Dirichlet forms and degenerate elliptic operators

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Dedicated to Philippe Clement on the occasion of his retirement

Abstract

It is shown that the theory of real symmetric second-order elliptic operators in divergence form on $\mathbb{R}^d$ can be formulated in terms of a regular strongly local Dirichlet form irregardless of the order of degeneracy. The behaviour of the corresponding evolution semigroup $S_t$ can be described in terms of a function $(A,B) \mapsto d(A;B) \in [0, \infty]$ over pairs of measurable subsets of $\mathbb{R}^d$. Then

$$|(\varphi_A, S_t \varphi_B)| \leq e^{-d(A;B)^2(4t)^{-1}} \|\varphi_A\|_2 \|\varphi_B\|_2$$

for all $t > 0$ and all $\varphi_A \in L^2(A)$, $\varphi_B \in L^2(B)$. Moreover $S_t L^2(A) \subseteq L^2(A)$ for all $t > 0$ if and only if $d(A;A^c) = \infty$ where $A^c$ denotes the complement of $A$.

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1 Introduction

The usual starting point for the analysis of second-order divergence-form elliptic operators with measurable coefficients is the precise definition of the operator by quadratic form techniques. Let \( c_{ij} = c_{ji} \in L_{\infty}(\mathbb{R}^d; \mathbb{R}) \), the real-valued bounded measurable functions on \( \mathbb{R}^d \), and assume that the \( d \times d \)-matrix \( C = (c_{ij}) \) is positive-definite almost-everywhere. Then define the quadratic form \( h \) by

\[
h(\varphi) = \sum_{i,j=1}^{d} (\partial_i \varphi, c_{ij} \partial_j \varphi)
\]

where \( \partial_i = \partial / \partial x_i \) and \( \varphi \in D(h) = W^{1,2}(\Omega) \). It follows that \( h \) is positive and the corresponding sesquilinear form \( h(\cdot, \cdot) \) is symmetric. Therefore if \( h \) is closed there is a canonical construction which gives a unique positive self-adjoint operator \( H \) such that \( D(h) = D(H^{1/2}) \), \( h(\varphi) = \| H^{1/2} \varphi \|_2^2 \) for all \( \varphi \in D(h) \) and \( h(\psi, \varphi) = (\psi, H \varphi) \) if \( \psi \in D(h) \) and \( \varphi \in D(H) \). Here and in the sequel \( \| \cdot \|_p \) denotes the \( L_p \)-norm. Formally one has

\[
H = - \sum_{i,j=1}^{d} \partial_i c_{ij} \partial_j.
\]

The critical point in the form approach is that \( h \) must be closed. But \( h \) is closed if and only if there is a \( \mu > 0 \) such that \( C \geq \mu I \) (see, for example, [ERZ], Proposition 2), i.e., if and only if the operator \( H \) is strongly elliptic. Therefore the construction of \( H \) is not directly applicable to degenerate elliptic operators. Nevertheless refinements of the theory of quadratic forms allow a precise definition of the elliptic operator. There are two distinct cases.

First, it is possible that \( h \) is closable on \( W^{1,2}(\Omega) \). Then one can repeat the previous construction to obtain the self-adjoint elliptic operator associated with the closure \( \overline{h} \) of \( h \). For example, let \( \Omega \) be an open subset of \( \mathbb{R}^d \) with \( |\partial \Omega| = 0 \). Suppose \( \text{supp} \ C = \overline{\Omega} \) and \( C(x) \geq \mu I > 0 \) uniformly for all \( x \in \Omega \). Then \( h \) is closable. Indeed the restriction of \( h \) to \( W^{1,2}(\Omega) \) is closed as a form on the subspace \( L_2(\Omega) \) and the corresponding self-adjoint operator \( H_\Omega \) can be interpreted as the strongly elliptic operator with coefficients \( C \) acting on \( L_2(\Omega) \) with Neumann boundary conditions. Then the form corresponding to the operator \( H_\Omega \oplus 0 \) on \( L_2(\mathbb{R}^d) \), where \( H_\Omega \) acts on the subspace \( L_2(\Omega) \), is the closure of \( h \).

Secondly, it is possible that \( h \) is not closable. There are many sufficient conditions for closability of the form \( \Pi \) (see, for example, [FOT], Section 3.1, or [Mar], Chapter II) and if \( d = 1 \) then necessary and sufficient conditions are given by [FOT], Theorem 3.1.6. The latter conditions restrict the possible degeneracy. But Simon [Sim3], Theorems 2.1 and 2.2, has shown that a general positive quadratic form \( h \) can be decomposed as a sum \( h = h_r + h_s \) of two positive forms with \( D(h_r) = D(h) = D(h_s) \) with \( h_r \) the largest closable form majorized by \( h \). Simon refers to \( h_r \) as the regular part of \( h \). Then one can construct the operator \( H_0 \) associated with the closure \( h_0 = \overline{h_r} \) of the regular part of \( h \). Note that if \( h \) is closed then \( h_0 = h \) and if \( h \) is closable then \( h_0 = \overline{h} \) so in both cases \( H_0 \) coincides with the previously defined elliptic operator. Therefore this method can be considered as the generic method of defining the operator associated with the form \( \Pi \).

There is an alternative method of constructing \( H_0 \) which demonstrates more clearly its relationship with the formal definition of the elliptic operator. Introduce the form \( l \) by
Then \( l \) is closed and the corresponding positive self-adjoint operator is the usual Laplacian \( \Delta \). Furthermore for each \( \varepsilon > 0 \) the form \( h_\varepsilon = h + \varepsilon l \) with domain \( D(h_\varepsilon) = D(h) \) is closed. The corresponding positive self-adjoint operators \( H_\varepsilon \) are strongly elliptic operators with coefficients \( c_{ij} + \varepsilon \delta_{ij} \). These operators form a decreasing sequence which, by a result of Kato [Kat], Theorem VIII.3.11, converges in the strong resolvent sense to a positive self-adjoint operator. The limit is the operator \( H_0 \) associated with the form \( h_0 \) by [Sim3], Theorem 3.2. The latter method of construction of \( H_0 \) as a limit of the \( H_\varepsilon \) was used in [ERZ] and [ERSZ] and motivated the terminology viscosity operator for \( H_0 \) and viscosity form for \( h_0 \). The form \( h_r \), constructed by Simon also occurs in the context of nonlinear phenomena and discontinuous media and is variously described as the regularization or relaxation of \( h \) (see, for example, [Bra] [EkT] [Jos] [Dal] [Mos] and references therein).

Although the viscosity form provides a basis for the analysis of degenerate elliptic operators its indirect definition makes it difficult to deduce even the most straightforward properties. Simon, [Sim2] Theorem 2, proved that \( D(h_0) \) consists of those \( \varphi \in L_2(\mathbb{R}^d) \) for which there is a sequence \( \varphi_n \in D(h) \) such that \( \lim_{n \to \infty} \varphi_n = \varphi \) in \( L_2 \) and \( \liminf_{n \to \infty} h(\varphi_n) < \infty \). Moreover, \( h_0(\varphi) \) equals the minimum of all \( \liminf_{n \to \infty} h(\varphi_n) \), where the minimum is taken over all \( \varphi_1, \varphi_2, \ldots \in D(h) \) such that \( \lim_{n \to \infty} \varphi_n = \varphi \) in \( L_2 \). (See [Sim2], Theorem 3.) It was observed in [ERZ] that this characterization allows one to deduce that \( h_0 \) is a Dirichlet form. The first purpose of this note is to use the theory of Dirichlet forms (see [FOT] [BoH] for background material) to strengthen the earlier conclusion. Specifically we establish the following result in Section 2.

**Theorem 1.1** The viscosity form \( h_0 \) is a regular local Dirichlet form.

The regularity is straightforward but the locality is not so evident and requires some control of the dissipativity of the semigroup \( S^{(0)}(t) \) generated by the viscosity operator \( H_0 \). Note that we adopt the terminology of [BoH] which differs from that of [FOT]. Locality in [BoH] corresponds to strong locality in [FOT] if the form is regular. (See [Sch] for a detailed study of the different forms of locality (in the sense of [FOT]).)

Our second purpose is to extend earlier results on \( L_2 \) off-diagonal bounds for the semigroup \( S^{(0)}(t) \) generated by the viscosity operator \( H_0 \). These bounds, which are also referred to as Davies–Gaffney estimates, integrated Gaussian estimates or an integrated maximum principle (see, for example, [Aus] [CGT] [Dav2] [Gri] [Stu1] [Stu2]), give upper bounds on the cross-norm \( \| S^{(0)}_t \|_{A \to B} \) of the semigroup \( S^{(0)}_t \) between subspaces \( L_2(A) \) and \( L_2(B) \) of the form

\[
\| S^{(0)}_t \|_{A \to B} \leq e^{-d(A;B)^2(4t)^{-1}}
\]

where \( d(A;B) \) is an appropriate measure of distance between the subsets \( A \) and \( B \) of \( \mathbb{R}^d \). In the case of strongly elliptic operators there is an essentially unique distance, variously referred to as the Riemannian distance, the intrinsic distance or the control distance, suited to the problem. But for degenerate elliptic operators, which exhibit phenomena of separation and isolation [ERSZ], the Riemannian distance is not necessarily appropriate.

We will establish estimates in terms of a set-theoretic function which is not strictly a distance since it can take the value infinity. The function is defined by a version of a
standard variational principle which has been used widely in the analysis of strongly elliptic operators and which was extended to the general theory of Dirichlet forms by Biroli and Mosco [BiM1] [BiM2]. In the case of strongly elliptic operators or subelliptic operators with smooth coefficients this method gives a distance equivalent to the Riemannian distance obtained by path methods (see, for example, [LaS1], Section 3). In the degenerate situation we make a choice of the set of variational functions which gives a distance compatible with the separation properties. In particular $d(A; A^c) = \infty$ if and only if $S_t(0) L_2(A) \subseteq L_2(A)$, where $A^c$ is the complement of $A$. Our choice is aimed to maximize the distance and thereby optimize the bounds (2). We carry out the analysis in a general setting of local Dirichlet forms which is applicable to degenerate elliptic operators defined as above but has a wider range of applicability.

Let $X$ be a topological Hausdorff space equipped with a $\sigma$-finite Borel measure $\mu$. Further let $E$ be a local Dirichlet form on $L_2(X)$ in the sense of [BoH]. First, for all $\psi, \phi \in D(E) \cap L_\infty(X; \mathbb{R})$ define $I_\psi^{(E)} : D(E) \cap L_\infty(X; \mathbb{R}) \to \mathbb{R}$ by

$$I_\psi^{(E)}(\phi) = E(\psi, \phi, \psi) - 2^{-1} E(\phi^2, \phi)$$

If no confusion is possible we drop the suffix and write $I_\psi(\phi) = I_\psi^{(E)}(\phi)$. If $\phi \geq 0$ it follows that $\psi \mapsto I_\psi(\phi) \in \mathbb{R}$ is a Markovian form with domain $D(E) \cap L_\infty(X; \mathbb{R})$ (see [BoH], Proposition I.4.1.1). This form is referred to as the truncated form by Roth [Rot], Theorem 5.

Secondly, define $D(E)_{loc}$ as the vector space of (equivalent classes of) all measurable functions $\psi : X \to \mathbb{C}$ such that for every compact subset $K$ of $X$ there is a $\hat{\psi} \in D(E)$ with $\psi|_K = \hat{\psi}|_K$. Since $E$ is local one can define $\hat{I}_\psi^{(E)} = \hat{I}_\psi : D(E) \cap L_{\infty,c}(X; \mathbb{R}) \to \mathbb{R}$ by

$$\hat{I}_\psi(\phi) = I_\psi^{(E)}(\phi)$$

for all $\psi \in D(E)_{loc} \cap L_\infty(X; \mathbb{R})$ and $\phi \in D(E) \cap L_{\infty,c}(X; \mathbb{R})$ where $\hat{\psi} \in D(E) \cap L_\infty(X; \mathbb{R})$ is such that $\psi|_{\text{supp} \; \phi} = \hat{\psi}|_{\text{supp} \; \phi}$. Here $L_{\infty,c}(X; \mathbb{R}) = \{ \phi \in L_\infty(X; \mathbb{R}) : \text{supp} \; \phi \text{ is compact} \}$ and $(\text{supp} \; \phi)^c$ is the union of all open subsets $U \subseteq X$ such that $\phi|_U = 0$ almost everywhere. Thirdly, for all $\psi \in D(E)_{loc} \cap L_\infty(X; \mathbb{R})$ define

$$||| \hat{I}_\psi ||| = \sup \{ ||\hat{I}_\psi(\phi)|| : \phi \in D(E) \cap L_{\infty,c}(X; \mathbb{R}), ||\phi||_1 \leq 1 \} \in [0, \infty]$$

Fourthly, for all $\psi \in L_\infty(X; \mathbb{R})$ and measurable sets $A, B \subseteq X$ introduce

$$d_\psi(A; B) = \sup \{ M \in \mathbb{R} : \psi(a) - \psi(b) \geq M \text{ for a.e. } a \in A \text{ and a.e. } b \in B \}$$

$$= \text{ ess inf } x \in A \psi(x) - \text{ ess sup } y \in B \psi(y) \in (-\infty, \infty]$$

(Recall that $\text{ ess sup } y \in B \psi(y) = \inf \{ m \in \mathbb{R} : |\{ y \in B : \psi(y) > m \} | = 0 \} \in [-\infty, \infty]$ and $\text{ ess inf } x \in A \psi(x) = -\text{ ess sup } x \in A - \psi(x)$.) Finally define the set theoretic distance

$$d(A; B) = d^{(E)}(A; B) = \sup \{ d_\psi(A; B) : \psi \in D_0(E) \}$$

where

$$D_0(E) = \{ \psi \in D(E)_{loc} \cap L_\infty(X; \mathbb{R}) : ||| \hat{I}_\psi ||| \leq 1 \}$$

3
A similar definition was given by Hino and Ramirez [HiR], but since they consider probability spaces the introduction of \( D(\mathcal{E})_{\text{loc}} \) is unnecessary in [HiR]. If, however, we were to replace \( D(\mathcal{E})_{\text{loc}} \) by \( D(\mathcal{E}) \) in the definition of \( d(A;B) \) then, since \( \psi \in L_2(X) \), one would obtain \( d_\psi(A;B) \leq 0 \) for all measurable sets \( A, B \subset X \) with \( |A| = |B| = \infty \) and this would give \( d(A;B) = 0 \). On the other hand the definition with \( D(\mathcal{E})_{\text{loc}} \) is not useful unless \( D(\mathcal{E}) \) contains sufficient bounded functions of compact support.

**Theorem 1.2** Let \( \mathcal{E} \) be a local Dirichlet form on \( L_2(X) \) with \( 1 \in D(\mathcal{E})_{\text{loc}} \) and such that \( D(\mathcal{E}) \cap L_{\infty,X}(X) \) is a core for \( \mathcal{E} \). Further let \( d(\cdot;\cdot) \) denote the corresponding set-theoretic distance. If \( S \) denotes the semigroup generated by the self-adjoint operator \( H \) on \( L_2(X) \) associated with \( \mathcal{E} \) and if \( A \) and \( B \) are measurable subsets of \( X \) then

\[
\langle \varphi_A, S_t(\varphi_B) \rangle \leq e^{-d(A;B)^2(4t)^{-1}} \|\varphi_A\|_2 \|\varphi_B\|_2
\]

for all \( \varphi_A \in L_2(A), \varphi_B \in L_2(B) \) and \( t > 0 \) with the convention \( e^{-\infty} = 0 \).

This applies in particular to the viscosity operator \( H_0 \). One has \( D(l) = D(h) \subseteq D(h_0) \).

Hence \( 1 \in D(l)_{\text{loc}} \subseteq D(h_0)_{\text{loc}} \).

**Theorem 1.2** will be proved in Section 3.

If \( h \) is closed, i.e., if the corresponding operator is strongly elliptic, then \( d(h)(\cdot;\cdot) \) is finite-valued. Nevertheless for degenerate operators one can have \( d(h)(A;B) = \infty \) with \( A, B \) non-empty open subsets of \( \mathbb{R}^d \) and the action of the semigroup \( S^{(0)} \) can be non-ergodic [ERSZ].

Specifically we establish the following result in Section 4.

**Theorem 1.3** Let \( \mathcal{E} \) be a local Dirichlet form on \( L_2(X) \) with \( 1 \in D(\mathcal{E})_{\text{loc}} \) and such that \( D(\mathcal{E}) \cap L_{\infty,X}(X) \) is a core for \( \mathcal{E} \). Further let \( d(\cdot;\cdot) \) denote the corresponding set-theoretic distance. If \( S \) denotes the semigroup generated by the self-adjoint operator \( H \) on \( L_2(X) \) associated with \( \mathcal{E} \) and \( A \subset X \) is measurable then the following conditions are equivalent.

I. \( S_tL_2(A) \subseteq L_2(A) \) for one \( t > 0 \).

II. \( S_tL_2(A) \subseteq L_2(A) \) for all \( t > 0 \).

III. \( d(A;A^c) = \infty \).

IV. \( d(A;A^c) > 0 \).

We conclude by deriving alternate characterizations of the distance and deriving some of its general properties in Section 5.

## 2 Dirichlet forms

In this section we prove that the form \( h_0 \) defined in the introduction is a regular strongly local Dirichlet form. First we recall the basic definitions.

A Dirichlet form \( \mathcal{E} \) on \( L_2(X) \) is called **regular** if there is a subset of \( D(\mathcal{E}) \cap C_c(X) \) which is a core of \( \mathcal{E} \), i.e., which is dense in \( D(\mathcal{E}) \) with respect to the natural norm \( \varphi \mapsto (\mathcal{E}(\varphi) + \|\varphi\|_2^2)^{1/2} \), and which is also dense in \( C_0(X) \) with respect to the supremum norm. There are three kinds of locality for Dirichlet forms [BoH] [FOT]. For locality we choose the definition of [BoH]. A Dirichlet form \( \mathcal{E} \) is called **local** if \( \mathcal{E}(\psi, \varphi) = 0 \) for all \( \varphi, \psi \in D(\mathcal{E}) \) and
$a \in \mathbb{R}$ such that $(\varphi + a \mathbb{1})\psi = 0$. Alternatively, the Dirichlet form $\mathcal{E}$ is called \textbf{FOT}-local if $\mathcal{E}(\psi, \varphi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ with supp $\varphi$ and supp $\psi$ compact and supp $\varphi \cap$ supp $\psi = \emptyset$. Moreover, it is called \textbf{FOT}-strongly local if $\mathcal{E}(\psi, \varphi) = 0$ for all $\varphi, \psi \in D(\mathcal{E})$ with supp $\varphi$ and supp $\psi$ compact and $\psi$ constant on a neighbourhood of supp $\varphi$. Every \textbf{FOT}-strongly local Dirichlet form is \textbf{FOT}-local and if $X$ satisfies the second axiom of countability then every local Dirichlet form (in the sense of [BoH]) is \textbf{FOT}-strongly local. If $X$ is a locally compact separable metric space and $\mu$ is a Radon measure such that supp $\mu = X$, then a regular \textbf{FOT}-strongly local Dirichlet form is local (in the sense of [BoH]) by [BoH] Remark I.5.1.5 and Proposition I.5.1.3 ($L_0$) $\Rightarrow$ ($L_2$).

**Lemma 2.1** The form $h_0$ is regular. Moreover, every core for the form $l$ of the Laplacian is a core for $h_0$.

**Proof** Let $h_r$ be the regular part of $h$ as in [Sim3]. Then $D(h_r) = D(h)$ and $h_0$ is the closure of $h_r$ by definition. So $D(l) = D(h) = D(h_r)$ is a core for $h_0$. Since $h_0 \leq h \leq \|C\| l$, where $\|C\|$ denotes the essential supremum of the matrix norms $\|C(x)\|$, it follows that any core for $l$ is also a core for $h_0$.

Finally, since $C^\infty_c(\mathbb{R}^d) \subset W^{1,2}(\mathbb{R}^d) = D(h) \subset D(h_0)$ the set $C^\infty_c(\mathbb{R}^d) \subset D(h_0) \cap C_0(\mathbb{R}^d)$ is a core of $h_0$ and is dense in $C_0(\mathbb{R}^d)$. \hfill $\Box$

Next we examine the locality properties.

**Proposition 2.2** The form $h_0$ is local.

**Proof** First note that the \textbf{FOT}-locality of $h_0$ is an easy consequence of Lemma 3.5 in [ERSZ]. Fix $\varphi, \psi \in D(h_0)$ with supp $\varphi \cap$ supp $\psi = \emptyset$. If $D$ is the Euclidean distance between the support of $\varphi$ and the support of $\psi$, then

$$|h_0(\psi, \varphi)| = \lim_{t \to 0} t^{-1}|(\psi, S_t(0)\varphi)| \leq \lim_{t \to 0} t^{-1}e^{-D^2(4\|C\|t)^{-1}}\|\psi\|_2\|\varphi\|_2 = 0$$

where we have used $(\psi, \varphi) = 0$ and the statement of [ERSZ], Lemma 3.5. The proof of \textbf{FOT}-strong local property is similar but depends on the estimates derived in the proof of Lemma 3.5.

Fix $\varphi, \psi \in D(h_0)$ with supp $\varphi$ and supp $\psi$ compact and $\psi = 1$ on a neighbourhood $U$ of supp $\varphi$. It suffices, by the remark preceding Lemma 2.1, to prove that $h_0(\psi, \varphi) = 0$. We may assume the support of $\psi$ is contained in the Euclidean ball $B_R$ centred at the origin and with radius $R > 0$. Set $\chi_R = \mathbb{1}_{B_R}$. Since $S_t(0)\mathbb{1} = \mathbb{1}$, by Proposition 3.6 of [ERSZ], one has $(\psi, \varphi) = (\mathbb{1}, \varphi) = (\mathbb{1}, S_t(0)\varphi)$. Therefore

$$t^{-1}(\psi, (I - S_t(0))\varphi) = t^{-1}((\chi_R - \psi), S_t(0)\varphi) - t^{-1}((\chi_R - \mathbb{1}), S_t(0)\varphi). \quad (3)$$

Now let $D$ denote the Euclidean distance from supp $\varphi$ to $U^c$. It follows by assumption that $D > 0$. Then by Lemma 3.5 of [ERSZ] there is a $c > 0$ such that

$$t^{-1}|((\chi_R - \psi), S_t(0)\varphi)| \leq t^{-1}e^{-D^2(4\|C\|t)^{-1}}\|\chi_R - \psi\|_2\|\varphi\|_2 \leq c R^{d/2} t^{-1}e^{-D^2(4\|C\|t)^{-1}} \quad (4)$$

uniformly for all large $R$ and all $t > 0$. The factor $R^{d/2}$ comes from the $L_2$-norm of $\chi_R$. Alternatively the estimate used in the proof of Proposition 3.6 in [ERSZ] establishes that there are $a, b > 0$ such that

$$|((\mathbb{1} - \chi_R), S_t(0)\varphi)| \leq a \sum_{n=2}^{\infty} R^{d/2} c^{-bn} R^{2l-1}\|\varphi\|_2$$
uniformly for all $R,t > 0$ such that $\text{supp} \varphi \subset B_{2^{-1}R}$. Next there is a $c' > 0$ such that 
\[ \sum_{n=0}^{\infty} n^{d/2} e^{-an^2} \leq c' \alpha^{-(d+2)/4} e^{-\alpha} \]
uniformly for all $\alpha > 0$. Hence there is a $c'' > 0$ such that 
\[ t^{-1} \| (1 - \varphi R), S_t^{(0)} \varphi \| \leq c'' R^{-1} t^{(d-2)/4} e^{-bR^2t^{-1}}. \]
Combining this with (3) and (4) one has 
\[ t^{-1} \| (\psi, (I-S_t^{(0)})\varphi) \| \leq c R^{d/2} t^{-1} e^{-D^2(4|C|)t^{-1}} + c'' R^{-1} t^{(d-2)/4} e^{-bR^2t^{-1}} \]
for all large $R$ and all $t > 0$. Taking the limit $t \to 0$ establishes that $h_0(\psi, \varphi) = 0$. Since $h_0$ is regular it follows that $h_0$ is local. \hfill $\Box$

### 3 L$_2$ off-diagonal bounds

In this section we prove the $L_2$ off-diagonal bounds of Theorem [12] Initially we assume that $\mathcal{E}$ is a local Dirichlet form on $L_2(X)$ without assuming any kind of regularity. The proof of the theorem follows by standard reasoning based on an exponential perturbation technique used by Gaffney [Gaf] and subsequently developed by Davies [Dav1] in the proof of pointwise Gaussian bounds. It is essential to establish that the perturbation of the Dirichlet form is quadratic. But this is a general consequence of locality. To exploit the latter property we use a result of Andersson [And] and [Rot].

**Proposition 3.1** Let $\mathcal{E}$ be a local Dirichlet form on $L_2(X)$ and $\varphi_1, \ldots, \varphi_n \in D(\mathcal{E}) \cap L_\infty(X: R)$. Then for all $i, j \in \{1, \ldots, n\}$ there exists a unique real Radon measure $\sigma^{(\varphi_1, \ldots, \varphi_n)}_{ij}$ on $R^n$ such that $\sigma^{(\varphi_1, \ldots, \varphi_n)}_{ij} = \sigma^{(\varphi_1, \ldots, \varphi_n)}_{ji}$ for all $i, j \in \{1, \ldots, n\}$ and 
\[ \mathcal{E}(F_0(\varphi_1, \ldots, \varphi_n), G_0(\varphi_1, \ldots, \varphi_n)) = \sum_{i,j=1}^{n} \int_{R^n} d\sigma^{(\varphi_1, \ldots, \varphi_n)}_{ij} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \] 
for all $F, G \in C^1(R^n)$ where $F_0 = F - F(0)$ and $G_0 = G - G(0)$. Let $K$ be a compact subset of $R^n$ such that $(\varphi_1(x), \ldots, \varphi_n(x)) \in K$ for a.e. $x \in X$. Then supp $\sigma^{(\varphi_1, \ldots, \varphi_n)}_{ij} \subseteq K$ for all $i, j \in \{1, \ldots, n\}$. In particular, if $i \in \{1, \ldots, n\}$ then $\sigma^{(\varphi_1, \ldots, \varphi_n)}_{ii}$ is a finite (positive) measure. Moreover, if $\mathcal{F}$ is a second local Dirichlet form with $\mathcal{E} \leq \mathcal{F}$ then 
\[ \int d\sigma^{(\mathcal{E}, \varphi_1, \ldots, \varphi_n)}_{ii} \chi \leq \int d\sigma^{(\mathcal{F}, \varphi_1, \ldots, \varphi_n)}_{ii} \chi \]
for all $\varphi_1, \ldots, \varphi_n \in D(\mathcal{F}) \cap L_\infty(X: R)$, $i \in \{1, \ldots, n\}$ and $\chi \in C_c(R^d)$ with $\chi \geq 0$.

**Proof** Theorem I.5.2.1 of [BoH], which elaborates a result of Andersson [And], establishes that there exist unique real Radon measures $\sigma^{(\varphi_1, \ldots, \varphi_n)}_{ij}$ such that $\sigma^{(\varphi_1, \ldots, \varphi_n)}_{ij} = \sigma^{(\varphi_1, \ldots, \varphi_n)}_{ji}$ for all $i, j \in \{1, \ldots, n\}$ and (5) is valid for all $F, G \in C^1_c(R^d)$. Moreover, $\sum_{i,j=1}^{n} \xi_i \xi_j \sigma^{(\varphi_1, \ldots, \varphi_n)}_{ij}$ is a (positive) measure for all $\xi \in R^n$. Let $\xi \in R^n$ and $\chi \in C^1_c(R^n)$. For all $\lambda > 0$ define $F_\lambda \in C^1_c(R^n)$ by $F_\lambda(x) = e^{\lambda x \cdot \xi} \chi(x)$. Then it follows from (5) that 
\[ \lim_{\lambda \to \infty} \lambda^{-2} \mathcal{E}(F_\lambda(\varphi_1, \ldots, \varphi_n)) = \int \sum_{i,j=1}^{n} \xi_i \xi_j d\sigma^{(\mathcal{E}, \varphi_1, \ldots, \varphi_n)}_{ij} |\chi|^2. \]
So if supp $\chi \subset K^c$ then the left hand side of (7) vanishes. Hence supp $\sigma_{ij}^{(\varphi_1,\ldots,\varphi_n)} \subseteq K$. But then (4) extends to all $F,G \in C^1(R^n)$. Moreover the measure $\sigma_{ii}^{(\varphi_1,\ldots,\varphi_n)}$ is finite for all $i \in \{1,\ldots,n\}$ since $\sigma_{ii}^{(\varphi_1,\ldots,\varphi_n)}$ is regular.

Finally, if $\mathcal{F}$ is a second local Dirichlet form with $\mathcal{E} \leq \mathcal{F}$ then it follows from (7), applied to $\mathcal{E}$ and $\mathcal{F}$, that

$$\int \sum_{i,j=1}^n \xi_i \xi_j d\sigma_{ij}^{(\mathcal{E},\varphi_1,\ldots,\varphi_n)} |\chi|^2 \leq \int \sum_{i,j=1}^n \xi_i \xi_j d\sigma_{ij}^{(\mathcal{F},\varphi_1,\ldots,\varphi_n)} |\chi|^2$$

for all $\chi \in C^1_c(R^n)$ and $\xi \in R^n$. Setting $\xi = e_i$ gives (6) by a density argument. 

Now we are prepared to prove the essential perturbation result for elements in $D(\mathcal{E}) \cap L_\infty(X:R)$.

**Proposition 3.2** If $\mathcal{E}$ is a local Dirichlet form then

$$\mathcal{E}(\varphi, \varphi) - \mathcal{E}(e^{-\psi} \varphi, e^\psi \varphi) = I^{(\mathcal{E})}(\varphi^2)$$

for all $\varphi, \psi \in D(\mathcal{E}) \cap L_\infty(X:R)$. Moreover, if $\mathcal{F}$ is a second local Dirichlet form with $\mathcal{E} \leq \mathcal{F}$ then

$$I^{(\mathcal{E})}(\varphi) \leq I^{(\mathcal{F})}(\varphi)$$

for all $\varphi, \psi \in D(\mathcal{F}) \cap L_\infty(X:R)$ with $\varphi \geq 0$.

**Proof** Observe that

$$\mathcal{E}(\varphi, \varphi) - \mathcal{E}(e^{-\psi} \varphi, e^\psi \varphi) = -\mathcal{E}((e^{-\psi} - 1)\varphi, (e^\psi - 1)\varphi)$$

$$- \mathcal{E}((e^{-\psi} - 1)\varphi, \varphi) - \mathcal{E}(\varphi, (e^\psi - 1)\varphi)$$

Now we can apply Proposition 3.1 with $n = 2$ to each term on the right hand side. First, one has

$$-\mathcal{E}((e^{-\psi} - 1)\varphi, (e^\psi - 1)\varphi) = \int d\sigma_{1,1}^{(\psi,\varphi)}(x_1, x_2) x_2^2$$

$$+ \int d\sigma_{1,2}^{(\psi,\varphi)}(x_1, x_2) (e^{-x_1}(e^x - 1) - e^{x_1}(e^{-x_1} - 1))x_2$$

$$- \int d\sigma_{2,2}^{(\psi,\varphi)}(x_1, x_2) (e^{x_1} - 1)$$

Secondly,

$$-\mathcal{E}((e^{-\psi} - 1)\varphi, \varphi) - \mathcal{E}(\varphi, (e^\psi - 1)\varphi) = \int d\sigma_{1,2}^{(\psi,\varphi)}(x_1, x_2) (e^{-x_1} - e^{x_1})x_2$$

$$- \int d\sigma_{2,2}^{(\psi,\varphi)}(x_1, x_2) ((e^{-x_1} - 1) + (e^{x_1} - 1))$$

Therefore, by addition,

$$\mathcal{E}(\varphi, \varphi) - \mathcal{E}(e^{-\psi} \varphi, e^\psi \varphi) = \int d\sigma_{1,1}^{(\psi,\varphi)}(x_1, x_2) x_2^2$$
Thirdly, two more applications of Proposition 3.1 give
\[ \mathcal{E}(\psi \varphi^2, \psi) = \int d\sigma_{1,1}^{(\psi,\varphi)}(x_1, x_2) x_1^2 + 2 \int d\sigma_{1,2}^{(\psi,\varphi)}(x_1, x_2) x_1 x_2 \]
and
\[ 2^{-1} \mathcal{E}(\psi^2, \varphi^2) = 2 \int d\sigma_{1,2}^{(\psi,\varphi)}(x_1, x_2) x_1 x_2. \]
Therefore, by subtraction,
\[ I_\psi(\varphi^2) = \mathcal{E}(\psi \varphi^2, \psi) - 2^{-1} \mathcal{E}(\psi^2, \varphi^2) = \int d\sigma_{1,1}^{(\psi,\varphi)}(x_1, x_2) x_2^2 \]
which gives the identity in the proposition.

Finally suppose \( \varphi \geq 0 \). Then one calculates similarly that
\[ I_\psi(\varphi) = \int d\sigma_{1,1}^{(\psi,\varphi)}(x_1, x_2) x_2. \]

But supp \( \sigma_{1,1}^{(\psi,\varphi)} \subseteq [-\|\psi\|_\infty, \|\varphi\|_\infty] \times [0, \|\varphi\|_\infty] \) by Proposition 3.1. Hence
\[ I_\psi(\varphi) = \int d\sigma_{1,1}^{(\psi,\varphi)}(x_1, x_2) (x_2 \vee 0). \]
Then the inequality follows immediately from the last part of Proposition 3.1. \( \square \)

The following lemma is useful.

**Lemma 3.3** Let \( \mathcal{E} \) be a local Dirichlet form.

I. If \( \psi_1, \psi_2, \varphi \in D(\mathcal{E}) \cap L_\infty(X: \mathbb{R}) \) with \( \varphi \geq 0 \) then
\[ I_{\psi_1 + \psi_2}(\varphi)^{1/2} \leq I_{\psi_1}(\varphi)^{1/2} + I_{\psi_2}(\varphi)^{1/2}. \]

II. If \( \chi, \psi, \varphi \in D(\mathcal{E}) \cap L_\infty(X: \mathbb{R}) \) then
\[ |I_{\chi \psi}(\varphi)|^{1/2} \leq I_{\psi}(\chi^2 |\varphi|)^{1/2} + I_{\chi}(\psi^2 |\varphi|)^{1/2}. \]

**Proof** For all \( \psi_1, \psi_2, \varphi \in D(\mathcal{E}) \cap L_\infty(X: \mathbb{R}) \) set
\[ I_{\psi_1, \psi_2}(\varphi) = 2^{-1} \mathcal{E}(\psi_1 \varphi, \psi_2) + 2^{-1} \mathcal{E}(\psi_2 \varphi, \psi_1) - 2^{-1} \mathcal{E}(\psi_1 \psi_2, \varphi). \]
Then
\[ (\psi_1, \varphi_1), (\psi_2, \varphi_2) \mapsto I_{\psi_1, \psi_2}(\varphi_1 \varphi_2) \]
is a positive symmetric bilinear form on \( (D(\mathcal{E}) \cap L_\infty(X: \mathbb{R}))^2 \) by \( \text{BoH} \) Proposition I.4.1.1. Hence Statement II follows by the corresponding norm triangle inequality applied to the vectors \((\psi_1, \varphi_1^{1/2})\) and \((\psi_2, \varphi_2^{1/2})\).

It follows as in the proof of Proposition 3.2 that the bilinear form satisfies the Leibniz rule
\[ I_{\psi_1, \psi_3, \psi_2}(\varphi) = I_{\psi_1, \psi_2}(\psi_3 \varphi) + I_{\psi_3, \psi_2}(\psi_1 \varphi). \]
for all $\psi_1, \psi_2, \psi_3, \varphi \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$. But the Cauchy–Schwarz inequality states that

$$|\mathcal{I}_{\psi_1, \psi_2}(\varphi_1 \varphi_2)| \leq \mathcal{I}_{\psi_1}(\varphi_1^2)^{1/2} \mathcal{I}_{\psi_2}(\varphi_2^2)^{1/2}$$

for all $\psi_1, \psi_2, \varphi_1, \varphi_2 \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$. Now let $\chi, \psi, \varphi \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$. Then

$$|\mathcal{I}_\chi(\varphi)| \leq \mathcal{I}_\chi(\varphi^2) = \mathcal{I}_\chi(\psi^2 |\varphi|) + 2 \mathcal{I}_\chi(\psi \varphi) + \mathcal{I}_\chi(\varphi^2)$$

$$\leq \left( \mathcal{I}_\chi(\chi^2 |\varphi|^2) + \mathcal{I}_\chi(\psi^2) |\varphi|^2 \right)^{1/2}$$

and Statement [II] follows.

We assume from now on in this and the next section that the Dirichlet form $\mathcal{E}$ is local, $\mathbf{1} \in D(\mathcal{E})_\text{loc}$ and $D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$ is a core for $\mathcal{E}$. These assumptions are valid if $X$ is a locally compact separable metric space and $\mu$ is a Radon measure such that $\text{supp } \mu = X$, the Dirichlet form is regular and $\text{[FOT]}$-strongly local.

**Lemma 3.4** If $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{E})_\text{loc} \cap L_\infty(X : \mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$ then $\psi \varphi \in D(\mathcal{E})$ and

$$\mathcal{E}(\psi \varphi)^{1/2} \leq |||\widehat{\mathcal{I}}_\psi|||^{1/2} \Vert \varphi \Vert_2 + \Vert \psi \Vert_\infty \mathcal{E}(\varphi)^{1/2}.$$  

**Proof** First assume that $\varphi \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$. Since $\mathbf{1} \in D(\mathcal{E})_\text{loc}$ there exists a $\chi \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$ such that $0 \leq \chi \leq 1$ and $\chi \text{supp } \varphi = 1$. Moreover, there is a $\hat{\psi} \in D(\mathcal{E})$ such that $\psi \text{supp } \varphi = \hat{\psi} \text{supp } \varphi$. We may assume that $\hat{\psi} \in L_\infty(X : \mathbb{R})$ and $\Vert \hat{\psi} \Vert_\infty \leq \Vert \psi \Vert_\infty$. Then it follows from locality and Lemma 3.3 [II] that

$$\mathcal{E}(\psi \varphi)^{1/2} = \mathcal{E}(\hat{\psi} \varphi)^{1/2} = \mathcal{I}_{\hat{\psi} \varphi}(\chi)^{1/2}$$

$$\leq \mathcal{I}_{\hat{\psi} \varphi}(\varphi^2 \chi)^{1/2} + \mathcal{I}_{\hat{\psi} \varphi}(\hat{\psi}^2 \chi)^{1/2} + \mathcal{I}_{\hat{\psi} \varphi}(\hat{\psi} \varphi^2 \chi)^{1/2}$$

$$\leq |||\widehat{\mathcal{I}}_\psi|||^{1/2} \Vert \varphi \Vert_2 + |||\widehat{\mathcal{I}}_\psi|||^{1/2} \Vert \varphi \Vert_2 + \Vert \psi \Vert_\infty \mathcal{E}(\varphi)^{1/2}.$$  

Now let $\varphi \in D(\mathcal{E})$. There exists a sequence $\varphi_1, \varphi_2, \ldots \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$ such that $\lim_{n \to \infty} \Vert \varphi - \varphi_n \Vert_2 = 0$ and $\lim_{n \to \infty} \mathcal{E}(\varphi - \varphi_n) = 0$. Then $\lim_{n \to \infty} \psi \varphi_n = \psi \varphi$ in $L_2(X)$. Moreover, it follows from the above estimates that $n \mapsto \psi \varphi_n$ is a Cauchy sequence in $D(\mathcal{E})$. Since $\mathcal{E}$ is closed one deduces that $\psi \varphi \in D(\mathcal{E})$ and the lemma is established.

**Corollary 3.5** If $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{E})_\text{loc} \cap L_\infty(X : \mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$ then $e^{\psi \varphi} \in D(\mathcal{E})$ and

$$\mathcal{E}(e^{\psi \varphi})^{1/2} \leq e^{\Vert \psi \Vert_\infty} \left( |||\widehat{\mathcal{I}}_\psi|||^{1/2} |||\varphi|||_2 + \mathcal{E}(\varphi)^{1/2} \right).$$  

**Proof** It follows by induction from Lemma 3.3 that

$$\mathcal{E}(e^{\psi \varphi})^{1/2} \leq n |||\widehat{\mathcal{I}}_\psi|||^{1/2} \Vert \psi \Vert_\infty^{n-1} \Vert \varphi \Vert_2 + \Vert \psi \Vert_\infty^n \mathcal{E}(\varphi)^{1/2}$$

for all $n \in \mathbb{N}$, $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{E})_\text{loc} \cap L_\infty(X : \mathbb{R})$ with $|||\widehat{\mathcal{I}}_\psi||| < \infty$. Since $\mathcal{E}(\tau)^{1/2} = \Vert H^{1/2} \tau \Vert_2$ for all $\tau \in D(\mathcal{E}) = D(\mathcal{E})$ and $H^{1/2}$ is self-adjoint it follows that $e^{\psi \varphi} \in D(\mathcal{E})$ and

$$\mathcal{E}(e^{\psi \varphi})^{1/2} \leq \sum_{n=0}^{\infty} n!^{-1} \left( n \Vert \widehat{\mathcal{I}}_\psi \Vert_2 \Vert \psi \Vert_\infty^{n-1} \Vert \varphi \Vert_2 + \Vert \psi \Vert_\infty^n \mathcal{E}(\varphi)^{1/2} \right)$$

$$= e^{\Vert \psi \Vert_\infty} \left( |||\widehat{\mathcal{I}}_\psi|||^{1/2} \Vert \varphi \Vert_2 + \mathcal{E}(\varphi)^{1/2} \right)$$

as required. 

□
Lemma 3.6 If $\psi \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R})$ with $|||^2 \lesssim < \infty$ then

$$-\mathcal{E}(e^{-\psi} \varphi, e^{\psi} \varphi) \leq |||^2 / 2$$

(8)

for all $\varphi \in D(\mathcal{E})$.

Proof It follows by definition together with Proposition 3.2 that

$$\mathcal{I}_\psi(\varphi^2) = \mathcal{E}(\varphi^2 \varphi, \psi) - 2^{-1} \mathcal{E}(\varphi^2, \psi^2) = \mathcal{E}(\varphi) - \mathcal{E}(e^{-\psi} \varphi, e^{\psi} \varphi)$$

for all $\varphi, \psi \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$. Therefore

$$-\mathcal{E}(e^{-\psi} \varphi, e^{\psi} \varphi) = -\mathcal{I}_\psi(\varphi^2) \leq \mathcal{I}_\psi(\varphi^2)$$

and (8) is valid for all $\psi \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R})$ and $\varphi \in D(\mathcal{E}) \cap L_\infty,\varepsilon(X : \mathbb{R})$. But if $\psi \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R})$ with $|||^2 < \infty$ then the maps $\varphi \mapsto e^{\psi} \varphi$ and $\varphi \mapsto e^{-\psi} \varphi$ are continuous from $D(\mathcal{E})$ into $D(\mathcal{E})$ by Corollary 3.5. Moreover, $D(\mathcal{E}) \cap L_\infty,\varepsilon(X)$ is dense in $D(\mathcal{E})$. Hence the bounds (8) extend to all $\varphi \in D(\mathcal{E})$ and $\psi \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R})$ with $|||^2 < \infty$.

Next for all $\psi \in L_\infty(X)$ define the multiplication operator $M_\psi : L_2(X) \rightarrow L_2(X)$ by $M_\psi \varphi = e^{\psi} \varphi$.

Proposition 3.7 If $\psi \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R})$ and $|||^2 < \infty$ then

$$\|M_\psi S_\varepsilon M_\psi^{-1}\|_{2 \rightarrow 2} \leq e^{|||^2 / 2}$$

for all $\varepsilon > 0$.

Proof Let $\varepsilon > 0$. It follows from Lemma 3.6 that

$$\|e^{\psi} S_\varepsilon \varphi\|^2 - \|e^{\psi} \varphi\|^2 = -2 \int_0^\varepsilon ds \mathcal{E}(S_\varepsilon \varphi, e^{2\psi} S_\varepsilon \varphi) \leq 2 \int_0^\varepsilon ds \|\mathcal{I}_\psi|||^2 \|e^{\psi} S_\varepsilon \varphi\|^2$$

for all $\varphi \in L_2(X)$. Then it follows from Gronwall’s lemma that

$$\|e^{\psi} S_\varepsilon \varphi\| \leq e^{|||^2 / 2} \|e^{\psi} \varphi\|$$

for all $\varepsilon > 0$.

Since $\psi \mapsto \mathcal{I}_\psi(\varphi)$ is a quadratic form one has $|||^2 \|\mathcal{I}_\rho \varphi\| = \rho^2 |||^2 \|\mathcal{I}_\rho \varphi\|$ for all $\rho \in \mathbb{R}$. It is now easy to complete the proof of the second theorem.

Proof of Theorem 1.2 Let $\psi \in D_0(\mathcal{E})$. Then $|||^2 \leq 1$ and

$$|(\varphi_A, S_\varepsilon \varphi_B)| \leq \|e^{-\rho_e} \varphi_A\| \|M_\rho \varphi S_\varepsilon M_\rho^{-1}\|_{2 \rightarrow 2} \|e^{\rho_e} \varphi_B\|$$

$$\leq \|\mathcal{I}_\rho e^{-\varepsilon \rho_e} \varphi_A\| \|\rho_e \varphi_B\| \leq \rho^2 \|e^{\rho_e} \varphi_B\|$$

for all $\rho > 0$. Minimizing over $\psi$ gives

$$|(\varphi_A, S_\varepsilon \varphi_B)| \leq e^\varepsilon \rho \rho^2 \|e^{\rho_e} \varphi_B\|$$

for all $\rho > 0$. If $d(A ; B) = \infty$ then $\langle \varphi_A, S_\varepsilon \varphi_B \rangle = 0$ and the theorem follows. Finally, if $d(A ; B) < \infty$ choose $\rho = (2\varepsilon)^{-1} d(A ; B)$.

One immediate corollary of the theorem is that the corresponding wave equation has a finite speed of propagation by the reasoning of [Sik].
Corollary 3.8 Let $H$ be the positive self-adjoint operator associated with a local Dirichlet form $E$ on $L_2(X)$ such that $1 \in D(E)_{loc}$ and $D(E) \cap L_{\infty,c}(X)$ is a core for $E$. If $A, B$ are measurable subsets of $X$ then

$$(\varphi_A, \cos(tH^{1/2})\varphi_B) = 0$$

for all $\varphi_A \in L_2(A)$, $\varphi_B \in L_2(B)$ and $t \in [-d(A;B), d(A;B)]$.

**Proof** This follows immediately from Theorem 1.2 and Lemma 3.3 of [ERSZ]. \qed

We have already remarked in the introduction that Theorem 1.2 applies directly to the second-order viscosity operators. Alternatively one may deduce off-diagonal bounds for operators on open subsets of $\mathbf{R}^d$ satisfying Dirichlet or Neumann boundary conditions. As an illustration consider the Laplacian.

**Example 3.9** Let $X$ be an open subset of $\mathbf{R}^d$. Define $E$ on $L_2(X)$ by $D(E) = W^{1,2}_0(X)$ and $E(\varphi) = \|\nabla \varphi\|_2^2$. In particular $D(E)$ is the closure of $C_c^\infty(X)$ with respect to the norm $\varphi \mapsto (\|\nabla \varphi\|_2^2 + \|\varphi\|_2^2)^{1/2}$. It follows that $E$ is a regular local Dirichlet form and the assumptions of Theorem 1.2 are satisfied. Therefore Theorem 1.2 gives off-diagonal bounds. The self-adjoint operator corresponding to $E$ is the Dirichlet Laplacian on $L_2(X)$.

**Example 3.10** Let $\Omega$ be an open subset of $\mathbf{R}^d$ with $|\partial\Omega| = 0$. Set $X = \overline{\Omega}$. Define the form $E$ on $L_2(X) = L_2(\Omega)$ by $D(E) = W^{1,2}(\Omega)$ and $E(\varphi) = \|\nabla \varphi\|_2^2$. Then $E$ is a local Dirichlet form. Fix $\tau \in C_c^\infty(\mathbf{R}^d)$ with $0 \leq \tau \leq 1$ and $\tau|_{B_1(0;1)} = 1$, where $B_1(0;1)$ is the Euclidean unit ball. For all $n \in \mathbf{N}$ define $\tau_n \in C_c^\infty(\mathbf{R}^d)$ by $\tau_n(x) = \tau(n^{-1}x)$. Then $\tau_n|_{\Omega} \in D(E)$ for all $n \in \mathbf{N}$ and therefore $1 \in D(E)_{loc}$. The space $D(E) \cap L_{\infty,c}(X)$ is dense in $D(E)$ by [FOT] Theorem 1.4.2.(iii). If $\varphi \in D(E) \cap L_{\infty,c}(X)$ then $\lim_{n \to \infty} \tau_n|_{\Omega} \varphi = \varphi$ in $W^{1,2}(\Omega) = D(E)$. But for all $n \in \mathbf{N}$ the support of the function $\tau_n|_{\Omega} \varphi$, viewed as an almost everywhere defined function on $X$, is closed in $X$, and therefore also closed in $\mathbf{R}^d$. Hence this support is compact in $\mathbf{R}^d$ and then also compact in $X$. So $\tau_n|_{\Omega} \varphi \in D(E) \cap L_{\infty,c}(X)$ and the space $D(E) \cap L_{\infty,c}(X)$ is dense in $D(E)$. Therefore Theorem 1.2 gives off-diagonal bounds. The self-adjoint operator corresponding to $E$ is the Neumann Laplacian on $L_2(\Omega)$.

Note that we do not assume that $\Omega$ has the segment property. In general the Dirichlet form $E$ is not regular on $X$.

4 Separation

In this section we give the proof of Theorem 1.3. In [ERSZ] we established that for degenerate elliptic operators phenomena of separation can occur. In particular the corresponding semigroup $S$ is reducible, i.e., it has non-trivial invariant subspaces. The theorem shows that such subspaces can be characterized by the set-theoretic distance $d(\cdot;\cdot)$.

For the proof of the implication $\mathbb{II} \Rightarrow \mathbb{I}$ in Theorem 1.3 we need a variation of an argument used to prove Theorem XIII.44 in [ReS] (see also [SimI]). The essence of the argument is contained in the following lemma.

**Lemma 4.1** Let $A, B$ be measurable subsets of $X$ with finite measure. If $(S_t1_A, 1_B) = 0$ for one $t > 0$ then $(S_t1_A, 1_B) = 0$ for all $t > 0$. 

11
Suppose that $A$ such that $\psi \in D(\mathcal{E})_{\text{loc}} \cap L^\infty(X : \mathbb{R})$ and $\lambda \in \mathbb{R}$ then $\psi + \lambda \mathbb{1} \in D(\mathcal{E})_{\text{loc}} \cap L^\infty(X : \mathbb{R})$ and $
abla_{\psi + \lambda \mathbb{1}} = \nabla_\psi$. In particular $|||\nabla_{\psi + \lambda \mathbb{1}}||| = |||\nabla_\psi|||$. 

II. If $\psi \in D(\mathcal{E})_{\text{loc}} \cap L^\infty(X : \mathbb{R})$ and $F$ is a normal contraction then $F \circ \psi \in D(\mathcal{E})_{\text{loc}} \cap L^\infty(X : \mathbb{R})$ and $|||\nabla_{F \circ \psi}||| \leq |||\nabla_\psi|||$. 

III. If $A, B$ are measurable subsets of $X$, $M \in [0, d(A ; B)] \cap \mathbb{R}$ and $\varepsilon > 0$ then there is a $\psi \in D_0(\mathcal{E})$ such that $0 \leq \psi \leq M$, $\psi(b) = 0$ for a.e. $b \in B$ and $\psi(a) \geq M - \varepsilon$ for a.e. $a \in A$. 

Proof Let $\varphi \in D(\mathcal{E}) \cap L^\infty_{\text{loc}, c}(X : \mathbb{R})$. There exist $\hat{\psi}, \hat{\chi} \in D(\mathcal{E}) \cap L^\infty(X : \mathbb{R})$ such that $\hat{\psi}|_{\text{supp } \varphi} = \varphi|_{\text{supp } \varphi}$ and $\hat{\chi}|_{\text{supp } \varphi} = 1$. Then the locality of $\mathcal{E}$ implies that 

\[
\nabla_{\psi + \lambda \chi}(\varphi) = \nabla_{\hat{\psi} + \lambda \chi}(\varphi) = \mathcal{E}((\hat{\psi} + \lambda \chi)\varphi, \hat{\psi} + \lambda \chi) - 2^{-1}\mathcal{E}(\varphi, (\hat{\psi} + \lambda \chi)^2) = \mathcal{E}(\varphi, \hat{\psi}) + \lambda \mathcal{E}(\varphi, \hat{\psi}) - 2^{-1}\left(\mathcal{E}(\varphi, \hat{\psi}^2) - 2\mathcal{E}(\varphi, \lambda \hat{\psi} \chi)\right) = \mathcal{E}(\varphi, \hat{\psi}) - 2^{-1}\mathcal{E}(\varphi, \hat{\psi}^2) = \nabla_\varphi(\varphi) = \nabla_{\hat{\psi}}(\varphi).
\]

This proves Statement I. 

If $F$ is a normal contraction on $\mathbb{R}$ then $\nabla_{F \circ \psi}(\varphi) \leq \nabla_\psi(\varphi)$ for all $\varphi, \psi \in D(\mathcal{E}) \cap L^\infty(X : \mathbb{R})$ with $\varphi \geq 0$ by [BoH] Proposition I.4.1.1. Then Statement II is an easy consequence. 

Finally, there exists a $\psi \in D_0(\mathcal{E})$ such that $d_\psi(A ; B) \geq M - \varepsilon$. We may assume that $|B| > 0$. Then $0 \lor (\psi + |||\mathbb{1} \psi|||_\infty) \land M$ satisfies the required conditions. 

Proof of Theorem 1.3 The implication II$\Rightarrow$III is now immediate. Let $B \subset A^c$, $A' \subset A$ and suppose that $A'$ and $B$ have finite measure. Then $(S_t \mathbb{1}_{A'}, \mathbb{1}_B) = 0$ for one $t > 0$ by assumption and for all $t > 0$ by Lemma 4.1. Hence $(S_t \varphi, \psi) = 0$ for all $t > 0$, $\varphi \in L^2(A)$ and $\psi \in L^2(A^c)$. Thus $S_tL_2(A) \subseteq L_2(A)$ for all $t > 0$. 

The converse implication III$\Rightarrow$I is obvious. 

Next we prove the implication III$\Rightarrow$I. For all $\varphi : X \to \mathbb{C}$ set $\bar{\varphi} = \mathbb{1}_A \varphi$. Then $\bar{\varphi} \in D(\mathcal{E})$ and $\mathcal{E}(\psi, \bar{\varphi}) = \mathcal{E}(\hat{\psi}, \varphi) = \mathcal{E}(\tilde{\psi}, \bar{\varphi})$ for all $\varphi, \psi \in D(\mathcal{E})$ by [ERSZ], Lemma 6.3. Hence
\[ \mathcal{I}_{\tilde{\psi}}(\varphi) = \mathcal{E}(\tilde{\psi}, \tilde{\psi}) - 2^{-1} \mathcal{E}(\varphi, \tilde{\psi}^2) = \mathcal{I}_{\psi}(\tilde{\varphi}) \] for all \( \varphi, \psi \in D(\mathcal{E}) \cap L_\infty(X : \mathbb{R}) \). Therefore, if \( \psi \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R}) \) and \( \varphi \in D(\mathcal{E}) \cap L_\infty,c(X : \mathbb{R}) \) then \( \tilde{\psi} \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R}) \) and \( \mathcal{I}_{\tilde{\psi}}(\varphi) = \mathcal{I}_{\psi}(\tilde{\varphi}) \). But \( \mathcal{I}_{\tilde{\varphi}} = 0 \) by Lemma 4.2.II since \( \mathcal{E} \) is local. So \( \mathcal{I}_{\tilde{\varphi}}(\varphi) = \mathcal{I}_{\varphi}(\tilde{\varphi}) = 0 \) for all \( \varphi \in D(\mathcal{E}) \cap L_\infty,c(X : \mathbb{R}) \). So \( ||| \mathcal{I}_{\tilde{\varphi}} ||| = 0 \). Then \( d(A; A^c) = \infty \).

The implication \( \text{III} \Rightarrow \text{IV} \) is trivial.

Finally suppose that \( \text{IV} \) is valid. Let \( \delta \in (0, \infty) \) such that \( 2\delta < d(A; A^c) \). By Lemma 4.2.III there exists a \( \psi \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R}) \) such that \( \psi(b) = 0 \) for a.e. \( b \in A^c \) and \( \psi(a) \geq \delta \) for a.e. \( a \in A \). Then \( 1_A = 0 \vee (\delta^{-1} \psi_2) \wedge 1 \) and since \( x \mapsto 0 \vee x \wedge 1 \) is a normal contraction it follows from Lemma 4.2.III that \( 1_A \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R}) \) and ||| \( \mathcal{I}_{\tilde{1}_A} ||| \leq \delta^{-2} < \infty \).

Now let \( \varphi \in D(\mathcal{E}) \). It follows from Lemma 3.4 that \( 1_A \varphi \in D(\mathcal{E}) \). Then also \( 1_{A^c} \varphi \in D(\mathcal{E}) \). For all \( t > 0 \) one has by Theorem 4.2 that

\[
|t^{-1} (1_A \varphi, (I - S_t)(1_{A^c} \varphi))| = t^{-1} |(1_A \varphi, S_t(1_{A^c} \varphi))| \leq t^{-1} e^{-\delta^2 t^{-1}} \|1_A \varphi\|_2 \|1_{A^c} \varphi\|_2 .
\]

Hence

\[
\mathcal{E}(1_A \varphi, 1_{A^c} \varphi) = \lim_{t \downarrow 0} t^{-1} (1_A \varphi, (I - S_t)(1_{A^c} \varphi)) = 0
\]

and

\[
\mathcal{E}(\varphi) = \mathcal{E}(1_A \varphi) + \mathcal{E}(1_{A^c} \varphi) .
\]

Then \text{ERSZ}, Lemma 6.3, implies that \( \text{II} \) is valid. \( \square \)

Note that the above proof shows that the equivalent statements of Theorem 1.3 are also equivalent with the statement \( 1_A \in D(\mathcal{E})_{\text{loc}} \cap L_\infty(X : \mathbb{R}) \) and ||| \( \mathcal{I}_{\tilde{1}_A} ||| < \infty \).

**Example 4.3** Let \( \delta \in [0, 1) \) and consider the viscosity form \( h \) on \( \mathbb{R}^d \) with \( d = 1 \) and coefficient \( c_{11} = c_\delta \) where

\[
c_\delta(x) = \left( \frac{x^2}{1 + x^2} \right)^\delta .
\]

Then \( h(\varphi) = \int |\varphi'(x)|^2 c_\delta(x) \) for all \( \varphi \in C_\infty^c(\mathbb{R}) \), the form \( h \) is closable and its closure is a Dirichlet form. If \( S \) is the semigroup generated by the operator associated with the closure and if \( \delta \in [1/2, 1) \) then \( S_t L_2(-\infty, 0) \subseteq L_2(-\infty, 0) \) and \( S_t L_2(0, \infty) \subseteq L_2(0, \infty) \) for all \( t > 0 \) by \text{ERSZ}, Corollary 2.4 and Proposition 6.5. The assumptions of Theorems 1.2 and 1.3 are satisfied and \( d(A; B) = \infty \) for each pair of measurable subsets \( A \subseteq (-\infty, 0) \) and \( B \subseteq (0, \infty) \). Note, however, that the corresponding Riemannian distance

\[
d(x; y) = \left| \int_y^x ds c_\delta(s)^{-1/2} \right|
\]

is finite for all \( x, y \in \mathbb{R} \). Therefore the Riemannian distance does not reflect the behaviour of the semigroup.

## 5 Distances

In this section we derive various general properties of the distance \( d(\mathcal{E})(\cdot; \cdot) \) and give several examples.
Let $\mathcal{E}$ be a local Dirichlet form on $L_2(X)$. If $A_1 \subseteq A_2$ and $B$ are measurable then $d_\psi(A_1; B) \geq d_\psi(A_2; B)$ for all $\psi \in L_\infty(X)$. Hence $d(A_1; B) \geq d(A_2; B)$. Also $d_\psi(B; A) = d_\psi(A; B)$ for all $\psi \in L_\infty(X)$. So $d(A; B) = d(B; A)$ and $d(A; B_1) \geq d(A; B_2)$ whenever $B_1 \subseteq B_2$ are measurable. Next we consider monotonicity of the distance as a function of the form.

If $h_1$ and $h_2$ are two strongly elliptic forms on $\mathbb{R}^d$ and $h_1 \geq h_2$ then the corresponding matrices of coefficients satisfy $C^{(1)} \geq C^{(2)}$. Therefore $(C^{(1)})^{-1} \leq (C^{(2)})^{-1}$. Hence the corresponding Riemannian distances satisfy $d_1(x; y) \leq d_2(x; y)$ for all $x, y \in \mathbb{R}^d$. Thus the order of the forms gives the inverse order for the distances. One can establish a similar result for general Dirichlet forms and the set-theoretic distance under subsidiary regularity conditions.

**Proposition 5.1** Let $\mathcal{E}$ and $\mathcal{F}$ be local Dirichlet forms with $\mathcal{E} \leq \mathcal{F}$. Assume $1 \in D(\mathcal{F})_{\text{loc}}$, the space $D(\mathcal{F}) \cap L_\infty,c(X)$ is a core for $D(\mathcal{E})$ and for each compact $K \subset X$ there is a $\chi \in D(\mathcal{F}) \cap L_\infty,c(X:\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi|_K = 1$ and $|||\widehat{\mathcal{I}_\chi(\mathcal{F})}||| < \infty$. Then

$$d(\mathcal{F})(A; B) \leq d(\mathcal{E})(A; B)$$

for all measurable $A, B \subseteq X$.

**Proof** It suffices to prove that $|||\widehat{\mathcal{I}_\psi(\mathcal{E})}||| \leq |||\widehat{\mathcal{I}_\psi(\mathcal{F})}|||$ for each $\psi \in D(\mathcal{F})_{\text{loc}} \cap L_\infty(X:\mathbb{R})$ because the statement of the proposition then follows from the definition of the distance.

If $\psi, \varphi \in D(\mathcal{F}) \cap L_\infty(X:\mathbb{R})$ with $\varphi \geq 0$ then $\mathcal{I}_\psi(\mathcal{E}) \leq \mathcal{I}_\psi(\mathcal{F})$ by Proposition 3.2. Therefore $\widehat{\mathcal{I}_\psi(\mathcal{E})}(\varphi) \leq \widehat{\mathcal{I}_\psi(\mathcal{F})}(\varphi)$ for all $\psi \in D(\mathcal{F})_{\text{loc}} \cap L_\infty(X:\mathbb{R})$ and $\varphi \in D(\mathcal{F}) \cap L_\infty,c(X:\mathbb{R})$ with $\varphi \geq 0$. Fix $\psi \in D(\mathcal{F})_{\text{loc}} \cap L_\infty(X:\mathbb{R})$.

If $|||\widehat{\mathcal{I}_\psi(\mathcal{F})}||| = \infty$ there is nothing to prove, so we may assume that $|||\widehat{\mathcal{I}_\psi(\mathcal{F})}||| < \infty$.

It follows as in the proof of Lemma 3.3 that

$$\mathcal{E}(\psi \varphi)^{1/2} \leq \widehat{\mathcal{I}_\psi(\mathcal{E})}(\varphi^2)^{1/2} + |||\mathcal{F}||| \mathcal{E}(\varphi)^{1/2}$$

$$\leq \widehat{\mathcal{I}_\psi(\mathcal{F})}(\varphi^2)^{1/2} + |||\mathcal{F}||| \mathcal{E}(\varphi)^{1/2} \leq |||\widehat{\mathcal{I}_\psi(\mathcal{F})}||| \varphi||_2 + |||\varphi|||_\infty \mathcal{E}(\varphi)^{1/2}$$

for all $\varphi \in D(\mathcal{F}) \cap L_\infty,c(X:\mathbb{R})$. Since $\mathcal{E}$ is closed and $D(\mathcal{F}) \cap L_\infty,c(X)$ is a core it then follows that $\psi \varphi \in D(\mathcal{E})$ for all $\varphi \in D(\mathcal{E})$ and

$$\mathcal{E}(\psi \varphi)^{1/2} \leq |||\widehat{\mathcal{I}_\psi(\mathcal{F})}||| \varphi||_2 + |||\varphi|||_\infty \mathcal{E}(\varphi)^{1/2}$$

for all $\varphi \in D(\mathcal{E})$.

Let $\varphi_0 \in D(\mathcal{E}) \cap L_\infty,c(X:\mathbb{R})$. By assumption there exists a $\chi \in D(\mathcal{F}) \cap L_\infty,c(X:\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi|_{\text{supp } \varphi_0} = 1$ and $|||\widehat{\mathcal{I}_\chi(\mathcal{F})}||| < \infty$. Then by the above argument with $\psi$ replaced by $\chi$ one deduces that

$$\mathcal{E}(\chi \varphi)^{1/2} \leq |||\widehat{\mathcal{I}_\chi(\mathcal{F})}||| \varphi||_2 + \mathcal{E}(\varphi)^{1/2}$$

for all $\varphi \in D(\mathcal{E})$. There exists a $\hat{\psi} \in D(\mathcal{E}) \cap L_\infty(X:\mathbb{R})$ such that $\hat{\psi}|_{\text{supp } \chi} = \psi|_{\text{supp } \chi}$. Then there is a $c > 0$ such that

$$|||\widehat{\mathcal{I}_\psi(\mathcal{E})(\chi \varphi)}||| = |||\mathcal{I}_\psi(\mathcal{E})(\chi \varphi)\mathcal{E}(\hat{\psi})^{1/2} \mathcal{E}(\chi \varphi)^{1/2} + 2^{-1}\mathcal{E}(\hat{\psi})^{1/2} \mathcal{E}(\chi \varphi)^{1/2}$$


uniformly for all \( \varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X: \mathbb{R}) \). Since the space \( D(\mathcal{F}) \cap L_{\infty,c}(X: \mathbb{R}) \) is dense in \( D(\mathcal{E}) \) there are \( \varphi_1, \varphi_2, \ldots \in D(\mathcal{F}) \cap L_{\infty,c}(X: \mathbb{R}) \) such that \( \lim_{n \to \infty}(\|\varphi_0 - \varphi_n\|_2 + \mathcal{E}(\varphi_0 - \varphi_n)) = 0 \). Then

\[
\lim_{n \to \infty} I^{(E)}_\psi(\chi \varphi_n) = I^{(E)}_\psi(\chi \varphi_0) = I^{(E)}_\psi(\varphi_0).
\]

On the other hand,

\[
|I^{(E)}_\psi(\chi \varphi_n)| \leq \|I^{(E)}_\psi\| \|\chi \varphi_n\|_1 \leq \|I^{(E)}_\psi\| \|\varphi_0\|_1 + \|I^{(E)}_\psi\| \|\chi\|_2 \|\varphi_0 - \varphi_n\|_2
\]

for all \( n \in \mathbb{N} \). Taking the limit \( n \to \infty \) gives \( |I^{(E)}_\psi(\varphi_0)| \leq \|I^{(E)}_\psi\| \|\varphi_0\|_1 \). The required monotonicity of the norms is immediate. \( \square \)

One can apply the proposition to a general elliptic form \( h \) as in (11) to obtain a lower bound on the distance. Then the viscosity form \( h_0 \) satisfies \( h_0 \leq h \leq \|C\| l \) where \( l \) is the form associated with the Laplacian. Therefore

\[
d^{(h_0)}(A; B) \geq \|C\|^{-1/2} d^{(l)}(A; B)
\]

for all measurable \( A, B \subseteq \mathbb{R}^d \). Alternatively if \( h \) is a strongly elliptic form then there is a \( \mu > 0 \) such that \( C \geq \mu I \) and \( \mu l \leq h \leq \|C\| l \). It then follows that \( d^{(h)}(\cdot; \cdot) \) is equivalent to the distance \( d^{(l)}(\cdot; \cdot) \). Specifically,

\[
\|C\|^{-1/2} d^{(l)}(A; B) \leq d^{(h)}(A; B) \leq \mu^{-1/2} d^{(l)}(A; B)
\]

for all measurable sets \( A \) and \( B \).

In the case of the Laplacian one can explicitly identify the distance.

**Example 5.2** If \( A \) and \( B \) are non-empty open subsets of \( \mathbb{R}^d \) then

\[
d^{(l)}(A; B) = \inf_{x \in A} \inf_{y \in B} |x - y|.
\]

First, observe that if \( \varphi, \psi \in D(l) \cap L_\infty(\mathbb{R}^d: \mathbb{R}) = W^{1,2} \cap L_\infty(\mathbb{R}^d: \mathbb{R}) \) then

\[
I^{(l)}_\psi(\varphi) = (\nabla(\varphi \psi), \nabla \psi) - 2 \langle \nabla \varphi, \nabla (\psi^2) \rangle = \int_{\mathbb{R}^d} dx \varphi(x) |(\nabla \psi)(x)|^2.
\]

Secondly, let \( \psi \in W^{1,2} \cap L_\infty(\mathbb{R}^d: \mathbb{R}) \) and \( K \subset \mathbb{R}^d \) compact. Choose \( \hat{\psi} \in W^{1,2} \cap L_\infty \) such that \( \psi|_K = \hat{\psi}|_K \). If \( \|I^{(l)}_\psi\| \leq 1 \) then

\[
\left| \int_K dx \varphi(x) |(\nabla \psi)(x)|^2 \right| = \left| \int_K dx \varphi(x) |(\nabla \hat{\psi})(x)|^2 \right| = |I^{(l)}_\psi(\varphi)| = |I^{(l)}_\psi(\varphi)| \leq \|\varphi\|_1
\]

for all \( \varphi \in W^{1,2} \cap L_\infty(\mathbb{R}^d: \mathbb{R}) \cap L_1 \) with \( \text{supp} \varphi \subseteq K \). Therefore \( \sup_{x \in K} |(\nabla \psi)(x)| \leq 1 \) uniformly for all \( K \) and \( \psi \in W^{1,\infty} \). Thus \( D_0(l) = \{\psi \in W^{1,\infty}(\mathbb{R}^d: \mathbb{R}) : \|\nabla \psi\|_\infty \leq 1\} \). But it is well known that

\[
|x - y| = \sup\{|\psi(x) - \psi(y)| : W^{1,\infty}(\mathbb{R}^d: \mathbb{R}), \|\nabla \psi\|_\infty \leq 1\}
\]

(see, for example, [JeS2], Proposition 3.1). Then (9) follows immediately.
The next proposition compares forms under a rather different regularity assumption. We define \( D(\mathcal{F}) \) to be an ideal of \( D(\mathcal{E}) \) if \( D(\mathcal{F}) \subseteq D(\mathcal{E}) \) and

\[
\left( D(\mathcal{F})_{\text{loc}} \cap L_\infty(X; \mathbb{R}) \right) \left( D(\mathcal{E}) \cap L_{\infty,c}(X; \mathbb{R}) \right) \subseteq \left( D(\mathcal{F})_{\text{loc}} \cap L_\infty(X; \mathbb{R}) \right).
\]

Note that if \( 1 \in D(\mathcal{F})_{\text{loc}} \) and if \( D(\mathcal{F}) \) is an ideal of \( D(\mathcal{E}) \) then

\[
\psi = 1 \psi \in \left( D(\mathcal{F})_{\text{loc}} \cap L_\infty(X; \mathbb{R}) \right) \left( D(\mathcal{E}) \cap L_{\infty,c}(X; \mathbb{R}) \right) \subseteq \left( D(\mathcal{F})_{\text{loc}} \cap L_{\infty,c}(X; \mathbb{R}) \right)
\]

for all \( \psi \in D(\mathcal{E}) \cap L_{\infty,c}(X; \mathbb{R}) \). So \( D(\mathcal{F})_{\text{loc}} \cap L_{\infty,c}(X; \mathbb{R}) = D(\mathcal{E})_{\text{loc}} \cap L_{\infty,c}(X; \mathbb{R}) \).

**Proposition 5.3** Let \( \mathcal{E} \) and \( \mathcal{F} \) be local Dirichlet forms such that \( \mathcal{E} \leq \mathcal{F} \). Assume \( 1 \in D(\mathcal{F})_{\text{loc}} \), the space \( D(\mathcal{F}) \) is an ideal of \( D(\mathcal{E}) \) and \( D(\mathcal{F}) \cap L_{\infty,c}(X) \) is dense in \( L_1 \). Then

\[
d(\mathcal{F})(A; B) \leq d(\mathcal{E})(A; B)
\]

for all measurable \( A, B \subset X \).

**Proof** It follows from the definition of the distance and Lemma 4.2 that it suffices to prove that \( ||| \widehat{\mathcal{T}}(\psi) ||| \leq ||| \widehat{\mathcal{T}}(\mathcal{E}) ||| \) for all positive \( \psi \in D(\mathcal{F})_{\text{loc}} \cap L_\infty(X; \mathbb{R}) \).

First, one has again \( \widehat{\mathcal{T}}(\psi)(\varphi) \leq \widehat{\mathcal{T}}(\mathcal{E})(\varphi) \) for all \( \psi \in D(\mathcal{F})_{\text{loc}} \cap L_\infty(X; \mathbb{R}) \) and \( \varphi \in D(\mathcal{F}) \cap L_{\infty,c}(X; \mathbb{R}) \) with \( \varphi \geq 0 \).

Fix a positive \( \psi \in D(\mathcal{F})_{\text{loc}} \cap L_\infty(X; \mathbb{R}) \). If \( ||| \widehat{\mathcal{T}}(\mathcal{E}) ||| \) = \( \infty \) there is nothing to prove. So we may assume \( ||| \widehat{\mathcal{T}}(\mathcal{E}) ||| < \infty \).

Since \( 1 \in D(\mathcal{F})_{\text{loc}} \) it follows that \( 1 \in D(\mathcal{E})_{\text{loc}} \). It then follows from Lemma 4.2 that

\[
\widehat{\mathcal{T}}(\psi)(\varphi) = \widehat{\mathcal{T}}(\mathcal{E})(\varphi) = 4 \widehat{\mathcal{T}}(\mathcal{E})(1 + \psi)(\varphi)
\]

for all \( \varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X; \mathbb{R}) \) by the Leibniz rule. But \( (1 + \psi) \varphi \in D(\mathcal{F})_{\text{loc}} \cap L_{\infty,c}(X; \mathbb{R}) \) by the ideal property. Therefore

\[
\widehat{\mathcal{T}}(\psi)(\varphi) \leq 4 \widehat{\mathcal{T}}(\mathcal{E})(1 + \psi)(\varphi)
\]

for all positive \( \varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X; \mathbb{R}) \). Hence

\[
||| \widehat{\mathcal{T}}(\psi) ||| \leq 4 (1 + ||| \psi ||| \infty) ||| \widehat{\mathcal{T}}(\mathcal{E})(1 + \psi)(1/2) ||| \leq 4 (1 + ||| \psi ||| \infty) ||| \widehat{\mathcal{T}}(\mathcal{E}) |||
\]

where the last inequality follows from Lemma 4.2. In particular \( ||| \widehat{\mathcal{T}}(\psi) ||| < \infty \).

Finally, since \( ||| \widehat{\mathcal{T}}(\psi) ||| < \infty \) one has

\[
|\widehat{\mathcal{T}}(\psi)(\varphi)| \leq \widehat{\mathcal{T}}(\psi)(||| \varphi |||) \leq ||| \widehat{\mathcal{T}}(\psi) ||| ||| \varphi |||_1
\]

for all \( \varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X; \mathbb{R}) \). But then there is a \( \Gamma \in L_\infty \) such that \( ||| \Gamma |||_\infty = ||| \widehat{\mathcal{T}}(\psi) ||| \) and

\[
\widehat{\mathcal{T}}(\psi)(\varphi) = \int \Gamma \varphi
\]
for all \( \varphi \in D(\mathcal{E}) \cap L_{\infty,c}(X; \mathbb{R}) \). Therefore

\[
\left| \int \Gamma \varphi \right| = |\tilde{T}_\psi^{(\mathcal{E})}(\varphi)| \leq |\tilde{T}_\psi^{(\mathcal{F})}(\varphi)| \leq |\tilde{T}_\psi^{(\mathcal{F})}(\varphi)| \leq \|\tilde{T}_\psi^{(\mathcal{F})}\| \|\varphi\|_1
\]

for all \( \varphi \in D(\mathcal{F}) \cap L_{\infty,c}(X; \mathbb{R}) \). Since the latter space is dense in \( L_1(X; \mathbb{R}) \) one deduces that \( \|\Gamma\|_\infty \leq \|\tilde{T}_\psi^{(\mathcal{F})}\| \). Hence \( \|\tilde{T}_\psi^{(\mathcal{E})}\| \leq \|\tilde{T}_\psi^{(\mathcal{F})}\| \). The required monotonicity of the distances is immediate. \( \square \)

The proposition allows comparison of elliptic operators with different boundary conditions.

**Example 5.4** Let \( X \) be an open subset of \( \mathbb{R}^d \). Define the forms of the Neumann and Dirichlet Laplacians on \( L_2(X) \) by \( h_N(\varphi) = \|\nabla \varphi\|_2^2 \) with \( D(h_N) = W^{1,2}(X) \) and \( h_D(\varphi) = \|\nabla \varphi\|_2^2 \) with \( D(h_D) = W^{1,2}_0(X) \). Then \( h_N \leq h_D, 1 \in D(h_D)_\text{loc} \), \( D(h_D) \) is an ideal of \( D(h_N) \) and \( D(h_D) \cap L_{\infty,c}(X) \) is dense in \( L_1(X) \). Therefore one deduces from Proposition 5.3 that \( d^{(h_D)}(A;B) \leq d^{(h_N)}(A;B) \) for all measurable \( A, B \subseteq X \). Note that Proposition 5.1 does not apply to this example since \( W^{1,2}_0(X) \cap L_{\infty,c}(X) \) is not a core for \( D(h_N) \).

The argument used in Example 5.2 allows one to identify the distances associated with \( h_N \) and \( h_D \) with the geodesic distance in the Euclidean metric. In particular the distance is independent of the boundary conditions.

**Example 5.5** Let \( X \) be an open subset of \( \mathbb{R}^d \) and \( A, B \) non-empty open subsets of \( X \). If \( X \) is disconnected then \( d^{(h_N)}(A;B) = \infty = d^{(h_D)}(A;B) \) whenever \( A \) and \( B \) are in separate components by Theorem 1.3. Hence we may assume \( X \) is connected.

Next if \( \psi, \varphi \in W^{1,2}(X) \cap L_\infty(X; \mathbb{R}) \) then by direct calculation

\[
I^{(h_N)}_{\psi}(\varphi) = (\nabla(\varphi \psi), \nabla \psi) - 2^{-1}(\nabla \varphi, \nabla \psi^2) = \int_X dx \varphi(x) |(\nabla \psi(x)|^2.
\]

Therefore if \( \psi \in W^{1,2}(X)_{\text{loc}} \cap L_\infty(X; \mathbb{R}) \) and \( \|\tilde{T}_\psi^{(h_N)}\| \leq 1 \) one finds as in Example 5.2 that \( \psi \in W^{1,\infty}(X; \mathbb{R}) \) and \( \|\nabla \psi\|_\infty \leq 1 \). Hence \( d^{(h_N)}(A;B) = \inf_{x \in A, y \in B} d(x; y) \) where

\[
d(x; y) = \sup \{|\psi(x) - \psi(y)| : W^{1,\infty}(X; \mathbb{R}), \|\nabla \psi\|_\infty \leq 1\}.
\]

But this is the geodesic distance, with the usual Euclidean metric, from \( x \) to \( y \). A similar conclusion follows for \( d^{(h_D)} \) by replacing \( W^{1,2} \) by \( W^{1,2}_0 \) in the argument. This replacement does not affect the identification with the geodesic distance. Therefore in both cases the set-theoretic distance between the sets is the geodesic distance in \( X \) equipped with the Euclidean Riemannian metric.

The definition of the distance in terms of the space \( D(\mathcal{E})_{\text{loc}} \) gives good off-diagonal bounds but it is somewhat complicated. One could ask whether it has any simpler characterization in terms of \( D(\mathcal{E}) \). One obvious choice is to set

\[
d'_1(A;B) = \sup \{d_{\psi}(A;B) : \psi \in D_1(\mathcal{E})\},
\]

for all measurable \( A, B \subseteq X \) with \( \overline{A}, \overline{B} \) compact, where

\[
D_1(\mathcal{E}) = \{\psi \in D(\mathcal{E}) \cap L_\infty(X; \mathbb{R}) : |||\tilde{T}_\psi||| \leq 1\}.
\]
Then one sets
\[
d_1^E(A;B) = d_1(A;B) = \inf \{ d'_1(A_0;B_0) : A_0 \subseteq A, \ B_0 \subseteq B \text{ and } \overline{A_0}, \overline{B_0} \text{ are compact} \}
\]
for all measurable \(A, B \subseteq X\). Note that \(d_1(A;B) = d'_1(A;B)\) if \(\overline{A}, \overline{B}\) are compact. It follows that in general \(d^E(A;B) \geq d_1^E(A;B)\) but one can have a strict inequality.

**Example 5.6** Let \(E\) be the form associated with the second-order operator \(-d^2/dx^2\) with Dirichlet boundary conditions on the interval \((0,1)\). If \(A = (0,a)\) and \(B = (b,1)\) with \(0 < a < b < 1\) then the boundary conditions ensure that \(d_\psi(A;B) \leq 2\varepsilon\) for all \(\psi \in D(E) \cap L_\infty((0,1);\mathbb{R})\) and small \(\varepsilon > 0\), where \(A_\varepsilon = (\varepsilon,a)\) and \(B_\varepsilon = (b,1-\varepsilon)\). Therefore \(d_1(A;B) = 0\). But if \(\psi(x) = 0 \vee (x-a) \wedge (b-a)\) then \(\psi \in D_0(E)\) and \(d(A;B) \geq d_\psi(A;B) = b - a\).

The two distances are, however, often equal.

**Proposition 5.7** Let \(E\) be a local Dirichlet form on \(L_2(X)\). Suppose that for all compact \(K\) and \(\varepsilon > 0\) there exists a \(\chi \in D(E) \cap L_{\infty,\varepsilon}(X;\mathbb{R})\) such that \(0 \leq \chi \leq 1, \chi|_K = 1\) and \(|||\hat{I}_\chi||| < \varepsilon\). Then \(d(A;B) = d_1(A;B)\) for all measurable \(A, B \subseteq X\) with \(\overline{A}, \overline{B}\) compact.

**Proof** We have to show that \(d(A;B) \leq d_1(A;B)\). Let \(M \in [0,\infty)\) and suppose that \(M \leq d(A;B)\). Let \(\varepsilon \in (0,1]\). By Lemma 4.2.11 there exists a \(\psi \in D_0(E)\) such that \(d_\psi(A;B) \geq M - \varepsilon\) and \(|||\hat{I}_\psi||| \leq M\). By assumption there exists a \(\chi \in D(E) \cap L_{\infty,\varepsilon}(X;\mathbb{R})\) such that \(0 \leq \chi \leq 1, \chi|_{\overline{A} \cup \overline{B}} = 1\) and \(|||\hat{I}_\chi||| < \varepsilon^2\). Then \(\chi \psi \in D(E) \cap L_{\infty}(X;\mathbb{R})\) and it follows from Lemma 3.3.31 that
\[
|||\hat{I}_{\chi \psi}||| \leq (1 + \delta) |||\chi|||^2 \sum |||\hat{I}_\psi||| + (1 + \delta^{-1}) |||\psi|||^2 \sum |||\hat{I}_\chi|||\]
for all \(\delta > 0\). Choosing \(\delta = \varepsilon\) gives
\[
|||\hat{I}_{\chi \psi}||| \leq (1 + \varepsilon) + (1 + \varepsilon^{-1})M^2 \varepsilon^2 \leq 1 + \varepsilon(1 + 2M^2) .
\]
But \(d_{\chi \psi}(A;B) = d_\psi(A;B) \geq M - \varepsilon\). So \(d_1(A;B) \geq (1 + \varepsilon(1 + 2M^2))^{-1}(M - \varepsilon)\). Hence \(d_1(A;B) \geq M\) and the proposition follows. \(\square\)

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**References**


