Semiclassical $L^p$ Estimates for Quasimodes on Submanifolds

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Declaration

Chapters 4 and 5 contain the primary new work of this thesis. Chapter 4 is based on material from a sole author publication “Semiclassical $L^p$ estimates of quasimodes on submanifolds” to appear in *Communications in Partial Differential Equations*. Chapter 5 is based on material from a joint paper with my supervisor Andrew Hassell, “Semiclassical $L^p$ estimates of quasimodes on curved hypersurfaces”.

The work in this thesis is my own except where otherwise stated.

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Abstract

Motivated by the desire to understand classical-quantum correspondences, we study concentration phenomena of approximate eigenfunctions of a semiclassical pseudodifferential operator $P(h)$. Such eigenfunctions appear as steady state solutions of quantum systems. Here we think of $h$ as being a small parameter such that $h^2$ is inversely proportional to the energy of such a system. As we understand classical mechanics to be the high energy (or small $h$) limit of quantum mechanics we expect the behaviour of eigenfunctions $u(h)$ for small $h$ to be related to properties of the associated classical system. In particular we study the connection between the classical flow and the quantum concentration properties.

The flow, $(x(t), \xi(t))$, of a classical system describes the system’s motion through phase space where $x(t)$ is interpreted as position and $\xi(t)$ is interpreted as momentum. In the quantum regime we think of an eigenfunction as being composed of highly localised packets moving along bicharacteristics of the classical flow. With this intuition we relate concentration of eigenfunctions in a region to the time spent by projections of bicharacteristics there.

We use the $L^p$ norm of $u$ when restricted to submanifolds as a measure of concentration. A high $L^p$ norm particularly for small $p$ is indicative of concentration near the submanifold.

We reduce the estimates on eigenfunctions to operator norm estimates on associated evolution operators. Using the semiclassical analysis methods developed in Chapter 3 we express these evolution operators as oscillatory integral operators. Chapter 2 covers the technical background needed to work with such operators. In Chapter 4 we determine eigenfunction estimates for eigenfunctions restricted to a smooth embedded submanifold $Y$ of arbitrary dimension. If $Y$ is a hypersurface, the greatest concentration occurs when there are bicharacteristics of the classical flow embedded in $Y$. In Chapter 5 we assume that projections of such bicharacteristics can be at worst simply tangent to $Y$ and thereby obtain better results for small values of $p$. 
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