Foundations of and Applications for the Abstract Boundary Construction for Space-Time

Benjamin Edward Whale

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I certify that the work contained in this thesis is my own original research, produced in collaboration with my supervisor, Prof. Susan Scott; it has not been submitted for any other degree. All material taken from other references is explicitly acknowledged as such.

Ben Whale
Dedication

For Him and her, the centres of my world.
Acknowledgements

Thank you to my supervisor, Prof. Susan Scott. You have provided me with a great deal of support and have always shown a large amount of faith in my ability to tackle the problems that you have given me and that I have encountered. In addition, thank you for your willingness to supervise me while I lived in New Zealand. Your ability to accommodate the changed circumstances of my life, is much appreciated.

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Abstract

The original content of this thesis is comprised of three parts.

First, we investigate the foundations of the Abstract Boundary. We start by presenting a one-to-one correspondence between the set of envelopments and a subset of the set of distances on our manifold. This correspondence allows us to define the Abstract Boundary in terms of mathematical structures defined on the manifold, rather than having to use structures additional to the manifold. We take the ideas used in the correspondence and generalise the Abstract Boundary to be applicable to any first countable topological space. Then, using the correspondence and the generalisation we give two alternative constructions for the Abstract Boundary. These new methods of construction allow us to bring many new tools to the analysis of the Abstract Boundary and thus enrich the subject and provide new avenues for research.

Second, we discuss how the limiting behaviour of curves relates to the Abstract Boundary. We restrict our attention to the manifold itself and give a classification of the behaviour of curves via the number of limit points they possess. As an application of the classification we weaken the causality assumption of the Abstract Boundary singularity theorem. As an illustration of the problems that curves in a certain class of the classification can cause we give a definition of causality for Abstract Boundary points. In the process of doing so we generalise the distinguishing and strong causality conditions for the boundaries of envelopments and the Abstract Boundary itself.

Third, we investigate the link between the Penrose-Hawking singularity theorems and the Krolak strong curvature condition. We review the singularity theorems and analyse their proofs to determine what can be said about the predicted incomplete geodesics. We see that the conclusions that can be made and the criteria for the Krolak strong curvature condition do not mesh easily. For this reason we present two necessary and sufficient conditions for a geodesic to satisfy the Krolak strong curvature condition, that provide a link between the conclusions and the Krolak condition. The result is that we need to investigate the limiting behaviour of ja-
cobi fields along conjugate point free geodesics. Hence we provide a preliminary result showing that maximal extension of the metric places real constraints on the behaviour of parallelly propagated frames. This material provides some interesting results, and opens the door to a number of new problems.
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Chapter 1

Introduction

1.1 Conventions

We shall include proofs of theorems in four cases: either, the proof is instructive, we use the details of the proof, the result is not obvious, or the result is original.

Throughout this thesis we take our manifolds to be paracompact, connected and hausdorff. We assume that the transition functions between charts are \( C^\infty \). A \( C^k \) (or \( C^{k-} \)) manifold is a manifold with a metric, \( g \), where \( g \) is \( C^k \) (or \( C^{k-} \)). Unless otherwise stated all indices giving components of tensors run across all dimensions.

A Lorentzian metric of dimension \( n \) is a metric with signature \( n-2 \). Thus if our manifold is 4 dimensional a Lorentzian metric would have signature 2 and for any point \( p \in \mathcal{M} \) there would exist coordinates so that,

\[
g_{ab}(p) = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We shall often consider ‘sequences’ or countable subsets of a topological space \( X \) as sets; that is as sequences with no ordering. We shall denote any such ‘sequence’ by \( s \), so that \( s \subset X \). For various proofs we shall sometimes need to impose an order on \( s \); in this case we shall write \( s = \{s_i\}_i \), where we consider that \( s_i = f(i) \) for some bijective function \( f : \mathbb{N} \to s \). Technical considerations about ordering only affect the results of chapter 5. In section 5.1.1 we discuss our notation and its justification further.

Where unambiguous we shall drop the brackets from expressions like \( \phi(\mathcal{M}) \) and write \( \phi \mathcal{M} \) instead. Likewise, when dealing with a function \( f : X \to Y \) we shall quite happily apply \( f \) to both elements of \( X \) as well as subsets. For example, if
A set $s = \{x_i\} \subset X$ is some sequence then $fs = \{f(x_i) : x_i \in s\}$, or if $U \subset X$ then $fU = \{f(x) : x \in U\}$.

Given a function $\phi : X \to Y$ between two topological spaces we shall often write $\partial \phi X$ to denote the boundary of the image of $X$ in $Y$. That is, $\partial \phi X = \overline{\phi(X)} - \phi(X)$.

We sometimes drop the $\circ$ from function compositions and write $\phi\psi$ rather than $\phi \circ \psi$.

Let $X$ be a set. Given a function $f : [a, b) \to X$, we will often treat $f$ as a subset of $X$. For example, given $x \in X$ we write $x \in f$ meaning $x \in f([a, b))$ and given $g : [c, d) \to X$ we will write $f \subset g$ rather than $f([a, b)) \subset g([c, d))$.

If $X$ is a topological space we shall denote the set of open neighbourhoods about a point $x \in X$ by $\mathcal{N}(x)$. If $U \subset X$ we shall write $\mathcal{N}(U)$ for the set of open neighbourhoods of $U$. If $f : X \to Y$ is a continuous map and $y \in Y$ we shall sometimes write $\mathcal{N}_f(y)$, rather than $\mathcal{N}(y)$, to emphasise that we are using the topology relative to $Y$. Similarly if $U \subset Y$ we shall write $\mathcal{N}_f(U)$ for the set of open neighbourhoods about $U$ in $Y$ respectively.

We shall always refer to a metric $d : X \times X \to \mathbb{R}$ on a topological space as a distance. We do this to distinguish a distance, $d : X \times X \to \mathbb{R}$, from the metric $g : TM \times TM \to \mathbb{R}$.

1.2 Thesis overview

1.2.1 Part one

The first part of this thesis gives a review of singularities and the Abstract Boundary.

Chapter two

We begin by discussing singularity theorems and the need for boundary constructions in the analysis of singularities. We spend a while doing this to highlight the distinction between incomplete geodesics and singularities.

We do not spend as much time reviewing boundary constructions since this has been done elsewhere [3, 101]. We do, however, focus on the ideological differences...
between boundary constructions and try to display the historical development of the ideas behind boundary constructions so that the reader can better understand why the Abstract Boundary is needed and how it arose.

Chapter three

We present here a review of the body of knowledge about the Abstract Boundary that is relevant to this thesis.

1.2.2 Part two

Part two gives the details of two alternative constructions for the Abstract Boundary, that reveal its mathematical structure more clearly than the original construction.

Chapter four

This chapter is the first of the original content, in this thesis. We present a one-to-one correspondence between equivalence classes of the set of envelopments, $\Phi(\mathcal{M})$ of a manifold $\mathcal{M}$ and a certain subset, $D(\mathcal{M})$, of the set of equivalence classes of topological metrics on the manifold. This chapter lays an important conceptual and mathematical framework for the next three chapters.

Chapter five

Chapter five continues the ideological arguments of chapter four and presents similar material. This time we abstract away from manifolds and distances and work instead only with the set of sequences without limit points and a set, $D_{\text{Seq}}(\mathcal{M})$, of functions on these sequences, which parallels the set $D(\mathcal{M})$. In the course of this discussion we show how to construct Abstract Boundary-like sets for any topological space.

Chapter six

We review and reformulate the Abstract Boundary construction, so that the underlying structure is made clearer. Then we use the material of chapters four and five to give two alternative constructions for the Abstract Boundary. By doing so we
show that the Abstract Boundary is fundamentally about sequences and the ‘distance’ between sequences. This in turn shows that the Abstract Boundary straddles the same fine line between topology and analysis as does the study of topological metrics.

Chapter seven

We discuss the important problem of classifying the metrics that belong to $D(M)$. This is a non-trivial problem and we make no attempt to solve it. We do, however, present a number of examples to illustrate metrics that do and do not belong to $D(M)$, as well as discuss several ways to attack the problem.

1.2.3 Part three

In this part we discuss the relationship between curves and the Abstract Boundary. Our motivation is to better meld the physical content of space-times with the Abstract Boundary as well as to illustrate a particular problem that can occur. We give two examples by strengthening the Abstract Boundary singularity theorem and by defining the past/future of Abstract Boundary points.

Chapter eight

We begin by discussing the importance and behaviour of ‘winding’ curves. We present a number of results pertaining to the classification of curves by the number of accumulation points they possess and then use them to improve the Abstract Boundary singularity theorem.

Chapter nine

As an example of the problems discussed in chapter eight, we give a definition of causal structure for the Abstract Boundary. We encounter a problem in the guise of ‘winding curves’ and generalise the strong causality condition to overcome it. We also discuss some of the differences between our definition and the usual definition of causality.
1.2 Thesis overview

1.2.4 Part four

In part four we build on the research program suggested by Ashley and Scott in [5], by investigating the problem of the physical properties of the incomplete geodesics predicted by the singularity theorems in [59]. While the material here does not directly refer to the Abstract Boundary, it is none-the-less intimately connected via the material in [5].

Chapter ten

We start by reviewing the details of the Penrose-Hawking singularity theorems. By analysing their proofs we are able to make a handful of conclusions about the properties of the predicted incomplete geodesics. We discuss these conclusions within the context of the increasing strength and the physicality of each singularity theorem’s assumptions. We finish the chapter by suggesting a set of conditions with which to start investigation of the physical properties of the predicted incomplete curves.

Chapter eleven

By comparing the conclusions of the singularity theorems, derived in chapter ten, to the Krolak strong curvature condition we observe that there is a problem with fitting them together. To solve this problem we prove necessary and sufficient conditions for a geodesic to satisfy the Krolak strong curvature condition. This reduces the problem of connecting physical properties to predicted incomplete geodesics to an analysis of the limiting behaviour of a single divergence along the geodesic. We then give a preliminary result which shows that maximal extension of a metric is related to the behaviour of parallelly propagated frames. Future work, linking parallelly propagated frames and jacobi fields would allow us to derive the necessary results about the limiting behaviour of divergence along geodesics.

1.2.5 Part five

Chapter twelve

We provide a final review of the material presented here, placing it within a single context and emphasising future research directions.
1.2.6 Appendix

Appendix A

We give a large number of standard results, for the benefit of the reader. For the most part, the results listed here are given in the thesis where they are first used.