eta w/N] \\ &= \sum_{w=0}^{N-1} \frac{1}{v_N(w)} \sum_{t=0}^{N-1} R_{b,a}(t) \exp[-i2\pi tw/N] \exp[i2\pi\eta w/N] \\ &= \sum_{t=0}^{N-1} R_{b,a}(t) \sum_{w=0}^{N-1} \frac{1}{v_N(w)} \exp[i2\pi(\eta - t)w/N] \\ &= \sum_{\tau=0}^{N-1} R_{b,a}(\eta - \tau) \sum_{w=0}^{N-1} \frac{1}{v_N(w)} \exp[i2\pi\tau w/N] \\ &= \sum_{\tau=0}^{N-1} R_{b,a}(\eta - \tau) H(\tau). \end{aligned} \quad (27)$$

So the multiplication of $\hat{R}_{b,a}(w)$ by $1/v_N(w)$ in the frequency domain corresponds to the 1D circular convolution of projections, $R_{b,a}$, with kernel, H , defined as

$$H(\tau) = \sum_{w=0}^{N-1} \frac{1}{v_N(w)} \exp[i2\pi\tau w/N]. \quad (28)$$

There are no singularities in this function (by design) as $v_N(w) \geq 1$ for all $w \in \mathbb{Z}_N$. An example of this filter for $N = 60$, based on the sampling function from Fig. 9, is presented in Fig. 10a. This can be compared to the Ram-Lak filter for $N = 60$ shown in Fig. 10b. While they appear very different, their purpose is the same. Both filters have a maximum at $w = 0$. The Ram-Lak filter has global minima at $|w| = \pm 1$ and minima of decreasing magnitude at $|w| = 3, 5, \dots$. This accounts for the over-representation upon back-projection immediately surrounding the point being back-projected due to the polar nature of the operation. The $1/v_N(w)$ filter has a global minimum at $|w| = 30 = N/2$ and minima of decreasing magnitude at $|w| \equiv 0 \pmod{N/3}, \pmod{N/4}, \pmod{N/5}, \dots$ (at the factors of N). This accounts for the multi-resolutional over-representation upon discrete back-projection about the point of back-projection.

Implementation of filtered back-projection using the result shown in (24) achieves exact image reconstruction from projections. This form of back-projection is completed in $O(N^2 Y(N))$ operations, regardless of performing the convolution in the frequency or spatial domain. The dominating computational cost is that of back-projection, requiring $O(Y(N))$ operations for each of the N^2 pixels.

5. Conclusion

An FRT based on modulo arithmetic that applies to $N \times N$ arrays for $N \in \mathbb{N}$ has been presented. It projects the 2D image into a set of $N\pi_{p|N}(1 + 1/p)$ projections of length N . A method to determine the required projection set was also presented.

Important properties of the continuous Radon transform are preserved in this discrete formalism. A discrete

Fig. 10. (a) Plot of the 1D filter obtained from the IDFT of $1/v_N(w)$ for $N = 2^2 \times 3 \times 5 = 60$. This enables exact image reconstruction from filtered projections by direct discrete back-projection. This filter is compared with (b) the standard Ram-Lak filter used to reconstruct images from acquired projections by filtered back-projection.

form of the Fourier slice theorem and the convolution property hold for this generalised FRT. These properties allow the FRT to be obtained in $O(N^2 \log N \pi_{p|N}(1 + 1/p))$ operations and make it a useful image processing tool, reducing 2D problems to a set of 1D problems.

Due to the compositeness of N , it is a redundant transform, as for the DPRT. The redundancy occurs in a multi-resolutional pattern based on the factors of N . A function, $v_N(x, y)$, which characterises this multi-resolutional redundancy was presented which gives the over-sampling of pixels upon back-projection of discrete line sums through the origin. Given the (x, y) symmetry of this function, it also gives the over-representation in the projections of each spatial frequency, at $(y, -x)$, in the 2D DFT of the image.

Determination of this over-representation function enabled the inverse FRT to be formulated. Three methods were presented; Fourier inversion, multi-resolutional corrected back-projection and filtered back-projection. The 2D over-representation function has the remarkable property of being identical along all 1D discrete lines of the form $ax \equiv by \pmod{N}$ for $\gcd(a, b) = 1$, implying the filter for each projection is the same regardless of direction.

The multi-resolutional nature of the transform may prove useful in image analysis, particularly for matching projected textures and patterns. Research into this aspect and the development of an orthogonal FRT over $N \times N$ and $M \times N$ are the subject of ongoing work. An investigation into the distribution of the discrete angles required, for $N \times N$ (as compared to the $p \times p$ set) and comparison with the uniformly distributed angle set for the continuous RT is also the subject of ongoing research.

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