Space-Time Coding and
Space-Time Channel Modelling
for Wireless Communications

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Declaration

The contents of this thesis are the results of original research and have not been submitted for a higher degree to any other university or institution.

Much of the work in this thesis has been published or has been submitted for publication as journal papers or conference proceedings. These papers are:


The research work presented in this thesis has been performed jointly with A/Prof. Thushara D. Abhayapala (The Australian National University), Prof. Rodney A. Kennedy (The Australian National University), Dr. Marvin K. Simon (NASA Jet Propulsion Laboratory, USA), Dr. Tony S. Pollock (National ICT Australia), Dr. Van K. Nguyen (Deakin University, Australia), Dr. Terence Betlehem (The Australian National University) and Dr. Jaunty Ho (Monash University, Australia). The substantial majority of this work was my own.
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Abstract

In this thesis we investigate the effects of the physical constraints such as antenna aperture size, antenna geometry and non-isotropic scattering distribution parameters (angle of arrival/departure and angular spread) on the performance of coherent and non-coherent space-time coded wireless communication systems. First, we derive analytical expressions for the exact pairwise error probability (PEP) and PEP upper-bound of coherent and non-coherent space-time coded systems operating over spatially correlated fading channels using a moment-generating function-based approach. These analytical expressions account for antenna spacing, antenna geometries and scattering distribution models. Using these new PEP expressions, the degree of the effect of antenna spacing, antenna geometry and angular spread is quantified on the diversity advantage (robustness) given by a space-time code. It is shown that the number of antennas that can be employed in a fixed antenna aperture without diminishing the diversity advantage of a space-time code is determined by the size of the antenna aperture, antenna geometry and the richness of the scattering environment.

In realistic channel environments the performance of space-time coded multiple-input multiple-output (MIMO) systems is significantly reduced due to non-ideal antenna placement and non-isotropic scattering. In this thesis, by exploiting the spatial dimension of a MIMO channel we introduce the novel use of linear spatial precoding (or power-loading) based on fixed and known parameters of MIMO channels to ameliorate the effects of non-ideal antenna placement on the performance of coherent and non-coherent space-time codes. The spatial precoder virtually arranges the antennas into an optimal configuration so that the spatial correlation between all antenna elements is minimum. With this design, the precoder is fixed for fixed antenna placement and the transmitter does not require any feedback of channel state information (partial or full) from the receiver. We also derive precoding schemes to exploit non-isotropic scattering distribution parameters of the scattering channel to improve the performance of space-time codes applied on MIMO systems in non-isotropic scattering environments. However, these schemes
require the receiver to estimate the non-isotropic parameters and feed them back to the transmitter.

The idea of precoding based on fixed parameters of MIMO channels is extended to maximize the capacity of spatially constrained dense antenna arrays. It is shown that the theoretical maximum capacity available from a fixed region of space can be achieved by power loading based on previously unutilized channel state information contained in the antenna locations. We analyzed the correlation between different modal orders generated at the transmitter region due to spatially constrained antenna arrays in non-isotropic scattering environments, and showed that adjacent modes contribute to higher correlation at the transmitter region. Based on this result, a power loading scheme is proposed which reduces the effects of correlation between adjacent modes at the transmitter region by nulling power onto adjacent transmit modes.

Furthermore, in this thesis a general space-time channel model for down-link transmission in a mobile multiple antenna communication system is developed. The model incorporates deterministic quantities such as physical antenna positions and the motion of the mobile unit (velocity and the direction), and random quantities to capture random scattering environment modeled using a bi-angular power distribution and, in the simplest case, the covariance between transmit and receive angles which captures statistical interdependency. The Kronecker model is shown to be a special case when the power distribution is separable and is shown to overestimate MIMO system performance whenever there is more than one scattering cluster. Expressions for space-time cross correlations and space-frequency cross spectra are given for a number of scattering distributions using Gaussian and Morgenstern’s family of multivariate distributions. These new expressions extend the classical Jake’s and Clarke’s correlation models to general non-isotropic scattering environments.
List of Acronyms

AOD  angle of departure
AOA  angle of arrival
AWGN additive white Gaussian noise
BER  bit-error rate
BPSK binary phase shift keying
CSI  channel state information
MGF  moment generating function
MISO multiple-input single-output
MIMO multiple-input multiple-output
OFDM orthogonal frequency-division multiplexing
PEP  pair-wise error probability
PSD  power spectral density
QPSK quadrature phase shift keying
SIMO single-input multiple-output
SISO single-input single-output
SNR  signal to noise ratio
STBC space-time block code
STTC space-time trellis code
UCA  uniform circular array
ULA  uniform linear array
UGA  uniform grid array
### Notations and Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>$A^\dagger$</td>
<td>complex conjugate transpose of matrix $A$</td>
</tr>
<tr>
<td>$a^\dagger$</td>
<td>complex conjugate transpose of vector $a$</td>
</tr>
<tr>
<td>$A^T$</td>
<td>transpose of matrix $A$</td>
</tr>
<tr>
<td>$a^T$</td>
<td>transpose of vector $a$</td>
</tr>
<tr>
<td>$A^*$</td>
<td>complex conjugate of matrix $A$</td>
</tr>
<tr>
<td>$a^*$</td>
<td>complex conjugate of vector $a$</td>
</tr>
<tr>
<td>$\bar{f}(\cdot)$</td>
<td>complex conjugate of scalar or function $f(\cdot)$</td>
</tr>
<tr>
<td>$|a|$</td>
<td>euclidian norm of vector $a$</td>
</tr>
<tr>
<td>$|A|^2$</td>
<td>squared norm of matrix $A$</td>
</tr>
<tr>
<td>$</td>
<td>A</td>
</tr>
<tr>
<td>$\text{tr}{A}$</td>
<td>trace of matrix $A$</td>
</tr>
<tr>
<td>$\text{vec}(A)$</td>
<td>matrix vectorization operator: stacks the columns of $A$</td>
</tr>
<tr>
<td>$\otimes$</td>
<td>matrix Kronecker product</td>
</tr>
<tr>
<td>$δ(\cdot)$</td>
<td>Dirac delta function</td>
</tr>
<tr>
<td>$\lceil\cdot\rceil$</td>
<td>ceiling operator</td>
</tr>
<tr>
<td>$\mathcal{E}{\cdot}$</td>
<td>mathematical expectation</td>
</tr>
<tr>
<td>$I_n$</td>
<td>$n \times n$ identity matrix</td>
</tr>
<tr>
<td>$\mathbf{1}$</td>
<td>vector of all ones</td>
</tr>
<tr>
<td>$S^1$</td>
<td>unit circle</td>
</tr>
<tr>
<td>$S^2$</td>
<td>unit sphere</td>
</tr>
<tr>
<td>$Q(x)$</td>
<td>Gaussian Q-function: $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2}du$</td>
</tr>
</tbody>
</table>
Contents

Declaration i
Acknowledgements v
Abstract vii
List of Acronyms ix
Notations and Symbols xi
List of Figures xix
List of Tables xxix

1 Introduction 1
  1.1 Motivation and Background ........................................... 1
  1.1.1 Mutual Information and Capacity of MIMO Channels .......... 3
  1.1.2 Space-Time Coding over Multi Antenna Wireless Channels . 9
  1.1.3 Space-Time Channel Modelling .................................... 15
  1.2 Questions to be Answered ............................................ 19
  1.3 Content and Contribution of Thesis ................................. 19

2 Orthogonal Space-Time Block Codes: Performance Analysis 23
  2.1 Introduction .......................................................... 23
  2.2 Spatial Channel Model ............................................... 24
  2.3 Transmitter and Receiver Spatial Correlation for General Distributions of Far-field Scatterers .................................. 26
  2.3.1 Two Dimensional Scattering Environment ........................ 29
  2.3.2 Non-isotropic Scattering Environments and Closed-Form Scattering Environment Coefficients ............................... 31
  2.4 Simulation Results: Alamouti Scheme ............................... 35

xiii
2.4.1 Generation of Correlated Channel Gains 36
2.4.2 Effects of Antenna Separation 36
2.4.3 Effects of Non-isotropic Scattering 38
2.4.4 A Rule of Thumb: Alamouti Scheme 40
2.4.5 Effects of Scattering Distributions 41
2.5 Analysis of Orthogonal STBC: A Modal Approach 43
2.6 Summary and Contributions 46

3 Performance Limits of Space-Time Codes in Physical Channels 49

Part I: Performance Limits of Coherent Space-Time Codes 51
3.1 Introduction 49
3.2 System Model: Coherent Space-Time Codes 51
3.3 Spatial Channel Model 52
3.3.1 Spatial Channel Decomposition 53
3.3.2 Transmitter and Receiver Modal Correlation 55
3.4 Exact PEP on Correlated MIMO Channels 57
3.4.1 Fast Fading Channel Model 58
3.4.2 Slow Fading Channel Model 62
3.4.3 Kronecker Product Model as a Special Case 64
3.5 PEP Analysis of Space-Time Codes in Physical Channel Scenarios 65
3.5.1 Diversity vs Antenna Aperture Size and Antenna Configuration 66
3.5.2 Diversity vs Non-isotropic Scattering 67
3.6 Exact-PEP in Closed-Form 69
3.6.1 Direct Partial Fraction Expansion 69
3.6.2 Partial Fraction Expansion via Eigenvalue Decomposition 70
3.7 Analytical Performance Evaluation: Examples 71
3.8 Effect of Antenna Separation 72
3.8.1 Slow Fading Channel 73
3.8.2 Fast Fading Channel 80
3.9 Effects of Non-isotropic Scattering 81
3.9.1 Slow Fading Channel 81
3.9.2 Fast Fading Channel 83
3.10 Extension of PEP to Average Bit Error Probability 87

Part II: Performance Limits of Non-coherent Space-Time Codes 88
3.11 System Model: Non-Coherent Space-Time Codes 88
3.12 Exact PEP of Differential Space-Time Codes 89
3.12.1 Exact-PEP for Uncorrelated Channels 92
3.12.2 Exact-PEP for Correlated Channels 92
3.13 Analytical Performance Evaluation ............................... 93
  3.13.1 Effects of Antenna Spacing .................................. 93
  3.13.2 Effects of Antenna Configuration ............................. 95
  3.13.3 Effects of Non-Isotropic Scattering .......................... 96
3.14 Summary and Contributions ........................................ 99

4 Spatial Precoder Designs: Based on Fixed Parameters of MIMO Channels 101
  4.1 Introduction ......................................................... 101
  4.2 System Model ....................................................... 103
    4.2.1 Coherent Space-Time Block Codes ............................. 103
    4.2.2 Differential Space-time Block Codes ......................... 104
  4.3 Problem Setup: Coherent STBC ................................... 105
    4.3.1 Optimum Spatial Precoder: Coherent STBC ................... 107
    4.3.2 MISO Channel ................................................ 110
    4.3.3 $n_T \times 2$ MIMO Channel ................................ 110
    4.3.4 $n_T \times 3$ MIMO Channel ................................ 110
    4.3.5 A Generalized Method ....................................... 111
    4.3.6 Spatially Uncorrelated Receive Antennas .................... 111
  4.4 Problem Setup: Differential STBC ................................ 112
    4.4.1 Optimum Spatial Precoder: Differential STBC ............... 113
  4.5 Simulation Results: Coherent STBC ............................... 114
    4.5.1 Performance in Non-isotropic Scattering Environments ... 116
  4.6 Simulation Results: Differential STBC .......................... 120
  4.7 Performance in other Channel Models ............................ 121
    4.7.1 Chen et al.’s MISO Channel Model .......................... 124
    4.7.2 Abdi et al.’s MIMO Channel Model .......................... 126
  4.8 Summary and Contributions ....................................... 128

5 Achieving Maximum Capacity: Spatially Constrained Dense Antenna Arrays 131
  5.1 Introduction ......................................................... 131
  5.2 System Model ....................................................... 132
  5.3 Capacity of Spatially Constrained Antenna Arrays ............... 133
  5.4 Optimization Problem Setup: Isotropic Scattering ............... 134
    5.4.1 Optimum input signal covariance ............................ 135
    5.4.2 Numerical Results ........................................... 138
    5.4.3 Capacity with Finite Number of Receiver Antennas ........... 139
5.4.4 Transmit Modes and Power Allocation .................................... 141
5.4.5 Effects of Non-isotropic Scattering ........................................ 144
5.5 Optimum Power Loading in Non-isotropic Scattering Environments 150
  5.5.1 Numerical Results ....................................................... 152
5.6 Power Loading Based on Mode Nulling .................................... 154
  5.6.1 Modal Correlation at the Transmitter .................................. 155
  5.6.2 Optimum Power Loading Scheme ...................................... 156
  5.6.3 Numerical Results ....................................................... 157
5.7 Summary and Contributions .................................................. 160

6 Space-Time Channel Modelling in General Scattering Environments 163
  6.1 Introduction ............................................................... 163
  6.2 Space-Time Channel Model ................................................ 164
  6.3 Space-Time and Space-Frequency Channel Correlation in General
      Scattering Environments .................................................. 169
    6.3.1 Space-Time Cross Correlation ...................................... 170
    6.3.2 Space-Frequency Cross Spectrum ................................... 172
    6.3.3 SISO Time-varying Channel: Temporal Correlation ............ 173
    6.3.4 Jake’s model for MIMO channels in isotropic scattering ..... 174
    6.3.5 Kronecker Model as a Special Case ................................ 174
  6.4 Non-isotropic Scattering Distributions .................................. 175
    6.4.1 Univariate Scattering Distributions ............................... 176
    6.4.2 Bivariate Scattering Distributions ................................ 178
  6.5 Simulation Examples ........................................................ 180
    6.5.1 Univariate Distributions: Space-Time Cross Correlation .... 180
    6.5.2 Uni-modal Distributed Field within a Limited Spread: Space-
         Time Cross Correlation and Space-Frequency Cross Spectrum 182
    6.5.3 Uni-modal vs Bi-modal Distributions: Spatial Correlation ... 184
    6.5.4 Validity of the Kronecker Channel Model ........................ 184
  6.6 Summary and Contributions ................................................ 189

7 Conclusions and Future Research Directions 193
  7.1 Conclusions .............................................................. 193
  7.2 Future Research Directions ............................................... 194

Appendices
Appendix A

A.1 Proof of the Matrix Proposition ................................................. 197
A.2 Error Events of 4-State QPSK STTC ........................................... 198
  A.2.1 Error Events of Length 2 .................................................. 198
  A.2.2 Error Events of Length 3 .................................................. 199
  A.2.3 Error Events of Length 4 .................................................. 200
A.3 Proof of the Conditional Mean and the Conditional Variance of $u = 2\text{Re}\{w(k)\Delta_{i,j}^\dagger y^\dagger(k - 1)\}$ ................................................................. 201
  A.3.1 Proof of the Conditional Mean ........................................... 201
  A.3.2 Proof of the Conditional Variance ........................................ 202

Appendix B

B.1 Proof of PEP Upper bound: Coherent Receiver ............................... 203
B.2 Proof of PEP Upper bound: Non-coherent Receiver ....................... 204
B.3 Proof of Generalized Water-filling Solution for $n_R = 2$ Receive Antennas ................................................................................. 206
B.4 Proof of Generalized Water-filling Solution for $n_R = 3$ Receive Antennas ................................................................................. 206
B.5 Optimum Precoder for Differential STBC ................................... 207
  B.5.1 MISO Channel ..................................................................... 207
  B.5.2 $n_T \times 2$ MIMO Channel .................................................. 207
  B.5.3 $n_T \times 3$ MIMO Channel .................................................. 208

Bibliography .............................................................. 209
# List of Figures

1.1 Illustration of a MIMO transmission system with $n_T$ transmit antennas and $n_R$ receive antennas ........................................ 4

1.2 Ergodic capacity of different multi-antenna systems when the channel is only known to the receiver: equal power-loading scheme. ... 7

1.3 A generic block diagram of space-time coding across a MIMO channel. 10

1.4 4-state QPSK space-time trellis code with two transmit antennas proposed by Tarokh et al. .................................................. 12

1.5 The two-branch diversity scheme with $n_R$ receive antennas proposed by Alamouti. .......................................................... 13

2.1 A General scattering model for a flat fading MIMO system. $r_T$ and $r_R$ are the radius of spheres which enclose the transmitter and the receiver antennas, respectively. $g(\hat{\phi}, \hat{\varphi})$ represents the gain of the complex scattering environment for signals leaving the transmitter scattering free region from direction $\hat{\phi}$ and entering at the receiver scattering free region from direction $\hat{\varphi}$. ........................................ 25

2.2 Spatial correlation between two receiver antenna elements for mean AOA $\varphi_0 = 90^\circ$ (broadside) and angular spread $\sigma = \{20^\circ, 5^\circ, 1^\circ\}$ against antenna separation for uniform-limited, truncated Gaussian, truncated Laplacian and von-Mises scattering distributions. ....... 34

2.3 Spatial correlation between two receiver antenna elements for mean AOA $\varphi_0 = 30^\circ$ ($60^\circ$ from broadside) and angular spread $\sigma = \{20^\circ, 5^\circ, 1^\circ\}$ against antenna separation for uniform-limited, truncated Gaussian, truncated Laplacian and von-Mises scattering distributions. ....... 35

2.4 BER performance vs receiver spatial separation for 2x2 orthogonal STBC and uncoded systems for a Uniform-limited distribution at the receiver antenna array. Mean AOA 0° from broadside, angular spread $\sigma = \{104^\circ, 20^\circ, 5^\circ\}$ and SNR = 10dB. ......................... 37
2.5 (a). Spatial correlation between two receiver antennas positioned on the x-axis for mean AOA 0° from broadside vs the spatial separation for a uniform-limited scattering distribution with angular spreads \( \sigma = [104°, 20°, 5°, 1°] \). (b). BER performance vs spatial separation for 2\( \times \)2 orthogonal STBC under the scattering environments given in (a) ......................................................... 39

2.6 (a). Spatial correlation between two receiver antennas positioned on the x-axis for mean AOA 60° from broadside against the spatial separation for a uniform-limited scattering distribution with angular spreads \( \sigma = [104°, 20°, 5°, 1°] \). (b). BER performance vs spatial separation for 2\( \times \)2 orthogonal STBC under the scattering environments given in (a) ......................................................... 40

2.7 Angular spread (\( \sigma \)) vs optimum antenna separation where the BER performance of 2\( \times \)2 orthogonal STBC is optimum for mean AOAs 0°, 30°, 45° and 60° from broadside. ................................. 41

2.8 BER performance of 2\( \times \)2 orthogonal STBC against the non-isotropic parameter for mean AOAs 0°, 30° and 60° from broadside, SNR 10dB and antenna separation \( \lambda/2 \): (a). uniform-limited (b). truncated Gaussian (c). von-Mises (d). truncated Laplacian ......................... 42

2.9 Radiation patterns and \( |a_m|^2 \) for orthogonal STBC with two transmit antennas: antenna separation 0.5\( \lambda \) (or \( r_T = 0.25\lambda \)) ......................... 44

2.10 Radiation patterns and \( |a_m|^2 \) for orthogonal STBC with two transmit antennas: antenna separation \( \lambda \) (or \( r_T = 0.5\lambda \)) ......................... 45

3.1 Trellis diagram for 4-state space-time code for QPSK constellation. 72

3.2 Exact pairwise error probability performance of the 4-state space-time trellis code with 2-transmit antennas and 1-receive antenna: length 2 error event, slow fading channel. ......................... 73

3.3 Exact PEP performance of the 4-state space-time trellis code with 2-transmit antennas and n-receive antennas: length 2 error event, slow fading channel. ................................. 75

3.4 Length 2 error event of 4-state QPSK space-time trellis code with two transmit antennas for an increasing number of receive antennas in an isotropic scattering environment. \( r_T = 0.5\lambda \), \( r_R = \{0.15\lambda, 0.25\lambda\} \) and SNR = 10dB; slow-fading channel. ................................. 76

3.5 The exact-PEP performance of the 16-state code with 3-transmit and 1-receive antennas for UCA and ULA transmit antenna configurations: length 3 error event, slow fading channel. ................................. 77
3.6 Frame error rate performance of the 16-state QPSK, space-time trellis code with three transmit antennas for UCL and ULA antenna configurations in an isotropic scattering environment; slow-fading channel. ........................................ 78

3.7 Frame error rate performance of the 64-state QPSK space-time trellis code with four transmit antennas for UCL and ULA antenna configurations in an isotropic scattering environment; slow-fading channel. ........................................ 79

3.8 Exact pairwise error probability performance of the 4-state space-time trellis code with 2-transmit antennas and 2-receive antennas-length two error event: fast fading channel. ........................................ 80

3.9 Length 2 error event of 4-state QPSK space-time trellis code with two transmit antennas for an increasing number of receive antennas in a non-isotropic scattering environment; $r_T = 0.5\lambda$, $r_R = 2\lambda$ and $\text{SNR} = 10\text{dB}$: slow-fading channel. ........................................ 82

3.10 Effect of receiver modal correlation on the exact-PEP of the 4-state QPSK space-time trellis code with 2-transmit antennas and 2-receive antennas for the length 2 error event. Uniform limited power distribution with mean angle of arrival 0° from broadside and angular spreads $\Delta_r = \{5^\circ, 30^\circ, 60^\circ, 180^\circ\}$; fast fading channel. ............... 83

3.11 Effect of receiver modal correlation on the exact-PEP of the 4-state QPSK space-time trellis code with 2-transmit antennas and 2-receive antennas for the length 2 error event. Uniform limited power distribution with mean angle of arrival 45° from broadside and angular spreads $\Delta_r = \{5^\circ, 30^\circ, 60^\circ, 180^\circ\}$; fast fading channel. ............... 84

3.12 Exact-PEP of the 4-state QPSK space-time trellis code with 2-transmit antennas and 2-receive antennas against the receive antenna separation at 8dB SNR. Uniform limited power distribution with mean angle of arrival 45° from broadside and angular spreads $\Delta_r = \{5^\circ, 30^\circ, 180^\circ\}$; fast fading channel ............... 85

3.13 Exact-PEP of the 4-state QPSK space-time trellis code with 2-transmit antennas and 2-receive antennas against the receive antenna separation at 10dB SNR. Uniform limited power distribution with mean angle of arrival 45° from broadside and angular spreads $\Delta_r = \{5^\circ, 30^\circ, 180^\circ\}$; fast fading channel ............... 86

3.14 Exact-PEP performance of DSTC scheme with two transmit and two receive antennas for transmit antenna separation 0.5\lambda and $\beta_{0,1} = 2$. 94
3.15 Exact-PEP performance of DSTC scheme with two transmit and three receive antennas for UCA and ULA receiver antenna configurations; $\beta_{0,1} = 2$. .......................................................... 95

3.16 Exact-PEP performance of the DSTC scheme with two transmit and two receive antennas against the receive antenna separation for a uniform limited power distribution at the receiver with mean angle of arrival $\varphi_0 = 45^\circ$ from broadside and $\Delta_r = [5^\circ, 30^\circ, 180^\circ]$ at 15dB SNR; Transmit antenna separation $0.5\lambda$ and $\beta_{0,1} = 2$. ............................. 96

3.17 Exact-PEP performance of the DSTC scheme with two transmit and two receive antennas against the receive antenna separation for a uniform limited power distribution at the receiver with mean angle of arrival $\varphi_0 = 45^\circ$ from broadside and $\Delta_r = [5^\circ, 30^\circ, 180^\circ]$ at 20dB SNR; Transmit antenna separation $0.5\lambda$ and $\beta_{0,1} = 2$. ............................. 97

3.18 Exact-PEP performance of DSTC scheme with two transmit and three receive antennas for UCA and ULA receiver antenna configurations for a uniform limited power distribution at the receiver with mean angle of arrivals $\varphi_0 = [60^\circ, 45^\circ, 15^\circ]$ from broadside and non-isotropic parameter $\Delta_r = 180^\circ$; Transmit antenna separation 0.15$\lambda$, receive antenna separation $0.15\lambda$ and $\beta_{0,1} = 2$. ............................. 98

4.1 Water level ($1/\upsilon_c$) for various SNRs for a MISO system. (a) $n_T = 2$, (b) $n_T = 3$ - UCA, (c) $n_T = 4$ - UCA, (d) $n_T = 3$ - ULA and (e) $n_T = 4$ - ULA for 0.2$\lambda$ minimum separation between two adjacent transmit antennas. .......................................................... 115

4.2 BER performance of the rate-1 coherent STBC (QPSK) with $n_T = 2$ and $n_R = 1,2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; transmit antenna separation 0.2$\lambda$. .......................................................... 117

4.3 BER performance of the rate-1 coherent STBC (BPSK) with $n_T = 4$ and $n_R = 1,2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; UCA transmit antenna configuration and 0.2$\lambda$ minimum separation between two adjacent transmit antenna elements......................... 118

4.4 BER performance of the rate-1 coherent STBC (BPSK) with $n_T = 4$ and $n_R = 1,2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; ULA transmit antenna configuration and 0.2$\lambda$ minimum separation between two adjacent transmit antenna elements......................... 119
4.5 BER performance of the rate-1 differential STBC (QPSK) with $n_T = 2$ and $n_R = 1, 2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; transmit antenna separation $0.1\lambda$. ................................................................. 121

4.6 BER performance of the rate-1 differential STBC (BPSK) with $n_T = 4$ and $n_R = 1, 2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; UCA transmit antenna configuration and $0.2\lambda$ minimum separation between two adjacent transmit antenna elements. ............................ 122

4.7 BER performance of the rate-1 differential STBC (BPSK) with $n_T = 4$ and $n_R = 1, 2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; ULA transmit antenna configuration and $0.2\lambda$ minimum separation between two adjacent transmit antenna elements. ............................ 123

4.8 Scattering channel model proposed by Chen et al. for three transmit and one receive antennas. ................................................................. 125

4.9 Spatial precoder performance with three transmit and one receive antennas for $0.2\lambda$ minimum separation between two adjacent transmit antennas placed in a uniform linear array, using Chen et al’s channel model: rate-3/4 coherent STBC. ............................ 126

4.10 Scattering channel model proposed by Abdi et al. for two transmit and two receive antennas. ................................................................. 127

4.11 Spatial precoder performance with two transmit and two receive antennas using Abdi et al’s channel model: rate-1 differential STBC. 128

5.1 Capacity comparison between spatial precoder and equal power loading ($Q = (P_T/n_T)I_{n_T}$) schemes for uniform circular arrays and uniform linear arrays in a rich scattering environment with transmitter aperture radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas ($r_R \to \infty$) for an increasing number of transmit antennas. Also shown is the maximum achievable capacity (5.14) from the transmitter region. ................................................................. 139

5.2 Simulated capacity of equal power loading and spatial precoding schemes for uniform circular arrays in a rich scattering environment with transmitter aperture radius $r_T = 0.5\lambda$ and receiver aperture radius $r_R = 5\lambda$ for an increasing number of transmit antennas. 140
xxiv

List of Figures

5.3

Simulated capacity of equal power loading and spatial precoding
schemes for uniform linear arrays in a rich scattering environment
with transmitter aperture radius rT = 0.5λ and receiver aperture
radius rR = 5λ for an increasing number of transmit antennas. . . 141

5.4

Average power allocated to each transmit mode for the UCA and
ULA antenna configurations, within a circular aperture of radius
0.5λ. PT = 10dB and nT = 80. . . . . . . . . . . . . . . . . . . . . 143

5.5

Capacity comparison between spatial precoding and equal power
loading schemes for a uniform limited scattering distribution at the
transmitter with mean AOD φ0 = 0◦ and angular spreads σ =
{104◦ , 30◦ , 15◦ , 5◦ }, for UCA transmit antenna configurations with
transmitter aperture radius rT = 0.5λ and a large number of uncorrelated receive antennas (rR → ∞), for increasing number of
transmit antennas. . . . . . . . . . . . . . . . . . . . . . . . . . . . 146

5.6

Capacity comparison between spatial precoding and equal power
loading schemes for a uniform limited scattering distribution at the
transmitter with mean AOD φ0 = 0◦ and angular spreads σ =
{104◦ , 30◦ , 15◦ , 5◦ }, for ULA transmit antenna configurations with
transmitter aperture radius rT = 0.5λ and a large number of uncorrelated receive antennas (rR → ∞), for increasing number of
transmit antennas. . . . . . . . . . . . . . . . . . . . . . . . . . . . 147

5.7

Capacity comparison between spatial precoding and equal power
loading schemes for a uniform limited scattering distribution at the
transmitter with mean AOD φ0 = 0◦ and increasing angular spread,
for UCA transmit antenna configurations with transmitter aperture
radius rT = 0.5λ and a large number of uncorrelated receive antennas
(rR → ∞), for nT = {10, 11, 25, 60, 80} transmit antennas. . . . . . 148

5.8

Capacity comparison between spatial precoding and equal power
loading schemes for a uniform limited scattering distribution at the
transmitter with mean AOD φ0 = 90◦ and increasing angular spread,
for UCA transmit antenna configurations with transmitter aperture
radius rT = 0.5λ and a large number of uncorrelated receive antennas
(rR → ∞), for nT = {10, 11, 25, 60, 80} transmit antennas. . . . . . 149


5.9 Capacity comparison between spatial precoding and equal power loading schemes for a uniform limited scattering distribution at the transmitter with mean AOD $\phi_0 = 0^\circ$ and increasing angular spread, for ULA transmit antenna configurations with transmitter aperture radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas $(r_R \to \infty)$, for $n_T = \{10, 11, 25, 60, 80\}$ transmit antennas.  

5.10 Capacity comparison between spatial precoding and equal power loading schemes for a uniform limited scattering distribution at the transmitter with mean AOD $\phi_0 = 90^\circ$ and increasing angular spread, for UCA transmit antenna configurations with transmitter aperture radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas $(r_R \to \infty)$, for $n_T = \{10, 11, 25, 60, 80\}$ transmit antennas.  

5.11 Capacity of different power loading schemes versus angular spread about the mean AOD $\phi_0 = 45^\circ$ at the transmitter for $n_T$ transmit antennas placed in a UCA within a spatial region of radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas $(r_R \to \infty)$: (a) $n_T = 11$, (b) $n_T = 25$, (c) $n_T = 80$ and (d) $n_T = 90$.  

5.12 Capacity of different power loading schemes versus angular spread about the mean AOD $\phi_0 = 45^\circ$ at the transmitter for $n_T$ transmit antennas placed in a ULA within a spatial region of radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas $(r_R \to \infty)$: (a) $n_T = 11$, (b) $n_T = 25$, (c) $n_T = 80$ and (d) $n_T = 90$.  

5.13 Capacity of ULA antenna systems versus angular spread about the mean AODs $\phi_0 = \{0^\circ, 30^\circ, 60^\circ, 90^\circ\}$ at the transmitter for 11 transmit antennas placed within a spatial region of radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas $(r_R \to \infty)$: (a) $\phi_0 = 0^\circ$, (b) $\phi_0 = 30^\circ$, (c) $\phi_0 = 60^\circ$ and (d) $\phi_0 = 90^\circ$.  

5.14 Modal correlation vs non-isotropic parameter $\Delta$ of a uniform limited azimuth power distribution at the transmitter region for a mean AOD $\phi_0 = 0^\circ$.  

5.15 Capacity comparison between power-loading scheme–1 and scheme–3 for a uniform limited azimuth power distribution at the transmitter with mean AOD $\phi_0 = 0^\circ$ for increasing angular spread: $n_T = 4$ transmit antennas.
5.16 Capacity comparison between power-loading scheme−1 and scheme−3 for a uniform limited azimuth power distribution at the transmitter with mean AOD $\phi_0 = 0^\circ$ for increasing angular spread: $n_T = 5$ transmit antennas. ................................................................. 159

5.17 Average power allocated to each effective transmit mode in a circular aperture of radius 0.25$\lambda$. $P_T = 10$dB: UCA antenna configuration, $n_T = 5$ transmit antennas. ......................................................... 160

5.18 Average power allocated to each effective transmit mode in a circular aperture of radius 0.25$\lambda$. $P_T = 10$dB: ULA antenna configuration, $n_T = 5$ transmit antennas. ......................................................... 161

6.1 General scattering model for a down-link MIMO communication system. $r_T$ and $r_R$ are the radius of spheres which enclose the transmitter and the receiver antennas, respectively. We demonstrate the generality of the model by showing three sample scatterers $S_1$, $S_2$ and $S_3$ which show a single bounce (reflection off $S_2$), multiple bounces (sequential reflection off $S_2$ and $S_3$), and wave splitting (with divergence at $S_2$), and also a direct path. ......................................................... 166

6.2 Space-time cross correlation between two MU receive antennas with $f_D T_S = 0.038$ against the spatial separation for Uniform-limited, truncated Gaussian, truncated Laplacian and von-Mises scattering distributions with angular spread $\sigma_r = \{20^\circ, 5^\circ, 2^\circ\}$ and mean AOA $\varphi_0 = 0^\circ$: (a) $\tau = 0$, (b) $\tau = 5T_S$, (c) $\tau = 20T_S$ and (d) $\tau = 30T_S$ .... 181

6.3 Space-time cross correlation between two MU receive antennas against $f_D T_S$ for Uniform-limited, truncated Gaussian, truncated Laplacian and von-Mises scattering distributions with angular spread $\sigma_r = \{20^\circ, 5^\circ, 2^\circ\}$ and mean AOA $\varphi_0 = 0^\circ$, for $\tau = 5T_S$: (a) $\|z_p - z'_p\| = 0.1\lambda$, (b) $\|z_p - z'_p\| = 0.25\lambda$, (c) $\|z_p - z'_p\| = 0.5\lambda$ and (d) $\|z_p - z'_p\| = \lambda$. 182

6.4 Magnitude of the space-time cross correlation function for $f_D = \omega_D/2\pi = 0.05$, $\varphi_v = 30^\circ$ and a Laplacian distributed field with mean AOA 60$^\circ$ from broadside and angular spread $\sigma_r = \{20^\circ, 10^\circ\}$. 183

6.5 Comparison of uni-modal and bi-modal von-Mises distributions. .. 185

6.6 Average mutual information of 3-transmit UCA and 3-receive UCA MIMO system in separable (Kronecker with $\rho = 0$) and non-separable ($\rho = 0.8$) scattering environments: bivariate truncated Gaussian azimuth field with mean AOD $= 90^\circ$, mean AOA $= 90^\circ$, transmitter angular spread $\sigma_t = 10^\circ$ and receiver angular spreads $\sigma_r = \{30^\circ, 10^\circ\}$. 186

6.7 An example multi-modal bivariate Gaussian distributed azimuth field. 187
6.8 Average mutual information of 3-transmit UCA and 3-receive UCA MIMO system for separable and non-separable scattering channel considered in Figure 6.7. .................................................. 188

6.9 Kronecker model PSD $\tilde{G}(\phi, \varphi) = P_{Tx}(\phi)P_{Rx}(\varphi)$ of the non-separable scattering distribution considered in Figure 6.7. .............................................. 189

6.10 Kronecker model PSD $\tilde{G}(\phi, \varphi) = P_{Tx}(\phi)P_{Rx}(\varphi)$ of the uni-modal non-separable scattering distribution used in the first example to obtain the results in Figure 6.6 for $\sigma_r = 10^\circ$. ........................................ 190
List of Tables

2.1 Maximum and minimum bit-error rates produced by coded and uncoded systems. ........................................ 38

4.1 Transmit antenna configuration details corresponding to water-filling scenarios considered in Figure 4.1. .................. 115

6.1 Scattering Coefficients $\beta^n$ for Uniform-limited, truncated Gaussian, cosine, von-Mises and truncated Laplacian univariate uni-modal power distributions ........................................ 176

A.1 4-state QPSK space-time trellis code: Error events of length two. ........................................ 198

A.2 4-state QPSK space-time trellis code: Error events of length three. ........................................ 199

A.3 4-state QPSK space-time trellis code: Error events of length four. ........................................ 200
Chapter 1

Introduction

1.1 Motivation and Background

In recent years, there has been an increasing demand for higher data rates in wireless communication systems to support emerging wireless applications, specifically real-time data and multimedia services. However, the bandwidth or the frequency spectrum is a limited resource and it cannot be increased to meet the demand correspondingly. Therefore, the wireless system designers face the challenge of designing wireless systems that are capable of providing increased data rates and improved performance while utilizing existing frequency bands and channel conditions.

Due to the nature of the wireless channel the design of wireless systems fundamentally differs from wired system designs. The wireless channel is much more unpredictable than the wired channel because of factors such as multipath, mobility of the user, mobility of the objects in the environment and delays arising from multipaths. Multipath is a phenomenon that occurs as a transmitted signal is reflected or diffracted by objects in the environment or refracted through the medium between the transmitter and the receiver. The net effect of these reflection, diffraction, and refraction on the transmitted signal is attenuation, phase change and delay, collectively called fading [1], which decreases the instantaneous signal-to-noise ratio (SNR) of the signal received, leading to performance degradation of wireless communication systems. In the early stage of wireless communication system designs the researchers mainly focused on mitigating or removing the fading effects of wireless channels. However, it was recently discovered that under certain conditions it is possible to exploit multipath fading channels to improve the performance of wireless communication systems. The underlying idea is to provide a number of different replicas of the same transmitted signal to the receiver and the receiver to combine these multiple replicas in some manner to improve the over-
all SNR and hence reliably detect the transmitted signal. The idea of conveying a number of different replicas of the same transmitted signal is called **diversity**. Some common diversity techniques used are:

- **temporal diversity**: the same information is transmitted at different time-slots where the duration of each time-slot exceeds the coherence time$^1$ of the channel [2];
- **frequency diversity**: information is transmitted on more than one carrier frequencies, where each carrier frequency is separated by more than the coherence bandwidth$^2$ of the channel [2];
- **polarization diversity**: consists of information transmission over a single antenna supporting orthogonal polarization to provide independently fading channels [3];
- **spatial diversity**: multiple transmit and/or receive antennas are used to obtain multiple replicas of the signal;

In this thesis we mainly focus on spatial diversity techniques.

Initial results from J. Winters [4] demonstrated that it is possible to exploit the multipath channel to improve the capacity gains of a wireless fading channel through spatial diversity. In [5, 6] Telatar and Foschini independently studied the information theoretic capacity of multiple-input multiple-output (MIMO) systems in wireless fading channels. It was shown that for a single-user system with $n_T$ transmit antennas and $n_R$ receive antennas, the channel capacity scales linearly with $\min(n_T, n_R)$ relative to a single-user system with single transmit and single receive antenna. These investigations have led to the development of space-time coding schemes [7–9] to provide high data rates and reliable communication over fading channels. The capacity analysis presented in [5, 6] and the space-time coding schemes proposed in [7–9] assumed independent and identically distributed (i.i.d.) flat fading channels corresponding to a rich scattering environment (isotropic scattering) surrounding the transmitter and receiver antenna arrays and sufficiently spaced antennas at both antenna arrays. Therefore, the performance improvements promised by MIMO systems are valid only under i.i.d. fading channel conditions.

In practice, the assumption of i.i.d. fading is often hard to satisfy. For example, the base station (BS) antennas in a mobile communication system are placed high

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$^1$coherence time: minimum time separation between independent channel fades

$^2$coherence bandwidth: minimum frequency separation between independent channel fades
above the ground and are not exposed to many local scatterers. As a consequence, the BS antennas receive signals mainly from a particular direction which leads to high signal correlation at the BS antennas. At the mobile unit (MU) it is often valid to assume the surrounding scattering environment is isotropic as the mobile unit is often surrounded by many local scatterers. However, the antennas at the MU cannot be sufficiently spaced apart due to the limited size of the MU. As a result, the spatial correlation limits the performance improvements promised by multi antenna systems.

In this thesis we primarily investigate the performance limits of space-time coding schemes in more realistic channel environments. In particular, we study the diversity advantage of space-time coding schemes when the antenna elements are placed in some configuration within a spatially constrained region and when there exists a non-isotropic scattering environment. We also focus on techniques that can be applied on MIMO systems to reduce the detrimental effects of the above mentioned physical factors.

The remainder of this chapter introduces basic concepts involved in capacity of MIMO systems and space-time coding, along with some space-time channel models found in the literature. For an in-depth study of MIMO systems including space-time channel modelling, the reader is referred to references [10–14].

1.1.1 Mutual Information and Capacity of MIMO Channels

Information theoretic studies of wireless fading channels give very useful results on the maximum information transfer rate between two points of a communication link. Furthermore, theoretical studies give a guideline to how well a particular design performs and how close the system operates to the ultimate Shannon limit.

We now discuss the multi antenna wireless communication systems from an information theoretic perspective. For the analysis in this section and also in the rest of the thesis, it is assume that no co-channel interferers (a single user channel) are present and that the noise is spatially and temporally white. Also, the transmitter is limited to a maximum output power of $P_T$.

Figure 1.1 illustrates the discrete time equivalent base-band model of a single user wireless communication system with $n_T$ transmit antennas and $n_R$ receive antennas. The input-output relationship for this model can be written as

$$y(\tau) = H(\tau) * s(\tau) + n(\tau),$$ (1.1)
Figure 1.1: Illustration of a MIMO transmission system with $n_T$ transmit antennas and $n_R$ receive antennas

where $H(\tau)$ is the channel impulse response matrix, $s(\tau)$ is the transmitted signal vector, $y(\tau)$ is the received signal vector, $n(\tau)$ is the additive what Gaussian noise and $*$ denotes the convolution operator. In this thesis, we only consider frequency flat fading channels (i.e., the signal bandwidth is sufficiently narrow so that the channel can be treated as approximately constant over frequency). Therefore, the corresponding input-output relationship can be written as

$$y = Hs + n,$$

where

$$H = \begin{bmatrix} h_{1,1} & \cdots & h_{1,n_T} \\ \vdots & \ddots & \vdots \\ h_{n_R} & \cdots & h_{n_R,n_T} \end{bmatrix},$$

is the $n_R \times n_T$ flat-fading channel gain matrix with coefficient $h_{p,q}$ representing the random complex channel gain between the $q$-th transmit antenna and the $p$-th receive antenna, $s = [s_1, s_2, \cdots, s_{n_T}]^T$, $n = [n_1, n_2, \cdots, n_{n_R}]^T$ with $\mathcal{E}\{nn^\dagger\} = N_0 I_{n_R}$, and $y = [y_1, y_2, \cdots, y_{n_R}]^T$.

When $s$ is circular symmetric complex Gaussian and $H$ is completely known at the receiver, it was shown in [5,6] that the mutual information is given by

$$I(s, y) = \log |I_{nr} + HQ_sH^\dagger|,$$

where $Q_s = \mathcal{E}\{ss^\dagger\}$ is the covariance of the transmitted signal vector $s$. The derivation of (1.3) assumed that the elements of the channel matrix $H$ are modelled...
by i.i.d. complex Gaussians with zero mean and unit variance. This assumption is valid only when there is rich scattering (isotropic scattering) and enough separation between the antennas at the transmitter and at the receiver.

The capacity of channel $H$ is defined as

$$C(H) \triangleq \max_{p(s), \text{tr}(Q_s) \leq P_T} I(s, y),$$

$$= \max_{p(s), \text{tr}(Q_s) \leq P_T} \log |I_{nr} + HQ_s H^\dagger|,$$  \hspace{1cm} (1.4)

where the maximization is taken over all possible input distributions $p(s)$ of transmitted signal vector $s$.

For a fading channel, the channel matrix $H$ is a random quantity and hence the associated channel capacity $C(H)$ is also a random variable. Note that, the capacity described by (1.4) is the instantaneous capacity for a given realization of $H$. In the literature, there are two ways to characterize the capacity of a MIMO fading channel: **ergodic capacity** [5] and **outage capacity** [6].

The **ergodic capacity** of the MIMO channel $H$ is defined by

$$C_{\text{erg}} \triangleq \mathcal{E} \left\{ \max_{p(s), \text{tr}(Q_s) \leq P_T} \log |I_{nr} + HQ_s H^\dagger| \right\},$$  \hspace{1cm} (1.5)

where the expectation is with respect to the distribution of $H$. The ergodic capacity represents the long term achievable bit-rate of the channel, average over the distribution of channel $H$. Note that the ergodic capacity is equivalent to the Shannon capacity of a channel.

The **outage capacity** is related to the outage probability which is defined as the fraction of time the capacity of the channel falls below a given threshold $C_{\text{out}}$. Therefore, the outage capacity is often a more realistic measure than the ergodic capacity (1.5) of $H$. The outage probability $p$ is defined as

$$p = \text{Prob}(C(H) \leq C_{\text{out}}).$$  \hspace{1cm} (1.6)

Note that the outage capacity is often presented in the form of a cumulative distribution function [6].

From (1.4), the channel capacity becomes a transmitter optimization problem subject to the transmit power constraint $\text{tr}(Q_s) \leq P_T$. Therefore, finding optimum $Q_s$ for various channel state conditions has been the subject of recent MIMO ca-
Capacity analysis research. We now proceed by distinguishing between the two cases with and without the full channel state information (CSI) available to the transmitter. In both cases, it is assumed that the receiver possesses full CSI (coherent detection).

**Channel Known at the Transmitter:** Assume that the full CSI is available to the transmitter via feedback from the receiver. Using the singular value decomposition theorem [15], the channel matrix $H$ can be decomposed as $H = UDV^\dagger$ where $U = [u_1, \cdots, u_n] \in \mathbb{C}^{n_R \times n_R}$, $V = [v_1, \cdots, v_{n_T}] \in \mathbb{C}^{n_T \times n_T}$ are unitary and $D = \text{diag}\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n}, 0, \cdots, 0\}$ with $\sqrt{\lambda_i}, i = 1, 2, \cdots, n$ are the singular values of $H$ (or the positive square root of the eigenvalues of $HH^\dagger$), and $n = \text{rank}(H) \leq \min(n_R, n_T)$. Now, we can write (1.2) as

$$\tilde{y} = D\tilde{s} + \tilde{n}, \quad (1.7)$$

where $\tilde{y} = U^\dagger y$, $\tilde{s} = V^\dagger s$ and $\tilde{n} = U^\dagger n$. Since $U$ and $V$ are unitary, the distributions of $\tilde{y}$, $\tilde{s}$ and $\tilde{n}$ remain the same as those of $y$, $s$ and $n$. Since $D$ is diagonal, we now have $n$ parallel and independent sub-channels with gains $\sqrt{\lambda_i}$, $i = 1, 2, \cdots, n$. These sub-channels are also referred to as “eigen-channels”.

When $H$ is completely known both at the transmitter and at the receiver, the optimum scheme for allocating power on to the $i$-th eigen-channel is commonly referred to as “water-filling” [5, 16, 17]. In this scheme, the power allocated to the $i$-th independent eigen-channel is $P_i = (\upsilon - \lambda_i^{-1})^+$ where $\upsilon$ is the water-level determined by the power constraint $\sum_i (\upsilon - \lambda_i^{-1})^+ \leq P_T$ and $a^+$ denotes $\max(a, 0)$. The power is allocated only to those eigen-channels in which $1/\lambda_i$ is less than the water filling level $\upsilon$ and zero power is allocated to the remaining eigen-channels. The corresponding channel capacity is given by [5],

$$C_{\text{wf}} = \sum_i \log(\upsilon\lambda_i^+). \quad (1.8)$$

**Channel Unknown at the Transmitter:** When the channel is completely known at the receiver, but it is unknown at the transmitter, it was shown in [5] that in i.i.d. Rayleigh fading channels, allocation of equal power on to each transmit antenna is optimal, i.e., $Q_s = (P_T/n_T)I_{n_T}$. In this case, the channel capacity becomes

$$C_{\text{eq}} = \log \left| I_{n_R} + \frac{\overline{\gamma}}{n_T} HH^\dagger \right|, \quad (1.9)$$

where $\overline{\gamma} = P_T/N_0$ is the average SNR at each receive antenna. Using the matrix
singular value decomposition (svd) of $H$, we can re-write (1.9) as

$$C_{eq} = \sum_{i=1}^{n} \log \left( 1 + \frac{\frac{\pi}{nT} \lambda_i}{n} \right),$$

(1.10)

which expresses the capacity of the MIMO channel as the sum of the capacities of $n$ sub-channels.

Figure 1.2: Ergodic capacity of different multi-antenna systems when the channel is only known to the receiver: equal power-loading scheme.

Figure 1.2 shows the ergodic capacity of multi-antenna systems in i.i.d. Rayleigh fading channels as a function of SNR for different transmit and receive antenna combinations, where the full CSI is only available to the receiver. The SISO channel capacity is also shown in Figure 1.2. From Figure 1.2 it can be seen that the ergodic capacity achieved from a system with $n_T = 3$ and $n_R = 1$ (MISO) is relatively small compared to that of the SISO system. This observation indicates that adding extra transmit antennas to the SISO system does not considerably improve the ergodic capacity of the system. However, the SIMO channel has a higher ergodic capacity than the MISO channel. It can be seen that at high SNR, for a SISO system the
increase in capacity is about 1 bit/s/Hz for a 3dB increase in SNR. In MIMO case, for e.g., \(n_T = 3\) and \(n_R = 3\), about 3 bits/s/Hz of increase in capacity is observed for 3dB increase in SNR. This observation indicates that if \(H\) is full rank and \(n_R = n_T\), then the channel capacity increases linearly by the number of antennas. However, in general, the capacity increases by the minimum of the number of transmit and receive antennas [5].

For any channel distribution, always \(C_{eq} \leq C_{wf}\), provided that accurate full CSI is available to the transmitter. However, in the event of inaccurate estimation of \(H\), the water-filling can lead to worse performance than equal power loading. The effect of incorrect channel estimation on water-filling has been studied in [18].

The improvements in capacity for MIMO systems discussed thus far assumed that channel gains are independent and identically distributed. More realistic evaluation of capacity for correlated MIMO channels were studied in [19–25]. These studies have given insights and bounds into the effects of correlated channels on the MIMO capacity. In particular they studied the “key-hole” (or pin-hole) effects, antenna spacing effects, antenna geometry effects, non-isotropic scattering effects (limited angular spread) on the MIMO capacity. In general, these studies showed that the correlation between elements of \(H\) reduces the capacity improvements promised by MIMO systems with perfect CSI on the receiver. In contrast, recently, [26] showed that when only the partial CSI is available at the receiver, channel correlation can significantly improve the capacity performance of MIMO systems.

In systems with channels that change rapidly, feedback of perfect CSI may not be possible. However, feedback of partial CSI such as mean and covariance of the channel may be possible as these partial CSI change more slowly than the channel itself. In [27], Narula et al. first introduced the idea of the use of partial channel knowledge at the transmitter to improve the capacity performance of wireless fading channels. Following this work, [28–34] studied several power loading schemes to improve the capacity of MIMO systems in correlated channel environments by exploiting the partial channel knowledge at the transmitter. The use of partial channel knowledge allows the transmitter to identify the dominant eigen-vectors of the channel and allocate additional power into these channels to improve the capacity of the system.

**Dense Antenna Arrays:** With the use of multiple antennas, the space becomes a new resource to be exploited towards increasing the capacity of wireless communication systems. However, in practical applications, the space allocated to transmitter and receiver devices is limited (e.g. mobile unit). As a result, we have
a fixed space to place the transmit/receive antennas. Recently, there has been interest in the capacity performance limits of spatially dense MIMO arrays. Dense arrays are created by packing a large number of antenna elements within a fixed region of space. Theoretical studies of [35–39] revealed that the capacity behavior of spatially dense MIMO systems is qualitatively different from unconstrained antenna arrays with unlimited antenna aperture sizes. In [37] it was shown that there exists a theoretical antenna saturation point for dense array MIMO systems, at which there is no capacity growth with increasing antenna numbers, and the capacity achieved from a fixed region of space is always lower than the theoretical maximum capacity available from that region. *In this thesis we address this issue and propose power loading schemes to achieve maximum capacity available from a fixed region of space.*

The capacity analysis presented thus far does not reflect the performance achieved by actual transmission systems and it only provides an upper bound at which information passes through error-free over a channel. To achieve the possible capacity increases promised by multi-antenna communication systems, a new two-dimensional encoding and decoding scheme, namely space-time coding, was introduced in the late 1990’s, which we outline in the next section.

### 1.1.2 Space-Time Coding over Multi Antenna Wireless Channels

Space-time coding is a transmit diversity scheme that can be applied on both MISO and MIMO systems. This transmit diversity scheme introduces spatial and temporal correlation between the signals transmitted from antennas in an intelligent manner to provide diversity at the receiver. Figure 1.3 shows a MIMO system with $n_T$ transmit antennas and $n_R$ receive antennas, utilizing a space-time coding scheme. At the transmitter, a block of information symbols $\mathbf{c} = [c_1, c_2, \cdots, c_T]$ of length $T$ is uniquely mapped to a $n_T \times L$ space-time codeword matrix $\mathbf{S}$

\[
\mathbf{S} = \begin{bmatrix}
s_1(1) & \cdots & s_1(L) \\
\vdots & \ddots & \vdots \\
s_{n_T}(1) & \cdots & s_{n_T}(L)
\end{bmatrix},
\]

where $s_q(k)$ is the code symbol, which belongs to a certain constellation, transmitted from $q$-th transmit antenna in the $k$-th symbol period. The received signal at
the $p$-th receive antenna in the $k$-th symbol period can be written as

$$y_p(k) = \sum_{q=1}^{n_T} h_{p,q}s_q(k) + n_p(k),$$  

(1.11)

$$p = 1, 2, \ldots, n_R, \quad n = 1, 2, \ldots, L,$$

where $n_p(k)$ is the additive noise on the $p$-th receive antenna at symbol interval $k$. The additive noise is assumed to be white and complex Gaussian distributed with mean zero and variance $N_0/2$ per dimension. Here the coefficient $h_{p,q}$ represents the random complex channel gain between the $q$-th transmit antenna and the $p$-th receive antenna, which undergoes flat-fading.

Using (1.11), the signals received at $n_R$ receive antennas over $L$ symbol periods can be written in matrix form as

$$Y = HS + N,$$

(1.12)

where $H$ denotes the $n_R \times n_T$ flat-fading channel gain matrix, $N = [n(1), n(2), \ldots, n(L)]^T$ with $n(k) = [n_1(k), n_2(k), \ldots, n_{n_R}(k)]$ and $Y = [y(1), y(2), \ldots, y(L)]^T$ with $y(k) = [y_1(k), y_2(k), \ldots, y_{n_T}(k)]^T$. For a given $Y$, the space-time decoder at the receiver will decode $c$ from $Y$ using the unique code mapping between $c$ and $S$. The generic model (1.12) can be used to represent most of the space-time coding schemes proposed in the literature by specifying different mapping structures between $c$ and $S$.

The three preliminary papers dealing with space-time coding over MIMO channels are attributed to Foschini [7], Tarokh et al. [8] and Alamouti [9]. The work of Foschini describes the layered space-time architecture to achieve the capacity improvements promised by MIMO systems. This space-time architecture utilizes

![Figure 1.3: A generic block diagram of space-time coding across a MIMO channel.](image-url)
a coding structure which divides the input source stream into sub-streams that are layered diagonally over space and time. This architecture is known as the Bell Labs Layered Space-Time architecture, more specifically the D-BLAST. The performance criteria for space-time codes in independent flat-fading channels were first established by Tarokh \textit{et al.} and they have proposed several hand-designed space-time codes (more specifically space-time trellis codes) for transmission using two transmit antennas. In [9] Alamouti introduced a two branch transmit diversity scheme, which he generalized for any number of receive antennas. Recognizing the simple encoding and decoding techniques used in this scheme, Tarokh \textit{et al.} extended Alamouti’s work by creating generalized space-time codes with orthogonal coding structure, called orthogonal space-time block codes [40].

The decoding in the above space-time coding schemes requires the receiver to estimate the channel gains between the transmitter and the receiver (coherent detection) either blindly or by using training symbols. In practice, accurate estimation of $H$ is difficult to obtain either due to the rapid changes in the channel or due to the higher overhead involved with MIMO systems. In view of this, several differential space-time coding schemes were introduced in the literature as extensions of the traditional differential phase shift keying (DPSK) [2] in flat-fading channels. Differential space-time coding schemes eliminate the necessity for channel estimation (non-coherent detection) at the receiver while maintaining the desired properties of space-time coding schemes. However, as a penalty, these differential schemes suffer a 3-dB performance loss compared to the space-time coding schemes with coherent detection at the receiver.

Following the pioneering work of Tarokh \textit{et al.}, a number of researchers have proposed numerous other types of space-time coding schemes to exploit the transmit diversity. Some major contributions found in the literature developed for coherent detection are: linear dispersion codes presented by Hassibi and Hochwald [41], super-orthogonal space-time trellis codes presented by Jafarkani and Seshadri [42], space-time turbo codes [43,44], constellation rotation codes [45], diagonal algebraic space-time (DAST) codes [46], universal space-time coding presented by Gamal and Damen [47], and developed for non-coherent detection are: unitary space-time codes by Hochwald and Sweldens [48] and cyclic and dicyclic codes by Hughes [49], generalized non-coherent orthogonal space-time block codes presented by Tarokh \textit{et al.} [50].

Space-time coding schemes proposed in the literature are derived assuming i.i.d. fading channels. However, in practice, received signals become correlated due to non-ideal antenna placement or non-isotropic scattering environment. As a re-
sult, diversity and coding gains promised by space-time coded MIMO systems are reduced, which is the primary focus of this thesis.

In following sections we briefly outline space-time trellis codes [8], orthogonal space-time block codes [9,40] and in Chapter 3, differential space-time block codes [50], which are being primarily used in this thesis. Detail derivations of these coding schemes can be found in references given.

**Space-Time Trellis Codes**

Space-time trellis codes (STTC) were originally proposed by Tarokh *et al.* in [8] as an extension to the delay diversity scheme proposed by Wittneben in [51], by removing the delay element in the transmitter. In [8], the performance criteria are established for code design assuming that the fading from each transmit antenna to each receive antenna is Rayleigh or Rician. It was shown that the delay diversity scheme is a specific case of space-time coding.

The diversity gain and the coding gain of STTCs are determined via a pairwise error probability (PEP) upper bound argument. The PEP expresses the probability of erroneously decoding the codeword \( \hat{\mathbf{S}} \) when the codeword \( \mathbf{S} \) was transmitted. It was shown in [8] that the performance of a space-time code applied on a i.i.d. MIMO fading channel is determined by the diversity advantage quantified by the rank of pair of distinct channel codeword matrices, and by the coding advantage that is quantified by the determinant of these codeword matrices.

![Figure 1.4: 4-state QPSK space-time trellis code with two transmit antennas proposed by Tarokh et al.](image)

The structure of space-time trellis codes is given by a trellis. Figure 1.4 depicts a trellis for 4-state QPSK space-time trellis code with two transmit antennas [8]. As in conventional trellis notation, each node in the trellis diagram is corresponding to a particular encoder state. In the example given in Figure 1.4 there are four
encoder states. The STTCs are decoded using the Viterbi Algorithm, which scales exponentially with the number of trellis states. Therefore, the decoder complexity increases exponentially with the diversity and the spectral efficiency of the scheme. This complexity is one of the main disadvantages of space-time trellis codes.

The original codes proposed by Tarokh et al. achieve the full diversity gain but not the optimal coding gain. Following this pioneer work, a number of researchers have searched space-time trellis coding structures that give optimal coding gains [52–54]. The coding gain achieved by these codes is around 1-2 dB higher than that of the original STTCs in [8].

**Space-Time Block Codes**

The Alamouti’s scheme [9] is the first and the most well known space-time block code which provides full transmit diversity for systems with two transmit antennas. It is well known for its simple encoding structure and fast maximum likelihood detection based on linear processing, and also its inherent protection against information loss due to spatially correlated fading. The code design followed an orthogonal block structure, providing a diversity advantage of $2n_R$, where $n_R$ is the number of receive antennas. A generalization to the Alamouti’s scheme is proposed by Tarokh et al. in [40] using the “Hurwitz-Radon Theory” on orthogonal designs where they have developed space-time block codes for both real and complex constellations for 2, 4 and 8 transmit antennas.

![Diagram of Alamouti's scheme](image)

*Figure 1.5: The two-branch diversity scheme with $n_R$ receive antennas proposed by Alamouti.*

Figure 1.5 shows the two-branch diversity scheme with $n_R$ receive antennas proposed by Alamouti in [9]. The input source to the space-time encoder is a stream of modulated symbols drawn from a real or complex constellation. In this scheme, the inputs to the space-time encoder is partitioned into groups of two symbols each. For example, two consecutive input symbols $s_1$ and $s_2$ form a group $\{s_1, s_2\}$. At a
given symbol interval, two information signals are transmitted simultaneously from two transmit antennas. During the first symbol interval, signal $s_1$ is transmitted from antenna 1 and signal $s_2$ is transmitted from antenna 2. During the second symbol interval, signal $-s_2^*$ is transmitted from antenna 1 and signal $s_1^*$ is transmitted from antenna 2. The transmitted codeword matrix $S$ over the two symbol periods can be written as

$$S = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}.$$  \hfill (1.13)

This coding scheme is capable of full-rate transmission, meaning that two symbols are transmitted over two consecutive symbol intervals (rate-1 code). Note that the two rows/columns of $S$ are orthogonal. Hence this scheme is also known as $2 \times 2$ orthogonal space-time block code.

Let $r_{11}$ and $r_{21}$ represent the received signals at receive antenna 1 and 2 during the first symbol interval, respectively and $r_{12}$ and $r_{22}$ represent the received signals at receive antenna 1 and 2 during the second symbol interval, respectively. The received signals at antennas 1 and 2, over two consecutive symbol intervals, can be written in matrix form as

$$y = Hs + n,$$  \hfill (1.14)

where $y = [y_{11}, y_{21}, y_{12}, y_{22}]^T$ is the received signal vector, $s = [s_1, s_2]^T$, $n = [n_{11}, n_{21}, n_{12}, n_{22}]^T$ is the complex white Gaussian noise random vector with zero-mean and variance $N_0/2$ per dimension, and the matrix $H$ is given by

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{12}^* & -h_{11}^* \\ h_{22}^* & -h_{21}^* \end{bmatrix}.$$  \hfill (1.15)

Note that two columns of $H$ are orthogonal. In effect, two input symbols are sent through two orthogonal vector channels. This is the main reason for two-branch STBC to provide full-rate transmission with two levels of diversity.

Assume that the receiver has perfect knowledge of the channel, then matched filtering is applied to the received signal vector $y$, giving the new signal vector $\hat{y}$

$$\hat{y} = H^*Hs + H^*n,$$

$$= H_F^2s + \bar{n},$$  \hfill (1.16)
where $H_F^2 = |h_{11}|^2 + |h_{12}|^2 + |h_{21}|^2 + |h_{22}|^2$ represents the squared Frobenius norm of the MIMO channel matrix $H$, and $\pi = \mathcal{H}^* n$. Since the column vectors of $\mathcal{H}$ are orthogonal, it can be easily shown that elements of $\pi$ are independent and identically distributed with zero-mean and variance $H_F^2 N_0$. This allows receivers to be designed with less complexity based on only the linear processing at the receiver. Assuming all symbol pairs are equiprobable and noise vector $n$ is Gaussian distributed, the maximum likelihood detection rule at the receiver is given by

$$\hat{s} = \arg \min_{\hat{s} \in \mathcal{S}} \| \hat{y} - H_F^2 \hat{s} \|.$$  \hspace{1cm} (1.17)

### Error Performance Analysis

When designing space-time codes, the main assumption being made is that the channel gains between the transmitter and the receiver antennas undergo independent flat-fading. However, as pointed out in Section 1.1, insufficient antenna spacing and lack of scattering cause the channel gains to be correlated. Therefore, the assumption of uncorrelated fading model will in general not be an accurate description of realistic fading channels in practice. Several approaches have been found in the literature, where the performance of space-time codes have been investigated for correlated fading channels [55–66]. However, none of these results explicitly address the effects of the physical constraints, such as antenna aperture size, antenna geometry and scattering distribution parameters, and also the independent effects of these physical constraints on the performance of space-time codes. In this thesis we explicitly address the individual effects of these physical constraints on the performance of both coherent and non-coherent space-time codes.

### 1.1.3 Space-Time Channel Modelling

To analyze the performance of MIMO systems under realistic channel conditions, a model is required to represent the underlying multipath channel between the transmitter and the receiver antenna arrays. A large number of modelling approaches have been presented in the literature. These modelling approaches may be divided into two general categories: non-physical models [5,6,20,67–70] and physical models [19,71–92]. We now present a brief overview of recent developments in modelling of multipath fading channels.

Non-physical models aimed at modelling the channel coefficients from each transmit to each receive antenna and the correlations between them. In general, non-physical models are developed based on the signal correlations at different
antennas at the receiver and transmitter arrays. In these models, the channel covariance matrix determines the diversity order of the system. A special case of the non-physical model is the i.i.d. channel (i.e., covariance matrix of the channel is given by a identity matrix), which was used in [5, 6] to analyze the capacity performances of MIMO systems. Here the channel is modelled based on the SISO multipath fading models where Rayleigh, Ricean, Nakagami distributions are used to model channel coefficients.

A widely used channel covariance matrix model for non-line-of-sight channels is the Kronecker model [20,67–70,90,93]. The channel covariance matrix is defined as

\[ R_H = \mathcal{E}\{h^\dagger h\} = R_{Rx} \otimes R_{Tx}, \]  

where \( h = (\text{vec}\{H^T\})^T \), \( R_{Rx} \) and \( R_{Tx} \) are the correlation matrices observed at the transmitter and receiver, respectively.

\[ R_{Rx} = \mathcal{E}\{h^j h^j\dagger\}, \quad \text{for } j = 1, 2, \cdots, n_T, \]
\[ R_{Tx} = \mathcal{E}\{h_i h_i\dagger\}, \quad \text{for } i = 1, 2, \cdots, n_R, \]

where \( h^j \) is the \( j \)-th column of \( H \) and \( h_i \) is the \( i \)-th row of \( H \). This correlation model leads to a narrowband channel model

\[ H = (R_{Rx})^{1/2}G(R_{Tx})^{T/2}, \]  

where \( G \) is a \( n_R \times n_T \) i.i.d. complex Gaussian random matrix with zero mean unit variance elements. In [68], this model was extended to a wide-band channel model using a tapped-delay-line approach.

The Kronecker model (1.18) simplifies the full channel covariance matrix with \( n_T^2 n_R^2 \) entries to a matrix with \( n_R^2 + n_T^2 \) entries by forcing the correlation at one end of the channel to be independent of the correlation at the other end of the channel (i.e., the channel is separable). However the channel measurement results presented in [94] showed that this separation of correlation results in some deficiencies of the model compared to the measured MIMO channels. Thus an immediate question to ask is: “Under what physical scattering conditions can the Kronecker model be used to represent the MIMO channel?” which will be addressed in this thesis from a theoretical perspective.

It should be noticed that non-physical models are easy to simulate and they provide accurate channel characterization supporting a particular modelling ap-
proach. However, non-physical channel models do not provide any useful insights to the physical characteristics of MIMO channels and their implications to the MIMO performance. In contrast, physical models focus on parameters such as angular and delay distributions of leaving and arriving signals [71–77, 80], distribution of scattering bodies surrounding the transmitter and receiver antenna arrays [19,78–90] and antenna configurations (or geometry) at the transmitter and receiver antenna arrays [79–83]. However, the disadvantage of physical models is that they are relatively complex and also complicated to parameterize. With some physical models e.g., [71–77, 80], the model is developed based on measurement data collected from field tests using radio equipments. Therefore they suffer from being specific to a particular test environment and the results also depend on the measuring equipments [95].

Two of the most well known physical narrowband MIMO channel models are the “one-ring” model and the “two-ring” model. In [79], a SIMO “one-ring” channel model has been proposed based on the Clarke/Jakes classic model [78] for a SISO channel assuming scatterers around the receiver (MU) are uniformly distributed\(^3\) on a ring and the transmitter (BS) is absent of local scatterers (BS antennas are elevated above the ground). In “two-ring” models, it is assumed that both the BS and MU are surrounded by local scatterers [85]. The “one-ring” model is generally applicable to microcellular/macrocellular channel scenarios whereas the “two-ring” model is more applicable to indoor wireless communication scenarios.

It has been argued in [96–98] that the assumption of uniform scattering is often hard to satisfy in real word scattering channel scenarios, and experimentally demonstrated in [99–101] that scattering encountered in many wireless channel environments is non-isotropic. Non-isotropic scattering distributions model the multipath as energy arriving from a particular direction (angle) with some angular spread\(^4\). Similar to the “one-ring” model proposed in [79], a narrowband space-time MIMO channel model\(^5\) was proposed in [80] where a von-Mises distribution is used to model the non-isotropic scattering at the MU. Several other scattering distributions (or angular power distributions) have been proposed in the literature to model non-isotropic scattering at antenna arrays. Some such distributions are: uniform-limited [102], Cosine [102, 103], truncated Laplacian [104] and truncated Gaussian [105].

\(^3\)each scatterer on the ring has an independent, uniformly distributed initial phase over \([-\pi, \pi]\]. In effect the impinging signal power is uniform over all angle of arrivals (isotropic scattering).

\(^4\)defined as the standard deviation of the scattering distribution.

\(^5\)channel models presented in [79] and [80] will be discussed in more details in Chapter 4.7.
propagation scenarios is the “distributed scattering model” [84], where signals pass from the transmit array, through local scatterers at the transmitter region, through local scatterers at the receiver region and to the receive array. In this model, the local scatterers at the transmit and receive side form virtual arrays with large spacing, and the array length is determined by the angular spread of the scatterers. The channel matrix is given by

$$H = \frac{1}{\sqrt{S}} R_{\theta_r,d_r}^{1/2} G_r R_{\theta_s,2D_r/s}^{1/2} G_t R_{\theta_t,d_t}^{T/2},$$

where $S$ is the number of scatterers at each end of the channel, $G_r$ and $G_t$ are random matrices with i.i.d. zero mean complex Gaussian elements, $R_{\theta_r,d_r}$ and $R_{\theta_t,d_t}$ are the correlation matrices observed from the transmitter and the receiver, respectively, and $R_{\theta_s,2D_r/s}$ is the correlation matrix that gives the angular diversity between the local scattering arrangements. Using this model it has been shown that when the angular spread of the impinging signal is small and/or the spacing between adjacent antenna elements in an array is small, the correlation matrix will lose rank, and as a result the MIMO channel will be rank deficient. Some other useful channel models found in the literature include: the virtual channel model [81], the extended Saleh-Valenzuela model [71], the Electro-Magnetic (EM) scattering model [91], the the COST 259 directional channel model [83]. The reader is referred to the given references for details regarding these channel models.

In this thesis we primarily use the continuous flat-fading spatial channel model proposed in [106] based on the underlying physics of the free space propagation. This spatial model separates the physical MIMO channel into three distinct regions of signal propagation: the scatterer free region around the transmitter antenna array, the scatterer free region around the receiver antenna array and the complex random scattering environment which is the complement of the union of two antenna array regions. With this separation of the physical channel, the MIMO channel matrix is decomposed into product of three matrices, where two of them are fixed and known for a given antenna placement and the other represents the parameters of the random scattering environment. A detail derivation of this spatial model is given in Chapters 2.2 and 3.3. In addition, we extend this continuous spatial channel model to a time-selective channel and address the issues such as the effect of Doppler spread (due to the movement of antenna arrays) in general scattering environments, the effect of multi-modal distributed scattering distributions and the effect of inter-dependency between transmit and receive angles.
1.2 Questions to be Answered

In this thesis following open questions are answered:

- What physical factors determine the performance in terms of diversity and coding gain of a space-time code and can we quantify the effects of these physical factors?

- Which antenna geometries (or configurations) do not diminish the diversity advantage promised by a space-time code in a general scattering environment?

- Can we eliminate the detrimental effects of non-ideal antenna placement and non-isotropic scattering on the performance of space-time communication systems?

- Is the popular Alamouti’s space-time block code susceptible to spatial fading correlation effects?

- Can we achieve the maximum theoretical capacity available from a fixed region of space?

- Does the feedback of partial CSI help to improve the capacity of dense MIMO systems in general scattering environments?

- Can we develop a space-time channel model that captures physical antenna positions, motion of the antenna arrays and joint statistical properties of scattering environments surrounding the transmitter and receiver regions?

- Under what circumstances can the Kronecker model be used to model the covariance matrix of the MIMO channel?

1.3 Content and Contribution of Thesis

Chapter 2 analyzes the performance of orthogonal space-time block codes in realistic propagation conditions using an analytical model for fading channel correlation which accounts for antenna separation, antenna placement, along with non-isotropic scattering environment parameters. This chapter begins by introducing the channel correlation model, which is derived based on a recently developed spatial channel model, and deriving channel correlation coefficients at the transmitter and the receiver for a number of commonly used scattering distributions. Using this channel correlation model we study the
impact of the space on the performance of orthogonal STBC. Furthermore, we analyze how the non-isotropic parameters of a scattering distribution effects the performance of orthogonal STBC. Finally, by applying the plane wave propagation theory in free space, the orthogonal STBC is analyzed from a physical perspective. Mainly we study the radiation patterns generated at the transmitter region for the Alamouti scheme with two transmit antennas. We show that radiation patterns generated in the transmit region over the two symbol intervals of the Alamouti code are orthogonal and also two different sets of transmit modes\textsuperscript{6} are excited during the two symbol intervals.

Chapter 3 derives analytical expressions for the exact pairwise error probability (PEP) and PEP upper-bound of coherent and non-coherent space-time coded MIMO systems operating over spatially correlated fading channels, using a moment-generating function-based approach. These analytical expressions fully account for antenna separation, antenna geometry (Uniform Linear Array, Uniform Grid Array, Uniform Circular Array, etc.) and surrounding scattering distributions, both at the receiver and the transmitter antenna array apertures. Therefore, these analytical expressions serve as a set of tools to analyze or predict the performance of space-time codes under realistic channel conditions. Using these new PEP expressions, we quantify the degree of the effect of antenna spacing, antenna geometry and angular spread on the diversity advantage given by a space-time code. It is shown that the number of antennas that can be employed in a fixed antenna aperture without diminishing the diversity advantage of a space-time code is determined by the size of the antenna aperture, antenna geometry and the richness of the scattering environment. PEP performance, BER performance and frame-error performance of coherent 4-state QPSK STTC with 2 transmit antennas, coherent 16-state QPSK STTC with 3 transmit antennas, coherent 64-state QPSK STTC with 4 transmit antennas and rate-1 2×2 differential STBC are investigated for a number of spatial scenarios at the receiver and the transmitter to support the theoretical analysis presented.

Chapter 4 introduces the novel use of linear spatial precoding (or power-loading) based on fixed and known parameters of MIMO channels to ameliorate the effects of \textit{non-ideal antenna placement} on the performance of coherent and non-coherent space-time codes by exploiting the spatial dimension of the MIMO channel model introduced in Chapter 2. Antenna spacing and an-

\textsuperscript{6}The set of modes form a basis of functions for representing a multipath wave field.
tenna placement are considered as fixed parameters, which are readily known at the transmitter. With this design, the precoder virtually arranges the antennas into an optimal configuration so that the spatial correlation between all antenna elements is minimum. We also derive precoding schemes to exploit non-isotropic scattering distribution parameters of the scattering channel to improve the performance of space-time codes in non-isotropic scattering environments. These schemes require the receiver to estimate the non-isotropic parameters and feed them back to the transmitter. The performance of both precoding schemes is assessed when applied on 1-D antenna arrays and 2-D antenna arrays.

Chapter 5 presents a fixed power loading scheme to maximize the capacity of spatially constrained dense antenna arrays. Similar to the fixed precoding scheme presented in Chapter 4 for space-time coded MIMO systems, the power loading is based on previously unutilized channel state information contained in the antenna locations. For a large number of transmit antennas, we numerically show that unlike the equal power loading scheme, the proposed fixed scheme is capable of achieving the theoretical maximum capacity available for a fixed region of space. We further develop a power loading scheme to exploit the non-isotropic scattering distribution parameters at the transmitter to improve the capacity performance of dense MIMO systems in non-isotropic scattering environments. We also analyze the correlation between different modal orders generated at the transmitter region due to the spatially constrained antenna arrays in non-isotropic scattering environments and show that adjacent modes significantly contribute to higher correlation at the transmitter region. Motivated by this observation, we propose a third power loading scheme which reduces the effects of correlation between adjacent modes at the transmitter region by nulling power onto adjacent transmit modes.

Chapter 6 develops a general non-separable space-time channel model for downlink transmission in a mobile multiple antenna communication system. The model is derived based on the theory of plane wave propagation in free-space. This chapter begins by deriving channel coefficients for a general scattering environment along with transmitter and receiver space-time cross correlation coefficients. Using a truncated modal expansion of plane wave in two-dimensional space, the space-time channel is separated into deterministic and random parts. The deterministic parts capture physical antenna
positions and the motion of the mobile unit (velocity and the direction), and the random part captures the random scattering environment modeled using a joint bi-angular power distribution parameterized by the transmit and receive angles. The well-known “Kronecker” model is recovered as a special case when this distribution is a separable function. Expressions for space-time cross correlation and space-frequency cross spectra are developed for a number of scattering distributions using Gaussian and Morgenstern’s family of multivariate distributions. We also introduce the concept of multi-modal power distributions surrounding the transmitter and receiver antenna arrays. Using our non-separable model we claim that well-known “Kronecker” model overestimates MIMO system performance whenever there is more than one scattering cluster (multi-modal distribution).

Chapter 7 gives an overview of the results presented and suggestions for future research work.
Chapter 2

Orthogonal Space-Time Block Codes: Performance Analysis

2.1 Introduction

The first space-time block coding (STBC) scheme was proposed by Alamouti in [9] and a generalization to this scheme was proposed by Tarokh et al. in [40] based on complex orthogonal designs. In [107] performance of the STBC is investigated for uncorrelated channel gains between transmit and receive antennas.

In general, the presence of spatial correlation between antenna elements will degrade the performance of space-time coding schemes. However, the orthogonal STBC has an inherent protection against information loss associated with spatially correlated fading [9]. This has motivated the investigation of the degree of fading resistance provided by orthogonal STBC when spatial correlation is present. Spatial correlation has two sources i) antenna placement (particularly antenna separation), and ii) scattering distribution (isotropic and non-isotropic).

There are few studies reported in the literature which consider the effects of spatial correlation on the performance of orthogonal space-time block codes [58, 108–110]. However these studies do not provide insights into the physical factors determining the performance of orthogonal space-time block codes operating over spatially correlated fading channels, in particular the effects of antenna spacing, spatial geometry of the antenna arrays and the scattering environment parameters such as mean angle of arrival (AOA), mean angle of departure (AOD) and the angular spreads of the azimuth power distributions at the receiver and transmitter antenna arrays.

In contrast, in this chapter the impact on the bit-error rate (BER) performance of orthogonal STBC due to spatial correlation is investigated using an analytic
model for spatial correlation which fully accounts for antenna separation, antenna placement, along with non-isotropic scattering environment parameters. The analytical model for spatial correlation between channel gains is derived based on the spatial channel model proposed in [106]. In this correlation model, channel correlation coefficients depend on the antenna spacing and antenna placement, along with the non-isotropic parameters of the scattering distribution. Following the work of receiver correlation modelling in [111], a closed-form series expansion for channel correlation coefficients is derived that converges in a low number of terms.

Using this spatial correlation model we show that the impact of the space is limited on the BER performance of orthogonal STBC. That is, most of the BER improvement of the orthogonal STBC can be attributed to “time-coding” rather than to “space-coding”. Also we investigate how the non-isotropic parameters of an azimuth power distribution effects the BER performance of orthogonal STBC. An empirical expression for the antenna separation is derived for a $2 \times 2$ MIMO system where the performance of orthogonal STBC is sufficiently close to the optimal under a given scattering environment. Finally, by applying the plane wave propagation theory in free space, we analyze the orthogonal STBC from a physical perspective. Mainly we study the radiation patterns generated at the transmitter when the Alamouti scheme with two transmit antennas is used. First we review the spatial channel model proposed in [106] for slow-fading channels.

### 2.2 Spatial Channel Model

Consider a MIMO system consisting of $n_T$ transmit antennas located at positions $\mathbf{x}_q, q = 1, 2, \cdots, n_T$ relative to the transmitter array origin, and $n_R$ receive antennas located at positions $\mathbf{z}_p, p = 1, 2, \cdots, n_R$ relative to the receiver array origin. $r_T \geq \max \| \mathbf{x}_q \|$ and $r_R \geq \max \| \mathbf{z}_p \|$ denote the radius of spheres that contain all the transmit and receive antennas, respectively. We assume that scatterers are distributed in the far field from the transmitter and receiver antennas and regions containing the transmit and receive antennas are distinct, as shown in Figure 2.1. Therefore, we define scatter-free transmitter and receiver spheres of radius $r_{TS}(> r_T)$ and $r_{RS}(> r_R)$, respectively.

Let $\mathbf{s} = [s_1, s_2, \cdots, s_{n_T}]^T$ be the column vector of baseband transmitted signals from $n_T$ transmit antennas over a single symbol interval. Then the signal leaving
2.2 Spatial Channel Model

Figure 2.1: A General scattering model for a flat fading MIMO system. $r_T$ and $r_R$ are the radius of spheres which enclose the transmitter and the receiver antennas, respectively. $g(\hat{\phi}, \hat{\varphi})$ represents the gain of the complex scattering environment for signals leaving the transmitter scattering free region from direction $\hat{\phi}$ and entering at the receiver scattering free region from direction $\hat{\varphi}$.

the scatter-free transmitter aperture along direction $\hat{\phi}$ is given by

$$\Phi(\hat{\phi}) = \sum_{q=1}^{n_T} s_q e^{ikx_q \cdot \hat{\phi}},$$  \hspace{1cm} (2.1)

where $k = 2\pi/\lambda$ is the wave number with $\lambda$ the wave length. The signal entering scatter-free receiver aperture from direction $\hat{\varphi}$ can be written as

$$\Psi(\hat{\varphi}) = \int_{S^2} \Phi(\hat{\phi}) g(\hat{\phi}, \hat{\varphi}) ds(\hat{\phi}),$$

$$= \sum_{q=1}^{n_T} s_q \int_{S^2} g(\hat{\phi}, \hat{\varphi}) e^{ikx_q \cdot \hat{\phi}} ds(\hat{\phi}),$$  \hspace{1cm} (2.2)

where $g(\hat{\phi}, \hat{\varphi})$ is the effective random scattering gain function for a signal leaving from the transmitter scatter-free aperture at a direction $\hat{\phi}$ and entering the receiver scatter-free aperture from a direction $\hat{\varphi}$ and $ds(\hat{\phi})$ is a surface element of the unit sphere $S^2$ with unit normal $\hat{\phi}$. Since the scatterers are assumed far-field to the
receiver region, signals arriving at the receiver array will be plane waves. Therefore, the received signal at the \( p \)-th receive antenna at position \( z_p \) is given by

\[
y_p = \int_{S^2} \Psi(\hat{\varphi}) e^{-ikz_p \hat{\varphi}} \text{ds}(\hat{\varphi}) + n_p,
\]

\[
= \sum_{q=1}^{n_T} s_q \int\!\!\!\!\!\!\int_{S^2 \times S^2} g(\hat{\phi}, \hat{\varphi}) e^{ikx_q \hat{\phi}} e^{-ikz_p \hat{\varphi}} \text{ds}(\hat{\phi}) \text{ds}(\hat{\varphi}) + n_p, \tag{2.3}
\]

where \( n_p \) is the additive white Gaussian noise at the \( p \)-th receiver antenna.

Let \( y = [y_1, y_2, \cdots, y_{n_R}]^T \) and \( n = [n_1, n_2, \cdots, n_{n_T}]^T \), then (2.3) can be written in vector form as

\[
y = Hs + n,
\]

where \( H \) represents the \( n_R \times n_T \) channel matrix with \( (p, q) \)-th element

\[
h_{p,q} = \int\!\!\!\!\!\!\int_{S^2 \times S^2} g(\hat{\phi}, \hat{\varphi}) e^{ikx_q \hat{\phi}} e^{-ikz_p \hat{\varphi}} \text{ds}(\hat{\phi}) \text{ds}(\hat{\varphi}), \tag{2.4}
\]

representing the complex channel gain between the \( p \)-th receive antenna and \( q \)-th transmit antenna. We assume that channel gains \( h_{p,q} \) are normalized such that \( \mathbb{E} \{ |h_{p,q}|^2 \} = 1 \), and hence the random scattering gains \( g(\hat{\phi}, \hat{\varphi}) \) are also normalized such that

\[
\int\!\!\!\!\!\!\int_{S^2 \times S^2} \mathbb{E} \{ |g(\hat{\phi}, \hat{\varphi})|^2 \} \text{ds}(\hat{\phi}) \text{ds}(\hat{\varphi}) = 1.
\]

### 2.3 Transmitter and Receiver Spatial Correlation for General Distributions of Far-field Scatterers

To define the spatial correlation coefficients at the transmitter and receiver antenna arrays we assume:

1. all antenna elements in the receiver and the transmitter antenna arrays have the same polarization and the same radiation pattern identical to each other.

2. correlation between two antenna elements in one array is independent of an antenna element selected from the other array as each element within an antenna array illuminate the same scattering environment. Therefore the
power arriving at the second array from each transmit antenna in the first array will have the same azimuth power distribution [69].

Note that the second assumption above leads to the well-known Kronecker channel correlation model [69]. In Chapter 7, we analyze the impact of this assumption on the MIMO system performance for various scattering distribution scenarios.

Using (2.4), the correlation coefficient between two arbitrary channel paths connecting two input-output pairs of antennas can be written as

$$
\rho_{p,p}^{q,q'} \triangleq \mathcal{E}\{h_{p,q}h_{p',q'}^*\},
= \int_4 \mathcal{E}\{g(\hat{\phi}, \hat{\psi})g^*(\hat{\phi}', \hat{\psi}')\} e^{-ik(z_p - z_p')\cdot \hat{\psi}'} e^{ik(x_q - x_q')\cdot \hat{\psi}} ds(\hat{\phi})ds(\hat{\psi})ds(\hat{\phi}')ds(\hat{\psi}'),
$$

(2.5)

where we have introduced the shorthand $\int_4 \triangleq \int\int\int\int_S^2 \times S^2 \times S^2 \times S^2$.

Assuming a wide-sense stationary zero-mean uncorrelated scattering environment, the second-order statistics of the scattering gain function $g(\hat{\phi}, \hat{\psi})$ can be defined as

$$
\mathcal{E}\{g(\hat{\phi}, \hat{\psi})g^*(\hat{\phi}', \hat{\psi}')\} \triangleq G(\hat{\phi}, \hat{\psi})\delta(\hat{\phi} - \hat{\phi}')\delta(\hat{\psi} - \hat{\psi}'),
$$

where $G(\hat{\phi}, \hat{\psi}) = \mathcal{E}\{|g(\hat{\phi}, \hat{\psi})|^2\}$ which satisfies the normalization

$$
\int S^2 \times S^2 G(\hat{\phi}, \hat{\psi})ds(\hat{\phi})ds(\hat{\psi}) = 1.
$$

With the above assumption, the correlation coefficient, $\rho_{q,q'}^{p,p'}$, can be simplified to

$$
\rho_{q,q'}^{p,p'} = \int S^2 \times S^2 G(\hat{\phi}, \hat{\psi})e^{-ik(z_p - z_p')\cdot \hat{\psi}'} e^{ik(x_q - x_q')\cdot \hat{\psi}} ds(\hat{\psi})ds(\hat{\phi}).
$$

Then, letting $q = q'$, the correlation between $p$-th and $p'$-th receive antennas due to the $q$-th transmit antenna is given by

$$
\rho^{p,p'} = \int S^2 \mathcal{P}_{\text{Rx}}(\hat{\psi})e^{-ik(z_p - z_p')\cdot \hat{\psi}} ds(\hat{\psi}),
$$

(2.6)

where $\mathcal{P}_{\text{Rx}}(\hat{\psi}) = \int G(\hat{\phi}, \hat{\psi})ds(\hat{\phi})$ is the normalized average power received from direction $\hat{\psi}$. Here we see that correlation coefficients at the receiver is independent of the antenna selected from transmit antenna array. Similarly, letting $p = p'$, we
can write the correlation between \( q \)-th and \( q' \)-th transmit antennas due to the \( p \)-th receive antenna as

\[
\rho_{q,q'} = \int_{\mathbb{S}^2} P_{\text{Tx}}(\hat{\phi}) e^{ik(x_q - x_{q'})} \hat{\phi} ds(\hat{\phi}),
\]  

(2.7)

where \( P_{\text{Tx}}(\hat{\phi}) = \int G(\hat{\phi}, \hat{\phi}) ds(\hat{\phi}) \) is the normalized average power transmitted in to the direction \( \hat{\phi} \). As for the receiver channel correlation, we can observe that channel correlation at the transmitter is independent of the antenna selected from receiver antenna array.

Denoting the \( q \)-th column of MIMO channel matrix \( H \) as \( H_q \), the \( n_R \times n_R \) receiver channel correlation matrix can be defined as

\[
R_{\text{Rx}} \triangleq \mathcal{E} \{ H_q H_q^\dagger \},
\]  

(2.8)

where \((p, p')\)-th element of \( R_{\text{Rx}} \) is given by (2.6) above. Similarly, the transmitter channel correlation matrix can be defined as

\[
R_{\text{Tx}} \triangleq \mathcal{E} \{ H_p^\dagger H_p \},
\]  

(2.9)

where \( H_p \) is the \( p \)-th row of \( H \). \((q, q')\)-th element of \( R_{\text{Tx}} \) is given by (2.7) and \( R_{\text{Tx}} \) is a \( n_T \times n_T \) matrix.

**Kronecker Model as a Special Case**

The correlation between two distinct antenna pairs can be written as the product of corresponding channel correlation at the transmitter and the channel correlation at the receiver, i.e.,

\[
\rho_{p,q',p',q} = \rho_{q,q'} \rho_{p,p'}.
\]  

(2.10)

Facilitated by (2.10), we can write the covariance matrix of the MIMO channel \( H \) as the Kronecker product between the receiver channel correlation matrix and the transmitter channel correlation matrix,

\[
R_H = \mathcal{E} \{ h^\dagger h \} = R_{\text{Rx}} \otimes R_{\text{Tx}},
\]  

(2.11)
where $\mathbf{h} = (\text{vec} \{ \mathbf{H}^T \})^T$ and $\otimes$ is the matrix Kronecker product, defined by
\[
\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix}
  a_{11} \mathbf{B} & a_{12} \mathbf{B} & a_{13} \mathbf{B} & \cdots \\
  a_{21} \mathbf{B} & a_{22} \mathbf{B} & a_{23} \mathbf{B} & \cdots \\
  a_{31} \mathbf{B} & a_{32} \mathbf{B} & a_{33} \mathbf{B} & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\] (2.12)

Note that (2.10) holds only for class of scattering environments where the power distribution of scattering channel satisfies [69,112]
\[
G(\hat{\phi}, \hat{\rho}) = P_{Tx}(\hat{\phi})P_{Rx}(\hat{\rho}).
\] (2.13)

### 2.3.1 Two Dimensional Scattering Environment

Consider the situation where the multipath is restricted to the azimuth plane only (2-D scattering environment), having no field components arriving at significant elevations. In this case, the modal expansion (also known as the Jacobi-Anger expansion) of plane wave $e^{ik\mathbf{y}\cdot\hat{\phi}}$ is given by [113, page 67]
\[
e^{ik\mathbf{y}\cdot\hat{\phi}} = J_0(k \| \mathbf{y} \|) + 2 \sum_{m=1}^{\infty} i^m J_m(k \| \mathbf{y} \|) \cos(m\theta),
\] (2.14a)
\[
= \sum_{m=-\infty}^{\infty} J_m(k \| \mathbf{y} \|) e^{-im(\phi_y - \frac{\pi}{2})} e^{im\phi},
\] (2.14b)

where $\theta$ denotes the angle between $\mathbf{y}$ and $\hat{\phi}$, $J_m(\cdot)$ is the integer order $m$ Bessel function, $\mathbf{y} \equiv (\| \mathbf{y} \|, \phi_y)$ and $\hat{\phi} \equiv (1, \phi)$ in the polar coordinate system. Bessel functions $J_m(\cdot)$ for $|m| > 0$ exhibit a spatially high pass character ($J_0(\cdot)$ is spatially low pass), that is, for fixed order $m$, $J_m(\cdot)$ starts small and reaches to its maximum at arguments $x \approx O(m)$ before starts decaying slowly. It was shown in [114] that $J_m(k \| \mathbf{y} \|) \approx 0$ for $|m| > ke \| \mathbf{y} \| / 2$. Therefore, we can truncate the series (2.14b) by $2M + 1$ terms where $M = \lceil ke \| \mathbf{y} \| / 2 \rceil$ with $e \approx 2.7183$.

Using the truncated expansion of plane wave $e^{ik\mathbf{y}\cdot\hat{\phi}}$ we can write
\[
e^{-ik(\mathbf{z}_p - \mathbf{z}_{p'})\cdot\hat{\phi}} = \sum_{m=-M_R}^{M_R} J_m(k \| \mathbf{z}_p - \mathbf{z}_{p'} \|) e^{im(\varphi_{p,p'} - \frac{\pi}{2})} e^{-im\varphi},
\] (2.15)

where $\varphi_{p,p'}$ denotes the angle of the vector connecting $\mathbf{z}_p$ and $\mathbf{z}_{p'}$, and $M_R =$
where $\phi_{q,q'}$ denotes the angle of the vector connecting $x_q$ and $x_{q'}$, and $M_T = \lceil ked_t/2 \rceil$ with $d_t \geq \|x_q - x_{q'}\|$. 

Substitution of (2.15) into (2.6) gives the channel correlation coefficients at the receiver for a 2-D scattering environment as

$$
\rho_{p,p'} = \sum_{m=-M_R}^{M_R} \alpha_m J_m(k \|z_p - z_{p'}\|) e^{im(\phi_{p,p'} - \frac{\pi}{2})}.
$$

(2.17)

The coefficients $\alpha_m$ characterize the 2-D scattering environment surrounding the receiver antenna array and are given by

$$
\alpha_m = \int_{S^1} P_{Rx}(\varphi) e^{-im\varphi} d\varphi,
$$

(2.18)

where $P_{Rx}(\varphi)$ is the normalized angular power distribution at the receiver antenna array. $P_{Rx}(\varphi)$ is commonly known as the power azimuth spectrum (PAS) [69], or power azimuth distribution (PAD) [11]. Substitution of (2.16) into (2.7) gives the channel correlation coefficients at the transmitter for a 2-D scattering environment as

$$
\rho_{q,q'} = \sum_{m=-M_T}^{M_T} \beta_m J_m(k \|x_q - x_{q'}\|) e^{im(\phi_{q,q'} - \frac{\pi}{2})}.
$$

(2.19)

The coefficients $\beta_m$ characterize the 2-D scattering environment surrounding the transmitter antenna array and are given by

$$
\beta_m = \int_{S^1} P_{Tx}(\phi) e^{im\phi} d\phi,
$$

(2.20)

where $P_{Tx}(\phi)$ is the normalized angular power distribution at the transmitter antenna array.

Note that, for a 3-D scattering environment, we can use the three dimensional Jacobi-Anger expansion of plane waves given in [113, page 32] to derive expressions for channel correlation coefficients at the transmitter and receiver antennas arrays. Related work can be found in [111]. In this thesis we constraint our investigations...
and analysis only to the 2-D scattering environment. However, following [111,113], our results can be extended to 3-D scattering environments.

### 2.3.2 Non-isotropic Scattering Environments and Closed-Form Scattering Environment Coefficients

Equations (2.17) and (2.19) are the general receiver and transmitter spatial correlation equations for any 2D scattering environment. The scattering environment surrounding the receiver and transmitter antenna arrays are characterized by $\alpha_m$ (2.18) and $\beta_m$ (2.20), respectively. Note that scattering environment coefficients $\alpha_m$ and $\beta_m$ are determined by the non-isotropic distribution (or the power azimuth distribution) functions at the receiver and transmitter regions, respectively.

Non-isotropic distributions are characterized by the mean angle of arrival $\varphi_0$ (or departure $\varphi_0$) and the angular spread $\sigma$, defined as the standard deviation of the distribution. Several azimuthal power distributions have been proposed in the literature for modelling the non-isotropic scattering in 2D space. Some commonly used non-isotropic scattering distributions include uniform limited [102], truncated Gaussian [105], truncated Laplacian [104] and von-Mises [115]. In the work on receiver correlation modelling, [111] has derived $\alpha_m$ in closed-form for these common distributions. In the following, we introduce these common distributions and corresponding scattering coefficients $\alpha_m$ and $\beta_m$ for each distribution. These distributions and scattering environment coefficients will be used later in this thesis to analyze the performance of space-time coded systems in non-isotropic scattering environments. In the following, all the distributions are defined for receiver side only, but scattering environment coefficients are given for both sides.

#### Uniform-limited Distributed Field

When the energy is arriving uniformly from a restricted range of azimuth angles $\pm \Delta$ around a mean angle of arrival (AOA) $\varphi_0 \in [-\pi, \pi)$, we have the uniform limited distribution [102]

$$P(\varphi) = \begin{cases} \frac{1}{2\Delta}, & |\varphi - \varphi_0| \leq \Delta; \\ 0, & \text{elsewhere}. \end{cases}$$

(2.21)

where $\Delta$ represents the non-isotropic parameter of the distribution, which is related to the standard deviation of the distribution (angular spread $\sigma = \Delta/\sqrt{3}$). Using
it is straight-forward to derive
\[ \alpha_m = e^{-im\phi_0 \text{sinc}(m\Delta)}. \] (2.22)

Similarly, scattering coefficients at the transmitter are given by
\[ \beta_m = e^{im\phi_0 \text{sinc}(m\Delta)}, \] (2.23)
where \( \phi_0 \) is the mean angle of departure (AOD).

**von-Mises Distributed Field**

The distribution function is given by [115]
\[ P(\varphi) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\varphi - \varphi_0)}, \quad \varphi \in [-\pi, \pi), \] (2.24)
where \( \kappa \geq 0 \) represents the non-isotropy factor of the distribution and \( I_m(\cdot) \) is the modified Bessel function of the first kind. Note that, \( \kappa = 0 \) represents isotropic scattering and the distribution becomes \( P(\varphi) = 1/2\pi \). The scattering coefficients \( \alpha_m \) at the receiver are given by [111]
\[ \alpha_m = e^{-im\phi_0} \frac{I_{-m}(\kappa)}{I_0(\kappa)}. \] (2.25)
Scattering coefficients \( \beta_m \) at the transmitter are given by
\[ \beta_m = e^{im\phi_0} \frac{I_m(\kappa)}{I_0(\kappa)}. \] (2.26)
Note that \( I_{-m}(\cdot) = I_m(\cdot) \).

**Truncated Laplacian Distributed Field**

The distribution function is given by [104]
\[ P(\varphi) = K_L e^{-\sqrt{2} |\varphi - \varphi_0|/\sigma_L}, \quad \varphi \in [-\pi, \pi), \] (2.27)
where \( K_L \) is the normalization constant such that \( \int_{-\pi}^{\pi} P(\varphi) d\varphi = 1 \), \( \sigma_L \) is the standard deviation of the non-truncated distribution, which is related to the angular spread \( \sigma \) at the receiver, and \( \varphi_0 \) is the mean AOA. In this case, the normalization
constant $K_L$ is given by

$$K_L = \frac{1}{\sqrt{2\sigma_L(1 - e^{-\sqrt{2}\pi/\sigma_L})}},$$

and the scattering coefficients $\alpha_m$ at the receiver are given by

$$\alpha_m = e^{-im\phi_0} \frac{\left(1 - e^{-\sqrt{2}\pi/\sigma_L}(\sqrt{2}\cos m\pi - m\sigma_L \sin m\pi)\right)}{\sqrt{2}(1 - e^{-\sqrt{2}\pi/\sigma_L})(1 + m^2\sigma_L^2/2)}. \quad (2.28)$$

Similarly, scattering coefficients $\beta_m$ at the transmitter are given by

$$\beta_m = e^{im\phi_0} \frac{\left(1 - e^{-\sqrt{2}\pi/\sigma_L}(\sqrt{2}\cos m\pi - m\sigma_L \sin m\pi)\right)}{\sqrt{2}(1 - e^{-\sqrt{2}\pi/\sigma_L})(1 + m^2\sigma_L^2/2)}. \quad (2.29)$$

**Truncated Gaussian Distributed Field**

The distribution function is given by [105]

$$P(\varphi) = K_G e^{-\frac{(\varphi - \varphi_0)^2}{2\sigma_G^2}}, \quad \varphi \in [-\pi, \pi), \quad (2.30)$$

where $\sigma_G$ is the non-isotropic parameter of the distribution, which is the standard deviation of the non-truncated distribution, and $K_G$ is the normalization constant

$$K_G = \frac{1}{\sqrt{2\pi\sigma_G\text{erf}(\pi/\sqrt{2}\sigma_G)}},$$

where erf($x$) is the error function, defined as $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}dz$. In this case, $\alpha_m$ is given by

$$\alpha_m = e^{-(m^2\sigma_G^2/2 + im\phi_0)} \frac{\text{Re}\left\{\text{erf}\left(\frac{\pi/2 + im\sigma_G^2}{\sqrt{2}\sigma_G}\right)\right\}}{\text{erf}(\pi/\sqrt{2}\sigma_G)}, \quad (2.31)$$

and

$$\beta_m = e^{-(m^2\sigma_G^2/2 - im\phi_0)} \frac{\text{Re}\left\{\text{erf}\left(\frac{\pi/2 - im\sigma_G^2}{\sqrt{2}\sigma_G}\right)\right\}}{\text{erf}(\pi/\sqrt{2}\sigma_G)}. \quad (2.32)$$

For narrow angular spreads, scattering environment coefficients can be well approximated by $\alpha_m \approx e^{-(m^2\sigma_G^2/2 + im\phi_0)}$ and $\beta_m \approx e^{-(m^2\sigma_G^2/2 - im\phi_0)}$ as the tails of distribution cause very little error [111].

We now explore the effects of mean AOA and angular spread on the spatial correlation between two antenna elements at the receiver for the above non-isotropic
Figure 2.2: Spatial correlation between two receiver antenna elements for mean AOA $\varphi_0 = 90^\circ$ (broadside) and angular spread $\sigma = \{20^\circ, 5^\circ, 1^\circ\}$ against antenna separation for uniform-limited, truncated Gaussian, truncated Laplacian and von-Mises scattering distributions.

distributions. The spatial correlation for mean AOA $\varphi_0 = 90^\circ$ (broadside) and $\varphi_0 = 30^\circ$ (60° from broadside) are shown in Figures 2.2 and 2.3, respectively, for uniform-limited, truncated Gaussian, truncated Laplacian and von-Mises scattering distributions. Here we set the angular spread $\sigma$ to $[20^\circ, 5^\circ, 1^\circ]$ for each distribution and position the two antennas on the x-axis. As we can see from the figures, the spatial correlation decreases as the antenna separation and angular spread increase. However the spatial correlation does not decrease monotonically with the increase in antenna separation. When the mean AOA moves away from the broadside angle, Figure 2.3, we see a significant increase in spatial correlation for all angular spreads and distributions for the same antenna separation. In general, we can observe that all scattering distributions give very similar spatial correlation values for a given angular spread, especially when the antenna separation is small. This

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1 Broadside angle is defined as the angle perpendicular to the line connecting the two antennas.
2.4 Simulation Results: Alamouti Scheme

Utilizing the tools developed in Section 2.3 we now investigate the performance of orthogonal STBC discussed in Chapter 1.1.2 for various spatial scenarios. In our simulations, modulated symbols $s_k$ are drawn from the normalized QPSK alphabet $\{\pm 1/\sqrt{2} \pm i/\sqrt{2}\}$. First we outline a method which could be used to generate correlated channel gains $h_{p,q}$ for our simulations.
2.4.1 Generation of Correlated Channel Gains

Here we briefly outline a method which could be used to generate correlated channel gains using the covariance matrix of the MIMO channel \( R_H \) and an uncorrelated MIMO channel matrix \( A \) with zero-mean independent and identically distributed random entries:

- Perform the standard Cholesky factorization on \( R_H \) to obtain the lower triangular matrix \( C \) such that \( R_H = CC^\dagger \), given that \( R_H \) is a positive definite matrix.

- Generate an independent and identically distributed (i.i.d.) \( n_R \times n_T \) channel matrix \( A \) where all the elements in \( A \) are complex Gaussian distributed with zero-mean and unit variance.

- The Correlated channel gains of the MIMO channel \( H \) are found by performing \( h = Ca \), where \( h = \text{vec}(H) \) and \( a = \text{vec}(A) \).

2.4.2 Effects of Antenna Separation

Now we investigate the effects of antenna separation on the performance of orthogonal STBC. We compare the bit-error rate (BER) performance of the Alamouti scheme applied on a two-transmit two-receive MIMO antenna system against the BER performance of an uncoded system. For simplicity, we assume that transmitter antennas are uncorrelated (i.e., isotropic scattering at the transmitter array and the two transmit antennas are placed far apart). Also assume a flat-fading scattering environment where the channel gains from transmitter antennas to receiver antennas remain constant over two consecutive symbol intervals.

We set the overall SNR, before detection of each symbol, to 10dB, mean AOA \( 0^\circ \) from broadside and angular spreads \( \sigma = [104^\circ, 20^\circ, 5^\circ] \) for a uniform-limited distribution\(^2\) and increase the separation distance between receiver antennas, which are positioned on the x-axis. Therefore the angle \( \varphi_{12} \) in (2.6) is zero. Note that the angular spread \( 104^\circ \) represents the isotropic scattering at the receiver antenna array.

The performance results for coded and uncoded systems are shown in Figure 2.4. For both systems, the bit-error rate decreases as the receiver antenna separation and the angular spread increase. Table-2.1 shows the minimum and maximum

\(^2\) We only consider the uniform-limited distribution as all other azimuth power distributions give the same spatial correlation result for the same angular spreads.
2.4 Simulation Results: Alamouti Scheme

Figure 2.4: BER performance vs receiver spatial separation for 2×2 orthogonal STBC and uncoded systems for a Uniform-limited distribution at the receiver antenna array. Mean AOA 0° from broadside, angular spread $\sigma = \{10^\circ, 20^\circ, 5^\circ\}$ and SNR = 10dB.

The BER performance of the orthogonal STBC varies from 0.007 to 0.002 with the increase in antenna separation. However, for a given SNR, the overall improvement is not that significant with the increase in antenna separation. Thus the antenna separation plays a secondary role in the performance of orthogonal STBC with two-transmit antennas.

As shown, the orthogonal coded system reaches its optimum performance, 0.002, when the antenna separation distances $\lambda, 1.5\lambda$ and $3\lambda$ for angular spreads 104°, 20° and 5°, respectively. Using these observations we can claim that the impact
of the space is limited on the performance of orthogonal STBC, even though the antenna separation is a main contributor to the spatial correlation. Here, most of the bit-error rate improvement can be attributed to “time-coding” rather than to “space-coding”. This corroborates the claim that the orthogonal STBC with two transmit antennas has good resistance against the spatially correlated fading.

2.4.3 Effects of Non-isotropic Scattering

We now investigate the effects of non-isotropic parameters on the performance of the Alamouti scheme applied on a two-transmit two-receive MIMO antenna system. The BER performance results of $2 \times 2$ Alamouti code for mean AOA $0^\circ$, from broadside, is shown in Figure 2.5. Here we have set the overall SNR to 10dB and angular spreads to $\sigma = [104^\circ, 20^\circ, 5^\circ, 1^\circ]$ for a Uniform-limited distribution where antennas are positioned on the $x$-axis. As shown, the BER decreases as the antenna spacing and the angular spread increase. Here we also see that the BER performance does not decrease monotonically with antenna separation, for example, when $\sigma = 104^\circ$ (isotropic distribution) and $20^\circ$. It is also observed that the performance of the orthogonal STBC is lower when the angular spread is smaller. This is due to the higher concentration of energy closer to the mean AOA for smaller angular spreads. Therefore, the angular spread of the power distribution is one of the major factors which governs the BER performance of the orthogonal STBC. This observation is not limited to orthogonal STBC. It is also valid for other space-time coding schemes [8, 49, 50, 116] found in the literature (Chapter 3). To achieve most of the performance gain from orthogonal STBC under the given scattering environment, as a rule of thumb, antennas in an aperture must be located at least $2.5\lambda$ apart from each other. This rule of thumb caters for narrow angular spreads like $5^\circ$ when the mean AOA is $0^\circ$ from broadside. Finally, we observe that
2.4 Simulation Results: Alamouti Scheme

Figure 2.5: (a). Spatial correlation between two receiver antennas positioned on the \(x\)-axis for mean AOA 0° from broadside vs the spatial separation for a uniform-limited scattering distribution with angular spreads \(\sigma = [104^\circ, 20^\circ, 5^\circ, 1^\circ]\). (b). BER performance vs spatial separation for 2\(\times\)2 orthogonal STBC under the scattering environments given in (a).

the BER performance is directly mapped to the squared absolute value of spatial correlation against the spatial separation for all angular spreads. In other words, BER performance has a strong correlation with the spatial correlation.

Figure 2.6 shows the performance results for mean AOA 60° from broadside. Here we observe similar results as for the mean AOA 0° case. In this case, a significant performance degradation is observed for all angular spreads for the same antenna separation as for previous results. So the performance of the orthogonal STBC is decreased as the mean AOA moves away from broadside. This can be justified by the reasoning that as the mean AOA moves away from broadside, there will be a reduction in the angular spread exposed to antennas and hence less signals being captured. Under this environment, antennas must be placed at least 4.5\(\lambda\) apart from each other to achieve most of the performances gain provided by the orthogonal STBC.
2.4.4 A Rule of Thumb: Alamouti Scheme

From Figures 2.5(b) and 2.6(b), we can observe that orthogonal STBC with two transmit antennas is capable of producing a minimum bit-error rate of 0.0015 for all scattering environments. The minimum antenna separation required to achieve this optimum bit-error rate varies with the angular spread and mean angle of arrival of impinging signals. Figure 2.7 shows the angular spread vs the minimum antenna separation, which gives the optimum error performance of orthogonal STBC for mean AOAs $0^\circ$, $30^\circ$, $45^\circ$ and $60^\circ$ from broadside. Based on the simulation results, an empirical relationship between angular spread ($\sigma$), mean AOA ($\varphi_0$) and minimum antenna separation distance ($d$) can be approximated as
\[ \frac{d}{\lambda} \approx \frac{0.25}{(2 - \frac{3}{\pi} \varphi_0)} \cdot (2.33) \]

Note that this approximation can be used to find the minimum distance between the two transmit antennas where the performance of Alamouti scheme is optimal for a given angular spread and a given mean AOA.

![Figure 2.7: Angular spread (σ) vs optimum antenna separation where the BER performance of 2×2 orthogonal STBC is optimum for mean AOAs 0°, 30°, 45° and 60° from broadside.](image)

2.4.5 Effects of Scattering Distributions

Now we consider the performance of orthogonal STBC for different scattering distributions against the non-isotropy factor and for mean AOA. A widely used rule of thumb is that half a wavelength is sufficient between two antennas in order to obtain the zero-correlation in an isotropic scattering environment. This distance
Figure 2.8: BER performance of 2×2 orthogonal STBC against the non-isotropic parameter for mean AOAs 0°, 30° and 60° from broadside, SNR 10dB and antenna separation λ/2. (a) uniform-limited (b) truncated Gaussian (c) von-Mises (d) truncated Laplacian.

requirement comes from the first null of the order zero spherical bessel function, which is the spatial correlation function for a three dimensional isotropic scattering environment [78, 111]. We fix the antenna separation at both ends of the link to half a wavelength (λ/2), and the overall SNR to 10dB. We assume scattering environment surrounding the transmitter region is isotropic.

Figure 2.8 shows the BER performance against the non-isotropic parameter at the receiver for Uniform-limited (Δ), truncated Gaussian (σ_G), von-Mises (κ) and truncated Laplacian (σ_L) distributions for mean AOAs 0°, 30° and 60° from broadside, where Δ, σ_G, κ and σ_L are the non-isotropic parameter related to the distribution. As shown, the bit-error rate decreases as the non-isotropic parameter increases, for the Uniform-limited, truncated Gaussian and truncated Laplacian distributions. It is also observed that, for these 3 distributions, the BER increases as the mean AOA moves away from broadside. For the von-Mises distribution, the
bit-error rate increases as the non-isotropic parameter ($\kappa$) increases. The lowest BER is observed when $\kappa = 0$. In fact $\kappa = 0$ represents the isotropic scattering for the von-Mises distribution. Therefore, the BER performance of the orthogonal STBC depends on the non-isotropic parameter and the mean AOA of the azimuth power distribution. Since the angular spread is a function of non-isotropic parameter and the mean AOA, the performance of the orthogonal STBC is directly dependent on the angular spread of the azimuth power distribution.

### 2.5 Analysis of Orthogonal STBC: A Modal Approach

Using the channel model introduced in Section 2.2 we now analyse the orthogonal STBC from a physical perspective. Recall, the signal leaving the scatter-free transmitter aperture along direction $\hat{\phi}$ is written as

$$\Phi(\hat{\phi}) = \sum_{q=1}^{n_T} s_q e^{ikx_q \cdot \hat{\phi}},$$

where $x_q$ is the location of the $q$-th transmit antenna relative to the transmitter array origin and $s_q$ is the baseband signal transmitted from $q$-th transmit antenna. Using the expansion (2.14b) of plane wave $e^{ikx_q \cdot \hat{\phi}}$ we can write

$$\Phi(\phi) = \sum_{q=1}^{n_T} \sum_{m=-\infty}^{\infty} s_q \mathcal{J}_m(x_q) e^{im\phi}, \quad (2.34a)$$

$$= \sum_{n=-\infty}^{\infty} a_m e^{im\phi}, \quad (2.34b)$$

where $\mathcal{f}(\cdot)$ is the complex conjugate of the function $f(\cdot)$,

$$a_m = \sum_{q=1}^{n_T} s_q \mathcal{J}_m(x_q), \quad (2.35)$$

is the $m$-th transmit mode excited by $n_T$ antennas in the scatter-free transmitter region and

$$\mathcal{J}_m(x_q) = J_m(k||x_q||) e^{im(\phi_q - \pi/2)}, \quad (2.36)$$
is the spatial-to-mode function which maps the antenna location \( \mathbf{x}_q \equiv (\|\mathbf{x}_q\|, \phi_q) \) in the polar coordinate system to the \( m \)-th communication mode\(^3\) of the transmit region [117]. Note that although there are infinite number of modes excited by an antenna array, there are only finite number of modes \((2M_T + 1)\) which have sufficient power to carry information. Also note that sum (2.34b) is in fact the Fourier series expansion of signal \( \Phi(\phi) \) with Fourier coefficients given by \( a_m \).

Suppose omni-directional antennas are used for transmission. Then from (2.34a), the radiation pattern generated at the scatter-free transmitter region during a symbol period can be shown to be

\[
P(\phi) = |\Phi(\phi)|^2 = \sum_{q=1}^{n_T} \sum_{q'=1}^{n_T} \sum_{m=-M_T}^{M_T} \sum_{m'=-M_T}^{M_T} s_q s_q' \mathcal{J}_m(\mathbf{x}_q) \mathcal{J}_{m'}(\mathbf{x}_{q'}) e^{i(m-m')\phi},
\]

\( (2.37) \)

\( ^3 \)The set of modes form a basis of functions for representing a multipath wave field.
2.5 Analysis of Orthogonal STBC: A Modal Approach

where $M_T = \lceil \pi r_T / \lambda \rceil$ with $r_T$ the radius of the transmitter aperture that encloses all the transmit antennas. From (2.35), the magnitude of the $m$-th transmit mode excited at the scatter-free transmit region during a symbol interval can be shown to be

$$P_m = |a_m|^2,$$

$$= \sum_{q=1}^{n_T} \sum_{q'=1}^{n_T} s_q s_{q'}^* \mathcal{J}_m(x_q) \mathcal{J}_m'(x_{q'}). \quad (2.38)$$

Figures 2.9 and 2.10 show the radiation patterns and squared magnitude of transmit modes $|a_m|^2$ for Alamouti scheme with two transmit antennas for antenna separations $0.5\lambda$ and $\lambda$, respectively. Note that antenna separations $0.5\lambda$ (or $r_T = 0.25\lambda$) and $\lambda$ (or $r_T = 0.5\lambda$) correspond to 7 and 11 effective modes at the scatter-free transmitter region, respectively. In these plots, modulated symbols $s_k$ of Alamouti scheme are drawn from a BPSK constellation.

Figure 2.10: Radiation patterns and $|a_m|^2$ for orthogonal STBC with two transmit antennas: antenna separation $\lambda$ (or $r_T = 0.5\lambda$)

From 2.9 it is observed that only the transmit mode set $\{-2, 0, 2\}$ is excited.
during the first symbol interval and the transmit mode set \{-3, -1, 1, 3\} is excited during the second symbol interval. Thus the orthogonal STBC with two transmit antennas excites two different set of modes over the two symbol intervals. As a result, in the radiation pattern plot we see energy is directed into different directions during the two symbol intervals, and the two beam patterns are orthogonal. Similar observations can be made when the antenna separation is 0.5\(\lambda\). In this case, transmit mode set \{-4, -2, 0, 2, 4\} is excited during the first symbol period and the set \{-5, -3, -1, 1, 3, 5\} is excited during the second symbol interval. Furthermore, we observe that as the antenna separation increases, the number of grating lobes in the radiation pattern is increased. This is due to the increase in number of effective modes in the region as the radius of the aperture increases.

In general, during the first symbol interval, orthogonal STBC with two transmit antennas excites all even numbered transmit modes including the 0-th mode out of \(2M_T + 1\) effective transmit modes associated with a region. During the second symbol interval, all odd numbered modes are excited. Therefore, the diversity gain is cleverly incorporated into the orthogonal STBC by activating different mode sets (or directing energy to different directions) over the two symbol intervals.

### 2.6 Summary and Contributions

This chapter has investigated the performance of orthogonal space-time block codes for realistic MIMO channel scenarios. In particular, we studied the performance of Alamouti scheme with two transmit antennas for antenna spacing and non-isotropic scattering environments.

Some specific contributions made in this chapter are:

1. An analytical model for spatial correlation between channel gains is derived which fully accounts for antenna separation, antenna placement and scattering environments surrounding the transmitter and receiver antenna arrays. This model has facilitated realistic modelling in an analytic framework.

2. Using the analytic correlation model we showed that Alamouti scheme provides a high degree of robustness against spatially correlated fading, in particularly for small antenna separations.

3. When the angular spread of the surrounding scattering distribution is small, the BER performance of the Alamouti scheme is reduced. Also the BER is
increased when the mean angle of arrival of an impinging signal moves away from the broadside.

4. An expression for the antenna separation is derived for a $2 \times 2$ MIMO system where the performance of orthogonal STBC sufficiently close to the optimal under a given scattering environment.

5. By applying the plane wave propagation theory in free space, the Alamouti scheme is analyzed from a physical perspective. We showed that radiation patterns generated in the transmit region over the two symbol intervals of the Alamouti scheme are orthogonal and also two different sets of transmit modes are excited during the two symbol intervals.
Chapter 3

Performance Limits of Space-Time Codes in Physical Channels

3.1 Introduction

Space-time coding combines channel coding with multiple transmit and multiple receive antennas to achieve bandwidth and power efficient high data rate transmission over fading channels. The performance criteria for coherent space-time codes have been derived in [8] based on the Chernoff bound applied to the pairwise error probability (PEP) assuming an independent identically distributed (i.i.d.) quasi-static fading channel. It was shown in [8] that the diversity advantage (robustness) and the coding gain of a space-time code in quasi-static fading channels is determined by the minimum rank and the minimum determinant of the distance matrix between two distinct codewords. Following this analysis, a rank determinant design criterion was proposed which involves maximizing the minimum rank and the minimum determinant of the distance matrix over all distinct pairs of codewords. Based on this design criterion, a number of QPSK and 8-PSK space-time trellis codes were constructed by hand in [8]. Following this pioneer work, a number of coherent space-time coding schemes have been proposed to exploit the potential increase in performance promised by multi-antenna communication systems [9, 40, 41, 52, 118].

The effectiveness of coherent space-time coding schemes heavily relies on the accuracy of the channel estimation at the receiver. Therefore, differential space-time coding schemes make an attractive alternative to combat inaccuracy of channel

\footnote{The channel state information (CSI) is fully known at the receiver.}
estimation in coherent space-time coding schemes. With differential space-time coding schemes, channel state information is not required at either end of the channel. Several differential space-time coding schemes for multi-antenna systems have been proposed in the literature [48–50].

The error performances of some of the coherent and non-coherent space-time coding schemes for i.i.d. fading channels have been investigated in [119–123]. In [119,120], the average bit error probability (BEP) of coherent space-time codes was evaluated using the traditional Chernoff bounding technique on the PEP. In general, the Chernoff bound is quite loose for low signal-to-noise ratios. In [121], the exact-PEP of coherent space-time codes operating over i.i.d. fast fading channels was derived using the method of residues. A simple method for exactly evaluating the PEP (and approximate BEP) based on the moment generating function associated with a quadratic form of a complex Gaussian random variable [124] is given in [122] for both i.i.d. slow and fast fading channels. In [123], a closed form expression for bit error probability of differential space-time block codes (DSTBC) based on Alamouti’s scheme was derived assuming fading channels are statistically independent.

When designing space-time codes, the main assumption being made is that the channel gains between the transmitter and the receiver antennas undergo independent fading. In practice, insufficient antenna spacing (physical size of the antenna array) and lack of scattering (limited angular spread) cause the channel gains to be correlated. Therefore, the assumption of uncorrelated fading model will in general not be an accurate description to realistic fading channels. Several approaches have been found in the literature, where the performance of space-time codes have been investigated for correlated fading channels [55–66]. However, none of these results explicitly address the effects of the physical constraints, such as antenna aperture size, antenna geometry and scattering distribution parameters, and also the independent effects of these physical constraints on the performance of space-time codes.

In this chapter we investigate the effects of the above mentioned physical constraints on the performance of both coherent and non-coherent space-time codes applied on spatially constrained MIMO channels. Using an MGF-based approach, first we derive analytical expressions for the exact-PEP (and approximate BEP) of a space-time coded system over spatially correlated fading channels. We also derive PEP upper-bounds for correlated fading channels. These PEP expressions fully account for antenna separation, antenna geometry (Uniform Linear Array, Uniform Grid Array, Uniform Circular Array, etc.) and surrounding azimuth power distri-
buttions, both at the receiver and the transmitter antenna array apertures. Using these generalized PEP expressions we quantify the degree of the effect of the size of the antenna aperture, antenna geometry and the angular spread of the scattering distribution surrounding the transmitter and receiver antenna apertures on the diversity advantage of a space-time code.

This chapter is divided into two parts. Part I: Performance Limits of Coherent Space-Time Codes and Part II: Performance Limits of Non-coherent Space-Time Codes. In Part I, we also introduce the spatial channel model that used to analyze the performance of both types of space-time coding schemes for physically realistic channel environments.

Part I: Performance Limits of Coherent Space-Time Codes

3.2 System Model: Coherent Space-Time Codes

Consider a MIMO system consisting of $n_T$ transmit antennas and $n_R$ receive antennas. Let $s_n = [s_1^{(n)}, s_2^{(n)}, \ldots, s_{n_T}^{(n)}]^T$ denotes the space-time coded signal vector transmitted from $n_T$ transmit antennas in the $n$-th symbol interval, where $s_q^{(n)}$ is a signal from a certain constellation with unit energy, and $S = [s_1, s_2, \ldots, s_L]$ denotes the space-time code representing the entire transmitted signal, where $L$ is the code length. The received signal at the $p$-th receive antenna in the $n$-th symbol interval is given by

$$r_p^{(n)} = \sqrt{E_s} \sum_{q=1}^{n_T} h_{p,q}^{(n)} s_q^{(n)} + \eta_p^{(n)},$$

$$p = 1, 2, \ldots, n_R, \ n = 1, 2, \ldots, L,$$  \hspace{1cm} (3.1)

where $E_s$ is the transmitted power per symbol at each transmit antenna and $\eta_p^{(n)}$ is the additive noise on the $p$-th receive antenna at symbol interval $n$. The additive noise is assumed to be white and complex Gaussian distributed with mean zero and variance $N_0/2$ per dimension. Here the coefficient $h_{p,q}^{(n)}$ represents the random complex channel gain between the $q$-th transmit antenna and the $p$-th receive antenna at symbol interval $n$. We assume fading coefficients remain constant during one symbol interval and change independently from one symbol interval to another (We classify this model as fast-fading channel model).
Let $H_n = [h_{p,q}^{(n)}]$ denotes the $n_R \times n_T$ channel gain matrix at the $n$-th symbol interval. By taking into account physical aspects of scattering, the channel matrix $H_n$ can be decomposed into deterministic and random parts as [90,106]

$$H_n = \Omega_R H_{s,n} \Omega_T,$$  

where $\Omega_R$ and $\Omega_T$ are deterministic and $H_{s,n}$ is a random matrix with complex normal Gaussian distributed entries. According to the channel model proposed in [90], $H_{s,n}$ is an i.i.d. channel matrix, which has zero-mean unit variance complex Gaussian entries, while $\Omega_R$ and $\Omega_T$ are associated to the receiver and transmitter antenna correlation matrices, respectively. In [106], $H_{s,n}$ represents the random non-isotropic scattering environment, while $\Omega_R$ and $\Omega_T$ represent the antenna geometries at the receiver and the transmitter antenna arrays, respectively.

In this work, we are interested in investigating the impact of antenna separation, antenna geometry and the scattering environment on the performance of space-time codes. The channel model given in [90] is restricted to a uniform linear array antenna configuration and a finite number of scatterers surrounding the transmitter and receiver antenna arrays. However, the channel decomposition given in [106], is capable of capturing different antenna geometries as well as various non-isotropic scattering distributions. In the next section we review the spatial channel model proposed in [106] for a 2-D scattering environment.

### 3.3 Spatial Channel Model

It was shown in Chapter 2.2 that by applying the underlying physics of free space propagation, the complex channel gain between the $p$-th receive antenna and the $q$-th transmit antenna at the $n$-th symbol interval can be written as

$$h_{p,q}^{(n)} = \int_{S^2} \int_{S^2} g_n(\hat{\phi}, \hat{\varphi}) e^{i k x_q \cdot \hat{\phi}} e^{-i k z_p \cdot \hat{\varphi}} ds(\hat{\phi}) ds(\hat{\varphi}), \quad (3.3)$$

where $x_q$ is the position of the $q$-th transmit antenna relative to the transmitter array origin, $z_p$ is the position of the $p$-th receive antenna relative to the receiver array origin, $g_n(\hat{\phi}, \hat{\varphi})$ is the effective random scattering gain function for a signal leaving from the transmitter scatter-free aperture at a direction $\hat{\phi}$ and entering the receiver scatter-free aperture from a direction $\hat{\varphi}$ at the $n$-th symbol interval and $S^2$ is the unit sphere. Although (3.3) allows us to model the spatial channel for any physical antenna configuration (or antenna geometry) and also for any general scattering distribution however it is difficult to evaluate or simulate due to
its integral representation. In the next section the channel is simplified to an easily computable form by expanding the plane waves $e^{ikx_q \cdot \hat{\phi}}$ and $e^{-ikz_p \cdot \hat{\phi}}$ in 2-D space.

### 3.3.1 Spatial Channel Decomposition

Using the modal expansion of plane waves for a 2-D scattering environment we can write

$$e^{ikx_q \cdot \hat{\phi}} = \sum_{n=-\infty}^{\infty} \mathcal{J}_n(x_q)e^{in\phi}, \quad (3.4)$$

where

$$\mathcal{J}_n(x_q) = J_n(k\|x_q\|)e^{in(\phi_q - \pi/2)}, \quad (3.5)$$

with $x_q \equiv (\|x_q\|, \phi_q)$ and $\hat{\phi} \equiv (1, \phi)$ in the polar coordinate system. In [114], it was shown that $J_n(r) \approx 0$ for $n > [re/2]$, then we can define

$$M_T \triangleq [\pi er_T/\lambda], \quad (3.6)$$

$$M_R \triangleq [\pi er_R/\lambda], \quad (3.7)$$

such that the expansions

$$e^{ikx_q \cdot \hat{\phi}} = \sum_{n=-M_T}^{M_T} \mathcal{J}_n(x_q)e^{in\phi} \quad (3.8)$$

and

$$e^{-ikz_p \cdot \hat{\phi}} = \sum_{m=-M_R}^{M_R} \mathcal{J}_m(z_p)e^{-im\varphi}, \quad (3.9)$$

hold for every antenna within the transmitter and receiver circular apertures of radii $r_T$ and $r_R$, respectively.

By substituting (3.8) and (3.9) into (3.3), MIMO channel $H_n$ can be decomposed as

$$H_n = J_R H_{s,n} J_T^\dagger, \quad (3.10)$$
where \( J_T \) is the \( n_T \times 2M_T + 1 \) deterministic transmitter configuration matrix

\[
J_T = \begin{bmatrix}
J_{-M_T}(x_1) & J_{-M_T+1}(x_1) & \cdots & J_{M_T-1}(x_1) & J_{M_T}(x_1) \\
J_{-M_T}(x_2) & J_{-M_T+1}(x_2) & \cdots & J_{M_T-1}(x_2) & J_{M_T}(x_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
J_{-M_T}(x_{n_T}) & J_{-M_T+1}(x_{n_T}) & \cdots & J_{M_T-1}(x_{n_T}) & J_{M_T}(x_{n_T})
\end{bmatrix}, \quad (3.11)
\]

\( J_R \) is the \( n_R \times (2M_R + 1) \) deterministic receiver configuration matrix,

\[
J_R = \begin{bmatrix}
J_{-M_R}(z_1) & J_{-M_R+1}(z_1) & \cdots & J_{M_R-1}(z_1) & J_{M_R}(z_1) \\
J_{-M_R}(z_2) & J_{-M_R+1}(z_2) & \cdots & J_{M_R-1}(z_2) & J_{M_R}(z_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
J_{-M_R}(z_{n_R}) & J_{-M_R+1}(z_{n_R}) & \cdots & J_{M_R-1}(z_{n_R}) & J_{M_R}(z_{n_R})
\end{bmatrix}, \quad (3.12)
\]

and \( H_{s,n} \) is a \((2M_R + 1) \times (2M_T + 1)\) random scattering channel matrix with \((\ell, m)\)-th element given by,

\[
\{H_{s,n}\}_{\ell,m} = \int_{S^1} \int_{S^1} g_n(\phi, \varphi)e^{-i(\ell-M_R-1)\phi}e^{i(m-M_T-1)\varphi}d\phi d\varphi, \quad (3.13)
\]

where \( S^1 \) is the unit circle.

Some remarks regarding the channel decomposition (3.10)

- The channel matrix decomposition (3.10) separates the channel into three distinct regions of interest: the scatter-free region around the transmitter antenna array, the scatter-free region around the receiver antenna array and the complex random scattering environment which is the complement of the union of two antenna array regions.

- The transmitter configuration matrix \( J_T \) captures the physical configuration of the transmitter antenna array (antenna positions and orientation relative to the transmitter origin) and it is fixed for a given transmitter antenna array geometry.

- The receiver configuration matrix \( J_R \) captures the physical configuration of the receiver antenna array (antenna positions and orientation relative to the receiver origin) and it is fixed for a given receiver antenna array geometry.

- \( H_{s,n} \) represents the complex scattering environment between the transmitter and the receiver antenna apertures. For a random scattering environment, \( \{H_{s,n}\}_{\ell,m} \) are random variables, and for an isotropic scattering environment, \( \{H_{s,n}\}_{\ell,m} \) are independent of each other.
3.3 Spatial Channel Model

- The size of $H_{s,n}$ is determined by the number of effective communication modes excited by the antenna arrays at the receiver and transmitter regions.

  The number of communication modes at the transmitter is determined by the size of the transmit region $r_T = \max \|x_q\|$ for $q = 1, \cdots, n_T$. At the receiver side, it is determined by the size of the receive region $r_R = \max \|z_p\|$ for $p = 1, \cdots, n_R$.

For the decomposition (3.10), the correlation matrix of the channel $H_n$ is

$$R_n = \mathcal{E} \{ h_n^\dagger h_n \} = (J_{R}^* \otimes J_T^*) R_{s,n} (J_T^T \otimes J_{R}^*) ,$$

where $h_n = (\text{vec} \{ H_{s,n}^T \})^T$ and $R_{s,n}$ the modal correlation matrix at the $n$-th symbol interval, which is defined as $R_{s,n} = \mathcal{E} \{ h_{s,n}^\dagger h_{s,n} \}$ with $h_{s,n} = (\text{vec} \{ H_{s,n}^T \})^T$.

### 3.3.2 Transmitter and Receiver Modal Correlation

Using (3.13), we define the modal correlation\(^3\) between complex scattering gains at the $n$-th symbol interval as

$$\gamma_{\ell,\ell'}^{m,m'}_{n} \triangleq \mathcal{E} \{ H_{s,n}^\ell m H_{s,n}^{\ell' m'} \} .$$

Assume that the scattering from one direction is independent of that from another direction for both the receiver and the transmitter apertures. Then the second order statistics of the scattering gain function $g_n(\phi, \varphi)$ can be defined as

$$\mathcal{E} \{ g_n(\phi, \varphi) g_n^* (\phi', \varphi') \} \triangleq G_n(\phi, \varphi) \delta(\phi - \phi') \delta(\varphi - \varphi'),$$

where $G_n(\phi, \varphi) = \mathcal{E} \{ |g_n(\phi, \varphi)|^2 \}$ with normalization $\iint G_n(\phi, \varphi) d\varphi d\phi = 1$. With the above assumption, the modal correlation coefficient, $\gamma_{m,m',n}^{\ell,\ell'}$ can be simplified to

$$\gamma_{m,m',n}^{\ell,\ell'} = \iint_{S^1 \times S^1} G_n(\phi, \varphi) e^{-i(\ell - \ell')\varphi} e^{i(m - m')\phi} d\varphi d\phi .$$

Then, at the $n$-th symbol interval, the correlation between $\ell$-th and $\ell'$-th modes at

\(^2\)Although there are infinite number of modes excited by an antenna array, there are only finite number of modes which have sufficient power to carry/receive information.

\(^3\)Correlation between modes generated at the transmitter and receiver regions. In Chapter 2, we considered spatial correlation between antenna elements at the transmitter and receiver antenna arrays, and expressions for spatial correlation were derived.
the receiver region due to the $m$-th mode at the transmitter region is given by

$$\gamma_{n}^{\ell, \ell'} = \int_{\mathbb{S}^1} P_{\text{Rx}, n}(\varphi) e^{-i(\ell - \ell')\varphi} d\varphi, \quad \forall m, \quad (3.15)$$

where $P_{\text{Rx}, n}(\varphi) = \int_{\mathbb{S}^1} G_n(\phi, \varphi) d\phi$ is the normalized azimuth power distribution of the scatterers surrounding the receiver antenna region at the $n$-th symbol interval. Here we see that modal correlation at the receiver is independent of the mode selected from the transmitter region. Similarly, the correlation between $m$-th and $m'$-th modes at the transmitter region due to the $\ell$-th mode at the receiver region is given by

$$\gamma_{m, m', n} = \int_{\mathbb{S}^1} P_{\text{Tx}, n}(\phi) e^{i(m - m')\phi} d\phi, \quad \forall \ell, \quad (3.16)$$

where $P_{\text{Tx}, n}(\phi) = \int_{\mathbb{S}^1} G_n(\phi, \varphi) d\varphi$ is the normalized azimuth power distribution at the transmitter region at the $n$-th symbol interval. As for the receiver modal correlation, we can observe that modal correlation at the transmitter is independent of the mode selected from the receiver region. Note that azimuth power distributions $P_{\text{Rx}, n}(\varphi)$ and $P_{\text{Tx}, n}(\phi)$ can be modeled using all common power distributions discussed in Chapter 2 such as Uniform, Gaussian, Laplacian and Von-Mises.

Denoting the $p$-th column of scattering matrix $H_{s, n}$ as $H_{s, n, p}$, the $(2M_R + 1) \times (2M_R + 1)$ receiver modal correlation matrix can be defined as

$$F_{R, n} \triangleq \mathbb{E}\{H_{s, n, p} H_{s, n, p}^\dagger\},$$

where $(\ell, \ell')$-th element of $F_{R, n}$ is given by (3.15) above. Similarly, the transmitter modal correlation matrix can be defined as

$$F_{T, n} \triangleq \mathbb{E}\{H_{s, n, q}^\dagger H_{s, n, q}\},$$

where $H_{s, n, q}$ is the $q$-th row of $H_{s, n}$. $(m, m')$-th element of $F_{T, n}$ is given by (3.16) and $F_{T, n}$ is a $(2M_T + 1) \times (2M_T + 1)$ matrix.

**Kronecker Model as a Special Case**

When the scattering channel $H_{s, n}$ is separable, i.e.,

$$G_n(\phi, \varphi) = P_{\text{Tx}, n}(\phi) P_{\text{Rx}, n}(\varphi), \quad (3.17)$$
correlation between two distinct modal pairs can be written as the product of corresponding modal correlation at the transmitter region and the modal correlation at the receiver region [69,112]. In this case,

\[ \gamma_{m,m',n}^{\ell,\ell'} = \gamma_n^{\ell,\ell'} \gamma_{m,m',n} \]  
(3.18)

Facilitated by (3.18), the modal correlation matrix of the scattering channel \( \mathbf{H}_{s,n} \) can be written as the Kronecker product between the receiver modal correlation matrix and the transmitter modal correlation matrix,

\[ \mathbf{R}_{s,n} = \mathcal{E} \{ \mathbf{h}_{s,n}^\dagger \mathbf{h}_{s,n} \} = \mathbf{F}_{R,n} \otimes \mathbf{F}_{T,n}, \]  
(3.19)

where \( \mathbf{h}_{s,n} = (\text{vec}(\mathbf{H}_{s,n}^T))^T \).

### 3.4 Exact PEP on Correlated MIMO Channels

Assume perfect channel state information (CSI) is available at the receiver and also a maximum likelihood receiver is employed at the receiver. Suppose the code word \( \mathbf{S} \) is transmitted but the maximum likelihood receiver erroneously selects the codeword \( \hat{\mathbf{S}} \). Then, the pair-wise error probability conditioned on the channel \( \mathbf{H}_n \) is given by [122]

\[ P(\mathbf{S} \rightarrow \hat{\mathbf{S}}|\mathbf{H}_n) = Q \left( \sqrt{\frac{E_s}{2N_0}d^2} \right) \],  
(3.20)

where \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}dy \), is the Gaussian Q-function and \( d \) is the Euclidian distance.

In the case of a time-varying fading channel,

\[ d^2 = \sum_{n=1}^L \|\mathbf{H}_n(\mathbf{s}_n - \hat{\mathbf{s}}_n)\|^2, \]

\[ = \sum_{n=1}^L \mathbf{h}_n [\mathbf{I}_{n_R} \otimes \mathbf{s}_n^\Delta] \mathbf{h}_n^\dagger, \]  
(3.21)

where \( \mathbf{s}_n^\Delta = (\mathbf{s}_n - \hat{\mathbf{s}}_n)(\mathbf{s}_n - \hat{\mathbf{s}}_n)^\dagger \) and \( \mathbf{h}_n = (\text{vec}(\mathbf{H}_{n}^T))^T \) is a row vector. For a slow fading channel (quasi-static fading), we would have \( \mathbf{H}_n = \mathbf{H} \) for \( n = 1, 2, \cdots, L \).
then $d^2$ simplifies to

$$
\begin{align*}
d^2 &= \|H(S - \hat{S})\|^2, \\
&= h[I \otimes S_\Delta]h^\dagger, \\
\end{align*}
$$

where $S_\Delta = (S - \hat{S})(S - \hat{S})^\dagger$ and $h = (\text{vec} \{H^T\})^T$ is a row vector. Note also that $S_\Delta = \sum_{n=1}^L s^\Delta_n$.

To compute the average PEP, we average (3.20) over the joint probability distribution of the channel gains. By using Craig’s formula for the Gaussian Q-function

$$
Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2 \sin^2 \theta}\right) d\theta
$$

and the MGF-based technique presented in [122], we can write the average PEP as

$$
\begin{align*}
P(S \rightarrow \hat{S}) &= \frac{1}{\pi} \int_0^{\pi/2} \int_0^\infty \exp\left(-\frac{\Gamma}{2 \sin^2 \theta}\right) p_\Gamma(\Gamma) d\Gamma d\theta, \\
&= \frac{1}{\pi} \int_0^{\pi/2} \mathcal{M}_\Gamma\left(-\frac{1}{2 \sin^2 \theta}\right) d\theta,
\end{align*}
$$

where $\mathcal{M}_\Gamma(\xi) \triangleq \int_0^\infty e^{\xi \Gamma} p_\Gamma(\Gamma) d\Gamma$ is the MGF of

$$
\Gamma = \frac{E_s}{2N_0} d^2
$$

and $p_\Gamma(\Gamma)$ is the probability density function (pdf) of $\Gamma$.

### 3.4.1 Fast Fading Channel Model

In this section, we derive the exact-PEP of a coherent space-time coded system applied to a spatially correlated fast fading MIMO channel.

Substituting (3.10) for $H_n$ in $h_n = (\text{vec} \{H^T_n\})^T$ and using the Kronecker product identity [15, page 180] $\text{vec} \{AXB\} = (B^T \otimes A) \text{vec} \{X\}$, we re-write (3.21) as

$$
\begin{align*}
d^2 &= \sum_{n=1}^L h_{s,n}(J_R^T \otimes J_T^\dagger)(I_{nR} \otimes s^\Delta_n)(J_R^* \otimes J_T)h_{s,n}^\dagger, \\
&= \sum_{n=1}^L h_{s,n} \left[(J_R^T J_R)^T \otimes (J_T^T s^\Delta_n J_T)\right] h_{s,n}^\dagger,
\end{align*}
$$
where $h_{s,n} = \left( \text{vec}\left\{ H_{s,n}^T \right\} \right)^T$ is a row vector and

$$G_n = (J_R^T J_R)^T \otimes (J_T^T s_{\Delta}^n J_T).$$ (3.27)

Note that, (3.26b) follows from (3.26a) via the identity [15, page 180] $(A \otimes C)(B \otimes D) = AB \otimes CD$, provided that the matrix products $AB$ and $CD$ exist. Substituting (3.26c) in (3.25), we obtain

$$\Gamma = \frac{E_s}{2N_0} \sum_{n=1}^{L} h_{s,n} G_n h_{s,n}^\dagger.$$ (3.28)

Since $h_{s,n}$ is a random row vector and $G_n$ is fixed as $J_T^T, J_R$ and $s_{\Delta}^n$ are deterministic matrices, then $\Gamma$ is a random variable too. In fact, $h_{s,n} G_n h_{s,n}^\dagger$ is a quadratic form of a random variable. Now we illustrate how one would find the MGF of $\Gamma$ in (3.28) for a fast fading channel.

Using the standard definition of the MGF, we can write

$$\mathcal{M}_\Gamma(\xi) = \mathcal{E}\left\{ \exp\left( \xi \frac{E_s}{2N_0} \sum_{n=1}^{L} h_{s,n} G_n h_{s,n}^\dagger \right) \right\},$$

$$= \mathcal{E}\left\{ \prod_{n=1}^{L} \exp\left( \xi \frac{E_s}{2N_0} h_{s,n} G_n h_{s,n}^\dagger \right) \right\}. \quad (3.29)$$

Assume that $h_{s,n}$ is a proper-complex Gaussian random row-vector (properties associated with proper-complex Gaussian vectors are given in [126]) with mean zero and covariance $R_{s,n}$ defined as $\mathcal{E}\{ h_{s,n}^\dagger h_{s,n} \}$. Let $p(h_{s,1}, h_{s,2}, \cdots, h_{s,L})$ denote the joint pdf of $h_s = (h_{s,1}, h_{s,2}, \cdots, h_{s,L})$. Then, we obtain

$$\mathcal{M}_\Gamma(\xi) = \int_V \prod_{n=1}^{L} \exp\left( \xi \frac{E_s}{2N_0} h_{s,n} G_n h_{s,n}^\dagger \right) p(h_{s,1}, h_{s,2}, \cdots, h_{s,L})dV,$$ \quad (3.30)

where we have introduced the following two shorthand notations

$$\int_V dV \triangleq \int_{V_1} \int_{V_2} \cdots \int_{V_L} dV_1 dV_2 \cdots dV_L,$$

$$dV_n = \prod_{\ell=1}^{K} \, dh_{s,n,\ell}^R dh_{s,n,\ell}^I,$$
with $h^{R}_{s,n,\ell}$ and $h^{I}_{s,n,\ell}$ are the real and imaginary parts of the $\ell$-th element of the vector $h_{s,n}$, respectively and $K = (2M_{R} + 1)(2M_{T} + 1)$ is the length of $h_{s,n}$.

In this work, we are mainly interested in investigating the spatial correlation effects of the scattering environment on the performance of space-time codes. Therefore, we can assume that the temporal correlation of the scattering environment is zero, i.e.

$$E\{h_{s,n}^{\dagger} h_{s,m}\} = \begin{cases} R_{s,n}, & n = m; \\ 0, & n \neq m. \end{cases}$$

for $n, m = 1, 2, \ldots, L$. (3.31)

Assuming now that the scattering environment is temporally uncorrelated, and as a result $p(h_{s,1}, h_{s,2}, \ldots, h_{s,L})$, we can write the MGF of $\Gamma$ as

$$M_{\Gamma}(\xi) = \prod_{n=1}^{L} \int_{V_{n}} \exp \left( \frac{\xi}{2N_{0}} h_{s,n} G_{n} h_{s,n}^{\dagger} \right) p(h_{s,n}) dV_{n},$$

$$= \prod_{n=1}^{L} M_{\Gamma_{n}}(\xi), \quad (3.32)$$

where

$$\Gamma_{n} = \frac{E_{s}}{2N_{0}} h_{s,n} G_{s} h_{s,n}^{\dagger}. \quad \text{Here the } 2L^{2} - \text{th order integral in (3.30) reduces to a product of } L \times 2K \text{-th order integrals, each corresponding to the MGF of one of the } \Gamma_{n}, \text{ where } \Gamma_{n} \text{ is a quadratic form of a random variable. The MGF associated with a quadratic random variable is readily found in the literature [124]. Here we present the basic result given in Turin [124] on MGF of a quadratic random variable as follows.}$$

Let $Q$ be a Hermitian matrix and $v$ be a proper complex normal zero-mean Gaussian row vector with covariance matrix $L = E\{v^{\dagger} v\}$. Then the MGF of the (real) quadratic form $f = vQv^{\dagger}$ is given by

$$M_{f}(\xi) = [\det (I - \xi L Q)]^{-1}. \quad (3.33)$$

In our case, $G_{n}$ is a Hermitian matrix (the proof is given in Appendix-A.1). Therefore, using (3.33) we write the MGF of $\Gamma_{n}$ as

$$M_{\Gamma_{n}}(\xi) = \left[ \det \left( I - \frac{\xi^{2}}{2} R_{s,n} G_{n} \right) \right]^{-1}. \quad (3.34)$$
where } γ = E_s/N_0 \text{ is the average symbol energy-to-noise ratio (SNR), } R_{s,n} \text{ is the covariance matrix of } h_{s,n} \text{ as defined in (3.31) and } G_n \text{ is given in (3.27). Substituting (3.34) in (3.32) and then the result in (3.24) yields the exact-PEP}

\[
P(S \rightarrow \hat{S}) = \frac{1}{\pi} \int_0^{\pi/2} \prod_{n=1}^L \left[ \det \left( I + \frac{\gamma}{4 \sin^2 \theta} R_{s,n} G_n \right) \right]^{-1} d\theta. \tag{3.35}
\]

**Remark 3.1** Eq. (3.35) is the exact-PEP\(^4\) of a coherent space-time coded system applied to a spatially-correlated fast fading channel following the channel decomposition (3.2).

Since the maximum of the integrand occurs at the upper limit, i.e., for } \theta = \pi/2, \text{ replacing the integrand by its maximum value immediately gives the Chernoff upper bound

\[
P(S \rightarrow \hat{S}) \leq \frac{1}{2} \prod_{n=1}^L \left[ \det \left( I + \frac{\gamma}{4 \sin^2 \theta} R_{s,n} G_n \right) \right]^{-1}, \tag{3.36}
\]

which is the PEP upper-bound between two distinct space-time codewords for a spatially-correlated fast-fading channel.

**Remark 3.2** When } R_{s,n} = I \text{ (i.e., correlation between different communication modes is zero), Eq. (3.35) above captures the effects due to antenna spacing and antenna geometry on the performance of a coherent space-time code over a fast fading channel.

**Remark 3.3** When the fading channels are independent (i.e., } R_{s,n} = I \text{ and } G_n = I_{n_R} \otimes s^n_{\Delta}), \text{ (3.35) simplifies to}

\[
P(S \rightarrow \hat{S}) = \frac{1}{\pi} \int_0^{\pi/2} \prod_{n=1}^L \left[ \det \left( I_{n_T} + \frac{\gamma}{4 \sin^2 \theta} s^n_{\Delta} \right) \right]^{-n_R} d\theta,
\]

which is the same as [122, Eq. (9)].

In the next section, we derive the exact-PEP of a coherent space-time coded system for a slow quasi-static fading channel. Note that, we are not able to use the fast fading result (3.35) to obtain the exact-PEP for a slow fading channel.

---

\(^4\)Eq. (3.35) can be evaluated in closed form using one of the analytical techniques discussed in Section 3.6.
This is because we derived (3.35) under the assumption of a temporally uncorrelated scattering environment. In contrast, for a slow fading channel, the scattering environment is fully temporally correlated.

### 3.4.2 Slow Fading Channel Model

For a slow fading channel, \( H_n = H \) independent of \( n \) in which case (3.25) becomes

\[
\Gamma = \frac{E_s}{2N_0} h_s G h_s^\dagger, \quad (3.37)
\]

where \( h_s = (\text{vec}(H_s^T))^T \) is a row vector with proper complex normal Gaussian distributed entries, \( H_s \) is the random scattering channel matrix with \( H_{s,n} = H_s \) for \( n = 1, \cdots, L \) in (3.2) and

\[
G = (J_R^\dagger J_R)^T \otimes (J_L^\dagger S \Delta J_L). \quad (3.38)
\]

As before, \( \Gamma \) is a random variable that has a quadratic form. Since \( G \) in (3.38) is Hermitian (as shown in Appendix A.1), using (3.33), we can write the MGF of \( \Gamma \) as

\[
\mathcal{M}_\Gamma(\xi) = \left[ \det \left( I - \frac{\xi}{4} R_s G \right) \right]^{-1}, \quad (3.39)
\]

where \( R_s = \mathbb{E} \{ h_s^\dagger h_s \} \). Substitution of (3.39) into (3.24) yields

\[
P(S \rightarrow \hat{S}) = \frac{1}{\pi} \int_0^{\pi/2} \left[ \det \left( I + \frac{\pi}{4 \sin^2 \theta} R_s G \right) \right]^{-1} \, d\theta. \quad (3.40)
\]

**Remark 3.4** Eq. (3.40) is the exact-PEP of a coherent space-time coded system applied to a spatially correlated slow fading MIMO channel following the channel decomposition (3.2).

Substitution of \( \theta = \pi/2 \) in (3.40) gives the PEP upper-bound for correlated slow-fading channels as

\[
P(S \rightarrow \hat{S}) \leq \frac{1}{2} \left[ \det \left( I + \frac{\pi}{4} R_s G \right) \right]^{-1}. \quad (3.41)
\]

**Remark 3.5** When the fading channels are independent (i.e., \( R_s = I \) and \( G = \).
\( I_{n_T} \otimes S_\Delta \), (3.40) simplifies to,

\[
P(S \rightarrow \hat{S}) = \frac{1}{\pi} \int_{0}^{\pi/2} \left[ \det \left( I_{n_T} + \frac{\gamma}{4\sin^2 \theta} S_\Delta \right) \right]^{-n_R} \, d\theta, \quad (3.42)
\]

which is the same as [122, Eq. (13)].

Substitution of \( \theta = \pi/2 \) in (3.42) gives PEP upper-bound

\[
P(S \rightarrow \hat{S}) \leq \frac{1}{2} \left[ \det \left( I_{n_T} + \frac{\gamma}{4} S_\Delta \right) \right]^{-n_R}, \quad (3.43)
\]

which is the global upper-bound derived by Tarokh et al. in [8] for i.i.d. fading channels.

**Space-Time Code Construction Criteria**

By construction, \( \text{rank}(S_\Delta) = \text{rank}(S - \hat{S}) \). If \( \beta \) is the rank of \( (S - \hat{S}) \), then exactly \( \beta \) eigenvalues are nonzero and exactly \( n_T - \beta \) eigenvalues are zero in \( S_\Delta \). Suppose \( \lambda_i \) is the \( i \)-th eigenvalue of \( S_\Delta \) arranged in descending order. For high SNR \( (E_s/N_0 \gg 1) \), the upper-bound (3.43) can be approximated as

\[
P(S \rightarrow \hat{S}) \leq \left( \frac{1}{\beta (S - \hat{S})(S - \hat{S})^\dagger} \right)^{n_R}, \quad (3.44a)
\]

\[
= \left( \frac{\gamma}{4 \prod_{i=1}^{\beta} \lambda_i} \right)^{-\beta n_R}, \quad (3.44b)
\]

From the exponent of the signal-to-noise ratio, the overall diversity advantage of the system is \( \beta n_R \) and from the multiplicative factor, the coding advantage is \( \left( \prod_{i=1}^{\beta} \lambda_i \right)^{1/\beta} \). The maximum diversity advantage \( n_R n_T \) is obtained when \( S_\Delta \) is full rank. The design criteria for space-time codes is then

- **The Rank Criterion**: To achieve maximize diversity advantage, the matrix \( S_\Delta = (S - \hat{S})(S - \hat{S})^\dagger \) must be of full rank for all pairs of distinct code words.

- **The Determinant Criterion**: The minimum product \( \prod_{i=1}^{\beta} \lambda_i = \det(S_\Delta) \) for all pairs of distinct code words must be maximized to give maximum coding gain.
For high SNR ($\gamma \gg 1$), the correlated upper-bound (3.41) can be approximated as

$$P(S \rightarrow \hat{S}) \leq \left( \frac{\gamma}{4} \right)^{-n_R n_T} \frac{1}{|R_H| \left| (S - \hat{S})(S - \hat{S})^\dagger \right|^n},$$

(3.45)

where $R_H = (J_R^* \otimes J_T^*) R_S (J_R^T \otimes J_T^\dagger)$ is the correlation matrix of the MIMO channel. Comparing (3.45) with (3.44a) we can observe that when $\text{rank}(R_H) \geq \text{rank}(S_\Delta)$, the above rank-determinant design criterion is independent of the fading channel correlation.

### 3.4.3 Kronecker Product Model as a Special Case

When the scattering distribution at the transmitter is independent of the scattering distribution at the receiver, the modal correlation matrix $R_{s,n}$ can be factored as [69,112]

$$R_{s,n} = \mathcal{E} \{ h_{s,n}^\dagger h_{s,n} \} = F_{R,n} \otimes F_{T,n},$$

(3.46)

where $F_{R,n}$ and $F_{T,n}$ are the transmitter and receiver modal correlation matrices associated with the $n$-th symbol interval. Substituting (3.46) in (3.35) and recalling the definition of $G_n$ in (3.27), we can simplify the exact-PEP for the fast fading channel to

$$P(S \rightarrow \hat{S}) = \frac{1}{\pi} \int_{0}^{\pi/2} \prod_{n=1}^{L} \left[ \text{det} \left( I + \frac{\gamma}{4 \sin^2 \theta} Z_n \right) \right]^{-1} d\theta,$$

(3.47)

where $Z_n = (F_{R,n} J_R^T J_R^*) \otimes (F_{T,n} J_T^\dagger S_\Delta J_T^T)$.

Similarly, for the slow fading channel, we can factor $R_s$ as

$$R_s = \mathcal{E} \{ h_s^\dagger s_s \} = F_R \otimes F_T,$$

(3.48)

and then the exact-PEP can be expressed as

$$P(S \rightarrow \hat{S}) = \frac{1}{\pi} \int_{0}^{\pi/2} \left[ \text{det} \left( I + \frac{\gamma}{4 \sin^2 \theta} Z \right) \right]^{-1} d\theta,$$

(3.49)

where $Z = (F_R J_R^T J_R^*) \otimes (F_T J_T^\dagger S_\Delta J_T^T)$. 

3.5 PEP Analysis of Space-Time Codes in Physical Channel Scenarios

Note that $(\ell, \ell')$-th element of $F_{R,n}$ and $F_{R}$ is given by (3.15) and $(m, m')$-th element of $F_{T,n}$ and $F_{T}$ is given by (3.16). The pairwise error probability expressions (3.47) and (3.49) will be used later in our simulations to investigate the effects of modal correlation on the performance of coherent space-time codes.

3.5 PEP Analysis of Space-Time Codes in Physical Channel Scenarios

Recall the PEP upper-bound (3.41) for slow-fading channels

$$P(S \rightarrow \hat{S}) \leq \frac{1}{2} \left[ \det \left( I_{n_T n_R} + \frac{1}{4} R_s \left( J^T_R J^*_R \otimes J^T_T S_{\Delta} J_T \right) \right) \right]^{-1}. \quad (3.50)$$

Note that PEP upper-bound (3.50) captures the properties of the space-time code used through $S_{\Delta}$, transmitter and receiver antenna geometries through $J_T$ and $J_R$ and the scattering environment surrounding the transmitter and receiver regions through $R_s$. This new upper-bound allows us to investigate the individual effects of antenna separation, antenna placement and the scattering distribution parameters. Note that upper-bounds found in [58, 60] do not allow one to analyze the individual effects of above mentioned deterministic and random factors on space-time codes.

In [8], Tarokh et. al. has used the PEP upper-bound for i.i.d. slow-fading channels to derive the design rules for space-time trellis codes, under the hypothesis of high SNR. In these design rules, the overall diversity advantage of the system, $d_g$, is associated with the rank of the code word difference matrix times the number of receiver antennas, i.e., $d_g = n_R \text{rank}(S_{\Delta})$. Using the new upper-bound, it can be shown that the quantitative degree to which the diversity advantage of a space-time code is reduced by the size of the antenna aperture, antenna geometry and scattering distribution parameters.
3.5.1 Diversity vs Antenna Aperture Size and Antenna Configuration

At high SNR, the upper-bound (3.50) becomes
\[
P(S \rightarrow \hat{S}) \leq \frac{1}{2} \left[ \det \left( \frac{7}{4} R_s \left( J_R^T J_R^* \otimes J_T^T S_{\Delta} J_T \right) \right) \right]^{-1}, \tag{3.51}
\]
and the rank of \( R_s(J_R^T J_R^* \otimes J_T^T S_{\Delta} J_T) \) gives the overall diversity advantage of the space-time coded system. To isolate the effects of antenna configuration and aperture size on the PEP, we assume isotropic scattering surrounding the transmitter and receiver apertures, i.e., \( R_s = I_{(2M_T+1)(2M_R+1)} \). In this case, the upper-bound (3.51) becomes
\[
P(S \rightarrow \hat{S}) \leq \frac{1}{2} \left[ \det \left( \frac{7}{4} \left( J_R^T J_R^* \otimes J_T^T S_{\Delta} J_T \right) \right) \right]^{-1},
\]
and the overall diversity advantage of the system is
\[
d_g = \text{rank}(J_R) \times \min\{\text{rank}(J_T), \text{rank}(S_{\Delta})\}.
\]

If rank(\( J_R \)) < \( n_R \) or rank(\( J_T \)) < rank(\( S_{\Delta} \)), then the diversity advantage provided by the space-time code is reduced by the transmitter and receiver antenna configuration matrices. Note that \( J_T \) is \( n_T \times (2M_T + 1) \) and \( J_R \) is \( n_R \times (2M_R + 1) \), where \( M_T \) and \( M_R \) are determined by the size of the transmitter and receiver apertures [114], but not by the number of antennas encompassed in the region. Therefore, it is possible to have a situation where the number of effective modes available in a region is less than the number of antennas used in that region. Therefore, in such a scenario, rank of the antenna configuration matrix is less than the number of antennas employed for transmission or reception, which will result in reduction of diversity advantage from the system.

Spatially Constrained Uniform Linear Array

Uniform linear array configuration is a commonly employed antenna array geometry. Due to the symmetry of uniform linear array, \( J_n(x_q) = J_n(x_{q'}) \), where \( x_q \equiv (\|x_q\|, 0) \) and \( x_{q'} \equiv (\|x_{q'}\|, \pi) \) are the antenna positions symmetric about the array origin. Therefore, there are at most \( m_{\text{ULA}} \) independent columns of antenna
configuration matrix $J$. For a ULA of aperture radius $r$,

$$m_r^{ULA} \leq \lceil \pi r / \lambda \rceil + 1.$$  

Hence, the rank of the receiver antenna configuration matrix $J_R$ is $\min\{n_R, m_r^{ULA}\}$ and the rank of the transmitter configuration matrix $J_T$ is $\min\{n_T, m_r^{ULA}\}$. Therefore, in the event of either $n_R > m_r^{ULA}$ or $n_T > m_r^{ULA}$, the diversity advantage of the space-time coded MIMO system is reduced due to the antenna configuration.

### 3.5.2 Diversity vs Non-isotropic Scattering

To investigate the effects of non-isotropic scattering on the diversity advantage of space-time codes employed on MIMO systems, we assume antenna configurations at the receiver and transmitter antenna arrays do not reduce the diversity of the system. With this assumption, we have $\text{rank}(J_R) = n_R$ and $\text{rank}(J_T) = n_T$. We also assume the scattering channel surrounding the transmitter and receiver regions satisfies the separability condition (3.17). This assumption allows to separate the modal correlation matrix $R_s$ as (Kronecker\(^5\) model)

$$R_s = (F_R \otimes F_T),$$  

(3.52)

where $F_R$ is the $(2M_R + 1) \times (2M_R + 1)$ receiver modal correlation matrix and $F_T$ is the $(2M_T + 1) \times (2M_T + 1)$ transmitter modal correlation matrix.

Substituting (3.52) in (3.51) yields the upper-bound at high SNR

$$P(S \rightarrow \hat{S}) \leq \frac{1}{\frac{E_b}{4N_0}(J_R^* F_R J_R^T) \otimes (J_T^* F_T J_T^T S_{\Delta})}$$

and the overall diversity advantage of the system is

$$d_g = \min\{n_R, \text{rank}(F_R)\} \times \min\{n_T, \text{rank}(F_T), \text{rank}(S_{\Delta})\}. \quad (3.53)$$

From the above expression it is evident that the rank of modal correlation matrices $F_R$ and $F_T$ directly affects the diversity order of the system. In Section 3.3.2 we showed that the $(\ell, \ell')$-th element of $(2M_R + 1) \times (2M_R + 1)$ receiver modal

\(^5\)It will be shown in Chapter 6 that the Kronecker modal is a good approximation to the actual scattering channel when the scattering distribution is uni-modal.
correlation matrix $F_R$ is given by

$$\gamma_{\ell,\ell'} = \int_{S^1} P_{Rx}(\varphi)e^{-i(\ell-\ell')\varphi} d\varphi,$$

and the $(m, m')$-th element of $(2M_T + 1) \times (2M_T + 1)$ transmitter modal correlation matrix $F_T$ is given by

$$\gamma_{m,m'} = \int_{S^1} P_{Tx}(\phi)e^{i(m-m')\phi} d\phi.$$

Azimuth power distributions $P_{Rx}(\varphi)$ and $P_{Tx}(\phi)$ are usually characterized by the mean angle-of-arrival ($\varphi_0$)/mean angle-of-departure ($\phi_0$) and the angular spreads $\sigma_r$ and $\sigma_t$ at the receiver and transmitter regions. In [127], it was shown that for an antenna aperture of fixed radius $r$ with angular spread $\sigma$ of the azimuth power distribution, the number of modes activated in the antenna aperture is given by

$$m_\sigma = 2\lceil \sigma r / \lambda \rceil + 1. \quad (3.54)$$

Note that $\sigma = \pi$ corresponds to the isotropic scattering surrounding the antenna aperture, which is the case considered for $M_R$ and $M_T$ in Section 3.3.1. Also, $m_\sigma$ is related to the number of non-zero eigen-values in the modal correlation matrix, which corresponds to the rank of modal correlation matrix.

Using (3.54), we can define

$$m_{\sigma_r} \triangleq 2\lceil \sigma_r r_R / \lambda \rceil + 1, \quad (3.55)$$

as the number of effective modes at the receiver aperture for fixed aperture radius $r_R$ and angular spread $\sigma_r$, and

$$m_{\sigma_t} \triangleq 2\lceil \sigma_t r_T / \lambda \rceil + 1, \quad (3.56)$$

as the number of effective modes at the transmitter aperture for fixed aperture radius $r_T$ and angular spread $\sigma_t$. Note that $m_{\sigma_r} \leq 2M_R + 1$ and $m_{\sigma_t} \leq 2M_T + 1$.

From (3.55) and (3.56) we can see that for a given aperture radius, the rank of modal correlation matrices $F_R$ and $F_T$ is constrained by the angular spread. As a result, if $m_{\sigma_r} < n_R$ or $m_{\sigma_t} < \min\{n_T, \text{rank}(S_{\Delta})\}$, the diversity advantage of the space-time coded MIMO system is reduced due to the limited angular spread.
3.6 Exact-PEP in Closed-Form

To calculate the exact-PEP, one needs to evaluate the integrals (3.47) and (3.49), either using numerical methods or analytical methods. In the following sections, we present two analytical techniques which can be employed to evaluate the integrals (3.47) and (3.49) in closed form, namely (a) Direct partial fraction expansion (b) Partial fraction expansion via eigenvalue decomposition. We shall use (3.49), which is the integral involved with the slow fading channel model, to introduce these two techniques. Note that both methods can be directly applied to evaluate the integral involved with the fast fading channel; therefore we omit the details here for the sake of brevity.

3.6.1 Direct Partial Fraction Expansion

Matrix $Z$ in (3.49) has size $m_R m_T \times m_R m_T$, where $m_R = 2M_R + 1$ and $m_T = 2M_T + 1$. Therefore, the integrand in (3.49) will take the form

$$\left[ \det \left( I + \frac{\gamma}{4 \sin^2 \theta} Z \right) \right]^{-1} = \frac{(\sin^2 \theta)^N}{\sum_{\ell=0}^{N} a_\ell (\sin^2 \theta)\ell}, \quad (3.57)$$

where $N = m_R m_T$ and $a_\ell$, for $\ell = 1, 2, \cdots, N$, are constants. Note that the denominator of (3.57) is an $N$-th order polynomial in $\sin^2 \theta$ (for the fast fading channel, it would be an $LN$-th order polynomial). To evaluate the integral (3.57) in closed form, we use the partial-fraction expansion technique given in [128, Appendix 5A] as follows.

First we begin by factoring the denominator of (3.57) into terms of the form $(\sin^2 \theta + c_\ell)$, for $\ell = 1, 2, \cdots, N$. This involves finding the roots of an $N$-th order polynomial in $\sin^2 \theta$ either numerically or analytically. Then (3.57) can be expressed in product form as

$$\frac{(\sin^2 \theta)^N}{\sum_{\ell=0}^{N} a_\ell (\sin^2 \theta)\ell} = \prod_{\ell=1}^{\Lambda} \left( \frac{\sin^2 \theta}{c_\ell + \sin^2 \theta} \right)^{m_\ell}, \quad (3.58)$$

where $m_\ell$ is the multiplicity of the root $c_\ell$ and $\sum_{\ell=1}^{\Lambda} m_\ell = N$. By applying the

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6 One would need to evaluate the determinant of $\left( I + \frac{\gamma}{4 \sin^2 \theta} Z \right)$ and then take the reciprocal of it to obtain the form (3.57) and coefficients $a_\ell$ in the denominator.
partial-fraction decomposition theorem to the product form (3.58), we obtain
\[
\prod_{\ell=1}^{\Lambda} \left( \sin^2 \theta \over c_\ell + \sin^2 \theta \right)^{m_\ell} = \sum_{\ell=1}^{\Lambda} \sum_{k=1}^{m_\ell} A_{k\ell} \left( \sin^2 \theta \over c_\ell + \sin^2 \theta \right)^{k}
\]
where the residual \( A_{k\ell} \) is given by [128, Eq. 5A.72]
\[
A_{k\ell} = \begin{cases}
\frac{d^{m_\ell-k}}{dx^{m_\ell-k}} \prod_{n=1, n \neq \ell}^{\Lambda} \left( 1 + c_n x \right)^{m_n} \\
(m_\ell - k)! c_\ell^{m_\ell-k}
\end{cases}
\]
(3.60)

Expansion (3.59) often allows integration to be performed on each term separately by inspection. In fact, each term in (3.59) can be separately integrated using a result found in [122], where
\[
P(c_\ell, k) = \frac{1}{\pi} \int_{0}^{\pi/2} \left( \frac{\sin^2 \theta}{c_\ell + \sin^2 \theta} \right)^{k} d\theta,
\]
\[
= \frac{1}{2} \left[ 1 - \sqrt{\frac{c_\ell}{1 + c_\ell}} \sum_{j=0}^{k-1} \binom{2j}{j} \left( \frac{1}{4(1 + c_\ell)} \right)^{j} \right].
\]
(3.61)

Now using the partial-fraction form of the integrand in (3.59) together with (3.61), we obtain the exact-PEP in closed form as
\[
P(S \rightarrow \hat{S}) = \frac{1}{\pi} \int_{0}^{\pi/2} \prod_{\ell=1}^{\Lambda} \left( \sin^2 \theta \over c_\ell + \sin^2 \theta \right)^{m_\ell} d\theta,
\]
\[
= \sum_{\ell=1}^{\Lambda} \sum_{k=1}^{m_\ell} A_{k\ell} P(c_\ell, k).
\]
(3.62)

For the special case of distinct roots, i.e., \( m_1 = m_2 = \cdots = m_N = 1 \), the exact-PEP is given by
\[
P(S \rightarrow \hat{S}) = \frac{1}{2} \sum_{\ell=1}^{N} \left( 1 - \sqrt{\frac{c_\ell}{1 + c_\ell}} \right) \prod_{n=1, n \neq \ell}^{N} \left( \frac{c_\ell - c_n}{c_\ell - c_n} \right).
\]

3.6.2 Partial Fraction Expansion via Eigenvalue Decomposition

The main difficulty with the above technique is finding the roots of an \( N \)-th order polynomial. Here we provide a rather simple way to evaluate the exact-PEP in
3.7 Analytical Performance Evaluation: Examples

In this section, we consider the following three space-time trellis codes as examples.

(a) 4-state QPSK STTC with two transmit antennas [8] as shown in Figure 3.1 where the labeling of the trellis branches follow [8]. The QPSK signal points are mapped to the edge label symbols as shown in Figure 3.1.

(b) 16-state QPSK STTC with three transmit antennas [118, Table 1].

(c) 64-state QPSK STTC with four transmit antennas [118, Table 1].

For a MIMO system with $n_R$ receive antennas, when the underlying MIMO channel is i.i.d., the diversity advantage obtained by applying code-(a) is $2n_R$, code-(b) is $3n_R$ and code-(c) is $4n_R$.

For the 4-state code, the exact-PEP results and approximate bit-error probability (BEP) results for $n_R = 1$ and $n_R = 2$ were presented in [121, 122] for i.i.d. fast fading and slow fading channels. In [59], the effects of fading correlation on the
average BEP were studied for $n_R = 1$ over a slow fading channel. In this chapter, we compare the i.i.d. channel performance results presented in [121,122] with our realistic exact-PEP results for different antenna spacing and scattering distribution parameters. We also compare the performance of the generalized PEP upper-bound (3.41) against that of the global upper-bound (3.43) derived by Tarokh et al. In addition, we use the 16-state code with three transmit antennas and the 64-state code with four transmit antennas to study the impact of transmit antenna geometry on the performance of coherent space-time codes.

In [121,122], performances were evaluated under the assumption that the transmitted codeword is the all-zero codeword. Here we also adopt the same assumption as we compare our results with their results. However, we are aware that space-time codes may, in general, be non-linear, i.e., the average BEP can depend on the transmitted codeword.

For the 4-state STTC, we have the shortest error event path of length $D = 2$, as illustrated by shading in Figure 3.1 and

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. $$

### 3.8 Effect of Antenna Separation

First we consider the effects of antenna aperture size and antenna configuration on the performance coherent space-time codes. To isolate the effects of antenna aperture size and antenna configurations, we assume the scattering environment surrounding the transmitter and receiver apertures is isotropic, i.e., $F_T = I_{2M_T + 1}$ and $F_R = I_{2M_R + 1}$ for the slow fading channel and $F_{T,n} = I_{2M_T + 1}$ and $F_{R,n} = I_{2M_R + 1}$ for the fast fading channel.
$I_{2M_R+1}$ for the fast fading channel.

### 3.8.1 Slow Fading Channel

Consider the 4-state STTC with $n_T = 2$ transmit antennas and $n_R = 1$ receive antenna. In this case, we place the two transmit antennas in a circular aperture of radius $r_T$ (antenna separation = $2r_T$). Since $n_R = 1$, there will only be a single communication mode available at the receiver aperture, hence $J_R = 1$.

Figure 3.2 shows the exact-PEP performance of the 4-state STTC for error event of length $D = 2$ and transmit antenna separations $0.1\lambda$, $0.2\lambda$, $0.5\lambda$ and $\lambda$, where $\lambda$ is the wave-length. Also shown in Figure 3.2 for comparison is the exact-PEP for the i.i.d. slow fading channel (Rayleigh) corresponding to $D = 2$.

![Figure 3.2: Exact pairwise error probability performance of the 4-state space-time trellis code with 2-transmit antennas and 1-receive antenna: length 2 error event, slow fading channel.](image)

As we can see from the figure, the effect of antenna separation on the exact-PEP is not significant when the transmit antenna separation is $0.5\lambda$ or higher. However, the effect is significant when the transmit antenna separation is small.
For example, at PEP $10^{-3}$, the realistic PEPs are 1dB and 3dB away from the i.i.d. channel performance results for $0.2\lambda$ and $0.1\lambda$ transmit antenna separations, respectively. From these observations, we can emphasize that the effect of antenna spacing on the performance of the 4-state STTC is minimum for higher antenna separations whereas the effect is significant for smaller antenna separations.

**Loss of Diversity Advantage due to a Region with Limited Size**

We now consider the diversity advantage of a coherent space-time coded system as the number of receive antennas increases while the receive antenna array aperture radius remains fixed. Figure 3.3 shows the exact-PEP of the 4-state STTC with two transmit antennas and $n_R$ receive antennas, where $n_R = 1, 2, \ldots, 10$. The two transmit antennas are placed in a circular aperture of radius $0.25\lambda$ (antenna separation $7 = 0.5\lambda$) and $n_R$ receive antennas are placed in a uniform circular array antenna configuration with radius $0.15\lambda$. In this case, the distance between two adjacent receive antenna elements is $0.3\lambda \sin(\pi/n_R)$.

The slope of the performance curve on a log scale corresponds to the diversity advantage of the code and the horizontal shift in the performance curve corresponds to the coding advantage. According to the code construction criteria given in [8], the diversity advantage promised by the 4-state STTC is $2n_R$. With the above antenna configuration setup, however, we observed that the slope of each performance curve remains the same when $n_R > 5$, which results in zero diversity advantage improvement for $n_R > 5$. Nevertheless, for $n_R > 5$, we still observe some improvement in the coding gain, but the rate of improvement is slower with the increase in number of receive antennas. Here the loss of diversity gain is due to the fewer number of effective communication modes available at the receiver region than the number of antennas available for reception. In this case, the receiver aperture of radius $0.15\lambda$ corresponds to $2 \left\lceil \pi e 0.15 \right\rceil + 1 = 5$ effective communication modes at the receiver region. Therefore when $n_R > 5$, the diversity advantage of the code is determined by the number of effective communication modes available at the receiver antenna region rather than the number of antennas available for reception. That is, the point where the diversity loss occurred is clearly related to the size of the antenna aperture, where smaller apertures result in diversity loss of

---

7In a 3-dimensional isotropic scattering environment, antenna separation $0.5\lambda$ (first null of the order zero spherical Bessel function) gives zero spatial correlation, but here we constraint our analysis to a 2-dimensional scattering environment. The spatial correlation function in a 2-dimensional isotropic scattering environment is given by a Bessel function of the first kind. Therefore, antenna separation $\lambda/2$ does not give zero spatial correlation in a 2-dimensional isotropic scattering environment.
3.8 Effect of Antenna Separation

Figure 3.3: Exact PEP performance of the 4-state space-time trellis code with 2-transmit antennas and n-receive antennas: length 2 error event, slow fading channel.

Figure 3.4 shows the PEP upper-bound for length 2 error event of 4-state STTC at 10dB SNR for apertures of radius 0.15\(\lambda\) and 0.25\(\lambda\) in isotropic scattering environment for increasing number of receive antennas. Vertical dashed lines indicate the number of effective modes in the receiver region for each aperture size. The global upper-bound corresponding to the i.i.d. channel is also shown in Figure 3.4.

It can be observed from Figure 3.4 that the global upper-bound is linearly decreased with increasing number of receive antennas. However, with both UCA and ULA antenna configurations PEP upper-bound is linearly decreased up until a certain number of receive antennas and thereafter a logarithmic reduction of PEP is observed with the increasing number of receive antennas. Due to the spatial correlation between the antenna elements, transition from linear to logarithmic occurs before the number of receive antennas equal the number of effective modes.
Figure 3.4: Length 2 error event of 4-state QPSK space-time trellis code with two transmit antennas for an increasing number of receive antennas in an isotropic scattering environment. $r_T = 0.5\lambda$, $r_R = \{0.15\lambda, 0.25\lambda\}$ and SNR = 10dB; slow-fading channel.

in the region for both antenna configurations.

**Effect of Transmit Antenna Configuration**

First we compare the exact-PEP performance of the 16-state QPSK STTC with three transmit antennas for different antenna configurations at the transmitter. Here we consider UCA and ULA antenna configurations as examples. Three transmit antennas are placed within a fixed circular aperture of radius $r_T (= 0.15\lambda, 0.25\lambda)$, where the antenna placements are shown in Figure 3.5. The exact-PEP performance for the shortest error event path of length three is also shown in Figure 3.5 for a single receive antenna.

From Figure 3.5, it is observed that at high SNRs the performance given by the UCA antenna configuration outperforms that of the ULA antenna configuration. For example, at 14dB SNR, the performance differences between UCA and ULA
3.8 Effect of Antenna Separation

Figure 3.5: The exact-PEP performance of the 16-state code with 3-transmit and 1-receive antennas for UCA and ULA transmit antenna configurations: length 3 error event, slow fading channel.

are 1.75dB with 0.15$\lambda$ transmitter aperture radius and 1dB with 0.25$\lambda$ transmitter aperture radius. From Figure 3.5, we observed that as the radius of the transmitter aperture decreases the diversity advantage of the code is reduced, particularly for the ULA antenna configuration. Here, the loss of diversity advantage is mainly due to the loss of rank of $\mathbf{J}_T$.

We now presents Monte Carlo simulation results of space-time trellis codes with three and four transmit antennas for a number of spatial scenarios. The performance is measured in terms of frame error rates. For simplicity, we assume that a single receive antenna is employed at the receiver and also assume isotropic scattering at the transmitter.

For the code-(b), we place the three transmit antennas in UCA and ULA configurations, and set the radius of the circular aperture to 0.1$\lambda$, corresponding to $2[\pi e0.1] + 1 = 3$ effective modes at the transmitter aperture. We found that $\text{rank}(\mathbf{J}_T) = 3 = \text{rank}(\mathbf{S}_\Delta)$ for UCA antenna configuration and $\text{rank}(\mathbf{J}_T) = 2(<$
rank(\(S_\Delta\))) for ULA antenna configuration. Frame-error rate performance results of code-(b) for these two antenna configurations are shown in Figure 3.6. On the same figure, the performance results of code-(b) for i.i.d. slow-fading channel is also shown.

Figure 3.6: Frame error rate performance of the 16-state QPSK, space-time trellis code with three transmit antennas for UCL and ULA antenna configurations in an isotropic scattering environment; slow-fading channel.

From Figure 3.6, it can be observed that at high SNR, the slope of the UCA performance curve is similar to that of i.i.d. channel. This observation indicates that UCA antenna configuration does not diminish the diversity advantage given by the space-time code. However, with the ULA antenna configuration, we observe that the slope of the ULA performance curve is not similar to that of i.i.d. channel at high SNR, hence reducing the overall diversity of the system. These observations indicate that at 0.1\(\lambda\) radius with three transmit antennas, the UCA antenna configuration is best suited to employ the QPSK 16-state STTC, as it does not diminish the diversity gain provided by the code, where as the ULA configuration is not suited as it reduces the diversity advantage given by the code since the rank
of $J_T$ is less than the rank of $S_\Delta$. It is also observed that there is a significant performance difference between the i.i.d. channel case and the UCA. The reason for this difference is that, in the i.i.d. channel case we assume transmit antennas are located far apart from each other, while in the UCA case all the transmit antennas are spatially constrained within a circular region of radius $0.1\lambda$. This will result in spatial correlation among transmit antenna elements and hence limiting the performance.

![Figure 3.7: Frame error rate performance of the 64-state QPSK space-time trellis code with four transmit antennas for UCL and ULA antenna configurations in an isotropic scattering environment; slow-fading channel.](image)

For the code-(c), we place the four transmit antennas in UCA and ULA configurations, and set the radius of the circular aperture to $0.2\lambda$, corresponding to $2[\pi e0.2] + 1 = 5$ effective modes at the transmit aperture. It is found that $\text{rank}(J_T) = 3(< \text{rank}(S_\Delta))$ for the ULA antenna configuration and $\text{rank}(J_T) = 4 = \text{rank}(S_\Delta)$ for the UCA antenna configuration. The frame error rate performance results of code-(c) for these two antenna configurations and also for i.i.d. slow-fading channel are shown in Figure 3.7. Similar performance results are ob-
served as for the code-(c). We observe that at $0.2\lambda$ radius with four transmit antennas, UCA antenna configuration is best suited to employ space-time trellis codes while ULA antenna configuration is not.

### 3.8.2 Fast Fading Channel

Consider the 4-state STTC with two transmit antennas and two receive antennas, where the two transmit antennas are placed in a circular aperture of radius $0.25\lambda$ (antenna separation $= 0.5\lambda$) and the two receive antennas are placed in a circular aperture of radius $r_R$ (antenna separation $= 2r_R$).

![Figure 3.8: Exact pairwise error probability performance of the 4-state space-time trellis code with 2-transmit antennas and 2-receive antennas-length two error event: fast fading channel.](image)

Figure 3.8 shows the exact pairwise error probability performance of the 4-state space-time trellis code with 2-transmit antennas and 2-receive antennas-length two error event: fast fading channel.

Similar results are observed as for the slow fading channel. For the fast fading channel, the effect of antenna separation is minimum when the antenna separation...
is higher and it is significant when the antenna separation is smaller (< 0.5λ). At 0.1λ receive antenna separation, the performance loss is 3dB and at 0.2λ the performance loss is 1dB for PEP of 10^{-5}. Note that the performance loss we observed here is mainly due to the insufficient antenna spacing.

In summary, the above results indicate that the diversity gain of a space-time coded system is governed by the rank of the antenna configuration matrix and the number of effective communication modes in the antenna aperture (directly related to the radius of the antenna aperture). In fact, the upper-limit for maximum number of antennas in an antenna aperture, without losing the diversity advantage of the space-time code, is given by the rank of $J_T$.

### 3.9 Effects of Non-isotropic Scattering

We now investigate the effects of non-isotropic scattering on the performance of space-time codes. For simplicity, we only consider non-isotropic scattering at the receiver region and assume isotropic scattering at the transmitter region. In Chapter 2 we observed that all azimuth power distributions (scattering distributions) give very similar correlation values for a given angular spread, especially for small antenna separations. Therefore, without loss of generality, we restrict our investigation only to the uniform limited azimuth power distribution. For this distribution, the modal correlation coefficients at the receiver region for a slow-fading scattering channel are given by

$$
\gamma_{\ell, \ell'} = \text{sinc}((\ell - \ell')\Delta_r)e^{-i(\ell - \ell')\varphi_0}, \quad \text{for } \ell, \ell' = 1, 2, \cdots, 2M_R + 1,
$$

(3.64)

where $\varphi_0$ is the mean angle of arrival (AOA) and $\Delta_r$ is the non-isotropic parameter of the azimuth power distribution, which is related to the angular spread $\sigma_r = \Delta_r/\sqrt{3}$.

#### 3.9.1 Slow Fading Channel

Figure 3.9 shows the PEP upper-bound for length 2 error event of 4-state QPSK STTC at 10dB SNR for mean AOA $\varphi_0 = 0^\circ$ and non-isotropic parameter $\Delta_r = \{2^\circ, 5^\circ, 10^\circ, 30^\circ\}$ for increasing number of receive antennas. We set $r_T = 0.5\lambda$ and $r_R = 2\lambda$, and position receiver antennas in a UCA configuration. Note that the size of the receiver aperture and the antenna configuration do not effect the diversity order of the system and the effects are mainly due to the non-isotropic scattering at the receiver. For comparison, the global upper-bound corresponding to the i.i.d.
slow-fading channel is also shown in Figure 3.9.

Using (3.55), we have \( m_{\sigma_r} = \{3, 3, 3, 5\} \) number of effective modes at the receiver region for receiver aperture radius \( 2\lambda \) and angular spread \( \sigma_r = \Delta_r/\sqrt{3} \approx \{1^\circ, 3^\circ, 6^\circ, 17^\circ\} \), respectively. It can be observed from Figure 3.9 that the global upper-bound is linearly decreased with increasing number of receive antennas, hence the diversity is increased linearly with the increasing number of receive antennas. However, in the presence of non-isotropic scattering, the PEP bound is decreased linearly with the increasing number of receive antennas for \( n_R \leq m_{\sigma_r} \) and there after a logarithmic decrease in PEP is observed with the increasing number of receive antennas. These observations indicate that the performance of the space-time codes are limited by the angular spread and the size of the antenna aperture.

![Figure 3.9: Length 2 error event of 4-state QPSK space-time trellis code with two transmit antennas for an increasing number of receive antennas in a non-isotropic scattering environment; \( r_T = 0.5\lambda, r_R = 2\lambda \) and SNR = 10dB: slow-fading channel.](image_url)
3.9.2 Fast Fading Channel

On a fast fading channel environment, we assume that the scattering gains change independently from symbol to symbol. It is also reasonable to assume that the statistics of the scattering channel remain constant over an interval of interest. Here we take the interval of interest as the length of the space-time codeword. Then we have, $R_{s,n} = R_s$ for $n = 1, 2, \cdots, L$ in (3.31) and the receiver modal correlation coefficients for a uniform limited distribution is given by (3.64).

![Figure 3.10: Effect of receiver modal correlation on the exact-PEP of the 4-state QPSK space-time trellis code with 2-transmit antennas and 2-receive antennas for the length 2 error event. Uniform limited power distribution with mean angle of arrival 0° from broadside and angular spreads $\Delta_r = \{5^\circ, 30^\circ, 60^\circ, 180^\circ\}$; fast fading channel.](image)

Consider the 4-state STTC with two transmit antennas and two receive antennas, where the two transmit antennas are separated by a distance of 0.5\(\lambda\) and also the two receive antennas are separated by a distance of 0.5\(\lambda\) (i.e., $r_R = r_T = 0.25\lambda$). Figure 3.10 shows the exact-PEP performances of the 4-state STTC for various receiver non-isotropic parameters $\Delta_r = \{5^\circ, 30^\circ, 60^\circ, 180^\circ\}$ (or receiver angular spreads $\sigma_r \approx \Delta_r/\sqrt{3} = \{3^\circ, 17^\circ, 35^\circ, 104^\circ\}$) about a mean AOA 0° from broadside.
Note that \( \Delta_r = 180^\circ \) represents the isotropic scattering environment. The exact-PEP performance for the i.i.d. fast fading channel (Rayleigh) is also plotted on the same graph for comparison.

![Graph showing the effect of receiver modal correlation on the exact-PEP of the 4-state QPSK space-time trellis code with 2-transmit antennas and 2-receive antennas for the length 2 error event. Uniform limited power distribution with mean angle of arrival 45° from broadside and angular spreads \( \Delta_r = \{5^\circ, 30^\circ, 60^\circ, 180^\circ\} \); fast fading channel.](image)

Figure 3.11: Effect of receiver modal correlation on the exact-PEP of the 4-state QPSK space-time trellis code with 2-transmit antennas and 2-receive antennas for the length 2 error event. Uniform limited power distribution with mean angle of arrival 45° from broadside and angular spreads \( \Delta_r = \{5^\circ, 30^\circ, 60^\circ, 180^\circ\} \); fast fading channel.

Figure 3.10 suggests that the performance loss incurred due to the modal correlation increases as the angular spread of the distribution decreases. For example, at PEP \( 10^{-5} \), the realistic PEP performance results obtained from (3.47) are 0.25dB, 2.5dB, 3.25dB and 7.5dB away from the i.i.d. channel performance results for non-isotropic parameters 180°, 60°, 30° and 5°, respectively. Therefore, in general, if the angular spread of the distribution is closer to isotropic scattering, then the loss incurred due to the modal correlation is insignificant, provided that the antenna spacing is optimal. However, for moderate angular spread values such as 35° and 17°, the performance loss is quite significant. This is due to the higher concentration of energy closer to the mean AOA for small angular spreads. It is also observed
that for large angular spread values, the diversity order of the code (slope of the performance curve) is preserved whereas for small and moderate angular spread values, the diversity order of the code is diminished.

Figure 3.11 shows the exact-PEP performance results of the 4-state STTC for a mean AOA 45° from broadside. Similar results are observed for the mean AOA 0° from broadside case. Comparing Figures 3.10 and 3.11 we observe that the performance loss is increased for all angular spreads as the mean AOA moves away from broadside. This can be justified by the reasoning that, as the mean AOA moves away from broadside, there will be a reduction in the angular spread exposed to the antennas and hence less signals being captured.

Finally, we consider the exact-PEP results for the length two error event against the receive antenna separation for a mean AOA 45° from broadside and non-isotropic parameters $\Delta_r = \{5°, 30°, 180°\}$. The results are plotted in Figures 3.12 and 3.13 for SNRs 8dB and 10dB, respectively.

![Figure 3.12: Exact-PEP of the 4-state QPSK space-time trellis code with 2-transmit antennas and 2-receive antennas against the receive antenna separation at 8dB SNR. Uniform limited power distribution with mean angle of arrival 45° from broadside and angular spreads $\Delta_r = \{5°, 30°, 180°\}$; fast fading channel](image-url)
It is observed that for a given SNR, the performance of the space-time code is improved as the receive antenna separation and the angular spread are increased. However, the performance does not improve monotonically with the increase in receive antenna separation. We also observed that when the angular spread is quite small (e.g. $3^\circ$), we need to place the two receive antenna elements at least several wavelengths apart in order to achieve the maximum performance gain given by the 4-state STTC.

Comparison of Figures 3.10, 3.11, 3.12 and 3.13 reveals that when the angular spread of the surrounding azimuth power distribution is closer to isotropic, the performance degradation of the code is mainly due to the insufficient antenna spacing. Therefore, employing multiple antennas on a Mobile-Unit (MU) will result in significant performance loss due to the limited size of the MU.

In summary, based on the results we obtained thus far, we can claim that, in
general, space-time trellis codes are susceptible to spatial fading correlation effects, in particular, when the antenna separation and the angular spread are small.

### 3.10 Extension of PEP to Average Bit Error Probability

An approximation to the average bit error probability (BEP) was given in [129] on the basis of accounting for error event paths of lengths up to $D$ as,

$$P_b(E) \approx \frac{1}{b} \sum_t q(S \rightarrow \hat{S})_t P(S \rightarrow \hat{S})_t,$$

(3.65)

where $b$ is the number of input bits per transmission, $q(S \rightarrow \hat{S})_t$ is the number of bit errors associated with the error event $t$ and $P(S \rightarrow \hat{S})_t$ is the corresponding PEP. In [122], it was shown that error event paths of lengths up to $D$ are sufficient to achieve a reasonably good approximation to the full upper (union) bound that takes into account error event paths of all lengths. For example, with the 4-state STTC, error event paths\(^8\) of lengths up to $D = 4$ and $D = 3$ are sufficient for the slow and fast fading channels, respectively.

The closed-form solution for average BEP of a space-time code can be obtained by finding closed-form solutions for PEPs associated with each error type, using one of the analytical techniques given in Section 3.6. In previous sections, we investigate the effects of antenna spacing, antenna geometry and modal correlation on the exact-PEP of a space-time code over fast and slow fading channels. The observations and claims which we made there, are also valid for the BEP case as the BEPs are calculated directly from PEPs. Therefore, to avoid repetition, we do not discuss BEP performance results here.

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\(^8\)The Appendix A.2 lists the all possible error events for the 4-state QPSK STTC up to $D = 4$. 
Part II: Performance Limits of Non-coherent Space-Time Codes

3.11 System Model: Non-Coherent Space-Time Codes

Consider a MIMO system consisting of $n_T$ transmit antennas and $n_R$ receive antennas within circular apertures of radius $r_T$ and $r_R$, respectively, along with the channel decomposition (3.10). Let $X(k)$ be the $k$-th $n_T \times L$ code matrix to be transmitted by $n_T$ transmit antennas over $L$ symbol intervals. At the start of the transmission, the transmitter sends the code matrix $X(0) = D$. Thereafter, information is differentially encoded according to the rule

$$X(k) = X(k-1)S_{\ell(k)}, \quad \text{for } k = 1, 2, \cdots$$  (3.66)

where $S_{\ell(k)} \in \mathbb{C}^{n_T \times L}$ is the $k$-th information matrix which is an element of a group of unitary space-time modulated constellation matrices $\mathcal{V}$ of size $T$ with unitary property $S_{\ell(k)}S_{\ell(k)}^\dagger = I$ for $\ell(k) = 0, 1, \cdots, L - 1$ [49]. This unitary space-time constellation can be constructed based on orthogonal designs [40] or group designs [48, 49]. Similar to [48, 49] we assume that $L = n_T$ and also $D = I_{n_T}$. As a result, $X(k)$ is also unitary.

Let $H \in \mathbb{C}^{n_R \times n_T}$ be the unknown fading channel gain matrix and $N(k) \in \mathbb{C}^{n_R \times n_T}$ be the additive noise matrix, then the received signal $Y(k) \in \mathbb{C}^{n_R \times n_T}$ corresponding to the $k$-th space-time codeword $X(k)$ can be written as

$$Y(k) = \sqrt{E_s}HX(k) + N(k), \quad \text{for } k = 0, 1, 2, \cdots$$  (3.67)

where $E_s$ is the average transmitted signal energy per symbol period. Each of the elements of $N(k)$ is assumed to be independently and identically distributed zero-mean complex Gaussian random variable with variance $\sigma_n^2/2$ per complex dimension. The $(p, q)$-th entry of $H$ is the complex channel fading gain from transmit antenna $q$ to receive antenna $p$ and fading gains are assumed to be quasi-static Rayleigh (slow-fading).

Differential Detection at the Receiver

At the receiver, the transmitted signal can be non-coherently demodulated by using two consecutive observations, $Y(k-1)$ and $Y(k)$. We assume that the channel
matrix $H$ remains constant for $Y(k-1)$ and $Y(k)$. Signals $Y(k-1)$ and $Y(k)$ can be expressed in vector form (row) as

$$
y(k-1) = \sqrt{E_s}hX(k-1) + n(k-1)
$$

$$
y(k) = \sqrt{E_s}hX(k) + n(k),
$$

where $y(k) = (\text{vec}\{Y^T(k)\})^T$, $X(k) = I_{nr} \otimes X(k)$, $h = (\text{vec}\{H^T\})^T$, $n(k) = (\text{vec}\{N^T(k)\})^T$, $S_{t(k)} = I_{nr} \otimes S_{t(k)}$ and $w(k) = n(k) - n(k-1)S_{t(k)}$. To obtain $y(k)$ and $y(k-1)$, we have used the vec{·} identity $\text{vec}\{AXB\} = (B^T \otimes A)\text{vec}\{X\}$. From (3.69), the transmitted data matrix is differentially detected using the following maximum likelihood receiver

$$
\hat{S} = \arg\min_{S \in V} \| y(k) - y(k-1)S \|^2
$$

$$
= \arg\max_{S \in V} \text{Re}\{y(k-1)Sy^\dagger(k)\}. \quad (3.70)
$$

### 3.12 Exact PEP of Differential Space-Time Codes

Based on (3.70), the receiver will erroneously select $S_j$ when $S_i$ was actually sent as the $k$-th information matrix if

$$
\| y(k) - y(k-1)S_j \|^2 \leq \| y(k) - y(k-1)S_i \|^2, \quad (3.71a)
$$

$$
y(k-1)D_{i,j}y^\dagger(k-1) \leq 2\text{Re}\{w(k)\Delta_{i,j}^\dagger y^\dagger(k-1)\}, \quad (3.71b)
$$

where $\Delta_{i,j} = S_j - S_i = I_{nr} \otimes (S_j - S_i)$ and $D_{i,j} = \Delta_{i,j}\Delta_{i,j}^\dagger = I_{nr} \otimes ((S_i - S_j)(S_i - S_j)^\dagger)$ is the code distance matrix. For given $y(k-1)$, the term on the left hand side of (3.71b) is a constant and the term on the right hand side is a Gaussian random variable.

Let $u = 2\text{Re}\{w(k)\Delta_{i,j}^\dagger y^\dagger(k-1)\}$, then in the Appendix A.3 we have shown that $u$ has the conditional mean

$$
\overline{m}_{u|y(k-1)} = \mathcal{E}\{u \mid y(k-1)\}, \quad (3.72)
$$

$$
= 2\text{Re}\{\overline{m}_{n(k-1)|y(k-1)}(I - S_jS_j^\dagger)y^\dagger(k-1)\}
$$

where $\overline{m}_{n(k-1)|y(k-1)} = \sigma_n^2 y(k-1)(\mathcal{X}^\dagger(k-1)R\mathcal{X}(k-1) + \sigma_n^2 I_{nrnn})^{-1}$ with $R$ the
correlation matrix of the MIMO channel $H$, and the conditional variance

$$
\sigma^2_{u|y(k-1)} = \mathcal{E} \{ \| u - \overline{m} \|^2 | y(k-1) \},
$$

(3.73)

$$
= 2y(k-1)\Delta_{i,j}(\sigma^2_nI + \Sigma_{i(k-1)|y(k-1)}\Delta_{i,j}^H y^H(k-1),
$$

where $\Sigma_{i(k-1)|y(k-1)} = \sigma^2_n(I - \sigma^2_n(E_sX^H(k-1)RX(k-1) + \sigma^2_nI)^{-1})$.

Let $d_{i,j}^2 = y(k-1)D_{i,j} y^H(k-1)$, the PEP is then given by

$$
P(S_i \rightarrow S_j \mid y(k-1)) = \Pr(u > d_{i,j}^2),
$$

(3.74)

$$
= \int_{d_{i,j}^2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{u|y(k-1)}} \exp \left( -\frac{(u - \overline{m}_{u|y(k-1)})^2}{2\sigma^2_{u|y(k-1)}} \right) du,
$$

$$
= Q \left( \frac{d_{i,j}^2 - \overline{m}_{u|y(k-1)}}{\sigma_{u|y(k-1)}} \right).
$$

(3.75)

By using Craig’s formula for the Gaussian Q-function (3.23) and the MGF-based technique presented in Part I of this chapter, we can write the average PEP as

$$
P(S_i \rightarrow S_j) = \frac{1}{\pi} \int_{0}^{\pi/2} \int_{0}^{\infty} \exp \left( -\frac{\Gamma}{2\sin^2 \theta} \right) p_{\Gamma}(\Gamma) d\Gamma d\theta,
$$

(3.75)

$$
= \frac{1}{\pi} \int_{0}^{\pi/2} \mathcal{M}_{\Gamma} \left( -\frac{1}{2\sin^2 \theta} \right) d\theta,
$$

(3.75)

where $\mathcal{M}_{\Gamma}(s) = \int_{0}^{\infty} e^{s\Gamma} p_{\Gamma}(\Gamma) d\Gamma$ is the MGF of

$$
\Gamma = \frac{(d_{i,j}^2 - \overline{m}_{u|y(k-1)})^2}{\sigma^2_{u|y(k-1)}}
$$

(3.76)

and $p_{\Gamma}(\Gamma)$ is the probability density function of $\Gamma$. Finding MGF of $\Gamma$ in (3.76) poses a much harder problem. However, at asymptotically high SNRs (i.e., keep $E_s$ constant and $\sigma^2_n \rightarrow 0$) the conditional mean and the conditional variance of $u$ reduce to $\overline{m}_{u|y(k-1)} = 0$ and $\sigma^2_{u|y(k-1)} = 4\sigma^2_n d_{i,j}^2$, respectively, and $\Gamma$ reduces to

$$
\Gamma = \frac{1}{4\sigma^2_n} y(k-1)D_{i,j} y^H(k-1).
$$

(3.77)

In this case $\Gamma$ is a quadratic form of a random variable since $y(k-1)$ is zero-mean complex Gaussian distributed random vector with covariance

$$
R_{y(k-1)} = E_sX^H(k-1)RX(k-1) + \sigma^2_n I
$$

(3.78)
and $D_{i,j}$ is Hermitian and also fixed for given two code words. Note that $R$ is the correlation matrix of the channel, defined by (3.14). Using (3.33), the MGF of $\Gamma$ can be written as

$$\mathcal{M}_\Gamma(s) = \left[ \det \left( I - \frac{s}{4\sigma_n^2} R_{y(k-1)} D_{i,j} \right) \right]^{-1}. \quad (3.79)$$

Recalling the definition of $R_{y(k-1)}$, we may write the MGF of $\Gamma$ as

$$\mathcal{M}_\Gamma(s) = \left[ \det \left( I - \frac{s}{4} (\gamma X(k-1) R X(k-1) + I) D_{i,j} \right) \right]^{-1}, \quad (3.80)$$

where $\gamma = E_s/\sigma_n^2$ is the average symbol energy-to-noise ratio, then from (3.79)

$$P(S_i \rightarrow S_j) = \frac{1}{\pi} \int_0^{\pi/2} \left[ \det \left( I + \frac{1}{8 \sin^2 \theta} (\gamma X(k-1) R X(k-1) + I) D_{i,j} \right) \right]^{-1} d\theta. \quad (3.81)$$

**Remark 3.6** Eq. (3.81) is the exact-PEP of a differential space-time coded system applied to a spatially correlated slow fading MIMO channel.

Eq. (3.81) reveals that the error performance of differentially space-time coded systems depends not only the channel correlation matrix $R$ and the code distance matrix $D_{i,j}$, but also on the previously transmitted code matrix $X(k-1)$.

Since the maximum of the integrand occurs at the upper limit, i.e., for $\theta = \pi/2$, replacing the integrand by its maximum value gives the Chernoff upper bound

$$P(S_i \rightarrow S_j) \leq \frac{1}{2} \left[ \det \left( I + \frac{1}{8} (\gamma X(k-1) R X(k-1) + I) D_{i,j} \right) \right]^{-1}. \quad (3.82)$$

In this work, we mainly focus on the space-time modulated constellations with the property

$$(S_i - S_j)(S_i - S_j) = \beta_{i,j} I_{nt}, \forall i \neq j, \quad (3.83)$$

where $\beta_{i,j}$ is a scalar. Space-time orthogonal designs [40] and some cyclic and dicyclic space-time modulated constellations in [49] are some examples which satisfy property (3.83) above. Applying (3.83) on (3.81) and using the unitary property of $X(k-1)$ and the determinant identity $|I + AB| = |I + BA|$, after straight
forward manipulations, we can simplify exact-PEP (3.81) to
\[
P(S_i \rightarrow S_j) = \frac{1}{\pi} \int_0^{\pi/2} \left| I + \frac{\beta_{i,j}}{8 \sin^2 \theta} (\gamma R + I) \right|^{-1} d\theta, \tag{3.84}
\]
and the Chernoff upper bound (3.82) to
\[
P(S_i \rightarrow S_j) \leq \frac{1}{2} \left( \frac{8 + \beta_{i,j}}{8} \right)^{-\eta T_{nr}} \left| I + \frac{\beta_{i,j} \gamma}{(8 + \beta_{i,j})} R \right|. \tag{3.85}
\]
With the property (3.83), it now becomes evident that error performance of DSTC is independent of the previously transmitted code matrix \(X(k-1)\).

### 3.12.1 Exact-PEP for Uncorrelated Channels

When the fading channels are independent and identically distributed (i.e., \(R = I\)), (3.84) simplifies to,
\[
P(S_i \rightarrow S_j) = \frac{1}{\pi} \int_0^{\pi/2} \left( I + \frac{\beta_{i,j}}{8 \sin^2 \theta} (\gamma + 1) \right)^{-n T_{nr}} d\theta,
\]
where \(\gamma = \frac{\beta_{i,j}(1 + \gamma)}{8}\). Using a result found in [128], integral (3.86) can be evaluated in closed form as
\[
P(S_i \rightarrow S_j) = \frac{1}{2} \left\{ 1 - \sqrt{\frac{\eta}{1 + \eta}} \sum_{\ell=0}^{n T_{nr}-1} \left( \frac{2\ell}{\ell} \right) \frac{1}{4^\ell(1 + \eta)^\ell} \right\}. \tag{3.87}
\]
This expression illustrates that the exact PEP of a differential space time code (DSTC) operating over an i.i.d. slow-fading channel depends on \(\gamma\) and \(\beta_{i,j}\), which is related to the code distance matrix \((S_i - S_j)(S_i - S_j)^\dagger\).

### 3.12.2 Exact-PEP for Correlated Channels

Let \(Z = \beta_{i,j}(\gamma R + I)/8\) in (3.84). Suppose matrix \(Z\) has \(K\) non-zero eigenvalues, including multiplicity, \(\lambda_1, \lambda_2, \cdots, \lambda_K\), and the decomposition \(Z = UDU^{-1}\), where \(U\) is the matrix of eigenvectors of \(Z\) and \(D\) is a diagonal matrix with the eigenvalues...
of $Z$ on the diagonal. Then (3.84) becomes

$$P(S_i \rightarrow S_j) = \frac{1}{\pi} \int_{0}^{\pi/2} \left| I + \frac{1}{\sin^2 \theta} Z \right|^{-1} d\theta,$$

$$= \frac{1}{\pi} \int_{0}^{\pi/2} \left| I + \frac{1}{\sin^2 \theta} D \right|^{-1} d\theta,$$

$$= \frac{1}{\pi} \int_{0}^{\pi/2} \prod_{\ell=1}^{K} \left( \frac{\sin^2 \theta}{\lambda_\ell \sin^2 \theta} \right)^{m_\ell} d\theta,$$

(3.88)

where $m_\ell$ is the multiplicity of eigenvalue $\lambda_\ell$. Using the partial fraction expansion technique given in Section 3.6.1, the integral in (3.88) can be evaluated in closed form.

Recall the definition of the channel correlation matrix $R$ given by (3.14). When $R_s = I$ (i.e., correlation between different communication modes is zero), Eq. (3.88) above captures the effects due to antenna spacing and antenna geometry on the performance of a differentially space-time coded communication system.

\section*{3.13 Analytical Performance Evaluation}

As an example, we consider the rate-1 $2 \times 2$ space-time modulated constellation set $\mathcal{V} \equiv \{ S_i | S_i S_i^\dagger = I, i = 0, \cdots, 3 \}$, derived in [40] based on orthogonal designs with

$$S_i = \begin{bmatrix} s_1 & -s_2 \\ s_2 & s_1 \end{bmatrix}, \text{for } i = 0, \cdots, 3,$$

(3.89)

where $s_i, i = 1, 2$ are symbols drawn from the normalized BPSK alphabet $\{ \pm 1/\sqrt{2} \}$. Let $S_0$ and $S_1$ correspond to the matrix with $(s_1, s_2) = (1/\sqrt{2}, 1/\sqrt{2})$ and $(s_1, s_2) = (1/\sqrt{2}, -1/\sqrt{2})$, respectively. In following sections we examine the probability that the receiver erroneously decides in favor of $S_1$ when $S_0$ was actually transmitted (i.e., $P(S_0 \rightarrow S_1)$) for various spatial scenarios. Note that in this case $\beta_{0,1} = 2$.

\subsection*{3.13.1 Effects of Antenna Spacing}

Consider a MIMO system with two transmit antennas and two receive antennas, where the two transmit antennas are placed in a circular aperture of radius $0.25\lambda$ (antenna separation $= 0.5\lambda$) and the two receive antennas are placed in a circular aperture of radius $r_R$ (antenna separation $= 2r_R$). To isolate the effects of antenna spacing, we assume an isotropic scattering environment surrounding the transmitter and the receiver regions.
Figure 3.14 shows the exact pairwise error probability performance of the DSTC for the error event $S_0 \rightarrow S_1$ and receive antenna separations $0.1\lambda$, $0.2\lambda$, $0.5\lambda$ and $\lambda$. Also shown in Figure 3.14 for comparison is the exact-PEP (3.87) for the i.i.d. slow fading channel corresponding to the error event $S_0 \rightarrow S_1$.

As we can see from the figure, the effect of antenna separation on the exact-PEP is not significant when the receiver antenna separation is $0.5\lambda$ or higher. However, the effect is significant when the receiver antenna separation is small. For example, at PEP $10^{-4}$, the realistic PEPs are about 1dB and 3dB away from the i.i.d. channel performance results for $0.2\lambda$ and $0.1\lambda$ receive antenna separations, respectively. From these observations, we can emphasize that the effect of antenna spacing on the performance of DSTC is minimum for higher antenna separations whereas the effect is significant for smaller antenna separations. We observed similar results with the coherent space-time codes discussed in Part I.
3.13.2 Effects of Antenna Configuration

In this section, we compare the PEP performance of the DSTC used in the previous section for different antenna configurations at the receiver antenna array. For example, we choose UCA and ULA antenna configurations. Consider a system with two transmit antennas and three receive antennas. The two transmit antennas are placed half wavelength ($\frac{\lambda}{2}$) distance apart and the three receive antennas are placed within a fixed circular aperture of radius $r_T (= 0.15\lambda, 0.25\lambda)$, as shown in Figure 3.15. The exact-PEP performance for the error event $S_0 \rightarrow S_1$, corresponding to $\beta_{0,1}$, is also plotted in Figure 3.15.

![Figure 3.15: Exact-PEP performance of DSTC scheme with two transmit and three receive antennas for UCA and ULA receiver antenna configurations; $\beta_{0,1} = 2$.](image)

From Figure 3.15, it is observed that at high SNRs the performance given by the UCA antenna configuration outperforms that of the ULA antenna configuration. For example, at PEP $10^{-6}$, the performance differences between UCA and ULA are about 2.5dB for 0.15$\lambda$ receiver aperture radius and about 2dB for 0.25$\lambda$ receiver aperture radius. Therefore, as we illustrated here, one can use the PEP expression (3.88) to determine the best antenna placement within a given region which gives
Figure 3.16: Exact-PEP performance of the DSTC scheme with two transmit and two receive antennas against the receive antenna separation for a uniform limited power distribution at the receiver with mean angle of arrival $\varphi_0 = 45^\circ$ from broadside and $\Delta_r = [5^\circ, 30^\circ, 180^\circ]$ at 15dB SNR; Transmit antenna separation $0.5\lambda$ and $\beta_{0,1} = 2$.

the maximum performance gain available from a DSTC scheme. Furthermore, from Figure 3.15, it is observed that as the radius of the receiver aperture decreases the diversity\(^9\) advantage of the DSTC scheme is reduced, particularly for the ULA antenna configuration. Here, the loss of diversity advantage is mainly due to the loss of rank of $J_R$.

### 3.13.3 Effects of Non-Isotropic Scattering

For simplicity, here we only consider the non-isotropic scattering effects at the receiver region and assume that the scattering environment surrounding the transmitter region is isotropic, i.e., $F_T = I_{2Mr+1}$. We assume an uniform limited az-

\(^9\)The slope of the performance curve on a log scale corresponds to the diversity advantage of the code
3.13 Analytical Performance Evaluation

Figure 3.17: Exact-PEP performance of the DSTC scheme with two transmit and two receive antennas against the receive antenna separation for a uniform limited power distribution at the receiver with mean angle of arrival $\varphi_0 = 45^\circ$ from broadside and $\Delta_r = [5^\circ, 30^\circ, 180^\circ]$ at 20dB SNR; Transmit antenna separation $0.5\lambda$ and $\beta_{0,1} = 2$.

...imuth power distribution at the receiver region. In this case, the $(\ell, \ell')$-th element of $(2M_R + 1) \times (2M_R + 1)$ receiver modal correlation matrix $F_R$ is given by (3.64).

We consider a MIMO system with two transmit and two receive antennas where the two transmit antennas are placed $0.5\lambda$ distance apart. Figures 3.16 and 3.17 show the exact-PEP results of the error event $S_0 \rightarrow S_1$ of rate-1 DSTC code considered in previous sections against the receiver antenna separation for a mean AOA $45^\circ$ from broadside and non-isotropic parameters $\Delta_r = [5^\circ, 30^\circ, 180^\circ]$ (or angular spreads $\sigma_r \approx [3^\circ, 17^\circ, 104^\circ]$) for 15dB and 20dB SNRs, respectively. Since the exact-PEP expression we derived is valid only at high SNRs, the PEP results are plotted for 15dB and 20dB SNRs.

From Figures 3.16 and 3.17 it is observed that for a given SNR, the perfor...
Figure 3.18: Exact-PEP performance of DSTC scheme with two transmit and three receive antennas for UCA and ULA receiver antenna configurations for a uniform limited power distribution at the receiver with mean angle of arrivals $\varphi_0 = [60^\circ, 45^\circ, 15^\circ]$ from broadside and non-isotropic parameter $\Delta_r = 180^\circ$; Transmit antenna separation $0.5\lambda$, receive antenna separation $0.15\lambda$ and $\beta_{0,1} = 2$.

Figure 3.18 illustrates the effects of mean AOA on the exact PEP of DSTC for UCA and ULA antenna configurations at the receiver. Antenna elements at the receiver are placed within a fixed circular aperture of radius $0.15\lambda$, similar to antenna configuration setup shown in Fig. 3.15 and the two transmit antennas are placed $0.5\lambda$ distance apart. As before, we consider a uniform limited azimuth power distribution at the receiver with mean AOAs $60^\circ$, $45^\circ$ and $15^\circ$ from broadside and non-isotropic parameter $\Delta_r = 180^\circ$.

Performance of the DSTC scheme is improved as the receiver antenna separation and the angular spread are increased. However, the performance does not improve monotonically with the increase in receiver antenna separation. We also observed that when the angular spread is quite small (e.g. $3^\circ$), we need to place the two receive antenna elements at least several wavelengths apart in order to achieve the maximum performance gain given by the DSTC scheme.
From Figure 3.18 we observed that the performance loss of the DSTC scheme is most pronounced for the ULA antenna configuration when the mean AOA is inline with the array. But, for the UCA antenna configuration, the performance loss is insignificant as the mean AOA moves away from broadside. This suggests that the UCA antenna configuration is less sensitive to change of mean AOA compared to the ULA antenna configuration. Hence, the UCA antenna configuration is best suited to employ a space-time code.

3.14 Summary and Contributions

In this chapter we have investigated the effects of physical constraints such as antenna spacing, antenna geometry and non-isotropic parameters (angular spread and mean AOA) on the performance of coherent and non-coherent space-time codes applied on spatially constrained MIMO channels.

Some specific contributions made in this chapter are:

1. Using an MGF-based approach, we have derived analytical expressions for the exact-PEP of coherent and non-coherent space-time coded systems operating over spatially correlated fading channels. Two analytical techniques are given which can be used to evaluate the exact-PEPs in closed form. Generalized PEP upper-bound of coherent and non-coherent space-time coded systems operating over spatially correlated fading channels is also derived.

2. Using these analytical PEP expressions we quantified the number of antennas that can be employed in a fixed antenna aperture, without diminishing the diversity advantage of a space-time code, and showed that the diversity advantage is upper-limited by the number of effective communication modes in the aperture, which is directly related to the size of the antenna aperture. We also quantified the degree of the effect of the angular spread of the scattering distribution surrounding the transmitter and receiver antenna apertures on the diversity advantage of a space-time code.

3. Considering a spatially constrained ULA antenna configuration, we analytically showed that the diversity advantage promised by a space-time code can be diminished by the antenna configuration. We also showed that UCA antenna configuration is less sensitive to change of mean AOA compared to ULA antenna configuration. Therefore, between UCA and ULA antenna configurations, UCA is best suited to apply a space-time code.
4. It is shown that i.i.d. channel models never be justified in realistic channel scenarios.

5. Using the results we obtained, it was shown that in general, both coherent and non-coherent space-time codes are susceptible to spatial fading correlation effects, in particular, when the antenna separation and the angular spread are small.
Chapter 4

Spatial Precoder Designs: Based on Fixed Parameters of MIMO Channels

4.1 Introduction

In practice, insufficient antenna spacing, non-ideal antenna placement and non-isotropic scattering environments lead to channels which exhibit correlated fades. As we saw in Chapter 3, correlated fading reduces the performance of multi-antenna wireless communication systems compared to the i.i.d. fading. This has motivated the design of linear precoders (or power loading schemes) for multi-antenna wireless communication systems by exploiting the statistical information of the MIMO channels [66,130–135]. In these designs, the receiver either feeds back the full channel state information (CSI) or the partial CSI (e.g., correlation coefficients of the channel) to the transmitter via a low rate feedback channel.

In [130], a joint transmit and receive optimization scheme for MIMO spatial multiplexing systems in narrow-band wireless channels is proposed by minimizing the mean square error of received signals. This scheme requires the receiver to feedback the full CSI to the transmitter. In [131], by minimizing the channel estimation error variance a general criteria to design optimal transmitter precoders is proposed for stationary random fading channels. The optimal design requires the knowledge of the channel’s correlation matrix. In [132–134], linear precoding schemes are developed based on channel correlation matrix for coherent space-time block coded wireless communication systems. In [132], the precoder is designed by minimizing the bit error rate and symbol error rate expressions of space-time block

\footnote{CSI is fully known at the receiver.}
coded (STBC) MISO systems. In [133, 134], the pair-wise error probability upper bound of STBC has been used as the cost function. In [133], the optimum precoder is derived in closed form for a MISO system and presented a numerical solution for MIMO systems assuming a Kronecker type scattering channel. In [134], the precoder is derived for a non-Kronecker type scattering channel. However, this design assumed a block diagonal structure for the correlation matrix of the MIMO channel. Linear precoding schemes for non-coherent differential space-time block coded systems are developed in [66, 135] based on channel correlation feedback. In [135], the Chernoff bound of approximate symbol error rate of differential STBC is minimized to obtain the precoder for a MISO system. Assuming an uncorrelated receiver antenna array and arbitrary correlation at the transmitter antenna array, [66] has derived a linear precoding scheme similar to that of [135].

In order to be cost effective and optimal, linear precoding schemes proposed in the literature assumed that the channel remains stationary (channel statistics are invariant) for a large number of symbol periods and the transmitter is capable of acquiring robust channel state information. However, when the channel is non-stationary or it is stationary for a small number of symbol periods, the receiver will have to feedback the channel information to the transmitter frequently. As a result, the system becomes costly and the optimum precoder design, based on the previously possessed information, becomes outdated quickly. In some circumstances feeding back channel information is not possible. These facts have motivated us to design a precoding scheme based on fixed and known parameters of the underlying MIMO channel.

In this chapter we introduce the novel use of linear spatial precoding based on fixed and known parameters of MIMO channels to improve the performance of both coherent and non-coherent space-time coded MIMO systems. Spatial precoding schemes are designed based on previously unutilized fixed and known parameters of MIMO channels, namely the antenna spacing and antenna placement (geometry) details. Both precoding schemes are fixed for fixed antenna placement and the transmitter does not require any form of feedback of channel state information (partial or full) from the receiver. Since the designs are fixed for given transmitter and receiver antenna configurations, these spatial precoders can be used in non-stationary channels as well as stationary channels. We derive the optimum precoders by minimizing the pair-wise error probability upper bound of coherent and non-coherent space-time codes derived in Chapter 3 subject to a transmit power constraint. Closed form solutions for both precoding schemes are presented for systems with up to three receive antennas and a generalized method is pro-
posed for more than three receive antennas. In addition, we develop precoding schemes to exploit the non-isotropic parameters to improve the performance of space-time coded systems applied on MIMO channels in non-isotropic scattering environments. Unlike in the first fixed scheme, this scheme requires the receiver to estimate the non-isotropic parameters of the scattering channel and feed them back to the transmitter. We use the coherent STBC and differential STBC to analyze the performance of proposed precoding schemes. We first derive precoders for coherent STBC and then followed with derivations of precoders for differential STBC.

4.2 System Model

At time instance $k$, the space time encoder at the transmitter takes a set of modulated symbols $C(k) = \{c_1(k), c_2(k), \cdots, c_K(k)\}$ and maps them onto an $n_T \times L$ code word matrix $S_{\ell(k)} \in \mathcal{V}$ of space-time modulated constellation matrices set $\mathcal{V} = \{S_1, S_2, \cdots, S_T\}$, where $L$ is the code length, $T = q^K$ and $q$ is the size of the constellation from which $c_n(k), n = 1, \cdots, K$ are drawn. By setting $|c_n(k)| = 1/\sqrt{K}$, each code word matrix $S_{\ell(k)}$ in $\mathcal{V}$ will satisfy the property

\begin{equation}
(S_i - S_j)(S_i - S_j)^\dagger = \beta_{i,j} I_{n_T}, \forall i \neq j,
\end{equation}

where $\beta_{i,j}$ is a scalar and $S_i, S_j \in \mathcal{V}$. Space-time orthogonal designs in [40] and some cyclic and dicyclic space-time modulated constellations in [49] are some examples which satisfy property (4.1) above.

4.2.1 Coherent Space-Time Block Codes

Let $s_n$ be the $n$-th column of $S_i = [s_1, s_2, \cdots, s_L] \in \mathcal{V}$. At the transmitter, each code vector $s_n$ is multiplied by a $n_T \times n_T$ fixed linear precoder matrix $F_c$ before transmitting out from $n_T$ transmit antennas. Assuming quasi-static fading, the signals received at $n_R$ receiver antennas during $L$ symbol periods can be expressed in matrix form as

\begin{equation}
Y(k) = \sqrt{E_s} HF_c S_{\ell(k)} + N(k),
\end{equation}
where $E_s$ is the average transmitted signal energy per symbol period, $N(k)$ is the $n_R \times L$ white Gaussian noise matrix in which elements are zero-mean independent Gaussian distributed random variables with variance $\sigma^2_n/2$ per dimension and $H$ is the $n_R \times n_T$ channel matrix. In this work, we use the spatial channel decomposition

$$H = J_R H_s J_T^\dagger$$  \hspace{1cm} (4.2)$$
given in Chapter 3 to represent the underlying MIMO channel $H$. The elements of the scattering channel matrix $H_s$ are modeled as zero-mean complex Gaussian random variables (Rayleigh fading) and assume a slow-flat fading scattering environment.

For coherent STBC, we assume that the receiver has perfect channel state information (CSI) and transmitter has partial CSI. At the receiver, the transmitted codeword is detected by applying the maximum likelihood detection rule:

$$\hat{S}_{\ell(k)} = \arg \min_{S_{\ell(k)} \in V} \| y(k) - \sqrt{E_s} \tilde{h} S_{\ell(k)} \|^2$$

$$= \arg \max_{S_{\ell(k)} \in V} \Re \{ \tilde{h} S_{\ell(k)} y(k)^\dagger \},$$  \hspace{1cm} (4.3)$$

where $y(k) = (\text{vec}\{Y^T(k)\})^T$, $S_{\ell(k)} = I_{n_R} \otimes S_{\ell(k)}$ and $\tilde{h} = (\text{vec}\{\tilde{H}^T\})^T$ with $\tilde{H} = HF_c$.

### 4.2.2 Differential Space-time Block Codes

In this scheme, codeword matrix $S_{\ell(k)}$ is differentially encoded according to the rule

$$X(k) = X(k-1)S_{\ell(k)}, \text{ for } k = 1, 2, \cdots$$

with $X(0) = I_{n_T}$. Then, each encoded $X(k)$ is multiplied by a $n_T \times n_T$ fixed linear precoder matrix $F_d$ before transmitting out from $n_T$ transmit antennas. Assuming quasi-static fading, the signals received at $n_R$ receiver antennas during $n_T$ symbol periods can be expressed in matrix form as

$$Y(k) = \sqrt{E_s} HF_d X(k) + N(k),$$

where $N(k)$ is the $n_R \times n_T$ white Gaussian noise matrix in which elements are zero-mean independent Gaussian distributed random variables with variance $\sigma^2_n/2$ per complex dimension and $H$ is the $n_R \times n_T$ channel matrix, which is modeled using (4.2).
Assume that the scattering channel matrix $H_s$ remains constant during the reception of two consecutive received signal blocks $Y(k - 1)$ and $Y(k)$, which can be expressed in vector (row) form as

\[ y(k - 1) = \sqrt{E_s} h \chi(k - 1) + n(k - 1), \]
\[ y(k) = \sqrt{E_s} h \chi(k) + n(k), \]
\[ = y(k - 1) S_{\ell(k)} + w(k), \]  
(4.4)

where $y(k) = (\text{vec} \{Y(k)^T\})^T$, $\chi(k) = I_{nr} \otimes (F_d X(k))$, $h = (\text{vec} \{H^T\})^T$, $n(k) = (\text{vec} \{N(k)^T\})^T$, $S_{\ell(k)} = I_{nr} \otimes S_{\ell(k)}$ and $w(k) = n(k) - n(k - 1) S_{\ell(k)}$.

For differential STBC, we assume that receiver has no CSI whilst transmitter has partial CSI. From (4.4), the transmitted code word matrix is detected differentially using the maximum likelihood detection rule:

\[ \hat{S}_{\ell(k)} = \arg \min_{S_{\ell(k)} \in \mathcal{V}} \| y(k) - y(k - 1) S_{\ell(k)} \|^2, \]
\[ = \arg \max_{S_{\ell(k)} \in \mathcal{V}} \text{Re} \{ y(k - 1) S_{\ell(k)} y(k)^\dagger \}. \]

### 4.3 Problem Setup: Coherent STBC

Assume that perfect CSI is available at the receiver and also maximum likelihood (ML) detection is employed at the receiver. Suppose codeword $S_i \in \mathcal{V}$ is transmitted, but the ML-decoder (4.3) chooses codeword $S_j \in \mathcal{V}$, then as shown in the Appendix B.1, the average pairwise error probability (PEP) is upper bounded by

\[ P(S_i \rightarrow S_j) \leq \frac{1}{\det \left( I_{ntm_n} + \frac{\tau}{4} R_H [I_{nr} \otimes S_{\Delta_F_c}] \right)}, \]  
(4.5)

where $S_{\Delta,F_c} = F_c (S_i - S_j)(S_i - S_j)^\dagger F_c^\dagger$, $\tau = E_s / \sigma_n^2$ is the average symbol energy-to-noise ratio (SNR) at each receive antenna and $R_H$ is the correlation matrix of the MIMO channel (4.2) given by

\[ R_H = \mathcal{E} \left\{ h^h h \right\}, \]
\[ = (J_R^\dagger \otimes J_T) R_s (J_R^T \otimes J_T^\dagger), \]  
(4.6)

where $h = (\text{vec} \{H^T\})^T$ and $R_s$ the modal correlation matrix defined as $R_s = \mathcal{E} \left\{ h_s^h h_s \right\}$ with $h_s = (\text{vec} \{H_s^T\})^T$. When the scattering channel is separable, from Chapter 3.3.2, $R_s$ can be separated as $R_s = F_R \otimes F_T$, where $F_R$ is the $(2M_R + 1) \times (2M_R + 1)$ receiver modal correlation matrix and $F_T$ is the $(2M_T +
1) \times (2M_T + 1) transmitter modal correlation matrix. In this case

\[
R_H = (J^*_R F_R J^T_R) \otimes (J^*_T F_T J^T_T).
\]

By applying the property (4.1) associated with orthogonal space-time block codes, we can simplify the PEP upper-bound (4.5) to

\[
P(S_i \rightarrow S_j) \leq \frac{1}{\det \left( I_{n_T n_R} + \frac{\tau_{R,i}}{4} R_H \left[ I_{n_R} \otimes (F_c F_c^\dagger) \right] \right)}.
\] (4.7)

In this work, our main objective is to find the optimum precoding scheme which reduces the spatial correlation effects on the performance of coherent STBC. We achieve this by minimizing the average PEP bound (4.7) subject to the transmit power constraint \( \text{tr}\{F_c F_c^\dagger\} = n_T \). Here we propose two schemes for the optimal precoder \( F_c \) by considering two scenarios for the channel correlation matrix \( R_H \). The two optimization problems can be stated as follows:

**Scheme 1 - Fixed scheme (coherent):** Find the optimum \( F_c \) that minimizes the average PEP upper bound (4.7) for coherent STBC, subject to the transmit power constraint \( \text{tr}\{F_c F_c^\dagger\} = n_T \), for given transmitter and receiver antenna configurations assuming a rich scattering environment (i.e., \( R_s = I \)).

In this case, the channel correlation matrix \(^3 R_H\) is given by,

\[
R_H = (J^*_R J^T_R) \otimes (J^*_T J^T_T).
\]

Since \( J_R \) and \( J_T \) are fixed and deterministic for given antenna configurations, the **precoder is fixed**. Therefore, in this scheme, the transmitter does not require any feedback information about the channel to derive the optimum precoder \( F_c \). This precoding scheme exploits the antenna placement information at both ends of the MIMO channel to compensate for any detrimental effects of non-ideal antenna placement on the performance of coherent space time block codes.

**Scheme 2 - Feedback scheme (coherent):** Find the optimum \( F_c \) that minimizes the average PEP upper bound (4.7) for coherent STBC, subject to the transmit power constraint \( \text{tr}\{F_c F_c^\dagger\} = n_T \), for given transmitter and receiver antenna configurations assuming the receiver estimates the non-isotropic distribution parameters and feeds them back to the transmitter.

\(^2\)The upper-bound (4.7) is derived assuming \( R_H \) is non-singular. Therefore, the precoder only exists when \( R_H \) is non-singular.

\(^3\)The Kronecker channel assumption can be relaxed in this case.
Note that the optimum precoder $F_c$ in scheme-2 exploits the non-isotropic scattering distribution parameters of the scattering channel and also the antenna placement information to improve the performance of differential STBC. However, the performance of this scheme profoundly relies on the accuracy of CSI received from the receiver.

### 4.3.1 Optimum Spatial Precoder: Coherent STBC

Since $\log(\cdot)$ is a monotonically increasing function, the logarithm of the PEP upper-bound (4.7) can be used as the objective function (or the cost function). The optimum linear precoder $F_c$ is found by solving the optimization problem

$$\min -\log \det \left( I_{nTnT} + \frac{\gamma \beta_{i,j}}{4} R_H \left[ I_{nR} \otimes (F_c F_c^\dagger) \right] \right)$$

subject to $\text{tr}\{F_c F_c^\dagger\} = n_T$. (4.8)

Note that different error event $(S_i \rightarrow S_j)$ will produce different value of $\beta_{i,j}$ and hence different PEP. As a result, we cannot design $F_c$ that minimizes the PEP of all error events. Since the performance of a communication system is mainly dependent on the PEP of dominant error events, we will design the precoder matrix $F_c$ using the value $\beta = \min_{i \neq j} \{\beta_{i,j}\}$. Consequently, the resulting precoder matrix $F_c$ minimizes the error probability of the dominant error events. The optimization problem (4.8) is similar to that considered in [133]. However, [133] derives the optimum precoder in closed form by considering a MISO channel.

Below we derive the optimal precoder $F_c$ for scheme-2. Note that the optimum precoder $F_c$ for scheme-1 can be easily derived from scheme-2 by letting $F_R = I$ and $F_T = I$.

Writing $J_R^* F_R J_R^T$ as the eigen-value decomposition (EVD) $J_R^* F_R J_R^T = U_R A_R U_R^\dagger$ and $J_T F_T J_T^\dagger$ as the EVD $J_T F_T J_T^\dagger = U_T A_T U_T^\dagger$, and using the Kronecker product identity $(A \otimes C)(B \otimes D) = AB \otimes CD$, we may write $R_H$ as

$$R_H = (U_R \otimes U_T) (A_R \otimes A_T) (U_R \otimes U_T)^\dagger. \quad (4.9)$$

Substituting (4.9) in (4.7), after straightforward manipulations using the matrix determinant identity $\det (I + AB) = \det (I + BA)$ and the Kronecker product identity $(A \otimes C)(B \otimes D) = AB \otimes CD$, we can simplify the objective function
of optimization problem (4.8) to
\[- \log \det \left( I_{n_T n_R} + \frac{\pi \beta}{4} (\Lambda_R \otimes \Lambda_T) (I_{n_R} \otimes U_T^\dagger F_c F_c^\dagger U_T) \right), \tag{4.10} \]
where \( \beta = \min_{i \neq j} \{ \beta_{i,j} \} \) over all possible codewords. Let
\[ Q_c = \frac{\pi \beta}{4} U_T^\dagger F_c F_c^\dagger U_T, \]
then the objective function (4.10) becomes
\[- \log \det \left( I_{n_T n_R} + \frac{\pi \beta}{4} (\Lambda_R \otimes \Lambda_T) (I_{n_R} \otimes Q_c) \right), \tag{4.11} \]
and \( Q_c \) must satisfy the power constraint \( \text{tr}\{Q_c\} = n_T \pi \beta / 4 \). It should be noted that \( Q_c \) in (4.11) is always positive semi-definite as \( Q_c = BB^\dagger \), with \( B = \sqrt{\pi \beta / 4} U_T^\dagger F_c \). The optimum \( Q_c \) is obtained by solving the optimization problem:
\[ \min \quad - \log \det \left( I_{n_T n_R} + (\Lambda_R \otimes \Lambda_T) (I_{n_R} \otimes Q_c) \right) \]
subject to \( Q_c \succeq 0, \text{tr}\{Q_c\} = n_T \pi \beta / 4 \). \( (4.12) \)

By applying Hadamard’s inequality on \( I_{n_T n_R} + (\Lambda_R \otimes \Lambda_T) (I_{n_R} \otimes Q_c) \) gives that this determinant is maximized when \((\Lambda_R \otimes \Lambda_T)(I_{n_R} \otimes Q)\) is diagonal \([5]\). Therefore \( Q_c \) must be diagonal as \( \Lambda_R \) and \( \Lambda_T \) are both diagonal. Since \((\Lambda_R \otimes \Lambda_T)(I_{n_R} \otimes Q_c)\) is a positive semi-definite diagonal matrix with non-negative entries on its diagonal, \( I_{n_T n_R} + (\Lambda_R \otimes \Lambda_T) (I_{n_R} \otimes Q_c) \) forms a positive definite matrix. As a result, the objective function of our optimization problem is convex \([136, \text{page 73}]\). Therefore the optimization problem (4.12) above is a convex minimization problem because the objective function and inequality constraints are convex and equality constraint is affine.

Let \( q_i = [Q_c]_{i,i}, t_i = [\Lambda_T]_{i,i} \) and \( r_j = [\Lambda_R]_{j,j} \). Optimization problem (4.12) then reduces to finding \( q_i > 0 \) such that
\[ \min \quad - \sum_{j=1}^{n_R} \sum_{i=1}^{n_T} \log(1 + t_i q_j r_j) \]
subject to \( q \succeq 0, \text{tr}\{q\} = n_T \pi \beta / 4 \). \( (4.13) \)

where \( q = [q_1, q_2, \cdots, q_{n_T}]^T \) and \( 1 \) denotes the vector of all ones.
4.3 Problem Setup: Coherent STBC

Introducing Lagrange multipliers $\lambda_c \in \mathbb{R}^{n_T}$ for the inequality constraints $-q \preceq 0$ and $v_c \in \mathbb{R}$ for the equality constraint $1^T q = n_T \gamma / 4$, we obtain the Karush-Kuhn-Tucker (K.K.T) conditions

$$q \succeq 0, \quad \lambda_c \succeq 0, \quad 1^T q = \frac{n_T \gamma}{4}$$

$$\lambda_i q_i = 0, \quad i = 1, 2, \ldots, n_T$$

$$-\sum_{j=1}^{n_R} \frac{r_j t_i}{1 + r_j t_i q_i} - \lambda_i + v_c = 0, \quad i = 1, 2, \ldots, n_T. \quad (4.14)$$

$\lambda_i$ in (4.14) can be eliminated since it acts as a slack variable\(^4\), giving new K.K.T conditions

$$q \succeq 0, \quad 1^T q = \frac{n_T \gamma}{4}$$

$$q_i \left( v_c - \sum_{j=1}^{n_R} \frac{r_j t_i}{1 + r_j t_i q_i} \right) = 0, \quad i = 1, \ldots, n_T, \quad (4.15a)$$

$$v_c \geq \sum_{j=1}^{n_R} \frac{r_j t_i}{1 + r_j t_i q_i}, \quad i = 1, \ldots, n_T. \quad (4.15b)$$

For $n_R = 1$, the optimal solution to (4.15) is given by the classical “water-filling” solution found in information theory [5]. The optimal $q_i$ for this case is given in Section 4.3.2. For $n_R > 1$, the main problem in finding the optimal $q_i$ for given $t_i$ and $r_j, j = 1, 2, \ldots, n_R$ is the case that, there are multiple terms that involve $q_i$ on (4.15a). Therefore we can view our optimization problem (4.13) as a generalized water-filling problem. In fact the optimum $q_i$ for this optimization problem is given by the solution to a polynomial obtained from (4.15a). In Sections 4.3.3 and 4.3.4, we provide closed form expressions for optimum $q_i$ for $n_R = 2$ and 3 receive antennas and a generalized method which gives optimum $q_i$ for $n_R > 3$ is discussed in Section 4.3.5.

As shown above, the optimal $Q_c$ is diagonal with $Q_c = \text{diag}\{q_1, q_2, \ldots, q_{n_T}\}$ and optimal spatial precoder $F_c$ is obtained by forming

$$F_c = \sqrt{\frac{4}{\beta \gamma}} U_T Q_c^{\frac{1}{2}} U_n^{\dagger},$$

where $U_n$ is any unitary matrix. In this work, we set $U_n = I_{n_T}$.

\(^4\)If $g(x) \leq v$ is a constraint inequality, then a variable $\lambda$ with the property that $g(x) + \lambda = v$ is called a slack variable [136].
4.3.2 MISO Channel

Consider a MISO channel where we have \( n_T \) transmit antennas and a single receive antenna. The optimization problem involved in this case is similar to the water-filling problem in information theory, which has the optimal solution

\[
q_i = \begin{cases} \frac{1}{\nu_c} - \frac{1}{t_i}, & \nu_c < t_i, \\ 0, & \text{otherwise}, \end{cases} \tag{4.16}
\]

where the water-level \( 1/\nu_c \) is chosen to satisfy

\[
\sum_{i=1}^{n_T} \max \left( 0, \frac{1}{\nu_c} - \frac{1}{t_i} \right) = \frac{n_T \gamma^/ \beta^4}{4}.
\]

4.3.3 \( n_T \times 2 \) MIMO Channel

We now consider the case of \( n_T \) transmit antennas and \( n_R = 2 \) receive antennas. As shown in the Appendix B.3, the optimum \( q_i \) for this case is

\[
q_i = \begin{cases} A + \sqrt{K}, & \nu_c < t_i(r_1 + r_2); \\ 0, & \text{otherwise}, \end{cases} \tag{4.17}
\]

where \( \nu_c \) is chosen to satisfy

\[
\sum_{i=1}^{n_T} \max \left( 0, A + \sqrt{K} \right) = \frac{n_T \gamma^/ \beta^4}{4},
\]

with

\[
A = \frac{2r_1r_2t_i^2 - \nu_c t_i(r_1 + r_2)}{2\nu_c r_1 r_2 t_i^2} \quad \text{and} \quad K = \frac{\nu_c^2 t_i^2 (r_1 - r_2)^2 + 4r_1^2 r_2^2 t_i^4}{2\nu_c r_1 r_2 t_i^2}. \tag{4.18}
\]

4.3.4 \( n_T \times 3 \) MIMO Channel

For the case of \( n_T \) transmit antennas and \( n_R = 3 \) receive antennas, the optimum \( q_i \) is given by

\[
q_i = \begin{cases} -\frac{\nu}{3r_3} + S + T, & \nu_c < t_i(r_1 + r_2 + r_3); \\ 0, & \text{otherwise}, \end{cases} \tag{4.19}
\]
where $v_c$ is chosen to satisfy

$$\sum_{i=1}^{n_T} \max \left(0, -\frac{a_2}{3a_3} + S + T\right) = \frac{n_T\gamma\beta}{4},$$

with

$$S + T = \left[R + \sqrt{Q^3 + R^2}\right]^\frac{1}{3} + \left[R - \sqrt{Q^3 + R^2}\right]^\frac{1}{3},$$

$$Q = \frac{3a_1a_3 - a_2^2}{9a_3^2}, \quad R = \frac{9a_1a_2a_3 - 27a_0a_3^2 - 2a_3^3}{54a_3^3},$$

$a_3 = v_c r_1 r_2 r_3 t_i^3$, $a_2 = v_c t_i^2 (r_1 r_2 + r_1 r_3 + r_2 r_3) - 3r_1 r_2 r_3 t_i^3$, $a_1 = v_c t_i (r_1 + r_2 + r_3) - 2t^2_i (r_1 r_2 + r_1 r_3 + r_2 r_3)$ and $a_0 = v_c - t_i (r_1 + r_2 + r_3)$. A sketch of the proof of (4.19) is given in the Appendix-B.4.

### 4.3.5 A Generalized Method

We now discuss a method which allows to find optimum solution to (4.13) for a system with $n_T$ transmit and $n_R$ receive antennas. The complementary slackness condition $\lambda_i q_i = 0$ for $i = 1, 2, \cdots, n_T$ states that $\lambda_i$ is zero unless the $i$-th inequality constraint is active at the optimum. Thus, from (4.15a) we have two cases: (i) $q_i = 0$ for $v_c > t_i \sum_{j=1}^{n_R} r_j$, (ii) $v_c = \sum_{j=1}^{n_R} r_j t_i / (1 + r_j t_i)$ for $q_i > 0$ [136, page 243]. For the later case, the optimum $q_i$ is found by evaluating the roots of $n_R$-th order polynomial in $q_i$, where the polynomial is obtained from $v_c = \sum_{j=1}^{n_R} r_j t_i / (1 + r_j t_i q_i)$. Since the objective function of the optimization problem (4.13) is convex for $q > 0$, there exist at least one positive root to the $n_R$-th order polynomial for $v_c < t_i \sum_{j=1}^{n_R} r_j$. In the case of multiple positive roots, the optimum $q_i$ is the one which gives the minimum to the objective function of (4.13). In both cases, $v_c$ is chosen to satisfy the power constraint $1^T q = n_T\gamma\beta/4$.

### 4.3.6 Spatially Uncorrelated Receive Antennas

If $n_R$ receive antennas are placed ideally within the receiver region such that the spatial correlation between antenna elements is zero (i.e., $J_R^T J_R = I$), then the cost function in (4.13) reduces to a single summation and the optimum $q_i$ is given by the water-filling solution (4.16) obtained for the MISO channel. This is not to say that such a placement is possible even approximately.
4.4 Problem Setup: Differential STBC

For the Differential STBC, we again use the average PEP upper bound to derive the optimum precoder $F_d$. At high SNR, as shown in Appendix B.2, the PEP which the receiver will erroneously select $S_j$ when $S_i$ was actually sent can be upper-bounded by

$$P(S_i \to S_j) \leq \frac{1}{\det\left(I + \frac{1}{8} \left(\bar{\gamma}X(k-1)\right)^\dagger QH \left(\frac{\bar{\gamma}X(k-1)}{F_d^\dagger F_d} + I_{n_{\text{nr}}} \otimes S_\Delta\right)\right)}, \tag{4.20}$$

where $S_\Delta = (S_i - S_j)(S_i - S_j)^\dagger$, $X(k) = I_{n_{\text{nr}}} \otimes (F_dX(k))$ and $\bar{\gamma} = E_s/\sigma_n^2$ is the average SNR at each receive antenna. As for the coherent STBC case, we mainly focus on the space-time modulated constellations with the property (4.1). Furthermore, similar to [48, 49] we assume that code length $L = n_T$. Under this assumption, each code word matrix $S_i$ in $V$ will satisfy the unitary property $S_i S_i^\dagger = I$ and $S_i^\dagger S_i = I$ for $i = 1, 2, \cdots, T$. As a result, $X(k)$ will also satisfy the unitary property $X(k)X(k)^\dagger = I$ and $X(k)X(k) = I$ for $k = 0, 1, 2, \cdots$. Applying (4.1) on (4.20) and then using the unitary property of $X(k-1)$ and the determinant identity $\det(I + AB) = \det(I + BA)$, after straightforward manipulations, we can simplify the PEP upper bound (4.20) to

$$P(S_i \to S_j) \leq \frac{\left(\frac{8+\beta_{i,j}}{8}\right)^{-n_T n_{\text{nr}}}}{\det\left(I_{n_{\text{nr}}} + \frac{\beta_{i,j}\bar{\gamma}}{(8+\beta_{i,j})} QH \left(I_{n_{\text{nr}}} \otimes \left(F_d^\dagger F_d\right)^\dagger\right)\right)} \tag{4.21}$$

Similar to the coherent STBC case considered previously, the optimal precoder $F_d$ for differential STBC is obtained by minimizing the maximum of all PEP upper-bounds subject to the power constraint $\text{tr}\{F_d F_d^\dagger\} = n_T$. In this case, by considering two scenarios for the channel correlation matrix $Q_H$, we can propose two schemes for optimum $F_d$.

**Scheme 3 - Fixed scheme (non-coherent):** Find the optimum $F_d$ that minimizes the average PEP upper bound (4.21) for differential STBC, subject to the transmit power constraint $\text{tr}\{F_d^\dagger F_d\} = n_T$, for given transmitter and receiver antenna configurations assuming a rich scattering environment (i.e., $R_s = I$).

**Scheme 4 - Feedback scheme (non-coherent):** Find the optimum $F_d$ that minimizes the average PEP upper bound (4.21) for differential STBC, subject to the transmit power constraint $\text{tr}\{F_d^\dagger F_d\} = n_T$, for given transmitter and receiver
antenna configurations assuming the receiver estimates the non-isotropic distribution parameters and feeds them back to the transmitter.

### 4.4.1 Optimum Spatial Precoder: Differential STBC

By taking the logarithm of PEP upper-bound (4.21) we can write the optimization problem for both above schemes as:

\[
\min - \log \det \left( I_{n_T n_R} + \frac{\beta \gamma}{(8 + \beta)} R_H \left( I_{n_R} \otimes \left( F_d F_d^\dagger \right) \right) \right)
\]

subject to \( \text{tr}\{F_d F_d^\dagger\} = n_T. \) (4.22)

where \( \beta = \min_{i \neq j}\{\beta_{i,j}\} \) over all possible codewords\(^5\). Substitute (4.9) for \( R_H \) in (4.22) and let

\[
P_d = \frac{\beta \gamma}{(8 + \beta)} U_T F_d F_d^\dagger U_T,
\]

then the optimum \( P_d \) (hence the optimum \( F_d \)) is obtained by solving the optimization problem

\[
\min - \log | I_{n_T n_R} + (\Lambda_R \otimes \Lambda_T)(I_{n_R} \otimes P_d)|
\]

subject to \( P_d \succeq 0, \text{tr}\{P_d\} = \frac{\beta \gamma n_T}{(8 + \beta)}. \)

The above optimization problem is identical to the optimization problem derived for coherent STBC, except a different scalar for the equality constraint. Therefore, following Section 4.3.1, here we present the final optimization problem and solutions to it without detail derivations.

Following Section 4.3.1, we can show that the optimum \( P_d \) is diagonal and diagonal entries of \( P_d \) are found by solving the optimization problem

\[
\min - \sum_{j=1}^{n_R} \sum_{i=1}^{n_T} \log(1 + t_i p_i r_j)
\]

subject to \( p \succeq 0, \quad 1^T p = \frac{\beta \gamma n_T}{(8 + \beta)}. \) (4.23)

where \( p_i = [P_d]_{i,i}, \quad t_i = [\Lambda_T]_{i,i}, \quad r_j = [\Lambda_R]_{j,j} \) and \( p = [p_1, p_2, \ldots, p_{n_T}]^T. \) The precoder

\(^5\)Setting \( \beta = \min_{i \neq j}\{\beta_{i,j}\} \) will minimize the error probability of the dominant error event(s).
$F_d$ is obtained by forming

$$F_d = \sqrt{\frac{8 + \beta}{\beta \gamma}} U_T P_d U_n^\dagger,$$

where $P_d = \text{diag}\{p_1, p_2, \ldots, p_{nt}\}$ and $U_n$ is any unitary matrix.

Similar to the coherent STBC case, when $n_R = 1$, the optimum power loading strategy is identical to the “water-filling” in information theory. When $n_R > 1$, a generalized water-filling strategy gives the optimum $P_d$. The Appendix B.5 gives the optimum $p_i$ for (4.23) for $n_R = 1, 2, 3$ receive antennas. For other cases, the generalized method discussed in Section 4.3.5 can be directly applied to obtain the optimum $p_i$.

### 4.5 Simulation Results: Coherent STBC

This section illustrates the performance improvements obtained from coherent STBC when the precoder $F_c$ derived in Section 4.3.1 is used. In particular, the performance is evaluated for small antenna separations and different antenna geometries at the transmitter and the receiver antenna arrays. In our simulations we use the rate-1 space-time modulated constellation constructed in [40] from orthogonal designs for two and four transmit antennas. Also use the rate $3/4$ STBC code for $n_T = 3$ transmit antennas given in [40]. When $n_T = 2$, the modulated symbols $c(k)$ are drawn from the normalized QPSK alphabet $\{\pm 1/\sqrt{2}, \pm i/\sqrt{2}\}$ and when $n_T = 3$ and 4, $c(k)$ are drawn from the normalized BPSK alphabet $\{\pm 1/\sqrt{2}\}$.

First we illustrate the water-filling concept for a MISO system with $n_T = 2, 3$ and 4 transmit antennas for scheme-1. The transmit antennas are placed in uniform circular array (UCA) and uniform linear array (ULA) configurations with 0.2λ minimum separation between two adjacent antenna elements, and we assume an isotropic scattering environment. For each transmit antenna configuration, Table-4.1 lists the radius of the transmit aperture, number of effective communication modes at the transmit region and the rank of the transmit side spatial correlation matrix $J_T J_T^\dagger$. Note that, in all spatial scenarios, we ensure that $J_T J_T^\dagger$ is full rank in order that the average PEP upper bound (4.7) to hold.

Figure 4.1 shows the water levels for various SNRs. For a given SNR, the optimal power value $q_i$ is the difference between water-level $1/\nu_c$ and base level $1/t_i$, whenever the difference is positive; it is zero otherwise. Note that, with this
Table 4.1: Transmit antenna configuration details corresponding to water-filling scenarios considered in Figure 4.1.

<table>
<thead>
<tr>
<th>Antenna Configuration</th>
<th>Tx aperture radius</th>
<th>Num. of modes</th>
<th>rank($J_T J_T^\dagger$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-Tx</td>
<td>0.1λ</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3-Tx UCA</td>
<td>0.115λ</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>3-Tx ULA</td>
<td>0.2λ</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>4-Tx UCA</td>
<td>0.142λ</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>4-Tx ULA</td>
<td>0.3λ</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

Figure 4.1: Water level ($1/\upsilon_c$) for various SNRs for a MISO system. (a) $n_T = 2$, (b) $n_T = 3$ - UCA, (c) $n_T = 4$ - UCA, (d) $n_T = 3$ - ULA and (e) $n_T = 4$ - ULA for 0.2λ minimum separation between two adjacent transmit antennas.

spatial precoder, the diversity order of the system is determined by the number of non-zero $q_i$’s. It is observed that at low SNRs, only one $q_i$ is non-zero for $n_T = 2$
and 3-UCA cases. In these cases, all the available power is assigned to the highest eigen-mode of $J^T J^\dagger T$ (or to the single dominant eigen-channel of $H$) and the system is operating in eigen-beamforming mode. With other cases, Figure 4.1(c), (d) and (e), systems are operating in between eigen-beam forming and full diversity for small SNRs as well as moderate SNRs. In these cases, the spatial precoder assigns more power to the higher eigen-modes of $J^T J^\dagger T$ (or to dominant eigen-channels of $H$) and less power to the weaker eigen-modes (or to less dominant eigen-channels of $H$).

### 4.5.1 Performance in Non-isotropic Scattering Environments

We now illustrate the performance improvements obtained using precoding scheme-1 and scheme-2 in non-isotropic scattering environments. Note that precoder $F_c$ in scheme-1 is derived based on the antenna configuration information and this scheme does not use any CSI feedback from the receiver. The scheme-2 uses both the antenna configuration details and the scattering environment parameters received from the receiver via feedback to derive the precoder $F_c$.

For simplicity, we only consider non-isotropic scattering at the transmitter region and assume the effective communication modes available at the receiver region are uncorrelated, i.e., $F_R = I_n$, for $n_R > 1$. Since all azimuth power distribution models give very similar correlation values for a given angular spread, especially for small antenna separations, we restrict only to the uniform-limited azimuth power distribution. In this case, the $(m, m')$-th entry of $F_T$ is given by

$$\gamma_{m,m'} = \text{sinc}((m - m')\Delta)e^{i(m-m')\phi_0}$$ \hfill (4.24)

where $\Delta$ represents the non-isotropic parameter of the azimuth power distribution (angular spread $\sigma_t = \Delta/\sqrt{3}$) and $\phi_0$ is the mean angle of departure (AOD). Note that, with scheme-2, transmitter only requires the knowledge of $\sigma_t$ and $\phi_0$ in order to build $F_T$ using (4.24), provided that the scattering distribution is uni-modal.

In our simulation, a realization of the underlying MIMO channel $H$ is generated by

$$\text{vec } (H) = R_H^{1/2} \text{vec } (H_{\text{iid}}),$$ \hfill (4.25)

where $R_H^{1/2}$ is the positive definite matrix square root of $R_H$ and $H_{\text{ iid}}$ is a $n_R \times n_T$
matrix which has zero-mean independent and identically distributed complex Gaussian random entries with unit variance. We use (4.6) and (4.25) to generate a realization of the underlying MIMO channel.

Figure 4.2: BER performance of the rate-1 coherent STBC (QPSK) with \(n_T = 2\) and \(n_R = 1, 2\) antennas for a uniform-limited azimuth power distribution with angular spread \(\sigma_t = 15^\circ\) and mean AOD \(\phi_0 = 0^\circ\); transmit antenna separation 0.2\(\lambda\).

Figure 4.2 illustrates the BER performance of the rate-1 coherent STBC with two-transmit antennas and \(n_R = 1, 2\) receive antennas for a uniform-limited azimuth power distribution at the transmitter with angular spread \(\sigma_t = 15^\circ\) about the mean AOD \(\phi_0 = 0^\circ\). When \(n_R = 2\), the two receiver antennas are placed \(\lambda\) apart, giving negligible spatial correlation effects at the receiver due to antenna spacing. From Figure 4.2, it is observed that both the fixed scheme (scheme-1) and the feedback scheme (scheme-2) provide significant BER improvements at low SNRs. In fact as discussed earlier, at very low SNRs, the optimum scheme is equivalent to eigen-beam forming.

Further we observe that as the SNR increases, the scheme-1 becomes redundant.
and the BER performance of scheme-1 approaches that of coherent STBC without precoding and the system is operating in full diversity. This also corroborates the claim that the STBC with two-transmit antennas has good resistance against spatial fading correlation at high SNRs as shown in Chapter 2. In contrast, scheme-2 provides significant BER improvements at high SNRs. However, we expect the performance of scheme-2 to converge to that of coherent STBC without precoding at higher SNRs.

![Figure 4.3: BER performance of the rate-1 coherent STBC (BPSK) with $n_T = 4$ and $n_R = 1, 2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; UCA transmit antenna configuration and 0.2$\lambda$ minimum separation between two adjacent transmit antenna elements.](image)

BER performance results of the rate-1 coherent STBC with 4-transmit UCA and 4-transmit ULA antenna configurations are shown in Figures 4.3 and 4.4, respectively for a uniform-limited azimuth power distribution at the transmitter with angular spread $\sigma_t = 15^\circ$ about the mean AOD $\phi_0 = 0^\circ$. For both antenna configurations, the minimum separation between two adjacent transmit antenna elements.

---

This precoder can be applied to any arbitrary antenna configuration.
4.5 Simulation Results: Coherent STBC

Figure 4.4: BER performance of the rate-1 coherent STBC (BPSK) with $n_T = 4$ and $n_R = 1, 2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; ULA transmit antenna configuration and $0.2\lambda$ minimum separation between two adjacent transmit antenna elements.

The BER performance of both precoding schemes is better than that of the non-precoded system. For example, when $n_R = 2$, it can be seen that at $10^{-3}$ BER, the performance of scheme-1 is about 2 dB and 2.5 dB better than that of the non-precoded system for UCA and ULA antenna configurations, respectively. Also, when $n_R = 2$, we observe that at BER of $10^{-3}$, the performance of scheme-2 is about 4 dB and 6 dB better than that of the non-precoded system for UCA and ULA antenna configurations, respectively. As before, we observe that the performance of scheme-1 converges to the performance of non-precoded system at high SNRs. A similar performance trend is observed with the scheme-2 at higher SNRs. However, with scheme-2, we observe significant BER improvements over all SNRs considered.
At high SNRs we observed that ULA antenna configuration provides better performance than UCA antenna configuration for both precoding schemes. This is because, the number of effective communication modes in the transmit region is higher for the ULA case (large aperture radius of ULA, c.f. Table 4.1) than the UCA case and both precoding schemes efficiently activate the transmit modes in the transmit region of ULA. This observation suggests that our precoding schemes give scope for improvement of ULA performance at high SNR, especially the fixed scheme.

4.6 Simulation Results: Differential STBC

We now demonstrate the performance advantage achieved from precoding schemes proposed in Section 4.4 for differential STBC. In our simulations we use the rate-1 space-time modulated constellations constructed in [40] from orthogonal designs for two and four transmit antennas. Normalized QPSK alphabet \(\{\pm 1/\sqrt{2}, \pm i/\sqrt{2}\}\) and normalized BPSK alphabet \(\{\pm 1/\sqrt{2}\}\) are used with two and four transmit antenna space-time block codes, respectively. As before, a realization of the underlying MIMO channel is simulated using (4.6) and (4.25).

Figure 4.5 illustrates the BER performance of the differential STBC with two-transmit antennas and \(n_R = 1, 2\) receive antennas for a uniform-limited azimuth power distribution at the transmitter with angular spread \(\sigma_t = 15^\circ\) about the mean AOD \(\phi_0 = 0^\circ\). In both cases, two transmit antennas are placed 0.1\(\lambda\) distance apart. When \(n_R = 2\), the two receiver antennas are placed \(\lambda\) apart. From Figure 4.5, it is observed that both the fixed scheme (scheme-3) and the feedback scheme (scheme-4) provide significant BER improvements at low SNRs. At moderate SNRs (e.g. 8 dB - 14 dB) we can observe that scheme-3 gives some BER performance when \(n_R = 2\). However as the SNR increases the BER performance of scheme-3 approaches that of differential STBC without precoding. In contrast, scheme-4 provides significant BER improvements at high SNRs and we expect the performance of this scheme to converge to that of differential STBC without precoding at higher SNRs.

BER performance results for 4-transmit UCA and 4-transmit ULA antenna configurations are shown in Figures 4.6 and 4.7, respectively for a uniform-limited azimuth power distribution at the transmitter with angular spread \(\sigma_t = 15^\circ\) about the mean AOD \(\phi_0 = 0^\circ\). For both antenna configurations, the minimum separation between two adjacent transmit antenna elements is set to 0.2\(\lambda\), corresponding to aperture radii 0.142\(\lambda\) and 0.3\(\lambda\) for UCA and ULA antenna configurations, respec-
Figure 4.5: BER performance of the rate-1 differential STBC (QPSK) with $n_T = 2$ and $n_R = 1, 2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; transmit antenna separation 0.1λ.

tively. As before, when $n_R = 2$, the two receiver antennas are placed $\lambda$ apart. For both transmit antenna configurations, simulation results show that the BER performance of both precoding schemes is better than that of non-precoded systems. For example, when $n_R = 2$, it can be seen that at $10^{-3}$ BER, the performance of scheme-3 is about 1.5dB and 2dB better than that of the non-precoded system, for UCA and ULA antenna configurations, respectively. As before, we can observe that the performance of the fixed scheme converges to the performance of the non-precoded system at high SNRs. With the feedback scheme, we observe significant BER improvements over all SNRs considered.

### 4.7 Performance in other Channel Models

Simulation results presented in previous sections used the channel model $H = J_R H_s J_T^*$, which is derived based on plane wave propagation theory, to simulate
Figure 4.6: BER performance of the rate-1 differential STBC (BPSK) with $n_T = 4$ and $n_R = 1, 2$ antennas for a uniform-limited azimuth power distribution with angular spread $\sigma_t = 15^\circ$ and mean AOD $\phi_0 = 0^\circ$; UCA transmit antenna configuration and $0.2\lambda$ minimum separation between two adjacent transmit antenna elements.
Figure 4.7: BER performance of the rate-1 differential STBC (BPSK) with \( n_T = 4 \) and \( n_R = 1, 2 \) antennas for a uniform-limited azimuth power distribution with angular spread \( \sigma_t = 15^\circ \) and mean AOD \( \phi_0 = 0^\circ \); ULA transmit antenna configuration and 0.2\( \lambda \) minimum separation between two adjacent transmit antenna elements.
the underlying channels between transmit and receive antennas. In this section we analyze the performance of fixed precoding scheme (both coherent and differential) derived in this chapter applied on other statistical channel models proposed in the literature. In particular we are interested in channel models that are consistent with plane wave propagation theory. MISO and MIMO channel models proposed by Chen et al. [79] and Abdi et al. [80], respectively are two such example channel models. Sections 4.7.1 and 4.7.2 provide simulation results of coherent STBC applied on Chen’s MISO channel model and differential STBC applied on Abdi’s MIMO channel model, respectively.

### 4.7.1 Chen et al.’s MISO Channel Model

Figure 4.8 depicts the MISO channel model proposed by Chen et al., where the space-time cross correlation between two antenna elements at the transmitter is given by

$$
[R(\tau)]_{m,n} = \exp \left[ j \frac{2\pi}{\lambda} (d_m - d_n) \right] \times \frac{J_0}{2\pi} \sqrt{\left( f_D \tau \cos \gamma + \frac{z_{mn}^c}{\lambda} \right)^2 + \left( f_D \tau \sin \gamma - \frac{z_{mn}^s}{\lambda} \right)^2},
$$

(4.26)

with

$$
z_{mn}^c = \frac{2a}{d_m + d_n} \left[ d_{mn}^{sp} - (d_m - d_n) \cos \alpha_{mn} \cos \beta_{mn} \right],
$$

$$
z_{mn}^s = \frac{2a}{d_m + d_n} (d_m - d_n) \cos \alpha_{mn} \sin \beta_{mn},
$$

where $a$ is the scatterer ring radius, $\gamma$ is the moving direction of the receiver with respect to the end-fire of the antenna array, $f_D$ is the Doppler spread and $d_{mn}$ is the receiver distance to the center of the transmit antenna pair $m, n$. All other geometric parameters are defined as in Figure 4.8.

Figure 4.9 shows the performance of the fixed precoding scheme (scheme 1) derived in Section 4.3.1 for rate-3/4 coherent STBC with three transmit antennas placed in a ULA configuration. In this simulation, we assume the time-varying channels are undergone Rayleigh fading at the fading rate $f DT = 0.001$, where $T$ is the codeword period. We set parameters $a = 30\lambda$, $d_{12}^{sp} = d_{23}^{sp} = 0.2\lambda$, $d_{12} = 1000\lambda$, $\gamma = 20^\circ$ and $\beta_{1,2} = 60^\circ$. All other geometric parameters of the model in Figure 4.8 can be easily determined from these parameters by using simple trigonometry. In this simulation, a realization of the underlying space-time MIMO
Figure 4.8: Scattering channel model proposed by Chen et al. for three transmit and one receive antennas.
channel is generated using (4.25) and (4.26). From Figure 4.9 we observed that proposed fixed precoding scheme gives significant performance improvements for time-varying channels. For example, at 0.05 BER, performance of the spatially precoded system is 1dB better than that of the non-precoded system.

![Spatial Precoder Performance](image)

Figure 4.9: Spatial precoder performance with three transmit and one receive antennas for 0.2\(\lambda\) minimum separation between two adjacent transmit antennas placed in a uniform linear array, using Chen et al.’s channel model: rate-3/4 coherent STBC.

### 4.7.2 Abdi et al.’s MIMO Channel Model

In this model, space-time cross correlation between two distinct antenna element pairs at the receiver and the transmitter is given by

\[
[R(\tau)]_{lp,mq} = \frac{\exp[jc_{pq}\cos(\alpha_{pq})]}{I_0(\kappa)} \times I_0 \left( \{ \kappa^2 - a^2 - b^2_{lm} - c_{pq}^2 \Delta^2 \sin^2(\alpha_{pq}) + 2ab_{lm}\cos(\beta_{lm} - \gamma) + 2c_{pq}\Delta \sin(\alpha_{pq}) \right) \\
\times \left[ a\sin(\gamma) - b_{lm}\sin(\beta_{lm}) \right]
\]
4.7 Performance in other Channel Models

\[-j2\kappa [a \cos(\varphi_0 - \gamma) - b_{lm} \cos(\varphi_0 - \beta_{lm}) \]
\[-c_{pq}\Delta \sin(\alpha_{pq}) \sin(\varphi_0)]^{1/2},\]  

(4.27)

where \(a = 2\pi f_D \tau\), \(b_{lm} = 2\pi d_{lm}/\lambda\), \(c_{pq} = 2\pi \delta_{pq}/\lambda\); \(f_D\) is the Doppler shift; \(\varphi_0\) is the mean angle of arrival at the receiver; \(\kappa\) controls the spread of the AOA; and \(\gamma\) is the direction of motion of the receiver. Other geometric parameters are defined in Figure 4.10. Note that this model also captures the non-isotropic scattering at the transmitter via \(\Delta\) and the model is valid only for small \(\Delta\) [80].

Figure 4.10: Scattering channel model proposed by Abdi et al. for two transmit and two receive antennas.

Figure 4.11 shows the performance of spatial precoder derived in Section 4.4.1 for rate-1 differential STBC with two transmit and two receive antennas for a stationary receiver (i.e. \(f_D = 0\)). In this simulation we set \(\delta_{12} = 0.1\lambda\), \(d_{12} = \lambda\) and \(\alpha_{12} = \beta_{12} = 0^\circ\). We assume the scattering environment surrounding the receiver antenna array is rich, i.e., \(\kappa = 0\) and the non-isotropic factor \(\Delta\) at the transmitter is \(10^\circ\). A realization of the underlying MIMO channel is generated using (4.25) and (4.27). It is observed that our precoding scheme based on antenna configuration details give promising improvements for low SNRs when the underlying channel is modeled using Abdi’s channel model.

Therefore, using the previous results from Chen’s channel model and the current results, we can come to the conclusion that our fixed spatial precoding scheme can be applied to any general wireless communication system. Furthermore, our precoder designs and simulation results provide an independent confirmation of the validity of the spatial channel decomposition \(H = J_R H_s J_T^H\) proposed in [106].
4.8 Summary and Contributions

In realistic channel scenarios the performance of space-time coded MIMO systems is significantly reduced due to the physical factors such as antenna spacing, antenna placement and non-isotropic scattering relative to the performance in i.i.d Rayleigh fading channels. This chapter proposed several linear precoding schemes to improve the performance of space-time coded MIMO systems, where both the antenna arrays and scattering are constrained.

Some specific contributions made in this chapter are:

- A fixed linear spatial precoding scheme is proposed which exploits the antenna placement information at both ends of the MIMO channel to ameliorate the effects of limited antenna separation and non-ideal antenna placement on the performance of coherent and non-coherent space-time coded systems. This scheme is designed based on previously unutilized fixed and known parameters of MIMO channels, the antenna spacing and antenna placement details.
The precoder is fixed for fixed antenna placement and the transmitter does not require any form of feedback of CSI (partial or full) from the receiver which is an added advantage over the other precoding schemes found in the literature.

- Proposed fixed scheme can be applied on uplink transmission of a wireless communications system as it can effectively reduce the effects due to insufficient antenna spacing and antenna placement at the mobile unit.

- Proposed a second linear precoding scheme which exploits the non-isotropic parameters of the scattering channel to improve the performance of space-time coded systems applied on MIMO channels in non-isotropic scattering environments. Unlike in the fixed scheme, this scheme requires the receiver to estimate the non-isotropic parameters of the scattering channel and feed them back to the transmitter.

- Performance of the feedback scheme is superior to that of the fixed scheme for all SNRs in non-isotropic scattering environments. At high SNRs, the fixed scheme provides very little performance improvements compared to the feedback scheme. Therefore, the exploitation of antenna locations (spatial dimension) does not warrant significant performance improvements at high SNRs.

- The performance of both precoding schemes is assessed when applied on 1-D antenna arrays (ULA) and 2-D antenna arrays (UCA). With 1-D antenna arrays, it is shown that both precoding schemes give scope for improvements than with 2-D antenna arrays.

- The precoder design is based on the spatial channel decomposition $H = J_R H_s J_T^\dagger$, but we showed that the performance of fixed precoding scheme does not depend on the channel model that is being used to model the underlying MIMO channel. Therefore, our design and simulation results provide an independent confirmation of the validity of the channel decomposition $H = J_R H_s J_T^\dagger$. 
Chapter 5

Achieving Maximum Capacity: Spatially Constrained Dense Antenna Arrays

5.1 Introduction

Multiple-input multiple-output (MIMO) wireless communication systems using multi-antenna arrays simultaneously during transmission and reception have generated significant interest in recent years. Theoretical work of [5] and [6] showed the potential for significant capacity increases in wireless channels via spatial multiplexing with sparse antenna arrays. However, in reality by increasing the number of antennas within a fixed region of space, the antenna array become dense and spatial correlation significantly limits the channel capacity [20]. The achievable capacities of MIMO channels and power allocation schemes to achieve these capacities under various assumptions of channel state information (CSI) has been the subject of recent research work in information theory.

Previous studies [19–25,37–39] have given insights and bounds into the effects of correlated channels and [35,37–39] have specifically studied the capacity of spatially constrained dense antenna arrays. The above studies have assumed that the perfect CSI is known only to the receiver. In [5,26–34] various power allocation schemes (or water filling strategies) have been derived assuming perfect CSI or partial CSI (e.g. channel covariance) is available at the transmitter through feedback. However, performance of these schemes heavily depends on the accuracy of the feedback information.

In [37] it was shown that there exists a theoretical antenna saturation point at which the maximum achievable capacity for a fixed region occurs, and further
increases in the number of antennas in the region will not give further capacity gains. However, it was also shown that due to non-ideal antenna placement, capacity achieved from a fixed region of space is always lower than the theoretical maximum capacity, and in this case the capacity achieved corresponds to a smaller region with optimally placed antennas within.

In contrast, in this chapter we show that the theoretical maximum capacity for a fixed region of space can be achieved via linear spatial precoding, which basically eliminates the detrimental effects of non-ideal antenna placement. Similar to the fixed linear spatial precoding scheme derived in Chapter 4 this scheme is also designed based on previously unutilized fixed and known parameters of a MIMO channel, the antenna spacing and antenna placement, assuming a isotropic scattering environment. Unlike the power loading schemes found in the literature [5, 27, 31, 33, 34] this new scheme does not require any feedback information from the receiver since the design is based on partial CSI contained in the antenna locations, which has previously been ignored. Furthermore, since this new power-loading scheme is fixed for a given antenna configuration, it can be used in non-stationary channels as well as stationary channels. This chapter also develops two other power-loading schemes specifically to improve the capacity performance of dense MIMO arrays in non-isotropic scattering environments.

5.2 System Model

Consider a MIMO system consisting of \( n_T \) transmit antennas and \( n_R \) receive antennas within circular apertures of radius \( r_T \) and \( r_R \), respectively, along with the channel decomposition given in Chapter 3. The original \( n_T \times 1 \) data vector sent from the transmitter is denoted by \( s \) with \( \mathbb{E}\{ss^\dagger\} = P_T/n_T I_{n_T} \), where \( P_T \) is the total transmit power. Before each data vector is transmitted, it is multiplied by a fixed linear spatial precoder matrix \( F \) of size \( n_T \times n_T \), so the \( n_R \times 1 \) received signal becomes

\[
y = Hx + w, \tag{5.1}
\]

where \( x = Fs \) is the \( n_T \times 1 \) baseband transmitted signal vector from \( n_T \) antennas with input signal covariance matrix

\[
Q = \mathbb{E}\{xx^\dagger\} = \frac{P_T}{n_T} FF^\dagger, \tag{5.2}
\]
$w$ is the $n_R \times 1$ white Gaussian noise matrix in which elements are zero-mean independent Gaussian distributed random variables with variance $1/2$ per dimension and $H$ is the $n_R \times n_T$ random flat fading channel matrix. Note that $P_T$ is also the average signal-to-noise (SNR) at each receiver antenna. In this work we adapt the spatial channel decomposition $H = J_R H_s J_T^\dagger$ introduced in Chapter 3 to represent $H$.

5.3 Capacity of Spatially Constrained Antenna Arrays

The ergodic capacity of $n_T$ transmit and $n_R$ receive antennas is given by [5],

$$\tilde{C} = \mathcal{E} \left\{ \log \left| I_{n_R} + HQH^\dagger \right| \right\},$$

where $Q = \mathcal{E} \{ xx^\dagger \}$ is the input signal covariance matrix. In the following we will assume that the channel matrix $H$ is fully known at the receiver and it is also partially known at the transmitter, where deterministic parts of the channel such as antenna spacing and antenna geometry are considered as partial channel information.

Consider the case where the receiver array consists of large number of receive antennas. It was shown in [35] that the total received power at the receiver array should remain a constant for a given region, regardless of the number of antennas in it. In this situation, the normalized ergodic capacity is given by

$$\tilde{C} = \mathcal{E} \left\{ \log \left| I_{n_R} + \frac{1}{n_R} HQH^\dagger \right| \right\}, \quad (5.3)$$

where the scaling factor $1/n_R$ scales the channel variances to $\mathcal{E} \{ |h_{r,t}|^2 \} / n_R$, which assures the total received power remains a constant as the number of antennas is increased.

Substitution of $H = J_R H_s J_T^\dagger$ into (5.3) gives the ergodic capacity

$$\tilde{C} = \mathcal{E} \left\{ \log \left| I_{n_R} + \frac{1}{n_R} J_R H_s J_T^\dagger Q J_T H_s J_T^\dagger \right| \right\},$$

$$= \mathcal{E} \left\{ \log \left| I_{n_T} + \frac{1}{n_T} Q J_T H_s J_T^\dagger J_R H_s J_T^\dagger \right| \right\}, \quad (5.4)$$

where the second equality follows from the determinant identity $|I + AB| = |I + BA|$.
Let $\tilde{H} = J_R H_s = [\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_{n_R}]^\dagger$, where $\tilde{h}_r$ is a $1 \times (2M_T + 1)$ row-vector of $\tilde{H}$, which corresponds to the complex channel gains from $(2M_T + 1)$ transmit modes to the $r$-th receiver antenna, then $(2M_T + 1) \times (2M_T + 1)$ transmitter modal correlation matrix can be defined as

$$R_{\tilde{H}} \triangleq \mathcal{E}\{\tilde{h}_r^\dagger \tilde{h}_r\}, \forall r \quad (5.5)$$

where $(m,m')$-th element of $R_{\tilde{H}}$ gives the modal correlation between $m$-th and $m'$-th modes in the transmit region.

We consider the situation where the receiver aperture of radius $r_R$ has optimally placed (uncorrelated) $n_R = 2M_R + 1$ antennas, which corresponds to independent $\tilde{h}_r$ vectors, then the sample transmitter modal correlation matrix is given by

$$\hat{R}_{\tilde{H}} = \frac{1}{n_R} \sum_{r=1}^{n_R} \tilde{h}_r^\dagger \tilde{h}_r.$$  

For a large number of receive antennas, the sample transmitter modal correlation matrix $\hat{R}_{\tilde{H}}$ converges to $R_{\tilde{H}}$ as $r_R \to \infty$. Since $\tilde{H}^\dagger \tilde{H} = \sum_{r=1}^{n_R} \tilde{h}_r^\dagger \tilde{h}_r$, then for a large number of uncorrelated receive antennas, the ergodic capacity (5.4) converges\(^1\) to the deterministic quantity $C$,

$$\lim_{r_R \to \infty} \bar{C} = C \triangleq \log \left| I_{n_T} + Q J_T R_{\tilde{H}} J_T^\dagger \right|. \quad (5.6)$$

This analytical capacity expression allows us to investigate the effects of transmit antenna configuration, scattering environment and the input signal covariance matrix $Q$ on the ergodic capacity. However, in this chapter, our main objective is to find the optimum transmit power loading scheme which maximizes the effects of non-ideal antenna placement on the capacity performance of dense MIMO systems. In other words, we wish to find the optimum $Q$ (and hence the linear spatial precoder $F$) which maximizes the deterministic capacity (5.6) for a given transmit antenna configuration assuming modes at the transmit aperture are uncorrelated.

### 5.4 Optimization Problem Setup: Isotropic Scattering

Assume that the scatterers generate an isotropic diffuse field at the transmitter, which corresponds to independent elements of scattering channel matrix $H_s$. With

\(^1\)When $n_R$ is small, the ergodic capacity can be bounded by the Jenson’s inequality.
this assumption we have $R_H = I_{2M_T+1}$ and (5.6) reduces to

$$C = \log \left| I_{n_T} + QJ_TJ_T^T \right|.$$  \hfill (5.7)

In this case, we see that the capacity obtained from a fixed region of space is dependent on the transmit antenna configuration and also on the input signal covariance matrix.

In (5.7), $(q, r)$-th element of scatter-free transmit matrix product $J_TJ_T^T$ is given by

$$\left\{J_TJ_T^T\right\}_{q,r} = \sum_{n=-M_T}^{M_T} J_n(u_q)\overline{J_n(u_r)},$$

$$= J_0(k \parallel u_q - u_r \parallel)$$

which follows from a special case of Gegenbauer’s Addition Theorem [137, page 363]. For a rich scattering environment, $J_0(k \parallel u_q - u_r \parallel)$ gives the spatial correlation between the complex envelopes of the transmitted signals from antennas $q$ and $r$ [78]. It is well known that the presence of spatial correlation between antenna elements limits the capacity of MIMO systems. So the main objective is to reduce the effects of spatial correlation (non-ideal antenna placement in our case) on MIMO capacity of dense antenna arrays by designing $Q$ (and hence the linear precoder $F$) to maximize the deterministic capacity (5.7) for a given antenna placement.

If the channel matrix $H$ is known only to the receiver, then as shown in [5], transmission of statistically independent equal power signals each with a Gaussian distribution will be optimal. In this case $Q = (P_T/n_T)I_{n_T}$. In what follows we will refer to this scheme as equal power loading.

### 5.4.1 Optimum input signal covariance

Writing $J_T$ as the singular value decomposition (svd) $J_T = U_T\Lambda_T V_T^\dagger$, then (5.7) becomes

$$C = \log \left| I_{n_T} + U_T^\dagger QU_T T \right|,$$

where $T = \Lambda_T\Lambda_T^\dagger$ is a diagonal matrix with squared singular values of $J_T$ (or the eigen-values of spatial correlation matrix $J_TJ_T^T$) on the diagonal.

The optimum input signal covariance $Q$ is obtained by solving the optimization
problem:

\[
\max \log \left| I_{n_T} + U_T^\dagger QU_T T \right|
\]
subject to \( Q \succeq 0, \tr\{Q\} = P_T, \)
\( \tr\{U_T^\dagger QU_T T\} = P_T, \)

(5.8)

where we assumed \( Q \) is non-negative definite (\( Q \succeq 0 \)). The power constraint \( \tr\{Q\} = P_T \) ensures the total power transmitted from \( n_T \) antennas in the dense transmit antenna array is \( P_T \) and the second power constraint \( \tr\{U_T^\dagger QU_T T\} = P_T \) ensures the total power assigned to effective modes at the scatter-free transmit region is also \( P_T \).

Let \( \tilde{Q} = U_T^\dagger QU_T \). Since \( U_T \) is unitary, maximisation/minimisation over \( Q \) can be carried equally well over \( \tilde{Q} \). Furthermore, \( \tilde{Q} \) is non-negative definite since \( Q \) is non-negative definite. Therefore, the optimization problem (5.8) becomes²

\[
\min - \log \left| I_{n_T} + \tilde{Q} T \right|
\]
subject to \( \tilde{Q} \succeq 0, \tr\{\tilde{Q}\} = P_T, \tr\{\tilde{Q} T\} = P_T. \)

(5.9)

By applying Hadamard’s inequality on \( |I_{n_T} + \tilde{Q} T| \) gives that this determinant is maximized when \( \tilde{Q} T \) is diagonal [5]. Therefore \( \tilde{Q} \) must be diagonal as \( T \) is diagonal. Since \( \tilde{Q} T \) is a non-negative definite diagonal matrix with non-negative entries on its diagonal, \( I + \tilde{Q} T \) forms a positive definite matrix. As a result, the objective function of our optimization problem is convex [136, page 73]. Therefore the optimization problem (5.9) above is a convex minimization problem because the objective function and the inequality constraint are convex and equality constraints are affine.

Let \( \tilde{q}_i = [\tilde{Q}]_{i,i} \) and \( t_i = [T]_{i,i} \). Optimization problem (5.9) then reduces to finding \( \tilde{q}_i > 0 \) such that

\[
\min - \sum_{i=1}^{n_T} \log(1 + t_i \tilde{q}_i)
\]
subject to \( \tilde{q} \succeq 0, \ \text{1}^T \tilde{q} = P_T, \ \text{1}^T \tilde{q} = P_T. \)

(5.10)

where \( \tilde{q} = [\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_{n_T}]^T, t = [t_1, t_2, \ldots, t_{n_T}]^T \) and \( \text{1} \) denotes the vector of all ones. Introducing Lagrange multipliers \( \lambda \in \mathbb{R}^{n_T} \) for the inequality constraint \(-q \preceq 0\) and \( \nu, \mu \in \mathbb{R} \) for equality constraints \( \text{1}^T \tilde{q} = P_T \) and \( t^T \tilde{q} = P_T \), respectively,
we obtain the Karush-Kuhn-Tucker (K.K.T) conditions

\[
\begin{align*}
\tilde{q} &\succeq 0, \quad \lambda \succeq 0, \quad 1^T\tilde{q} = P_T, \quad t^T\tilde{q} = P_T \\
\lambda_i\tilde{q}_i & = 0, \quad i = 1, 2, \cdots, n_T \\
-\frac{t_i}{1 + t_i\tilde{q}_i} - \lambda_i + v + \mu t_i & = 0, \quad i = 1, 2, \cdots, n_T.
\end{align*}
\] (5.11)

Note that \(\lambda_i\) in (5.11) can be eliminated since it acts as a slack variable, giving new K.K.T conditions

\[
\begin{align*}
\tilde{q} &\succeq 0, \quad 1^T\tilde{q} = P_T, \quad t^T\tilde{q} = P_T \\
\tilde{q}_i \left( v + \mu t_i - \frac{t_i}{1 + t_i\tilde{q}_i} \right) & = 0, \quad i = 1, \cdots, n_T, \quad (5.12a) \\
v + \mu t_i & \geq \frac{t_i}{1 + t_i\tilde{q}_i}, \quad i = 1, \cdots, n_T. \quad (5.12b)
\end{align*}
\]

The complementary slackness condition \(\lambda_i\tilde{q}_i = 0\) for \(i = 1, 2, \cdots, n_T\) states that \(\lambda_i\) is zero unless the \(i\)-th inequality constraint is active at the optimum. Therefore, from (5.12a) we obtain optimum \(\tilde{q}_i\)

\[
\tilde{q}_i = \begin{cases} 
\frac{1 - \mu}{v + \mu t_i} - \frac{v}{t_i(v + \mu t_i)} & , t_i > \frac{v}{1 - \mu}; \\
0, & \text{otherwise},
\end{cases} 
\] (5.13)

where \(v\) and \(\mu\) are constants chosen to satisfy two power constraints

\[
\begin{align*}
\sum_{i=1}^{n_T} \max \left( 0, \frac{1 - \mu}{v + \mu t_i} - \frac{v}{t_i(v + \mu t_i)} \right) & = P_T, \\
\sum_{i=1}^{n_T} t_i \max \left( 0, \frac{1 - \mu}{v + \mu t_i} - \frac{v}{t_i(v + \mu t_i)} \right) & = P_T
\end{align*}
\]

and \(\tilde{Q} = \text{diag}(\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_{n_T})\). Therefore, the optimum input signal covariance matrix \(Q = U_T\tilde{Q}U_T^\dagger\). From (5.2), the linear spatial precoder

\[
F = \sqrt{\frac{P_T}{nT}}U_T\tilde{Q}^{1/2}U_n^\dagger,
\]

where \(U_n\) is an arbitrary unitary matrix.
5.4.2 Numerical Results

We now present numerical results to illustrate the capacity improvements obtained from the spatial precoder derived in the previous section. The performance of the precoder is compared with the equal power loading scheme.

We consider a MIMO system with \( n_T \) transmit antennas constrained within a scatter-free circular region of radius \( r_T = 0.5\lambda \) and a large number of uncorrelated receive antennas for a total power budget of \( P_T = 10\text{dB} \). Figure 5.1 shows the capacity results for 2-D antenna arrays (Uniform Circular Arrays) and 1-D antenna arrays (Uniform Linear Arrays) using the linear spatial precoder \( F \) and equal power allocation scheme \( Q = (P_T/n_T)I_{n_T} \) for increasing the number of transmit antennas in the transmitter region. Also shown is the maximum achievable capacity from the transmit region when all the \( n_T \) antennas are placed optimally such that the spatial correlation is zero between all the antennas. In this case, the maximum achievable capacity from the transmitter region is given by [37, Eq. 35],

\[
C_{\text{max}}(r_T) = n_{\text{sat}}(r_T) \log \left( 1 + \frac{P_T}{n_{\text{sat}}(r_T)} \right),
\]

where \( n_{\text{sat}}(r_T) = 2M_T + 1 \) is the antenna saturation point for the region which also corresponds to the number of effective modes in the scatter-free transmit region. In our case, from (3.6), \( n_{\text{sat}}(r_T = 0.5\lambda) = 11 \), which is shown by the vertical dashed line in Figure 5.1.

It is observed that with the equal power loading scheme, capacity performance of both the Uniform Circular array (UCA) and Uniform Linear Array (ULA) does not reach the maximum achievable capacity \( C_{\text{max}}(r_T) \) from the region as the number of antennas is increased. This is because both the UCA and ULA do not optimally place the antennas within the given region. Furthermore, with this scheme capacity is saturated even before \( n_T \) approaches \( n_{\text{sat}} \) for both antenna configurations. In fact the capacity achieved with this scheme corresponds to a region of smaller radius with optimally placed antennas within. Let \( \tilde{n}_{\text{sat}}( < n_{\text{sat}}) \) be the new antenna saturation point for a given antenna configuration. Therefore, with equal power loading one cannot achieve further capacity gains by increasing the number of antennas beyond \( \tilde{n}_{\text{sat}} \).

In contrast, spatially precoded systems give significant capacity improvements as the number of antennas are increased beyond \( \tilde{n}_{\text{sat}} \). For \( n_T > 80 \), we see the capacity of the precoded UCA system reaches \( C_{\text{max}}(r_T) \), which corresponds to 1.2bps/Hz capacity gain over the equal power loading scheme. In this case, spatial precoder virtually arranges the antennas into an optimal configuration as such the virtual ar-
5.4 Optimization Problem Setup: Isotropic Scattering

Figure 5.1: Capacity comparison between spatial precoder and equal power loading \((Q = (P_T/n_T)I_{n_T})\) schemes for uniform circular arrays and uniform linear arrays in a rich scattering environment with transmitter aperture radius \(r_T = 0.5\lambda\) and a large number of uncorrelated receive antennas \((r_R \to \infty)\) for an increasing number of transmit antennas. Also shown is the maximum achievable capacity (5.14) from the transmitter region.

Arrangement gives the optimum capacity performance. In the case of precoded ULA, it requires a large number of transmit antennas to achieve \(C_{\text{max}}(r_T)\). However, as we can see, the spatial precoder still provides significant capacity gains over the equal power loading scheme for any \(n_T > \tilde{n}_{\text{sat}}\). We also observed that precoding does not provide significant capacity gains for lower number of transmit antennas. This is mainly due to the low spatial correlation between antenna elements in the transmit array for lower number of antennas.

5.4.3 Capacity with Finite Number of Receiver Antennas

Capacity results obtained in the previous section assumed that the receiver consists of a large number of uncorrelated receive antennas \((r_R = \infty)\) and also the
communication modes at the receiver region are uncorrelated. In this section, we present Monte-Carlo simulations to show the capacity achieved through precoding when there are finite number of receive antennas in a region with finite size.

![Graph showing capacity vs. number of transmit antennas for different scenarios](image)

Figure 5.2: Simulated capacity of equal power loading and spatial precoding schemes for uniform circular arrays in a rich scattering environment with transmitter aperture radius $r_T = 0.5\lambda$ and receiver aperture radius $r_R = 5\lambda$ for an increasing number of transmit antennas.

As before, we consider the effect of increasing the number of transmit antennas $n_T$ constrained within a scatter-free circular region of radius $r_T = 0.5\lambda$, for a fixed number of receive antennas constrained within a scatter-free region of radius $r_R = 5\lambda$ (choose $n_R = 2M_R + 1 = 87$) for SNR of 10dB. Figures 5.2 and 5.3 show the simulated capacity of equal power loading and spatial precoding schemes for UCA and ULA using the channel model presented in Chapter 3 and assuming an isotropic scattering environment. Also shown is the maximum achievable capacity (5.14) from the transmitter region and upper bound on capacity of both schemes for a large number of optimally placed uncorrelated receive antennas ($r_R = \infty$).

As expected, spatially precoded antenna systems provide significant capacity
Improvements compared to the equally power loaded antenna systems. Previously, we observed that with a large number of uncorrelated receiver antennas, the capacity of the spatially precoded UCA system approaches $C_{\text{max}}(r_T)$ for $n_T > 80$. However, from Figure 5.2, it is observed that when there are finite number of receive antennas in the system, the capacity of the precoded system does not reach $C_{\text{max}}(r_T)$ as the $n_T$ increases. This is due to the presence of spatial correlation at the receiver array.

### 5.4.4 Transmit Modes and Power Allocation

In this section we compare the average power allocated to modes in the transmit region for the two power loading schemes we considered and follow with some analysis.
From Chapter 2.2, the signal leaving the scatter-free transmit region along direction $\hat{\phi}$ is written as

$$\Phi(\hat{\phi}) = \sum_{t=1}^{n_T} x_t e^{i k u_t \cdot \hat{\phi}}, \quad (5.15)$$

where $x_t$ is the signal transmitted from $t$-th transmit antenna and $u_t$ is the location of it. Using the 2-D modal expansion of the plane wave $e^{i k u_t \cdot \hat{\phi}}$, given by (3.4), $\Phi(\hat{\phi})$ can be written as

$$\Phi(\hat{\phi}) = \sum_{n=-\infty}^{\infty} \sum_{t=1}^{n_T} x_t \mathcal{J}_n(u_t) e^{i n \phi}, \quad (5.16a)$$

$$= \sum_{n=-\infty}^{\infty} a_n e^{i n \phi}, \quad (5.16b)$$

where $\hat{\phi} \equiv (1, \phi)$ in polar coordinates system and $a_n = \sum_{t=1}^{n_T} x_t \mathcal{J}_n(u_t)$ is the $n$-th transmit mode excited by $n_T$ antennas. Note that sum (5.16b) in fact is the Fourier series expansion of signal $\Phi(\hat{\phi})$ with Fourier coefficients $a_n$. The average power allocated to the $n$-th transmit mode is then given by

$$\sigma_n^2 = \mathcal{E}\{|a_n|^2\} = \sum_{t=1}^{n_T} \sum_{t'=1}^{n_T} \mathcal{E}\{x_t x_{t'}\} \mathcal{J}_n(u_t) \mathcal{J}_n(u_{t'}), \quad (5.17)$$

where $\mathcal{E}\{x_t x_{t'}\}$ is the $(t, t')$-th entry of $Q$. For the equal power loading scheme, (5.17) simplifies to

$$\sigma_n^2 = \frac{P_T}{n_T} \sum_{t=1}^{n_T} J_n^2(k \|u_t\|).$$

As described in Chapter 3.3.1, the number of effective modes excited by a spatially constrained antenna array is limited by the size of the aperture and is independent of number of antennas packed into the aperture. Figure 5.4 shows the average power allocation to the first 11 effective transmit modes for the two antenna configurations considered in the previous section. The results shown here are for $n_T = 80$ and $P_T = 10$dB.

Thus far we have assumed that the receiver has the full knowledge of the channel matrix $H = J_R H_s J_T^\dagger$ and the transmitter has the knowledge of antenna configuration matrix $J_T$. Since the scattering channel matrix $H_s$ is not known to the transmitter, the maximum capacity will occur for equal power allocation to the full set of uncorrelated transmit modes available for the given region, i.e.,
\[ \sigma_n^2 = \frac{P_T}{2M_T + 1}. \]

From Figure 5.4, for both antenna configurations, equal power loading scheme assigns different power levels to modes in the transmit region, and as a result, both configurations fail to achieve the maximum capacity available from the region (Figure 5.1). However, in the case of spatially precoded UCA, precoder assigns equal power to all available modes in the transmit region. In this case, precoder makes the transmitter scatter-free matrix product \( J_T^* J_T = I \) by correctly allocating power into each transmit antenna and utilizes the full set of uncorrelated communication modes between regions to achieve the theoretical maximum capacity \( C_{\max}(r_T) \). With spatially precoded ULA, we see that lower order modes (except the 0-th order mode) receive equal power while higher order modes receive unequal power. However, for a large number of transmit antennas, spatial precoder assigns equal power to all effective modes in the transmit region and thus achieves the theoretical maximum capacity \( C_{\max}(r_T) \).
5.4.5 Effects of Non-isotropic Scattering

We now investigate the effects of non-isotropic scattering at the transmitter on the capacity performance of dense MIMO systems when the spatial precoding scheme derived in Section 5.4.1 is used. The ergodic capacity of the system is calculated using (5.6).

First we derive the modal correlation matrix at the transmitter for any general scattering environment. Recall the definition of the transmitter modal correlation matrix

\[ \mathbf{R}_{\tilde{H}} \triangleq \mathcal{E}\left\{ \tilde{h}_r^\dagger \tilde{h}_r \right\}, \]

for \( r = 1, 2, \ldots, n_R \),

where \( \tilde{h}_r \) is the \( r \)-th row of \( \tilde{H} = J_R \mathbf{H}_s \), which corresponds to the complex channel gains from \((2M_T + 1)\) transmit modes in the scatter-free transmit region to the \( r \)-th receiver antenna. \( 1 \times (2M_T + 1) \) row-vector \( h_r \) takes the form

\[ h_r = \left[ \sum_{n=-N_R}^{N_R} \mathcal{J}_n(v_r) s_{n,-N_T}, \ldots, \sum_{n=-N_R}^{N_R} \mathcal{J}_n(v_r) s_{n,m}, \ldots, \sum_{n=-N_R}^{N_R} \mathcal{J}_n(v_r) s_{n,N_T} \right], \]

where \( s_{n,m} \) is the complex scattering gain between the \( m \)-th mode of the transmitter region and \( n \)-th mode of the receiver region, which is given by (3.13). Now the \((m, m')\)-th element of \( \mathbf{R}_{\tilde{H}} \), which is the correlation between \( m \)-th and \( m' \)-th modes at the transmitter region due to the \( r \)-th receive antenna, is given by

\[ \{ \mathbf{R}_{\tilde{H}} \}_{m,m'} = \mathcal{E}\left\{ \sum_{n=-M_R}^{M_R} \sum_{n'=-M_R}^{M_R} \mathcal{J}_n(v_r) \mathcal{J}_{n'}(v_r) s_{n,m} s_{n,m'}^* \right\}, \]

\[ = \sum_{n=-M_R}^{M_R} \sum_{n'=-M_R}^{M_R} \mathcal{J}_n(v_r) \mathcal{J}_{n'}(v_r) \gamma_{m,m'}^{n,n'}, \quad (5.18) \]

where

\[ \gamma_{m,m'}^{n,n'} = \mathcal{E}\left\{ s_{n,m} s_{n',m'}^* \right\} \]

(5.19)

is the correlation between two distinct modal pairs at the transmitter and the receiver regions. As we showed in Chapter 3.3.2, when the scattering from one direction is independent of that from another direction for both the receiver and the transmitter regions, (5.19) can be written as

\[ \gamma_{m,m'}^{n,n'} = \int_{S^1 \times S^1} G(\phi, \varphi) e^{i(m-m')\phi} e^{-i(n-n')\varphi} d\phi d\varphi, \quad (5.20) \]
where $G(\phi, \varphi) = \mathcal{E} \{ |g(\phi, \varphi)|^2 \}$ is the normalized joint azimuth power distribution of the scatterers surrounding the transmitter and receiver regions. Also, when the scattering channel is separable, i.e.,

$$G(\phi, \varphi) = P_{\text{Tx}}(\phi) P_{\text{Rx}}(\varphi),$$

we can write the correlation between two distinct modal pairs as the product of corresponding modal correlations at the transmitter and the receiver regions

$$\gamma_{m,m'} = \gamma_{n,n'} \gamma_{m,m'},$$

where $\gamma_{n,n'}$ is the correlation between $n$-th and $n'$-th modes at the receiver region given by (3.15) and $\gamma_{m,m'}$ is the correlation between $m$-th and $m'$-th modes at the transmitter region given by (3.16). Condition (5.21) also yields that:

- modal correlation at the transmitter $\gamma_{m,m'}$ is independent of the mode selected from the receiver region and
- modal correlation at the receiver $\gamma_{n,n'}$ is independent of the mode selected from the transmitter region.

In the current problem, we assumed that modes at the receiver region are uncorrelated, i.e., $\gamma_{n,n'} = 0$ for $n \neq n'$ and $\gamma_{n,n'} = 1$ for $n = n'$. Thus, applying (5.22) on (5.18) gives the correlation between $m$-th and $m'$-th modes at the transmit region due to the $r$-th receiver as

$$\{ \tilde{R}_H \}_{m,m'} = \gamma_{m,m'} \sum_{n=-M_R}^{M_R} |J_n(v_r)|^2.$$

From Gegenbauer’s Addition Theorem [137, page 363] we have

$$\lim_{M_R \to \infty} \sum_{n=-M_R}^{M_R} |J_n(v_r)|^2 = 1,$$

then $\{ \tilde{R}_H \}_{m,m'}$ simplifies to

$$\{ \tilde{R}_H \}_{m,m'} = \int_{\mathbb{S}^1} P_{\text{Tx}}(\phi) e^{i(m-m')\phi} d\phi.$$

Eq. (5.24) suggests that when the modes at the receiver region are uncorrelated, the correlation between different modes at the transmitter is independent of the
receive antenna selected from the receiver array. Note that, $P_{Tx}(\phi)$ can be modeled using all common azimuth power distributions such as Uniform, Laplacian, Gaussian, von-Mises, discussed in Chapter 2.

Figures 5.5 and 5.6 show the capacity performance of the two antenna configurations considered in Section 5.4.2 for a uniform limited azimuth power distribution at the transmitter for various angular spreads $\sigma = \{104^\circ, 30^\circ, 15^\circ, 5^\circ\}$ at the transmitter about the mean AOD $\phi_0 = 0^\circ$. Note that $\sigma = 104^\circ$ represents isotropic scattering at the transmitter for the uniform limited azimuth power distribution.

![Graph showing capacity comparison between spatial precoding and equal power loading schemes.](image)

Figure 5.5: Capacity comparison between spatial precoding and equal power loading schemes for a uniform limited scattering distribution at the transmitter with mean AOD $\phi_0 = 0^\circ$ and angular spreads $\sigma = \{104^\circ, 30^\circ, 15^\circ, 5^\circ\}$, for UCA transmit antenna configurations with transmitter aperture radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas ($r_R \to \infty$), for increasing number of transmit antennas.

From Figures 5.5 and 5.6 it can be observed that for $n_T > n_{sat}$, linear spatial precoding scheme, based on fixed parameters of underlying MIMO channel, provides significant capacity gains compared to the equal power loading scheme, in the presence of non-isotropic scattering at the transmitter. Furthermore, we observe
5.4 Optimization Problem Setup: Isotropic Scattering

Figure 5.6: Capacity comparison between spatial precoding and equal power loading schemes for a uniform limited scattering distribution at the transmitter with mean AOD $\phi_0 = 0^\circ$ and angular spreads $\sigma = \{104^\circ, 30^\circ, 15^\circ, 5^\circ\}$, for ULA transmit antenna configurations with transmitter aperture radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas ($r_R \to \infty$), for increasing number of transmit antennas.

that with the UCA antenna configuration, capacity performance of the spatial precoding scheme does not reach the maximum achievable capacity $C_{\text{max}}(r_T)$ when the angular spread of the APD is small.

To further illustrate the effects of angular spread and mean AOD, we consider the capacity performance of both power loading schemes with increasing angular spread about the mean AOD $\phi_0 = \{0^\circ, 90^\circ\}$. Figures 5.7 and 5.8 show the capacity performance of UCA antenna configuration for $\phi_0 = 0^\circ$ and $90^\circ$, respectively. For a given $n_T$, comparing Figures 5.7 and 5.8, we can observe that the capacity of UCA antenna configuration is independent of the mean AOD for both power loading schemes. Therefore, UCA antenna configuration is less sensitive to the variation of mean AOD. Furthermore it is observed that for $n_T > n_{\text{sat}}$, the capacity of the spatially precoded UCA system is increased with increasing number of transmit antennas.
Figure 5.7: Capacity comparison between spatial precoding and equal power loading schemes for a uniform limited scattering distribution at the transmitter with mean AOD $\phi_0 = 0^\circ$ and increasing angular spread, for UCA transmit antenna configurations with transmitter aperture radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas ($r_R \rightarrow \infty$), for $n_T = \{10, 11, 25, 60, 80\}$ transmit antennas.

In Figures 5.9 and 5.10 the capacity performance of ULA antenna configuration is shown for $\phi_0 = 0^\circ$ and $90^\circ$ (broadside), respectively. It is observed that the capacity of both power loading schemes is decreased as the mean DOA moves away from the broadside angle for all angular spreads, except at isotropic scattering. Furthermore, as the mean AOD moves towards the broadside angle, a saturation in capacity is observed with increasing angular spread. For example, when the mean AOD $\phi_0 = 90^\circ$ (Figure 5.10), the capacity of both power loading schemes is saturated for angular spread values $\sigma > 50^\circ$. In contrast to the UCA antenna configuration, the capacity performance of the equal power loading scheme is de-
creased as the number of antennas in the transmit region is increased for a given angular spread. Therefore, with equal power loading scheme, by increasing the number of antennas in the transmit region beyond $n_{\text{sat}}$ will decrease the capacity performance of 1-D arrays when the scattering environment around the transmit array is non-isotropic. In contrast to the equal power loading scheme, it is observed that spatially precoded ULA system provides capacity improvements as the number of antennas in the transmit region is increased.

Figure 5.8: Capacity comparison between spatial precoding and equal power loading schemes for a uniform limited scattering distribution at the transmitter with mean AOD $\phi_0 = 90^\circ$ and increasing angular spread, for UCA transmit antenna configurations with transmitter aperture radius $r_T = 0.5 \lambda$ and a large number of uncorrelated receive antennas ($r_R \to \infty$), for $n_T = \{10, 11, 25, 60, 80\}$ transmit antennas.
5.5 Optimum Power Loading in Non-isotropic Scattering Environments

Thus far we have seen that compared to the equal power loading scheme, the spatial precoding scheme designed based on antenna spacing and antenna placement details provides significant capacity improvements on spatially constrained dense MIMO systems in isotropic scattering environments. In this section, to further improve the capacity performance of such MIMO systems, we incorporate the second order statistics of the scattering channel (modal correlation) to derive a second power loading scheme (precoding scheme) that reduces the effects of non-ideal antenna placement and non-isotropic scattering at the transmitter.
In this case, to obtain the optimal power loading scheme, we maximize the deterministic capacity (5.6)

\[ C = \log \left| \mathbf{I}_{n_T} + \mathbf{Q} \mathbf{J}_T \mathbf{R}_{\tilde{H}} \mathbf{J}_T^\dagger \right|, \quad (5.25) \]

subject to the power constraints considered previously in optimization problem (5.9). Unlike in the previous scheme, this scheme requires the knowledge of the transmit modal correlation matrix \( \mathbf{R}_{\tilde{H}} \) to be available at the transmitter via feedback from the receiver. As we showed in Chapter 2, all uni-modal azimuth power distributions give very similar modal correlation values for a given angular spread about a mean AOD. Therefore the transmitter only requires the knowledge of \( \sigma \) and \( \phi_0 \) in order to build \( \mathbf{R}_{\tilde{H}} \) using (5.24).
Writing $J_T R_H J_T^\dagger$ as the eigen-decomposition $J_T R_H J_T^\dagger = U_H^\dagger T_H U_H^\dagger$ and taking $\tilde{Q} = U_H^\dagger Q U_H$, then the objective function (5.25) becomes

$$C = \log \left| I_{n_T} + \tilde{Q} T_H \right|.$$  \hspace{1cm} (5.26)

Using Section 5.4.1, it can be shown that the optimization problem in this case is identical to (5.9) with optimum solution given by (5.13), where in this case, $t_i = [T_H]_{i,i}$, and the optimum precoder $F$ is given by

$$F = \sqrt{P_T / n_T} U_H \tilde{Q}^{1/2} U_n^\dagger,$$

with arbitrary unitary matrix $U_n$ and diagonal $\tilde{Q}$.

### 5.5.1 Numerical Results

We now compare the capacity performance of different power loading schemes considered thus far. This allows us to study the effectiveness of CSI feedback on the capacity performance of dense antenna arrays.

Figures 5.11 and 5.12 show the capacity performance for increasing angular spread about the mean AOD $\phi_0 = 45^\circ$ and $n_T = \{11, 25, 80, 90\}$ transmit antennas placed in UCA and ULA configurations within spatial regions of radius $r_T = 0.5 \lambda$, for SNR = 10dB. In these plots, scheme $-1$ refers to the power loading based on antenna placement information and scheme $-2$ refers to the power loading based on antenna placement and scattering distribution information.

From Figures 5.11 and 5.12 it is observed that at small angular spread values the capacity achieved from scheme $-2$ is significant compared to that of scheme $-1$, for both antenna configurations. With scheme $-2$, a linear growth in capacity is observed in the range $0 < \sigma \lesssim 10^\circ$. Thereafter, a logarithmic growth in capacity is observed with the increase in angular spread. However, for both antenna configurations, a saturation in capacity is seen at higher angular spread values. For UCA, saturation occurs when the angular spread at the transmitter is approximately $100^\circ$ (close to isotropic scattering) and for ULA, the saturation occurs when $\sigma \approx 75^\circ$. Furthermore, we observe that this saturation point is independent of the number of antennas in the transmit array for $n_T > n_{sat}$.

With the first two power loading schemes we have seen that capacity of the UCA antenna system is unaffected by the variation of mean AOD. Similar results are observed with the scheme $-2$ applied on UCA antenna systems (results are not shown here). Figure 5.13 shows the capacity performance of scheme $-2$ along
5.5 Optimum Power Loading in Non-isotropic Scattering Environments

Figure 5.11: Capacity of different power loading schemes versus angular spread about the mean AOD $\phi_0 = 45^\circ$ at the transmitter for $n_T$ transmit antennas placed in a UCA within a spatial region of radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas ($r_R \to \infty$): (a) $n_T = 11$, (b) $n_T = 25$, (c) $n_T = 80$ and (d) $n_T = 90$.

with other two schemes applied on ULA antenna systems with $n_T = 11$ for mean AODs $\phi_0 = \{0^\circ, 30^\circ, 60^\circ, 90^\circ\}$. All four cases show no capacity growth for angular spreads greater than approximately $95^\circ$, $84^\circ$, $70^\circ$ and $50^\circ$ for mean AODs $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$, respectively. At saturation point ($\sigma_{sat}$) and there after, we observe that capacity given by scheme $-2$ is identical to that of scheme $-1$ for all four cases. This observation suggests that when $\sigma > \sigma_{sat}$, scheme $-2$ will completely eliminate the detrimental effects due to non-isotropic scattering at the transmitter, and the capacity saturation seen is due to the limited size of the transmit region. More interestingly, these results also reveal that when $\sigma > \sigma_{sat}$, exploitation of scattering distribution information does not give any capacity benefits compared to the scheme $-1$ applied on 1-D antenna arrays. Hence, in such cases, we can avoid
Figure 5.12: Capacity of different power loading schemes versus angular spread about the mean AOD $\phi_0 = 45^\circ$ at the transmitter for $n_T$ transmit antennas placed in a ULA within a spatial region of radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas ($r_R \to \infty$): (a) $n_T = 11$, (b) $n_T = 25$, (c) $n_T = 80$ and (d) $n_T = 90$.

the use of feedback, instead can use the spatial precoding scheme based on antenna placement information to achieve optimum capacity from the given region of space. However, with UCA systems, we can see scheme $-2$ provides extra capacity benefits over the scheme $-1$ for all angular spreads. Therefore, with UCA systems (2-D antenna arrays), the accurate feedback of scattering distribution information helps to improve the capacity performance of dense MIMO systems significantly for all angular spread values.

5.6 Power Loading Based on Mode Nulling

In this section we propose another power loading scheme which provides significant capacity improvements of dense MIMO arrays at small angular spreads. First we discuss the motivation behind the proposed scheme.
Figure 5.13: Capacity of ULA antenna systems versus angular spread about the mean AODs $\phi_0 = \{0^\circ, 30^\circ, 60^\circ, 90^\circ\}$ at the transmitter for 11 transmit antennas placed within a spatial region of radius $r_T = 0.5\lambda$ and a large number of uncorrelated receive antennas ($r_R \to \infty$): (a) $\phi_0 = 0^\circ$, (b) $\phi_0 = 30^\circ$, (c) $\phi_0 = 60^\circ$ and (d) $\phi_0 = 90^\circ$.

5.6.1 Modal Correlation at the Transmitter

Figure 5.14 shows the modal correlation at the transmitter region for a uniform limited azimuth power distribution with mean AOD $\phi_0 = 0^\circ$. Note that the transmitter modal correlation coefficients $\gamma_{m,m'}$ are calculated from

\[
\gamma_{m,m'} = \gamma_{m-m'} = \text{sinc}((m - m')\Delta)e^{i(m-m')\phi_0},
\]

where $\Delta$ is the non-isotropic parameter of the azimuth power distribution, which is related to the angular spread $\sigma = \Delta/\sqrt{3}$. From Figure 5.14 it is observed that the modal correlation decreases as the non-isotropic parameter increases. Also we observe a rapid reduction of modal correlation for well separated mode orders, e.g. for large $m - m'$. More importantly, we can observe that adjacent modes contribute...
to higher correlation than well separated mode orders, e.g. \( \gamma_1 \) for \( m - m' = 1 \). Therefore the goal is to eliminate the correlation between adjacent modes by allocating zero power to every second effective mode at the transmitter region.

![Modal correlation vs non-isotropic parameter \( \Delta \) of a uniform limited azimuth power distribution at the transmitter region for a mean AOD \( \phi_0 = 0^\circ \).](image)

**5.6.2 Optimum Power Loading Scheme**

Recall from Section 5.4.1, the svd of \( J_T = U_T \Lambda_T V_T^\dagger \) and \( Q = \mathcal{E} \{ xx^\dagger \} \). Using (5.1) and the channel decomposition (3.10), the received signal vector at the \( 2M_T + 1 \) effective transmit modes can be written as \( z = J_T^\dagger x \) and the covariance matrix of \( z \) is given by

\[
M_P = \mathcal{E} \{ zz^\dagger \},
\]
\[
= J_T^\dagger Q J_T,
\]
\[
= V_T \Lambda_T^\dagger U_T^\dagger Q U_T \Lambda_T V_T^\dagger,
\]
\[
= V_T \Lambda_T^\dagger Q \Lambda_T V_T^\dagger,
\]
where \( \widetilde{Q} = U_T^\dagger QU_T \). Note that the \((m, m)\)-th diagonal element of \( M_T \) gives the average power allocated to the \( m \)-th mode in the transmitter region.

Now, based on the mode nulling concept discussed above and also using the optimization problem (5.8) that was developed for precoding based on \( J_T \) (scheme - 1), we can write the new optimization problem as

\[
\min \quad -\log \left| I_{nt} + \widetilde{Q}T \right|
\]

subject to \( \widetilde{Q} \succeq 0 \),

\[
\text{tr}\{\widetilde{Q}\} = P_T, \quad \text{tr}\{\widetilde{Q}T\} = P_T,
\]

\[
\left[ V_T\Lambda_T^\dagger \widetilde{Q}\Lambda_T V_T^\dagger \right]_{m,m} = 0,
\]

(5.28)

where \( T = \Lambda_T \Lambda_T^\dagger \) and \( m \in [1, \cdots, 2M_T + 1] \) is the transmit mode (or modes) subject to power nulling. The closed form solution to this problem is unknown. However, we can find the solution for \( \widetilde{Q} \) by numerical methods such as Sequential Quadratic Programming [138]. Results of the numerical scheme are provided in Section 5.6.3 for several spatial scenarios.

Similar to the scheme - 1, this new power loading scheme is also fixed for fixed antenna placement and it does not require any feedback of CSI from the receiver. In what follows we will refer to this new power-loading scheme as scheme - 3.

### 5.6.3 Numerical Results

We now illustrate the capacity benefits obtained by applying the scheme - 3 on spatially constrained antenna arrays. We consider a MIMO system with \( n_T = \{4, 5\} \) transmit antennas constrained within a scatter-free circular region of radius \( r_T = 0.25\lambda \), corresponding to 7 effective modes at the transmitter region, and a large number of uncorrelated receive antennas for a total power budget of \( P_T = 10\text{dB} \). As before, transmit antennas are placed in UCA and ULA configurations. Figures 5.15 and 5.16 shows the capacity comparison between power-loading based on scheme - 1 and scheme - 3 for a uniform limited azimuth power distribution at the transmitter with mean AOD \( \phi_0 = 0^\circ \) for 4 and 5 transmit antennas, respectively. For scheme - 3, we set the 4-th element of the diagonal of \( V_T\Lambda_T^\dagger \widetilde{Q}\Lambda_T V_T^\dagger \) in (5.28) to zero, i.e., allocate zero power to the 0-th mode at the transmitter region.

From Figures 5.15 and 5.16 it is observed that scheme - 3 provides significant capacity improvements at small angular spread values, in particularly for the ULA transmit antenna configuration. However, in comparison with the capacity performance of scheme - 1, we observe a capacity loss from scheme - 3 at high angular
spread values for both antenna configurations. For example, \textit{scheme} – 3 gives poor capacity performance when $\sigma > 55^\circ$ for UCA with 4-transmit antennas and $\sigma > 30^\circ$ for ULA with 4-transmit antennas. To further investigate this capacity loss, we now consider the power assignment to each mode at the transmitter region by \textit{scheme} – 1 and \textit{scheme} – 3.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure515.png}
\caption{Capacity comparison between power-loading \textit{scheme} – 1 and \textit{scheme} – 3 for a uniform limited azimuth power distribution at the transmitter with mean AOD $\phi_0 = 0^\circ$ for increasing angular spread: $n_T = 4$ transmit antennas.}
\end{figure}

Figures 5.17 and 5.18 show the average power assigned to the first 7 effective transmit modes for the case of $n_T = 5$ for UCA and ULA transmit antenna configurations, respectively. It is observed that with the ULA antenna configuration \textit{scheme} – 3 allocates considerable amount of power on to the transmit mode set $\{-3, -1, 1, 3\}$ and almost zero power to the rest of the available transmit modes. However, in the case of UCA, \textit{scheme} – 3 allocates considerable amount of power on to the transmit mode set $\{-3, -2, -1, 1, 2, 3\}$ and almost zero power to the 0-th mode. Therefore, at high angular spread values, \textit{scheme} – 3 does not utilize the full set of uncorrelated (or near uncorrelated) modes available at the region.
for transmission. As a result, \textit{scheme} – 3 gives poor capacity performance at high angular spread values. It is further observed that for 1-D arrays (ULA) \textit{scheme} – 3 gives scope for improvement at low angular spread values, but for 2-D arrays (UCA) little capacity improvements are seen at low angular spread values. However, with the UCA, we observe some capacity improvements using \textit{scheme} – 3 for moderate angular spread values as it utilizes a larger set of transmit modes for transmission with the UCA than the ULA.
Figure 5.17: Average power allocated to each effective transmit mode in a circular aperture of radius $0.25\lambda$. $P_T = 10\text{dB}$: UCA antenna configuration, $n_T = 5$ transmit antennas.

### 5.7 Summary and Contributions

The pioneer work by Telatar and independently by Foshini and Gans has shown that when the wireless fading channels are statistically independent and known at the receiver, the information theoretic capacity of wireless fading channels increases linearly with the smaller of the number of transmit and receive antennas. However, in reality by increasing the number of antennas in a fixed region of space will limit the channel capacity due to the increase in spatial correlation between antenna elements. Recently it was shown that due to the non-ideal antenna placement the capacity achieved from a spatially constrained dense antenna array is always lower than the theoretical maximum capacity available from the same region. In contrast, in this chapter we showed through simulation that the theoretical maximum capacity for a fixed region of space can be achieved via linear spatial precoding, which eliminates the detrimental effects of non-ideal antenna placement.

Some specific contributions made in this chapter are:
1. A fixed power loading scheme (or a linear precoding scheme) is proposed by considering a spatial dimension of a MIMO channel, assuming a isotropic scattering environment. The proposed power loading scheme eliminates the detrimental effects of non-ideal antenna placement and improves the capacity performance of spatially constrained dense MIMO systems. The design is based on readily available antenna configuration details (antenna spacing and placement), therefore the precoder is fixed and transmitter does not require any feedback of channel state information from the receiver.

2. For a large number of transmit antennas, we numerically showed that unlike the equal power loading scheme the proposed scheme has the potential of achieving the theoretical maximum capacity available for a fixed region of space.

3. It is shown that spatial precoding can provide significant capacity gains by adding two to three more antennas in to the fixed region than the number...
which saturates the equal power loading scheme.

4. A second precoding scheme is proposed which exploits the non-isotropic scattering distribution parameters at the transmitter to improve the capacity performance of dense MIMO systems in non-isotropic scattering environments. This scheme requires the receiver to estimate the scattering distribution parameters at the transmitter and feed them back to the transmitter periodically.

5. It is shown that accurate feedback of scattering distribution parameters always helps to improve the capacity of 2-D antenna arrays for all angular spread values while 1-D antenna arrays do not provide capacity improvements for large angular spread values, suggesting the use of feedback free first scheme.

6. Analyzed the correlation between different modal orders generated at the transmitter region due to spatially constrained antenna arrays. It is shown that adjacent modes contribute to higher modal correlation at the transmitter region.

7. A third power loading scheme is proposed which reduces the effects of correlation between adjacent modes at the transmitter region by nulling power onto adjacent transmit modes. Similar to the first fixed scheme, this scheme is also fixed for a given fixed antenna configuration and it does not require any feedback of CSI from the receiver. This scheme gives capacity improvements only at small angular spread values and it suffers capacity loss at higher angular spread values as the scheme does not utilize full set of near uncorrelated transmit modes available for transmission at higher angular spread values.

8. It is shown that the third power loading scheme gives scope for capacity improvements of 1-D arrays at low angular spread values than for 2-D arrays.
Chapter 6

Space-Time Channel Modelling in General Scattering Environments

6.1 Introduction

Wireless channel modelling has received much attention in recent years since space-time processing using multiple antennas is becoming one of the most promising areas for improvements in performance of mobile communication systems [5,8]. Major challenges facing MIMO system researchers are to understand the characteristics of wireless channel and to develop realistic channel models that can efficiently and accurately predict the performance of a wireless system. The wireless channel is distinct and much more unpredictable than the wired channel because of factors such as multipath, mobility of the user, mobility of the objects in the environment and delays arising from multipaths. Multipath is a phenomenon that occurs as a transmitted signal is reflected or diffracted by objects in the environment or refracted through the medium between the transmitter and the receiver [14]. The net effect of these reflection, diffraction, and refraction on the transmitted signal is attenuation, phase change and delay, collectively called fading. Another important property of wireless channels is the presence of Doppler shifts, which are caused by the motion of the transmitter, the receiver, and any other objects in the channel environment. Fading is usually divided into fading based on multipath delay spread and Doppler spread. There are two types of fading based on multipath delay spread: flat fading and frequency-selective fading, and two types of fading based on Doppler spread: fast fading and slow fading [1]. The fading based on Doppler spread is also known as time-selective fading.

Several time-selective fading channel models have been proposed in the literature. However, most of these channel models have one or more of the following
limitations:

- the directions of arrival of multipaths are assumed to be uniformly distributed in a circle, e.g., [78, 79, 86];
- particular scattering distribution, e.g., [79, 80, 85];
- particular antenna array geometry, e.g., [79, 81, 84, 139];
- co-located antennas, e.g., [82];
- a single cluster\(^1\) of far-field scatterers, e.g., [79, 80, 85, 86, 140];
- rely on measured data and object databases, e.g., [72, 74, 83, 87–89, 92, 141–144];
- signal arrival angles at the receiver are independent\(^2\) from the signal departure angles at the transmitter, e.g., [67–69, 85, 90, 93].

In this chapter, a general space-time channel model for down-link transmission in a mobile multiple antenna communication system is developed to overcome most of the limitations described above. The proposed space-time channel model is derived based on the underlying physics of the free space propagation theory together with appropriate parameterizations. The model incorporates deterministic quantities such as physical antenna positions and the motion of the mobile unit (velocity and the direction), and random quantities to capture random scattering environment modeled using a bi-angular power distribution and, in the simplest case, the covariance between transmit and receive angles which captures statistical interdependency.

### 6.2 Space-Time Channel Model

We consider a down-link MIMO transmission system, where the Base Station (BS) consists of \(n_T\) transmit antennas and the Mobile Unit (MU) consists of \(n_R\) receive antennas. Suppose \(n_T\) transmit antennas are located at positions \(\mathbf{x}_q\), \(q = 1, 2, \ldots, n_T\) relative to a transmitter array origin, and \(n_R\) receive antennas are located at positions \(\mathbf{z}_p\), \(p = 1, 2, \ldots, n_R\) relative to a receiver array origin. Quantities \(r_T \geq \max \|\mathbf{x}_q\|\) and \(r_R \geq \max \|\mathbf{z}_p\|\) denote the radius of spheres that contain all the transmit and receive antennas, respectively. Scatterers are assumed

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\(^1\)leads to a uni-modal power distribution around the antenna array.

\(^2\)leads to the well known Kronecker model.
6.2 Space-Time Channel Model

to be in the far-field of the transmitter and receiver regions. Therefore, we define scatter-free transmitter and receiver spheres of radius \( r_{TS} > r_T \) and \( r_{RS} > r_R \), respectively.

Let \( s = [s_1, s_2, \cdots, s_{n_T}]^T \) be the column vector of baseband transmitted signals from \( n_T \) transmit antennas transmitted over a single symbol interval. Then the signal leaving the scatter-free transmit aperture along direction \( \hat{\phi} \) is given by

\[
\Phi(\hat{\phi}) = \sum_{q=1}^{n_T} s_q e^{ikx_q \cdot \hat{\phi}},
\]

where \( k = 2\pi/\lambda \) is the wave number with \( \lambda \) the wave length. The signal entering scatter-free receiver aperture from direction \( \hat{\phi} \) can be written as

\[
\Psi(\hat{\phi}) = \int_{S} \Phi(\hat{\phi}) g(\hat{\phi}, \hat{\varphi}) d\hat{\phi} = \sum_{q=1}^{n_T} s_q \int_{S} g(\hat{\phi}, \hat{\varphi}) e^{ikx_q \cdot \hat{\varphi}} d\hat{\phi},
\]

where \( g(\hat{\phi}, \hat{\varphi}) \) is the effective random scattering gain function for a signal leaving the transmitter scatter-free aperture at a direction \( \hat{\phi} \) (relative to the transmitter array origin) and entering the receiver scatter-free aperture from a direction \( \hat{\varphi} \) (relative to the receiver array origin). Note that \( S \) is the unit sphere in the case of a 3-dimensional multipath environment or unit circle in the 2-dimensional case.

Suppose the MU is moving at constant velocity \( \upsilon \) in the direction of \( \hat{\upsilon} \). At time \( t \), the received signal \( y_p \) at the \( p \)-th receive antenna at position \( z_p \) is given by

\[
y_p(t) = \int_{S} \Psi(\hat{\varphi}) e^{i\omega_d(\hat{\varphi}) t} e^{-ikz_p \cdot \hat{\varphi}} d\hat{\varphi} + n_p,
\]

\[
= \sum_{q=1}^{n_T} s_q \int_{S} g(\hat{\phi}, \hat{\varphi}) e^{i\omega_d(t)} e^{-ik(z_p \cdot \hat{\varphi})} d\hat{\varphi} + n_p,
\]

where \( \omega_d(\hat{\varphi}) = 2\pi/\lambda\hat{\upsilon} \cdot \hat{\varphi} \) is the angular Doppler spread,

\[
u_p(t) = z_p - t\upsilon \hat{\upsilon},
\]

is the position of the \( p \)-th receive antenna at time \( t \) and \( n_p \) is the additive white Gaussian noise at the \( p \)-th receive antenna. Let \( y(t) = [y_1(t), y_2(t), \cdots, y_{n_R}(t)]^T \) and \( n = [n_1, n_2, \cdots, n_{n_R}]^T \), then (6.3) can be written in vector form as

\[
y(t) = H(t)s + n,
\]
where $H(t)$ represents the $n_R \times n_T$ channel matrix at time $t$, with $(p, q)$-th element

$$h_{p,q}(t) = \int \int_{S \times S} g(\hat{\phi}, \hat{\varphi}) e^{ikx_q \cdot \hat{\phi}} e^{-iku_{p}(t) \cdot \hat{\varphi}} d\hat{\phi} d\hat{\varphi}, \quad (6.5)$$

representing the complex channel gain between the $p$-th receive antenna and the $q$-th transmit antenna at time $t$.

Figure 6.1: General scattering model for a down-link MIMO communication system. $r_T$ and $r_R$ are the radius of spheres which enclose the transmitter and the receiver antennas, respectively. We demonstrate the generality of the model by showing three sample scatterers $S_1, S_2$ and $S_3$ which show a single bounce (reflection off $S_2$), multiple bounces (sequential reflection off $S_2$ and $S_3$), and wave splitting (with divergence at $S_2$), and also a direct path.

Equation (6.5) subsumes the Double Directional Channel Model (DDCM) [142], where the channel is expressed in terms of only $L$ of propagation paths:

$$h_{p,q}(t) = \sum_{\ell=1}^{L} g_{\ell} e^{ikx_q \cdot \hat{\phi}_{\ell}} e^{-iku_{p}(t) \cdot \hat{\varphi}_{\ell}} \quad (6.6)$$

where $g_{\ell} = g(\hat{\phi}_{\ell}, \hat{\varphi}_{\ell})$ is the gain for the multipath propagating out of the transmitter aperture in direction $\hat{\phi}_{\ell}$ and into the receiver aperture in direction $\hat{\varphi}_{\ell}$. As can be seen from Figure 6.1 the DDCM, which is a specific case of the general model with $g(\hat{\phi}, \hat{\varphi}) = \sum_{\ell=1}^{L} g_{\ell} \delta(\hat{\phi} - \hat{\phi}_{\ell}) \delta(\hat{\varphi} - \hat{\varphi}_{\ell})$, accommodates wave phenomena such as single bounces, multiple bounces, wave splits, and the direct (unscattered) path.

Consider when the multipath is restricted to the azimuth plane only (2-D scattering). A 3-D version of this space-time channel model can be derived by expanding the plane wave.
terring environment, having no field components arriving at significant elevations. The modal expansion of plane wave $e^{iky\hat{\phi}}$ is given by [113, page 67]

\[ e^{iky\hat{\phi}} = \sum_{m=-\infty}^{\infty} J_m(k\|y\|)e^{-im(\phi_y-\pi/2)}e^{im\phi}, \quad (6.7) \]

where $\theta$ denotes the angle between $y$ and $\hat{\phi}$, $J_m(\cdot)$ is the integer order $m$ Bessel function, $y \equiv (\|y\|, \phi_y)$ and $\hat{\phi} \equiv (1, \phi)$ in polar coordinates.

Bessel functions $J_m(\cdot)$ for $|m| > 0$ exhibit a spatially high pass character ($J_0(\cdot)$ is spatially low pass), that is, for fixed order $m$, $J_m(\cdot)$ starts small and reaches to its maximum at arguments $x \approx O(m)$ before starts decaying slowly. It was shown in [114] that $J_m(k\|y\|) \approx 0$ for $|m| > ke\|y\|/2$ with $e = 2.7183\ldots$. Therefore, we can truncate the series (6.7) to $2\lceil ke\|y\|/2 \rceil + 1$ terms.

Using the truncated modal expansion of plane wave $e^{iky\hat{\phi}}$ we can write

\[ e^{ikx_q\hat{\phi}} = \sum_{m=-M_T}^{M_T} J_m(k\|x_q\|)e^{-im(\phi_q-\pi/2)}e^{im\phi}, \quad (6.8) \]

where $x_q \equiv (\|x_q\|, \phi_q) = (x_q, \phi_q)$ in the transmitter polar coordinates and $M_T = \lceil ker_T/2 \rceil$ the transmitter region dimensionality with $r_T \geq \max \|x_q\|$. Similarly,

\[ e^{-iku_p(t)\hat{\phi}} = \sum_{n=-M_R}^{M_R} J_n(k\|u_p(t)\|)e^{in(\varphi_p(t)-\pi/2)}e^{-in\varphi}, \quad (6.9) \]

where $u_p(t) \equiv (\|u_p(t)\|, \varphi_p(t)) = (u_p(t), \varphi_p(t))$ in the receiver polar coordinates and $M_R = \lceil keR(t)/2 \rceil$ the receiver region dimensionality with $d_R(t) \geq \max \|z_p - tv\hat{u}\|$ the maximum receiver antenna distance from the origin at time $t$.

By substituting (6.8) and (6.9) into (6.5), we can decompose the space-time MIMO channel $H(t)$ as

\[ H(t) = J_R(t)H_sJ_T^\dagger, \quad (6.10) \]

where $J_T$ is the $n_T \times (2M_T + 1)$ deterministic transmitter configuration matrix,

\[ J_T = \begin{bmatrix} J_{-M_T}(x_1) & \cdots & J_{M_T}(x_1) \\ \vdots & \ddots & \vdots \\ J_{-M_T}(x_{n_T}) & \cdots & J_{M_T}(x_{n_T}) \end{bmatrix}, \quad (6.11) \]

in 3-D space using spherical harmonics on a sphere [113].
\( J_R(t) \) is the \( n_R \times (2M_R + 1) \) deterministic receiver configuration matrix at time \( t \),

\[
J_R(t) = \begin{bmatrix}
J_{-M_R}(u_1(t)) & \cdots & J_{M_R}(u_1(t)) \\
\vdots & \ddots & \vdots \\
J_{-M_R}(u_{n_R}(t)) & \cdots & J_{M_R}(u_{n_R}(t))
\end{bmatrix},
\]

(6.12)

with

\[
J_n(y) \triangleq J_n(k\|y\|)e^{in(\phi_y - \pi/2)},
\]

(6.13)

and \( H_s \) is a \((2M_R + 1) \times (2M_T + 1)\) scattering channel matrix with \((\ell, m)\)-th element given by

\[
\{H_s\}_{\ell,m} = \int \int_{S \times S} g(\phi, \varphi) e^{-i(\ell-M_R-1)\phi} e^{i(m-M_T-1)\varphi} d\varphi d\phi.
\]

(6.14)

Some remarks regarding the down-link\(^4\) space-time channel model given by (6.10):

1. Decomposition (6.10) separates the channel into deterministic and random parts.

2. The transmitter configuration matrix \( J_T \) captures the physical configuration of the transmitter antenna array (antenna positions and orientation relative to the transmitter origin) and it is fixed for a given transmitter antenna array geometry.

3. The receiver configuration matrix \( J_R(t) \) captures the physical configuration of the receiver antenna array and the time-varying nature of the channel (velocity and the direction of motion). \( J_R(t) \) is deterministic for given receiver antenna array geometry and receiver movement information.

4. \( H_s \) represents the complex scattering environment between the transmit and the receive antenna apertures. For a random scattering environment, \( \{H_s\}_{\ell,m} \) are random variables, and for an isotropic scattering environment, \( \{H_s\}_{\ell,m} \) are independent of each other.

5. The size of \( H_s \) is determined by the number of effective communication modes excited by the antenna arrays at the receiver and transmitter regions. The number of communication modes at the transmitter is determined by the size

\(^4\)Following the wave propagation approach presented in this work, a space-time channel model for up-link transmission can be easily derived.
of the transmitter region $r_T = \max \|\mathbf{x}_q\|$ for $q = 1, \ldots, n_T$. At the receiver side, it is determined by the maximum length of the vector $\mathbf{z}_p - tv\hat{\mathbf{u}}$, i.e., $d_R(t) = \max \|\mathbf{z}_p - tv\hat{\mathbf{u}}\|$ for $p = 1, \ldots, n_R$. Since $d_R(t)$ changes with time, the number of effective communication modes at the receiver region changes with time. Thus, the size of $H_s$ and $J_R(t)$ also change with time (but are bounded given bounded velocity).

6. When $\tau = 0$ or the MU is stationary, the channel decomposition (6.10) simplifies to the spatial decomposition given in [106].

7. The proposed channel model allows investigation of the individual effects of antenna spacing, antenna placement (linear array, circular array, grid array, etc.), movement and non-isotropic scattering on the performance of MIMO communication systems. This flexibility and breadth gives the proposed model advantages over other space-time channel models proposed in the literature.

### 6.3 Space-Time and Space-Frequency Channel Correlation in General Scattering Environments

In this section, we quantify the correlation properties of a MIMO channel in a general (random) scattering environment. The covariance matrix of the MIMO channel $\mathbf{H}(t)$ can be defined as

$$R_H(\tau) \triangleq \mathbb{E}\{\mathbf{h}(t)\mathbf{h}^\dagger(t - \tau)\},$$

where $\mathbf{h}(t) = \text{vec}\{\mathbf{H}(t)\}$. Each element of matrix $R_H(\tau)$ consists of a space-time cross correlation between the channel gains $h_{p,q}$ and $h_{p',q'}$ connecting two pairs of antennas:

$$\rho_{p,q'}(\tau) \triangleq \mathbb{E}\{h_{p,q}(t)h_{p',q'}^*(t - \tau)\}.$$

The related space-frequency cross spectrum is computed

$$S_{p,q'}(\omega) \triangleq \mathcal{F}\{\rho_{p,q'}(\tau)\} = \int_{-\infty}^{\infty} \rho_{p,q'}(\tau)e^{-j\omega\tau}d\tau.$$

Below, we derive expressions for the space-time and space-frequency correlations between the channel gains in any scattering environment, for when the MU is moving. These expressions are shown to subsume several popular correlation models.
in the recent literature, namely the Kronecker model [69], von Mises distributed scatterer model [145], Jake's spatial correlation model and Clarke's Doppler fading model [78]. In Section 6.5 these expressions are used to characterize space-time correlation properties in a wide range of scattering environments.

### 6.3.1 Space-Time Cross Correlation

From (6.5), the space-time cross correlation can be written as

\[
\rho_{q,q'}(\tau) = \int_{S \times S} \mathcal{E}\left\{ g(\hat{\phi}, \hat{\varphi}) g^*(\hat{\phi}', \hat{\varphi}') \right\} e^{ik(x_q - x_{q'} \cdot \hat{\phi})} e^{-ik(u_p(t) \cdot \hat{\phi} - u_{p'}(t - \tau) \cdot \hat{\varphi}')}
\]

\[
\times d\hat{\phi} d\hat{\varphi} d\hat{\phi}' d\hat{\varphi}',
\]

where we have introduced the shorthand \( \int_{S \times S} \equiv \int_{S} \int_{S} \).

Assuming a wide-sense stationary zero-mean uncorrelated scattering environment, the second-order statistics of the scattering gain function \( g(\hat{\phi}, \hat{\varphi}) \) can be defined as

\[
\mathcal{E}\left\{ g(\hat{\phi}, \hat{\varphi}) g^*(\hat{\phi}', \hat{\varphi}') \right\} \equiv G(\hat{\phi}, \hat{\varphi}) \delta(\hat{\phi} - \hat{\phi}') \delta(\hat{\varphi} - \hat{\varphi}'),
\]

where \( G(\hat{\phi}, \hat{\varphi}) = \mathcal{E}\left\{ |g(\hat{\phi}, \hat{\varphi})|^2 \right\} \) characterizes the joint power spectral density (PSD) surrounding the transmitter and receiver apertures. The correlation, \( \rho_{q,q'}^{p,p'} \), then simplifies to

\[
\rho_{q,q'}^{p,p'}(\tau) = \int_{S \times S} G(\hat{\phi}, \hat{\varphi}) e^{-iku_{p,p'}(\tau) \cdot \hat{\varphi}'} e^{ik(x_q - x_{q'} \cdot \hat{\phi})} d\hat{\phi} d\hat{\varphi}.
\]  

(6.17)

where \( u_{p,p'}(\tau) = z_p - z_{p'} + \tau \hat{v} \).

For 2-D scattering environments, applying (6.7) in (6.17) gives

\[
\rho_{q,q'}^{p,p'}(\tau) = \sum_{m_T=-m_T}^{m_T} \sum_{n_R=-m_R}^{m_R} \beta_m^n J_m(k \|x_q - x_{q'}\|) J_n(k \|u_{p,p'}(\tau)\|) 
\]

\[
\times e^{-im(\phi_q, \phi_{q'} - \pi/2)} e^{im(\varphi_{p,p'}(\tau) - \pi/2)},
\]

(6.18)

where \( x_q - x_{q'} \equiv (\|x_q - x_{q'}\|, \phi_q, \phi_{q'}) \) and \( u_{p,p'}(\tau) \equiv (\|z_p - z_{p'} + \tau \hat{v} \|, \varphi_{p,p'}(\tau)) \) in polar coordinates, \( m_T = \lceil ke \|x_q - x_{q'}\| / 2 \rceil, m_R = \lceil ke \|z_p - z_{p'} + \tau \hat{v} \| / 2 \rceil \) and the coefficients \( \beta_m^n \) characterize the 2-D scattering environment surrounding the
transmitter and receiver antenna apertures and are given by

\[ \beta_{m}^{n} = \int_{S} \int_{S} G(\phi, \varphi) e^{-im\phi} e^{im\varphi} d\varphi d\phi. \] (6.19)

Since the scattering gain function \( g(\phi, \varphi) \) is periodic in both \( \phi \) and \( \varphi \), the joint PSD \( G(\phi, \varphi) \) is also periodic in both \( \phi \) and \( \varphi \). Therefore, using the orthogonal circular harmonics\(^5\) as the basis set, \( G(\phi, \varphi) \) can be expanded in a 2-D Fourier series as

\[ G(\phi, \varphi) = \frac{1}{4\pi^2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \beta_{m}^{n} e^{-im\phi} e^{im\varphi}. \] (6.20)

Note that (6.19) and (6.20) form a Fourier transform pair. Also note that scattering coefficients \( \beta_{m}^{n} \) are independent of the speed \( \upsilon \) of MU. Hence \( \beta_{m}^{n} \) are invariant to Doppler effects and are fixed for a given scattering distribution type.

One can see from (6.17) that channel gains possess the same expected energy:

\[ \mathcal{E} \{ |h_{p,q}(t)|^2 \} = \rho_{p,p'}(0) = \int_{S} \int_{S} G(\hat{\phi}, \hat{\varphi}) d\hat{\phi} d\hat{\varphi}. \] (6.21)

Without loss of generality, we normalize the PSD so that \( \int_{S} \int_{S} G(\hat{\phi}, \hat{\varphi}) d\hat{\phi} d\hat{\varphi} = 1 \). The average energy of each channel gain then unity and each antenna correlation \( \rho_{p,p'}(\tau) \) represents a correlation coefficient.

**Space-Time Cross Correlation at the Receiver**

Using (6.17), the space-time cross correlation between \( p \)-th and \( p' \)-th receiver antennas due to the \( q \)-th transmitter antenna can be written as

\[ \rho_{p,q}^{p',q'}(\tau) \triangleq \rho_{q,q'}^{p,p'}(\tau) = \int_{S} P_{Rx}(\hat{\varphi}) e^{-iku_{p,p'}(\tau)\hat{\varphi}} d\hat{\varphi}, \quad \forall q, \] (6.22)

where \( P_{Rx}(\varphi) \) is the average power density of the scatterers surrounding the receiver region in each direction \( \hat{\varphi} \), given by the marginalized PSD

\[ P_{Rx}(\varphi) = \int_{S} G(\phi, \varphi) d\phi. \]

Here we see that correlation coefficients at the receiver is independent of the antenna selected from transmit antenna array. Also it is independent of the power distribution at the transmit antenna region.

\(^5\)Circular harmonics \( e^{in\theta} \) form a complete orthogonal function basis set on the unit circle \( S^1 \). Orthogonality is with respect to the inner product \( \langle f, g \rangle = \int_{S^1} f(\varphi) g^*(\varphi) d\varphi \).
Applying (6.7) on (6.22) gives
\[ \rho_{p,p'}(\tau) = \sum_{n=-m_R}^{m_R} \beta_n^p J_n(k\|u_{p,p'}(\tau)\|) e^{in(\varphi_{p,p'}(\tau)-\pi/2)}, \]  
(6.23)
where the coefficients \( \beta_n \) characterize the scattering environment surrounding the receiver antenna array and are given by
\[ \beta_n^p = \int_{S} P_{Rx}(\varphi) e^{-in\varphi} d\varphi. \]  
(6.24)

**Space-Time Cross Correlation at the Transmitter**

Similarly, the space-time cross correlation between \( q \)-th and \( q' \)-th transmitter antennas due to the \( p \)-th receiver antenna can be written as
\[ \rho_{p,p}^{q,q'}(\tau) \equiv \rho_{q,q'}(\tau) = \int_{S\times S} G(\hat{\phi}, \hat{\varphi}) e^{ik(x_q-x_{q'})\hat{\varphi}} e^{-ik\tau v\hat{\varphi}} d\hat{\phi} d\hat{\varphi}, \quad \forall \ p. \]  
(6.25)

As for the receiver channel correlation, we can observe that channel correlation at the transmitter is independent of the antenna selected from the receiver antenna array. However, due to the motion of the MU, the space-time cross correlation at the transmitter depends on the joint power distribution at the transmitter and the receiver apertures.

Applying (6.7) on (6.25) gives
\[ \rho_{q,q'}^{p,p}(\tau) = \sum_{m=-m_T}^{m_T} \sum_{n=-D_v}^{D_v} \beta_m^n J_m(k\|x_q-x_{q'}\|) J_n(k\tau v) \times e^{-im(\phi_{q,q'}-\pi/2)} e^{in(\varphi_{q,q'}-\pi/2)}, \]  
(6.26)
where \( \hat{\varphi} \equiv (1, \varphi_v) \) in polar coordinates, \( D_v = [ke\tau v/2] \) and the scattering coefficients \( \beta_m^n \) are given by (6.19).

### 6.3.2 Space-Frequency Cross Spectrum

To evaluate the space-frequency correlation in (6.16), we first expand the term \( J_n(k\|u_{p,p'}(\tau)\|) e^{in\varphi_{p,p'}(\tau)} \) in (6.18) as follows.

Recall \( u_{p,p'}(\tau) \equiv (\|\mathbf{z}_p - \mathbf{z}_{p'} + \tau v \hat{\mathbf{v}}\|, \varphi_{p,p'}(\tau)) \). Let \( \mathbf{z}_{p',p} = \mathbf{z}_p - \mathbf{z}_{p'} \equiv (\|\mathbf{z}_{p,p'}\|, \phi_x) \) and \( \tau v \hat{\mathbf{v}} \equiv (\tau v, \varphi_v) \) in polar coordinates, \( \theta_{p,p'} = \varphi_v - \phi_x \) with maximum angular Doppler spread \( \omega_D = vk \) (maximum Doppler frequency \( f_D = v/\lambda \)). By applying the summation theorem for Bessel functions [146, 8.530] on the argument of
$J_n(k\|u_{p,p'}(\tau))$, we can write

$$J_n(k\|u_{p,p'}(\tau))e^{in\varphi_{p,p'}(\tau)} = e^{in\phi_x}J_n(k\|u_{p,p'}(\tau))e^{in(\varphi_{p,p'}(\tau)-\phi_x)},$$

$$= e^{in\phi_x} \sum_{\ell=-\infty}^{\infty} J_\ell(\omega_D \tau) J_{n+\ell}(k\|z_{p,p'}\|) e^{-i\ell(\theta_{p,p'}-\pi)}. \quad (6.27)$$

Substituting (6.27) in (6.18) and then taking the Fourier transform with respect to $\tau$ yields

$$S_{p,p'}^{q,q'}(\omega) = \sum_{m=-m_T}^{m_T} \sum_{n=-m_R}^{m_R} e^{-im(\phi_{q,q'}-\pi/2)} e^{im(\phi_x-\pi/2)} \beta_m^n J_m(k\|x_q - x_{q'}\|)$$

$$\times \sum_{\ell=-\infty}^{\infty} e^{-i\ell(\theta_{p,p'}-\pi)} F_\ell(\omega) J_{n+\ell}(k\|z_{p,p'}\|). \quad (6.28)$$

where $F_\ell(\omega) = \mathcal{F}\{J_\ell(\omega_D \tau)\}$. Now for $|\omega| \leq \omega_D$, using a result found in [147, page 66],

$$F_\ell(\omega) = \begin{cases} 
\Lambda(\omega) \cos\{\ell \sin^{-1}(\frac{\omega}{\omega_D})\}, & \ell \geq 0; \\
(-1)^{\ell} \Lambda(\omega) \cos\{\ell \sin^{-1}(\frac{\omega}{\omega_D})\}, & \ell < 0,
\end{cases} \quad (6.29)$$

where $\Lambda(\omega) = (\omega_D^2 - \omega^2)^{-\frac{1}{2}}$ and scattering coefficients $\beta_m^n$ are given by (6.19).

The space-frequency cross spectrum at the receiver $S_{p,p'}^{q,q'}(\omega)$ can be derived by applying the expansion (6.27) on (6.23) and then taking the Fourier transform with respect to $\tau$. The space-frequency cross spectrum at the transmitter $S_{q,q'}(\omega)$ can be derived by directly taking the Fourier transform of (6.26) with respect to $\tau$. For brevity, here we omit the derivation of these expressions.

### 6.3.3 SISO Time-varying Channel: Temporal Correlation

We now recover the Clarke’s temporal correlation model from (6.18). By substituting $\|x_q - x_{q'}\| = 0$, $\phi_{q,q'} = 0$, $\|z_p - z_{p'}\| = 0$ and $\varphi_{p,p'}(\tau) = \varphi_v$ in (6.18), gives the temporal correlation of a signal received at the receiver as

$$\rho(\tau) = \sum_{n=-D_v}^{D_v} \beta_0^n J_n(k\tau\nu)e^{in(\varphi_v-\pi/2)}, \quad (6.30)$$

where $D_v = \lceil ke\nu/2 \rceil$ and

$$\beta_0^n = \iint_{S\times S} G(\phi, \varphi)e^{-in\varphi}d\varphi d\phi,$$
characterize the scattering environment surrounding the transmit and receive antennas. For an isotropic scattering environment (i.e., $P_{\text{Rx}}(\varphi) = 1/2\pi$), $\beta_0^n = 1$ for $n = 0$ and 0 elsewhere. In this case, the temporal correlation $\rho(\tau)$ reduces to

$$\rho(\tau) = J_0(k\tau\dot{\upsilon}),$$

which is the classical Clarke’s temporal correlation model [78].

### 6.3.4 Jake’s model for MIMO channels in isotropic scattering

We now recover the classical Jake’s model from (6.18) for a stationary MU. For an isotropic scattering environment (i.e., $G(\phi, \varphi) = 1/4\pi^2$), we have

$$\beta^m_n = \begin{cases} 1, & m = n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

and correlation coefficient

$$\rho_{p,p'}^{q,q'}(\tau) = J_0(k\|x_q - x_{q'}\|)J_0(k\|z_p - z_{p'}\|),$$

which is the Jake’s model for a MIMO channel in an isotropic scattering environment. Note also that if the MU is moving and the scattering environment is isotropic, then (6.18) simplifies to

$$\rho_{q,q'}^{p,p'}(\tau) = J_0(k\|x_q - x_{q'}\|)J_0(k\|z_p - z_{p'} + \tau\dot{\upsilon}\dot{\upsilon}\|),$$

which extends the MISO space-time correlation model proposed by Chen et al. in [79] to the MIMO case.

### 6.3.5 Kronecker Model as a Special Case

In some circumstances the correlation between two distinct antenna pairs can be written as the product of corresponding channel correlation at the transmitter and the channel correlation at the receiver, i.e.,

$$\rho_{q,q'}^{p,p'}(\tau) = \rho_{q,q'}(\tau)\rho_{q',q'}^{p,p'}(\tau).$$
Facilitated by (6.34), we can write the covariance matrix of the space-time MIMO channel $\mathbf{H}(t)$ as the Kronecker product between the receiver channel correlation matrix $\mathbf{R}_{\text{Rx}}(\tau)$ and the transmitter channel correlation matrix $\mathbf{R}_{\text{Tx}}(\tau)$,

$$\mathbf{R}_H(\tau) = \mathbf{R}_{\text{Rx}}(\tau) \otimes \mathbf{R}_{\text{Tx}}(\tau).$$

(6.35)

where $(p, p')$-th element of $\mathbf{R}_{\text{Rx}}(\tau)$ is given by (6.23) and $(q, q')$-th element of $\mathbf{R}_{\text{Tx}}(\tau)$ is given by (6.26). In this case, diagonal blocks $\mathbf{R}_{H,q,q}$ and off-diagonal blocks $\mathbf{R}_{H,q,q'}$ of $\mathbf{R}_H(\tau)$ are given by $\mathbf{R}_{\text{Rx}}(\tau)$ and $\rho_{q,q'}(\tau)\mathbf{R}_{\text{Rx}}(\tau)$, respectively.

Note that (6.34) holds only for a class of scattering environments where the power distribution of the scattering channel satisfies [69,112]

$$G(\phi, \varphi) = \mathcal{P}_{\text{Tx}}(\phi)\mathcal{P}_{\text{Rx}}(\varphi)$$

(6.36)

and these types of channels are known as separable channels. Substitution of (6.36) in (6.19) gives

$$\beta^m_n = \beta_m \beta^n,$$

where the coefficients $\beta_m$ characterize the scattering environment surrounding the transmitter antenna array and are given by

$$\beta_m = \int_\mathcal{S} \mathcal{P}_{\text{Tx}}(\phi)e^{im\phi}d\phi,$$

(6.37)

with $\mathcal{P}_{\text{Tx}}(\phi)$ the average power density of the scatterers surrounding the transmitter region, given by the marginalized PSD

$$\mathcal{P}_{\text{Tx}}(\phi) = \int_\mathcal{S} G(\phi, \varphi)d\varphi,$$

and the coefficients $\beta^n$ characterize the scattering environment surrounding the receiver antenna array and are given by (6.24).

### 6.4 Non-isotropic Scattering Distributions

A number of univariate azimuthal power distributions have been proposed in the literature [102,104,105,115] for modelling the non-isotropic scattering distributions $\mathcal{P}_{\text{Tx}}(\phi)$ and $\mathcal{P}_{\text{Rx}}(\varphi)$ at the transmitter and the receiver, respectively. Here we propose an extension of the univariate distributions to the more general bivariate case.
In the next two sub-sections we discuss a number of univariate and bivariate scattering distributions and give the corresponding scattering distribution coefficients in closed-form.

### 6.4.1 Univariate Scattering Distributions

Some commonly used univariate uni-modal non-isotropic scattering distributions include uniform-limited [102], truncated Gaussian [105], truncated Laplacian [104], Cosine [102,103] and von-Mises [115]. These distributions were used to model the non-isotropic scattering environment surrounding either the receive or transmit antenna arrays. The univariate distributions are characterized by the mean angle of arrival $\varphi_0$ (or departure $\phi_0$) and the angular spread $\sigma$. In the work on receiver correlation modelling, [111] has derived $\beta^n$ in closed-form for these common distributions. Table 6.1 provides a summary of the distributions relevant for this work and gives the corresponding receiver scattering coefficients $\beta^n$ in closed-form (See Chapter 2.3.2 for more details). Scattering coefficients $\beta_m$ at the transmitter can

<table>
<thead>
<tr>
<th>PSD</th>
<th>Distribution Function $\mathcal{P}(\varphi)$</th>
<th>$\beta^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform limited</td>
<td>$\frac{1}{2\triangle_r}$ when $\varphi \in (\varphi_0 - \Delta_r, \varphi_0 + \Delta_r)$ and zero elsewhere</td>
<td>$\exp(-in\varphi_0)\text{sinc}(n\Delta_r)$</td>
</tr>
<tr>
<td>Gaussian</td>
<td>$K_G \exp\left{-(\varphi - \varphi_0)^2/2\sigma_G^2\right}$, $</td>
<td>\varphi - \varphi_0</td>
</tr>
<tr>
<td></td>
<td>$K_G = \frac{1}{\sqrt{2\pi\sigma_G}} \text{erf}(\varphi/\sqrt{2\sigma_G})$</td>
<td>$\Gamma(\cdot)$</td>
</tr>
<tr>
<td></td>
<td>$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$</td>
<td>$I_n(\cdot)$ : Gamma function</td>
</tr>
<tr>
<td>cosine</td>
<td>$K_C \cos^2\left(\frac{\varphi - \varphi_0}{\sigma_C}\right)$, $</td>
<td>\varphi - \varphi_0</td>
</tr>
<tr>
<td></td>
<td>$K_C = \frac{2\sigma_C^{-1}\Gamma(\varphi_0 + n\sigma_C)}{\pi I_0(\varphi_0 + n\sigma_C)}$</td>
<td>$\Gamma(\cdot)$</td>
</tr>
<tr>
<td>von-Mises</td>
<td>$K_v e^{\kappa \cos(\varphi - \varphi_0)}$, $</td>
<td>\varphi - \varphi_0</td>
</tr>
<tr>
<td></td>
<td>$K_v = \frac{1}{2\pi I_0(\kappa \sigma_v)}$</td>
<td>$I_n(\cdot)$ : modified Bessel function of the first kind.</td>
</tr>
<tr>
<td>Laplacian</td>
<td>$K_L e^{-\sqrt{2}\frac{</td>
<td>\varphi - \varphi_0</td>
</tr>
<tr>
<td></td>
<td>$K_L = \frac{1}{\sqrt{2\sigma_L(1-e^{-\sqrt{2}\sigma_L})}}$</td>
<td>$\xi = \exp\left(-\frac{\sqrt{2}}{\sigma_L}\right)$</td>
</tr>
</tbody>
</table>

Table 6.1: Scattering Coefficients $\beta^n$ for Uniform-limited, truncated Gaussian, cosine, von-Mises and truncated Laplacian univariate uni-modal power distributions
be obtained by taking the conjugate of the receiver scattering coefficients $\beta^m$. In following we consider two special cases to demonstrate the strength of the modal approach we considered to derive space-time cross correlation functions.

**A Uni-modal Distributed Field within a Limited Spread**

Scattering coefficients given in Table-6.1 and the results in [79,80] have been based on the distribution of scatterers over all angles. However, using the method we presented in this work, it is possible to explicitly account for the impact of limited spreads with an arbitrary distribution within the limited spread. For example, we consider a Laplacian distributed field within the limited spread ($-\theta_0, \theta_0$). The truncated Laplacian distribution function is given by,

$$P(\phi) = \frac{K_L}{\sqrt{2}\sigma_L} e^{-\sqrt{2}|\phi-\phi_0|/\sigma_L}, \quad |\phi - \phi_0| \leq \theta_0, \quad |\theta_0| \leq \pi,$$

where $K_L$ is the normalization constant such that $\int P(\phi) d\phi = 1$, $\sigma_r$ is the angular spread at the receiver, and $\phi_0$ is the mean AOA. In this case, the scattering coefficients $\beta^n$ at the receiver are

$$\beta^n = e^{-in\phi_0} \frac{(1 - \xi (\cos n\theta_0 - n\alpha \sin n\theta_0))}{(1 - \xi)(1 + n^2\alpha^2)},$$

where $\alpha = \sigma_L/\sqrt{2}$ and $\xi = e^{-\sqrt{2}\theta_0/\sigma_L}$. The scattering coefficients $\beta_m$ at the transmitter can be obtained by taking the conjugate of $\beta^n$.

**Multi-modal Distributed Field**

A multi-modal azimuth power distribution arises when there are two or more strong multipaths exist in a fading channel. This may be the result of multiple remote macroscopic scattering regions. A multi-modal azimuth distribution can be constructed from a mixture of uni-modal azimuth distributions. For example, here we construct a multi-modal distribution from a mixture of von-Mises distributions, where each mode (strong multipath) is represented by a mixture component with a mean value $\phi_\ell$ and a concentration parameter $\kappa_\ell$:

$$P(\phi) = \frac{1}{2\pi} \sum_{\ell=1}^{M} \frac{\gamma_\ell}{I_0(\kappa_\ell)} e^{\kappa_\ell \cos(\phi - \phi_\ell)}, \quad |\phi - \phi_\ell| \leq \pi,$$
with $\sum_{\ell=1}^{M} \gamma_{\ell} = 1$, where $M$ is the number of modes and $\gamma_{\ell}$ is the mixing coefficient which is independent of $\varphi$. For $M = 1$, the distribution becomes uni-modal von-Mises (Table-6.1). Using [146, 3.937], scattering coefficients at the receiver region are given by

$$
\beta_n = \sum_{\ell=1}^{M} \frac{\gamma_{\ell}}{2\pi I_0(\kappa_{\ell})} \int_{0}^{2\pi} e^{\kappa_{\ell}\cos(\varphi_{\ell})} e^{in\varphi} d\varphi,
$$

$$
= \sum_{\ell=1}^{M} \frac{\gamma_{\ell}I_n(\kappa_{\ell})}{I_0(\kappa_{\ell})} e^{-in\varphi_{\ell}}.
$$

(6.38)

### 6.4.2 Bivariate Scattering Distributions

The bivariate power distributions are characterized by the mean departure and arrival angles $\phi_0$, $\varphi_0$, angular spreads $\sigma_t$ and $\sigma_r$ for marginal distributions at the transmitter and receiver apertures, and the covariance

$$
\gamma = \text{cov}(\phi, \varphi) \triangleq \frac{E\{\phi\varphi\} - \phi_0\varphi_0}{\sigma_t\sigma_r},
$$

(6.39)

between transmit and receive angles. We now outline a number of examples of bivariate angular scattering distributions and corresponding scattering coefficients $\beta_m^n$ in closed-form for each distribution.

**Uniform-limited azimuth field**

When the energy leaves uniformly to a restricted range of azimuth $(\phi_0 - \Delta_t, \phi_0 + \Delta_t)$ at the transmitter and to arrive at the receiver uniformly from $(\varphi_0 - \Delta_r, \varphi_0 + \Delta_r)$, then following Morgenstern’s family of multivariate distributions [148], we have the joint uniform limited azimuth scattering distribution

$$
G_U(\phi, \varphi) = \frac{1}{4 \Delta_t \Delta_r} - \frac{\gamma(\phi - \phi_0)(\varphi - \varphi_0)}{4 \Delta_t^2 \Delta_r^2},
$$

for $|\phi - \phi_0| \leq \Delta_t$ and $|\varphi - \varphi_0| \leq \Delta_r$, and 0 elsewhere. The parameter $\gamma$ is the covariance between $\phi \in [-\pi, \pi]$ and $\varphi \in [-\pi, \pi]$, which controls the flatness of $G_U(\phi, \varphi)$. Using (6.19), we can derive without any approximation:

$$
\beta_{m}^{n} = \begin{cases} 
\text{sinc}(m\Delta_t)e^{in\phi_0}, & \text{if } n = 0 \\
\text{sinc}(n\Delta_r)e^{-in\varphi_0}, & \text{if } m = 0 \\
e^{i(m\phi_0 - n\varphi_0)}\Gamma_m^n, & \text{otherwise}
\end{cases}
$$
where $\Gamma_{nm}^n$ is given by

$$
\Gamma_{nm}^n = \text{sinc}(m\Delta_t)\text{sinc}(n\Delta_r) + \frac{\gamma}{nm\Delta_t\Delta_r} [\cos(m\Delta_t) - \text{sinc}(m\Delta_t)] \\
\times [\text{sinc}(n\Delta_r) - \cos(n\Delta_r)].
$$

Note that $G_U(\phi, \varphi)$ has marginal distributions $P_{\text{Tx}}(\phi) = 1/2\Delta_t$ for $\phi \in (\phi_0 - \Delta_t, \phi_0 + \Delta_t)$ and zero elsewhere, and $P_{\text{Rx}}(\varphi) = 1/2\Delta_r$ for $\varphi \in (\varphi_0 - \Delta_r, \varphi_0 + \Delta_r)$ and zero elsewhere (independent of $\gamma$). For $\Delta_t = \pi$ and $\Delta_r = \pi$ with $\gamma = 0$ (isotropic scattering), we have uniform PSD $G_U(\phi, \varphi) = 1/4\pi^2$ and scattering coefficients become

$$
\beta_m^n = \delta_m\delta_n,
$$

where $\delta_m$ is the Dirac delta function.

**Truncated Gaussian Distributed Field**

A distribution that can be used to model the joint PSD is the truncated Gaussian bivariate distribution, defined as

$$
G_G(\phi, \varphi) = \Omega_G \exp\left[\frac{-Q(\phi, \varphi)}{2(1 - \gamma^2)}\right], \quad \phi, \varphi \in [-\pi, \pi),
$$

where $\Omega_G$ is a normalization constant such that $\int_{S \times S} G_G(\phi, \varphi) d\phi d\varphi = 1$ and

$$
Q(\phi, \varphi) = \frac{(\phi - \phi_0)^2}{\sigma_t^2} - \frac{2\gamma(\phi - \phi_0)(\varphi - \varphi_0)}{\sigma_t\sigma_r} + \frac{(\varphi - \varphi_0)^2}{\sigma_r^2}
$$

with $\phi_0$ the mean AOD at the transmitter, $\sigma_t$ the standard deviation of the non-truncated marginalized PSD at the transmitter, $\varphi_0$ the mean AOA at the receiver, $\sigma_r$ the standard deviation of the non-truncated marginalized PSD at the receiver and $\gamma$ the covariance between receive and transmit angles, as defined by (6.39). In this case, finding scattering coefficients in closed-form poses a much harder problem.$^6$ However, if the angular spread at the both ends of the channel is small, then a good approximation for the truncated Gaussian case can be obtained by integrating over the domain $(-\infty, \infty)$, since the tails of marginalized PSDs cause a very little error. Using a result found in [149],

$$
\beta_m^n \approx \exp\left(i(m\phi_0 - n\varphi_0) - \frac{1}{2} \left(\sigma_t^2 m^2 - 2\gamma\sigma_t\sigma_r mn + \sigma_r^2 n^2\right)\right).
$$

$^6$A numerical method can be applied to evaluate the integral (6.19).
Truncated Laplacian Distributed Field

Similar to the truncated Gaussian distribution, an elliptical truncated bivariate Laplacian distribution can be defined as

\[ G_L(\phi, \varphi) = \Omega_L K_0 \left( \frac{2Q(\phi, \varphi)}{1 - \gamma^2} \right), \quad \phi, \varphi \in [-\pi, \pi), \]

where \( \Omega_L \) is a normalization constant such that \( \int_{\mathbb{S} \times \mathbb{S}} G_L(\phi, \varphi) d\phi d\varphi = 1 \), \( Q(\phi, \varphi) \) is as for the Gaussian case and \( K_0(\cdot) \) is the modified Bessel function of the second kind of order zero. Assuming tails of the marginalized PSDs cause a very little error, the scattering coefficients (6.19) for this distribution are given by

\[ \gamma_{nm}^\tau \approx \exp \left( \frac{\exp(im\phi_0 - in\varphi_0)}{\sigma_t^2 m^2 - 2\gamma \sigma_t \sigma_r mn + \sigma_r^2 n^2 + 1} \right). \]

6.5 Simulation Examples

In this section we discuss examples to demonstrate the utility of our proposed space-time MIMO channel model and the correlation coefficient expressions.

6.5.1 Univariate Distributions: Space-Time Cross Correlation

First we explore the effects of angular spread and Doppler frequency \( f_D \) on the space-time cross correlation for the univariate uni-modal scattering distributions discussed in Section 6.4.1 (Table-6.1). In the first part of simulations, we set \( f_D T_S = 0.038 \), where \( T_S \) is the symbol period. This value of \( f_D T_S \) represents a realistic value expected in a Hyperlan-2 standard [151] with a Doppler frequency of 38 Hz, which corresponds to MU velocity of 2 ms\(^{-1}\) for a carrier frequency of 5.725 GHz. Figure 6.2 shows the space-time cross correlation between two receive antennas placed on the \( x \)-axis against the spatial separation for \( \tau = \{0, 5T_S, 20T_S, 30T_S\} \). For each distribution, we set the angular spread at the receiver to \( \sigma_r = \{20^\circ, 5^\circ, 2^\circ\} \) and mean AOA \( \varphi_0 = 0^\circ \). As shown, the space-time cross correlation decreases as the antenna spacing, angular spread and the number of symbol periods increases. More interestingly, all distributions give very similar correlation values for the same angular spread, especially for small antenna separations and for small number of symbol periods. This observation indicates that the choice of scattering distribution (uni-modal) is unimportant as \( \sigma_r \) dominates the space-time cross correlation at
small antenna separations and small number of symbol periods.

Figure 6.2: Space-time cross correlation between two MU receive antennas with $f_D T_S = 0.038$ against the spatial separation for Uniform-limited, truncated Gaussian, truncated Laplacian and von-Mises scattering distributions with angular spread $\sigma_r = \{20^\circ, 5^\circ, 2^\circ\}$ and mean AOA $\varphi_0 = 0^\circ$: (a) $\tau = 0$, (b) $\tau = 5T_S$, (c) $\tau = 20T_S$ and (d) $\tau = 30T_S$.

We now explore the effect of Doppler frequency on the space-time cross correlation for the distributions considered in the previous example. Figure 6.3 shows the space-time cross correlation between two receive antennas placed on the $x$-axis for increasing Doppler frequency. In this simulation, we set $\tau = 5T_S$ and antenna separation $\|z_p - z'_p\| = \{0.1\lambda, 0.2\lambda, 0.5\lambda, \lambda\}$. Similar to previous example, we set the angular spread at the receiver to $\sigma_r = \{20^\circ, 5^\circ, 2^\circ\}$ and mean AOA $\varphi_0 = 0^\circ$ for each distribution. It is observed that correlation decreases as the antenna spacing, angular spread and $f_D T_S$ increases. Here we also see that over the range of $f_D T_S$ considered all distributions give very similar correlation values for the same angular spread, particularly for small angular spreads and small antenna separations. From Figures 6.2 and 6.3 we can observe that, for all Doppler frequencies, if the scattering distribution surrounding the receiver array is uni-modal, and the antenna
separation and angular spread are small, then the choice of non-isotropic distribution is unimportant to model the space-time cross correlation at the receiver over a small number of symbol intervals.

6.5.2 Uni-modal Distributed Field within a Limited Spread:

Space-Time Cross Correlation and Space-Frequency Cross Spectrum

Figure 6.4 shows the magnitude of the space-time correlation function (6.23) at the receiver for a Laplacian distributed field with mean AOA 60° from broadside, limited spread θ₀ = 90° around the mean AOA and angular spreads σ_τ = {20°, 10°}, varying the receiver antenna separation ||z_p - z'_p|| and τ. Here we assumed that two receive antennas are placed on the x-axis, the traveling direction of the MU is
$\varphi_v = 30^\circ$ from end-fire of the receiver antennas and maximum Doppler frequency $f_D = \omega_D/2\pi = 0.05$. In this case, the MU is traveling directly towards the strongest signal reception direction, which is the mean AOA of the distribution.

From Figure 6.4, it is observed that after $\tau = 20$ time samples, space-time cross correlation is insignificant ($|\rho^{p,p'}(\tau)| < 0.3$) for both angular spreads when the receive antenna separation is small. Furthermore, for all values of $\tau$, the space-time cross correlation is negligible when the receiver antenna separation is larger than $0.75\lambda$ and $1.5\lambda$ for angular spreads $20^\circ$ and $10^\circ$, respectively. In general, we can observe that, $|\rho^{p,p'}(\tau)|$ increases as the angular spread and antenna separation decreases and also with small number of time samples.

Figure 6.4: Magnitude of the space-time cross correlation function for $f_D = \omega_D/2\pi = 0.05$, $\varphi_v = 30^\circ$ and a Laplacian distributed field with mean AOA $60^\circ$ from broadside and angular spread $\sigma_r = \{20^\circ, 10^\circ\}$. 
6.5.3 Uni-modal vs Bi-modal Distributions: Spatial Correlation

We now investigate the correlation effects due to uni-modal and bi-modal distributions at the receiver aperture. Figure 6.5(a) and 6.5(b) depict the bi-modal von-Mises distributions for mean AOA $\varphi_0 = 0^\circ$ and non-isotropic parameters (concentration parameters) $\kappa_1 = \kappa_2 = 200$. In Figure 6.5(a), modes are located at $\varphi_1 = -25^\circ$ and $\varphi_2 = 25^\circ$, and in Figure 6.5(b), modes are located at $\varphi_1 = -15^\circ$ and $\varphi_2 = 15^\circ$. For both cases we set mixture coefficients $\gamma_1 = \gamma_2 = 0.5$. In the first case, the angular spread $\sigma_r$ at the receiver is $25^\circ$ and in the second case it is $15^\circ$. Also shown in Figure 6.5(a) and 6.5(b) are the uni-modal von-Mises distributions with mean AOA $\varphi_0 = 0^\circ$ and the receiver angular spread $25^\circ(\kappa = 6)$ and $15^\circ(\kappa = 14)$, respectively. Figure 6.5(c) and 6.5(d) show the corresponding spatial correlation between two receive antennas against the spatial separation for $\tau = 0$. Scattering coefficients $\beta_n$ and correlation coefficients $\rho_{pp'}(0)$ are calculated using (6.38) and (6.23), respectively. From Figures 6.5(c) and 6.5(d) we can observe that bi-modal distributions give slightly less spatial correlation than uni-modal distributions for small antenna separations. However, at large antenna separations ($\|z_p - z_{p'}\| > \lambda$), the spatial correlation results from bi-modal distributions is significant compared to that of uni-modal distributions.

6.5.4 Validity of the Kronecker Channel Model

Now we compare the performance of MIMO communication systems operating in separable (Kronecker channel with $\gamma = \text{cov}(\phi, \varphi) = 0$ in (6.39)) and non-separable scattering environments. Suppose the frequency nonselective channel between transmitter and receiver array is such that the symbol duration is much less than the coherence time $1/f_D$ of the channel. In this situation, we can consider the channel matrix $H(t)$ as a random constant matrix $H$ over several frames of data. Performance of the system is measured in terms of the average mutual information. Here we assume transmitter has no knowledge about the channel and the receiver has the full knowledge about the channel. In this case, the average mutual information is given by [5],

$$I = \mathcal{E} \left\{ \log_2 \left| \frac{1}{n_T} \mathbf{H}^* \mathbf{H} \right| \right\},$$

where $\mathcal{E}$ is the average symbol energy-to-noise ratio at each receive antenna.

We consider transmit and receive apertures of radius $0.5\lambda$, corresponding to
2\lceil \pi e^0.5 \rceil + 1 = 11 effective communication modes at each aperture. Within each aperture, we place three antennas in a uniform circular array (UCA) configuration (3 × 3 MIMO channel).

Figure 6.6 shows the average mutual information for a bivariate truncated Gaussian distributed azimuth field with \( \rho = 0.8 \). In Chapter 4 we showed that the performance of UCA antenna configuration is less sensitive to change of mean AOD (\( \phi_0 \)) and mean AOA (\( \varphi_0 \)). Therefore, without loss of generality, we set \( \phi_0 = \varphi_0 = 90^\circ \).

Also, in this simulation, we set transmitter angular spread \( \sigma_t = 10^\circ \) and receiver angular spreads \( \sigma_r = \{30^\circ, 10^\circ\} \). For comparison, also shown is the average mutual information of the 3 × 3 i.i.d. MIMO channel. We observe that when \( \sigma_r = 30^\circ \), both models give very similar performance for all SNRs. When the angular spread at the receiver is small, e.g. \( \sigma_r = 10^\circ \), we can observe that the Kronecker model gives slightly higher performance than the non-separable model for higher SNRs. However, the margin of capacity over estimation is insignificant in comparison with the i.i.d. channel capacity performance. Therefore, the Kronecker model provides a good estimation to the actual scattering channel when the joint scattering distri-
Figure 6.6: Average mutual information of 3-transmit UCA and 3-receive UCA MIMO system in separable (Kronecker with $\rho = 0$) and non-separable ($\rho = 0.8$) scattering environments: bivariate truncated Gaussian azimuth field with mean AOD = 90°, mean AOA = 90°, transmitter angular spread $\sigma_t = 10°$ and receiver angular spreads $\sigma_r = \{30°, 10°\}$.

Reasoning for this claim will be discussed later.

Figure 6.7 shows a multi-modal bivariate Gaussian distributed azimuth field with 3 modes centered around $(\phi_0, \varphi_0) = \{(-40°, 40°), (0°, -40°), (50°, 0°)\}$, each mode with angular spreads $\sigma_r = \sigma_t = 5°$ and $\rho = 0.8$. Note that, in this case the effective angular spreads at the receiver and the transmitter are larger than 5°.

We now consider the $3 \times 3$ antenna configuration setup discussed in the previous example. Figure 6.8 shows the average mutual information with the multi-modal scattering distribution shown in Figure 6.7. It is observed that Kronecker model tends to overestimate the average mutual information at high SNRs. Unlike in the uni-modal case considered previously, the margin of error seen here is quite significant, especially at high SNRs. We now provide reasons why the Kronecker model overestimates the mutual information for the scattering distribution shown in Figure 6.9.
The joint PSD of the Kronecker model is given by \[ \tilde{G}(\phi, \varphi) = P_{\text{Tx}}(\phi)P_{\text{Rx}}(\varphi), \]
where \( P_{\text{Tx}}(\phi) \) and \( P_{\text{Rx}}(\varphi) \) are the transmit and receive power distributions, generated by marginalizing \( G(\phi, \varphi) \). Figure 6.9 shows the Kronecker model PSD \( \tilde{G}(\phi, \varphi) \) of the scattering channel considered in Figure 6.7. Comparing Figure 6.9 with Figure 6.7 we can observe that \( \tilde{G}(\phi, \varphi) \) consist of six extra modes, corresponding to additional six scattering clusters. Therefore, Kronecker model introduces virtual scattering clusters located at the intersection of the actual scattering clusters. As a result, Kronecker model will increase the effective angular spread at the transmit and receive apertures (lower modal correlation) and hence improved system performance. Therefore, the popular Kronecker model does not model the MIMO channel accurately when there exist multiple scattering clusters in the channel. These observations match the measurement results published in [94].

Now we consider the uni-modal PSD used in our first simulation example. Figure 6.10 shows the corresponding Kronecker Model PSD \( \tilde{G}(\phi, \varphi) \) for this channel, for \( \sigma_r = 10^\circ \) and \( \sigma_t = 10^\circ \). In this case the Kronecker model does not introduce any additional virtual scattering clusters into the channel. As a result, there is no increase in the number of multipaths of the channel. Hence both models give very
Figure 6.8: Average mutual information of 3-transmit UCA and 3-receive UCA MIMO system for separable and non-separable scattering channel considered in Figure 6.7.

similar performance.
6.6 Summary and Contributions

A space-time channel model for down-link transmission is proposed. The proposed model captures the antenna geometry at the receiver and transmitter antenna arrays, movement of the MU and joint correlation properties of the scattering channel.

Some specific contributions made in this chapter are:

1. A new space-time channel model for down-link transmission is proposed. It separates the space-time channel into deterministic and random parts. The deterministic part captures the physical antenna placements (linear array, circular array, grid array, etc.) and the motion of the mobile unit (velocity and the direction). The random part captures the random scattering environment.

2. The random scattering environment is modeled using a joint bi-angular power distribution parameterized by the transmit and receive angles. The well-known “Kronecker” model is recovered as a special case when this distribution is a separable function.
3. The simplest non-trivial, non-separable bi-angular power distribution is developed which consist of models parameterized by the angular power distribution at the transmitter, angular power distribution at the receiver and the covariance between transmit and receive angles which captures their statistical interdependency. We proposed a number of bi-angular power distributions to model realistic scattering channels.

4. We showed that Kronecker model is a good approximation to an actual channel only when the scattering channel consists of a single scattering cluster. When the scattering channel consists of multiple scattering clusters, we demonstrated the Kronecker model over-estimates the performance of MIMO systems because it includes phantom scattering paths. This significant deficiency is addressed in our model by use of a non-separable bi-angular power distribution.

5. We derived expressions for space-time cross correlation and space-frequency cross spectra for a number of scattering distributions. This generalizes the

Figure 6.10: Kronecker model PSD $\tilde{G}(\phi, \varphi) = P_{\text{Tx}}(\phi)P_{\text{Rx}}(\varphi)$ of the uni-modal non-separable scattering distribution used in the first example to obtain the results in Figure 6.6 for $\sigma_r = 10^\circ$. 
limited number of special cases available in the literature.

6. We introduced the concept of multi-modal power distributions surrounding
the transmitter and receiver antenna arrays. An example is discussed using
a mixture of von-Mises distributed components.

7. Using the proposed model we showed that for all Doppler frequencies, the
choice of power distribution is not critical to model the space-time cross cor-
relation at the receiver over a small number of symbol intervals when the
antenna separation and angular power distribution spread are small.
Chapter 7

Conclusions and Future Research Directions

In this chapter we state the general conclusions drawn from this thesis. The summary of contributions can be found at the end of each chapter and are not repeated here. We also outline some future research directions arising from this work.

7.1 Conclusions

This thesis has been primarily concerned with the performance of space-time coding schemes applied on a single user, narrowband wireless communications link utilizing multiple transmit and receive antennas. Motivated by the performance improvements promised by space-time coded MIMO communication systems in i.i.d. fading channels, this thesis investigated the performance for more physically realistic environments, where both the antenna arrays and scattering are constrained.

By introducing spatial aspects (antenna spacing and antenna geometry) and scattering distribution parameters (angular spread, mean angle of departure, mean angle of arrival), performance bounds of space-time coded systems were derived for spatially constrained antenna arrays operating in non-isotropic scattering environments. The most significant result was that the number of antennas that can be employed in a fixed antenna aperture without diminishing the diversity advantage (robustness) of a space-time code is determined by the size of the antenna aperture, antenna geometry and the richness of the scattering environment.

Classical MIMO results rely implicitly on sparse spatial sampling along with rich scattering between transmit and receive antenna arrays. In this thesis it was shown that these i.i.d. models never be justified in realistic scenarios as even knowing where antennas are holds valuable information that can be exploited through spatial
precoding. However, with dense antenna arrays, it was shown that exploitation of space through precoding can significantly improve the capacity performance of dense MIMO systems.

This thesis has also shown that widely used “Kronecker” model is a good approximation to an actual channel only when the scattering channel consists of a single scattering cluster. When the scattering channel consists of multiple scattering clusters, the Kronecker model over-estimates the performance of MIMO systems.

### 7.2 Future Research Directions

In this section we outline a number of future research directions to arise from the work presented in this thesis.

**Performance analysis of space-time codes:** In this thesis, we considered communication between circular shaped antenna apertures in 2-D space\(^1\) and analyzed the performance of space-time codes for various spatial scenarios in 2-D scattering environments. Communication between arbitrary shaped antenna apertures in space is a more general problem to consider. In this case, performance analysis of space-time codes can be divided into two parts: i) wave propagation in free-space and ii) wave propagation in random scattering environments. Such analysis would reveal the properties that a continuous MIMO channel must have in order to achieve full diversity advantage and coding gain given by a space-time code. For arbitrary apertures, finding the eigenfunctions to represent the wave field poses a much harder problem. This work will require in depth knowledge of functional analysis, operator theory and Hilbert space theory.

**Space-time code designs for dense antenna arrays:** It was shown in Chapter 5 that with dense antenna arrays, exploitation of space through precoding can significantly improve the capacity performance of dense MIMO systems. The capacity results presented in Chapter 5 does not reflect the performance achieved by an actual transmission system and it only provides an upper bound at which information passes through error-free over a channel. Therefore it is of interest to study the performance of dense MIMO systems which apply space-time codes along with spatial precoders. This study also requires the design of new space-time

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\(^1\)The performance analysis presented in this thesis can be easily extended to spherical shaped antenna apertures in 3-D space by using the three dimensional Jacobi-Anger expansion of plane waves given in [113, page 32].
codes for a large number of transmit antennas.

**Spatial precoder designs:** The precoders proposed in this thesis have been for single-user systems. A possible extension is to design spatial precoders for multi-user systems. In a multi-user system the performance is limited by interference from other users (co-channel users) as well as spatially correlated multipath fading. In [152, 153] it was shown that by combining interference suppression and ML decoding scheme for space-time block codes can effectively suppress interference from other co-channel users while providing each user with a diversity benefit. These results assumed that channel gains are uncorrelated and also the interfering signals are uncorrelated. Following [152, 153] and the work presented in this thesis it is of interest to design precoders for multi-user mobile communication systems in spatially correlated non-isotropic scattering environments, in particular fixed schemes for up-link communications.

**Time-selective fading channels - Performance analysis:** In Chapter 6 we have developed a general space-time channel model for down-link transmission in a mobile multi-antenna communication system. However, in this thesis we did not utilize the proposed model to investigate the performance of space-time communication systems in time-selective fading channels. Therefore, it would be of further research interest to study the performance of space-time communication systems using the proposed general space-time channel model in time-selective fading channel environments. Such a study will reveal the impact of joint correlation properties of scattering environment, antenna spacing, antenna placement and MU motion (Doppler effects) on the performance of space-time communication systems.

**Space-time-frequency channel modelling and validation:** In Chapter 6 we have assumed that the system bandwidth is low compared to the coherence bandwidth of the channel, which has led to frequency-flat fading approximation of the received signal. However, the time-selective channel model proposed in Chapter 6 can be extended to a frequency-selective channel model by introducing a propagation delay to the signal leaving transmit aperture at direction $\hat{\phi}$ and arriving in direction $\hat{\phi}$. The development of such an analytical model will benefit the performance investigation of MIMO OFDM systems in realistic channel scenarios, in particular to understand the effects of physical factors such as antenna spacing, antenna geometry, non-isotropic scattering distributions, arrival time distributions and inter-dependency between angle of arrival, angle of departure and arrival time.
In addition, it is of interest to see how the proposed analytical channel models would scale and parameterise to actual channel measurements. Furthermore, in this thesis we have only considered narrow-band channels. Therefore, another obvious extension is to propose analytical models for wide-band channels.

**Near field channel modelling:** The space-time channel model derived in Chapter 6 assumes that the impinging and outgoing waves are plane. This assumption is reasonable for most outdoor scattering channels. However, it does not always valid in indoor scattering scenarios. The near field effects need to be taken into account when the scatterers are very close to either the transmitter or the receiver. Therefore, another obvious extension is to propose channel models for near field scattering channels.
Appendix A

A.1 Proof of the Matrix Proposition

The following three properties of Hermitian matrices will be used to prove that \( G_n \) in (3.27) and \( G \) in (3.38) are Hermitian.

Property A.1.1 If \( H \) is any \( m \times n \) matrix, then \( HH^\dagger \) and \( H^\dagger H \) are Hermitian.

Property A.1.2 If \( A \) is a Hermitian matrix and \( H \) is any matrix, then \( HAH^\dagger \) and \( H^\dagger AH \) are Hermitian.

Property A.1.3 Kronecker product between two Hermitian matrices are always Hermitian.

Proposition A.1 Matrices \( G_n = (J_R^\dagger J_R)^T \otimes (J_T^\dagger s_\Delta J_T) \) and \( G = (J_R^\dagger J_R)^T \otimes (J_T^\dagger S_\Delta J_T) \) are Hermitian, where \( s_\Delta = (s_n - \hat{s}_n)(s_n - \hat{s}_n)^\dagger \) and \( S_\Delta = (X - \hat{X})(X - \hat{X})^\dagger \).

Proof

From property-A.1.1, matrices \( J_R^\dagger J_R, s_\Delta \) and \( S_\Delta \) are Hermitian. Therefore, property-A.1.2 implies that \( J_T^\dagger s_\Delta J_T \) and \( J_T^\dagger S_\Delta J_T \) are Hermitian. Thus, from property-A.1.3, \( G_n \) and \( G \) are Hermitian.
A.2 Error Events of 4-State QPSK STTC

A.2.1 Error Events of Length 2

Table A.1: 4-state QPSK space-time trellis code: Error events of length two.

<table>
<thead>
<tr>
<th>Type (t)</th>
<th>$S_\Delta = \sum_{n=1}^{L} s^n_\Delta$</th>
<th>Total bit-errors $q(S \rightarrow \hat{S})_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\begin{bmatrix} 2 &amp; 0 \ 0 &amp; 2 \end{bmatrix}$</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>$\begin{bmatrix} 4 &amp; 0 \ 0 &amp; 4 \end{bmatrix}$</td>
<td>1</td>
</tr>
</tbody>
</table>
### A.2.2 Error Events of Length 3

Table A.2: 4-state QPSK space-time trellis code: Error events of length three.

<table>
<thead>
<tr>
<th>Type (t)</th>
<th>$S_\Delta = \sum_{n=1}^{L} s^n_\Delta$</th>
<th>Total bit-errors $q(S \rightarrow \hat{S})_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\begin{bmatrix} 4 &amp; 2 \ 2 &amp; 4 \end{bmatrix}$</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} 6 &amp; 2 - 2j \ 2 + 2j &amp; 6 \end{bmatrix}$</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} 4 &amp; -2j \ 2j &amp; 4 \end{bmatrix}$</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>$\begin{bmatrix} 6 &amp; 2 + 2j \ 2 - 2j &amp; 6 \end{bmatrix}$</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>$\begin{bmatrix} 8 &amp; 4 \ 4 &amp; 8 \end{bmatrix}$</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>$\begin{bmatrix} 4 &amp; 2j \ -2j &amp; 4 \end{bmatrix}$</td>
<td>3</td>
</tr>
</tbody>
</table>
### A.2.3 Error Events of Length 4

Table A.3: 4-state QPSK space-time trellis code: Error events of length four.

<table>
<thead>
<tr>
<th>Type (t)</th>
<th>$\mathbf{s}<em>n = \sum</em>{l=1}^{n} s_{n-l}^l$</th>
<th>$n(S \rightarrow \hat{S})_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>$[6 \ 4 \ 4 \ 6]$</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>$[8 \ 4 + 2j \ 4 - 2j \ 8]$</td>
<td>8</td>
</tr>
<tr>
<td>11</td>
<td>$[6 \ 2 + 2j \ 2 - 2j \ 6]$</td>
<td>9</td>
</tr>
<tr>
<td>12</td>
<td>$[8 \ 4 \ 4 \ 8]$</td>
<td>8</td>
</tr>
<tr>
<td>13</td>
<td>$[10 \ 6 + 2j \ 6 - 2j \ 10]$</td>
<td>7</td>
</tr>
<tr>
<td>14</td>
<td>$[8 \ 4 + 4j \ 4 - 4j \ 8]$</td>
<td>4</td>
</tr>
<tr>
<td>15</td>
<td>$[6 \ 0 \ 0 \ 6]$</td>
<td>9</td>
</tr>
<tr>
<td>16</td>
<td>$[8 \ 2 \ 2 \ 8]$</td>
<td>16</td>
</tr>
<tr>
<td>17</td>
<td>$[8 \ 4 + 2j \ 4 - 2j \ 8]$</td>
<td>8</td>
</tr>
<tr>
<td>18</td>
<td>$[10 \ 4 \ 4 \ 10]$</td>
<td>7</td>
</tr>
<tr>
<td>19</td>
<td>$[10 \ 6 + 2j \ 6 - 2j \ 10]$</td>
<td>7</td>
</tr>
<tr>
<td>20</td>
<td>$[12 \ 8 \ 8 \ 12]$</td>
<td>3</td>
</tr>
<tr>
<td>21</td>
<td>$[6 \ 2 + 2j \ 2 - 2j \ 6]$</td>
<td>9</td>
</tr>
<tr>
<td>22</td>
<td>$[8 \ 4 + 4j \ 4 - 4j \ 8]$</td>
<td>4</td>
</tr>
</tbody>
</table>
A.3 Proof of the Conditional Mean and the Conditional Variance of $u = 2\text{Re}\{w(k)\Delta_{i,j}^\dagger y^\dagger (k - 1)\}$

A.3.1 Proof of the Conditional Mean

Mean of $u$ condition on the received signal $y(k - 1)$ can be written as

$$m_{u|y(k-1)} = \mathcal{E}\left\{2\text{Re}\left\{w(k)\Delta_{i,j}^\dagger y^\dagger (k - 1)\right\} \mid y(k - 1)\right\},$$

$$= 2\text{Re}\left\{\mathcal{E}\left\{w(k) \mid y(k - 1)\right\}\Delta_{i,j}^\dagger y^\dagger (k - 1)\right\}. \quad (A.1)$$

Substituting $w(k) = n(k) - n(k - 1)S_i$ and noting $\mathcal{E}\{n(k) \mid y(k - 1)\} = 0$, (A.1) can be simplified to

$$m_{u|y(k-1)} = -2\text{Re}\left\{m_{n(k-1)}|y(k-1)}S_i\Delta_{i,j}^\dagger y^\dagger (k - 1)\right\},$$

$$= 2\text{Re}\left\{m_{n(k-1)}|y(k-1)}(I - S_iS_i^T)y^\dagger (k - 1)\right\}, \quad (A.2)$$

where $m_{n(k-1)}|y(k-1)} = \mathcal{E}\{n(k - 1) \mid y(k - 1)\}$. Using the minimum mean square error estimator results given in [154, Section 2.3], we obtain

$$m_{n(k-1)}|y(k-1)} = \mathcal{E}\{n(k - 1)\} + [y(k - 1) - \mathcal{E}\{y(k - 1)\}]\Sigma_{y(k-1),y(k-1)}^{-1}y^\dagger y(k - 1),$$

where

$$\Sigma_{y(k-1),y(k-1)} = \mathcal{E}\{y^\dagger (k - 1)y(k - 1)\},$$

$$= E_s\mathcal{X}(k - 1)^\dagger R\mathcal{X}(k - 1) + \sigma_n^2 I_{n_T n_R}, \quad (A.3)$$

and

$$\Sigma_{y(k-1),n(k-1)} = \mathcal{E}\{y^\dagger (k - 1)n(k - 1)\},$$

$$= \sigma_n^2 I_{n_T n_R}, \quad (A.4)$$

Since $\mathcal{E}\{n(k - 1)\} = 0$ and $\mathcal{E}\{y(k - 1)\} = 0$, we have

$$m_{n(k-1)}|y(k-1)} = \sigma_n^2 y(k - 1) \left(E_s\mathcal{X}(k - 1)^\dagger R\mathcal{X}(k - 1) + \sigma_n^2 I\right)^{-1}. \quad (A.5)$$

Substituting (A.5) for $m_{n(k-1)}|y(k-1)}$ in (A.2) gives the conditional mean $m_{u|y(k-1)}$. 


A.3.2 Proof of the Conditional Variance

Variance of $u$ condition on the received signal $y(k-1)$ can be written as

$$
\sigma_u^2 | y(k-1) = \mathcal{E} \left\{ \| u - \bar{m}_u | y(k-1) \|^2 | y(k-1) \right\} \quad (A.6)
$$

$$
= \mathcal{E} \left\{ (u - \bar{m}_u | y(k-1))^\dagger (u - \bar{m}_u | y(k-1)) | y(k-1) \right\}.
$$

After some straightforward manipulations we can show

$$
u - \bar{m}_u | y(k-1) = 2 \text{Re} \left\{ (n(k) - [n(k-1) - \bar{m}_n(k-1)|y(k-1)] \mathcal{S}_i) \Delta_{i,j}^\dagger y^\dagger(k-1) \right\}. \quad (A.7)
$$

Substituting (A.7) for $u - \bar{m}_u | y(k-1)$ in (A.6) gives

$$
\sigma_u^2 | y(k-1) = \mathcal{E} \left\{ [2 \text{Re} \left\{ (n(k) - [n(k-1) - \bar{m}_n(k-1)|y(k-1)] \mathcal{S}_i) \Delta_{i,j}^\dagger y^\dagger(k-1) \right\}]^\dagger \times [2 \text{Re} \left\{ (n(k) - [n(k-1) - \bar{m}_n(k-1)|y(k-1)] \mathcal{S}_i) \times \Delta_{i,j}^\dagger y^\dagger(k-1) \right\}] | y(k-1) \right\}, \quad (A.8a)
$$

$$
= 2 y(k-1) \Delta_{i,j} \left[ n(k) - \mathcal{S}_i \Sigma_n(k-1) y^\dagger(k-1) \mathcal{S}_i \right] \Delta_{i,j}^\dagger y^\dagger(k-1), \quad (A.8b)
$$

where $\Sigma_n(k-1) = \mathcal{E} \{ n^\dagger(k) n(k) \} = \sigma_n^2 I$ and

$$
\Sigma_n(k-1) y(k-1) = \mathcal{E} \left\{ \| n(k-1) - \bar{m}_n(k-1)|y(k-1) \|^2 | y(k-1) \right\}
$$

is the covariance of the noise vector $n(k-1)$ condition on $y(k-1)$. Using the minimum mean square error estimator results given in [154], we can write

$$
\Sigma_n(k-1) y(k-1) = \Sigma_n(k-1) n(k-1) - \Sigma_{y(k-1),n(k-1)} \Sigma_n^{-1}(k-1) y(k-1) \Sigma_{y(k-1),n(k-1)} \Sigma_n^{-1}(k-1),
$$

$$
= \sigma_n^2 \left[ I - \sigma_n^2 \Sigma_{y(k-1),y(k-1)} \right]. \quad (A.9)
$$

Substituting (A.3) for $\Sigma_{y(k-1),y(k-1)}$ in (A.9) and then the result in (A.8b) gives the conditional variance $\sigma_u^2 | y(k-1)$. 
Appendix B

B.1 Proof of PEP Upper bound: Coherent Receiver

The conditional average pairwise error probability \( P(S_i \rightarrow S_j) \), defined as the probability that the receiver erroneously decides in favor of \( S_j \) when \( S_i \) was actually transmitted for a given channel realization, is upper bounded by the Chernoff bound \cite{8}

\[
P(S_i \rightarrow S_j|h) \leq \exp \left( -\frac{\gamma}{4} d_h^2(S_i, S_j) \right),
\]

where \( d_h^2(S_i, S_j) = h[I_{nR} \otimes S_{\Delta, F_c}]h^\dagger, S_{\Delta, F_c} = F_c(S_i - S_j)(S_i - S_j)^\dagger F_c^\dagger, h = (\text{vec} \{ H^T \})^T \) a row vector and \( \gamma = E_s/\sigma_n^2 \) is the average SNR at each receive antenna. To compute the average PEP, we average (B.1) over the joint distribution of \( h \). Assume \( h \) is a proper complex\(^1\) \( n_{T_R} \)-dimensional Gaussian random vector with mean \( 0 \) and covariance matrix \( R_H = \mathcal{E} \{ h^\dagger h \} \), then the pdf of \( h \) is given by \cite{155}

\[
p(h) = \frac{1}{\pi^{n_{T_R} n_{R}}} \det (R_H) \exp \{-hR_H^{-1}h^\dagger\},
\]

provided that \( R_H \) is non-singular. Then the average PEP is bounded as follows

\[
P(S_i \rightarrow S_j) \leq \frac{1}{\pi^{n_{T_R} n_{R}}} \det (R_H) \int \exp \{-hR_H^{-1}h^\dagger\} dh
\]

where \( R_H^{-1} = (\gamma/4 I_{n_R} \otimes S_{\Delta, F_c} + R_H^{-1}) \). Assume \( R_H \) is non-singular (positive definite), therefore the inverse \( R_H^{-1} \) is positive definite, since the inverse matrix of a positive definite matrix is also positive definite \cite[page 142]{15}. Also note that

\(^1\)To be proper complex, the mean of both the real and imaginary parts of \( H_S \) must be zero and also the cross-correlation between real and imaginary parts of \( H_S \) must be zero.
$S_{\Delta,F_c}$ is Hermitian and it has positive eigenvalues (through code construction, e.g. [8]), therefore $S_{\Delta,F_c}$ is positive definite, hence $I_{nr} \otimes S_{\Delta,F_c}$ is also positive definite. Therefore $R_0^{-1}$ is positive definite and hence $R_0$ is non-singular. Using the normalization property of Gaussian pdf

$$
\frac{1}{\pi^{n_{TR}} \det (R_0)} \int \exp \{-h R_0^{-1} h^\dagger\} dh = 1,
$$

we can simplify (B.2) to

$$
P(S_i \rightarrow S_j) \leq \frac{\det (R_0)}{\det (R_H)} = \frac{1}{\det (R_0^{-1} R_H)}.
$$

or equivalently

$$
P(S_i \rightarrow S_j) \leq \frac{1}{\det (I_{n_{TR}} + \frac{\gamma}{4} R_H [I_{nr} \otimes S_{\Delta,F_c}])}.
$$

## B.2 Proof of PEP Upper bound: Non-coherent Receiver

At asymptotically high SNRs, the PEP condition on the received signal $y(k - 1)$ is given by

$$
P(S_i \rightarrow S_j \mid y(k - 1)) = Q \left( \sqrt{\frac{d_{ij}^2}{4\sigma_n^2}} \right).
$$

Now using the Chernoff bound

$$
Q(x) \leq \frac{1}{2} \exp \left( \frac{-x^2}{2} \right),
$$

the conditional PEP can be upper bounded by

$$
P(S_i \rightarrow S_j \mid y(k - 1)) \leq \frac{1}{2} \exp \left( \frac{-d_{ij}^2}{8\sigma_n^2} \right). \quad (B.3)
$$

To compute the average PEP, we average (B.3) over the joint distribution of $y(k - 1)$. Assume $y(k - 1)$ is a proper complex Gaussian random vector that has mean
\( \mathcal{E} \{ y(k-1) \} = 0 \) and covariance

\[
R_y(k-1) \triangleq \mathcal{E} \{ y(k-1)^\dagger y(k-1) \} = E_s \mathcal{X}(k-1)^\dagger R_H \mathcal{X}(k-1) + \sigma_n^2 I_{nTnR} \tag{B.4}
\]

If \( R_y(k-1) \) is non-singular, then the pdf of \( y(k-1) \) is given by

\[
p(y(k-1)) = \frac{\pi^{-nTnR}}{\det (R_y(k-1))} \exp \left\{ -y(k-1)R_y^{-1}y(k-1)^\dagger \right\}.
\]

Averaging (B.3) over the pdf of \( y(k-1) \), we obtain

\[
P(S_i \rightarrow S_j) \leq \frac{\pi^{-nTnR}}{2 \det (R_y(k-1))} \int \exp \left\{ -y(k-1)R_d^{-1}y(k-1)^\dagger \right\} dy(k-1),
\]

where

\[
R_d^{-1} = R_y^{-1} + \frac{1}{8\sigma_n^2} D_{i,j}.
\]

Assume \( R_H \) is non-singular (positive definite). It can be shown that both \( R_y(k-1) \) and \( D_{i,j} \) are positive definite. Therefore, \( R_d \) is non-singular. Using the normalization property of Gaussian pdf

\[
\frac{1}{\pi^{nTnR} \det (R_d)} \int \exp \left\{ -y(k-1)R_d^{-1}y(k-1)^\dagger \right\} dy(k-1) = 1,
\]

we can simplify (B.5) to

\[
P(S_i \rightarrow S_j) \leq \frac{\det (R_d)}{2 \det (R_y(k-1))} = \frac{1}{2 \det (R_d^{-1}R_y(k-1))},
\]

or equivalently

\[
P(S_i \rightarrow S_j) \leq \frac{1}{2 \det \left( I + \frac{1}{8} \left( \mathcal{X}(k-1)^\dagger R_H \mathcal{X}(k-1) + I_{nTnR} \right) D_{i,j} \right)}.
\]
B.3 Proof of Generalized Water-filling Solution for $n_R = 2$ Receive Antennas

Let $n_R = 2$ in (4.15b), then we obtain the second-order polynomial $r_1r_2v_c t_1^2 q_i^2 + (v_c t_i(r_1 + r_2) - 2r_1r_2 t_1^2) q_i + (v_c - r_1 t_i - r_2 t_i)$ in $q$ which has roots $q_{i,1} = A + \sqrt{K}$ and $q_{i,2} = A - \sqrt{K}$, where $A$ and $K$ are given by (4.18). Then the product $q_{i,1} q_{i,2} = (v_c - r_1 t_i - r_2 t_i)$.

Case 1: $q_{i,1} q_{i,2} > 0 \Rightarrow v_c > t_i(r_1 + r_2)$. In this case, both roots are either positive or negative. Let $v_c = \alpha t_i(r_1 + r_2)$, where $\alpha > 1$. Then $A = -t_i^2 \alpha [(r_1 + r_2)^2 - 2r_1r_2/\alpha] < 0$ for all $\alpha > 1$. Since $K > 0$, $q_{i,2} < 0$, thus $q_{i,1}$ must also be negative to hold $v_c > t_i(r_1 + r_2)$. Therefore, when $v_c > t_i(r_1 + r_2)$, the optimum $q_i$ is zero to hold the inequality constraints of (4.13).

Case 2: $q_{i,1} q_{i,2} < 0 \Rightarrow v_c < t_i(r_1 + r_2)$. In this case, we always have one positive root and one negative root. Assume $q_{i,1} > 0$ and $q_{i,2} < 0$ and let $v_c = \alpha t_i(r_1 + r_2)$, where $0 < \alpha < 1$. For $q_{i,1}$ to positive, we need to prove that $\sqrt{K} > t_i^2 \alpha [(r_1 + r_2)^2 - 2r_1r_2/\alpha]$ for $0 < \alpha < 1$. Instead, we show that

$$\sqrt{K} < t_i^2 \alpha [(r_1 + r_2)^2 - 2r_1r_2/\alpha], \quad (B.6)$$

only when $\alpha > 1$. Note that, since $K > 0$, (B.6) can be squared without affecting to the inequality sign. Therefore squaring (B.6) and further simplification to it yields $\alpha > 1$. This proves that $q_{i,1} > 0$ and $q_{i,2} < 0$ when $v_c < t_i(r_1 + r_2)$ and the optimum solution to (4.13) is given by $q_{i,1}$.

B.4 Proof of Generalized Water-filling Solution for $n_R = 3$ Receive Antennas

Let $n_R = 3$ in (4.15b), then we obtain the third-order polynomial $a_3 q_i^3 + a_2 q_i^2 + a_1 q_i + a_0$ in $q_i$ which has roots [156]

$$q_{i,1} = -\frac{a_2}{3} + (S + T),$$

$$q_{i,2} = -\frac{a_2}{3} - \frac{1}{2}(S + T) + \frac{\sqrt{3}}{2}(S - T),$$

$$q_{i,3} = -\frac{a_2}{3} - \frac{1}{2}(S + T) - \frac{\sqrt{3}}{2}(S - T),$$
where $S \pm T = \left[ R + \sqrt{Q^3 + R^2} \right]^{1/3} \pm \left[ R - \sqrt{Q^3 + R^2} \right]^{1/3}$ and all other variables are as defined in Section 4.3.4, then the product $q_{i,1}q_{i,2}q_{i,3} = (r_1t_i + r_2t_i + r_3t_i - \nu_c)/r_1r_2r_3\nu_c t_i^3$.

**Case 1:** $q_{i,1}q_{i,2}q_{i,3} < 0 \Rightarrow \nu_c > t_i(r_1 + r_2 + r_3)$. Let $\nu_c = \alpha t_i(r_1 + r_2 + r_3)$, where $\alpha > 1$. For $\alpha > 1$, it can be shown that $(Q^3 + R^2) > 0$, hence $q_{i,1} < 0$ and $q_{i,2}, q_{i,3} \in \mathbb{C}$. Therefore, when $\nu_c > t_i(r_1 + r_2 + r_3)$, the optimum $q_i$ is zero.

**Case 2:** $q_{i,1}q_{i,2}q_{i,3} > 0 \Rightarrow \nu_c < t_i(r_1 + r_2 + r_3)$. Let $\nu_c = \alpha t_i(r_1 + r_2 + r_3)$, where $0 < \alpha < 1$. For $0 < \alpha < 1$, it can be shown that $(Q^3 + R^2) < 0$ and $R^4 > \frac{Q^3}{R}$, hence we get two negative roots $q_{i,2}, q_{i,3} < 0$ and one positive root $q_{i,1} > 0$ as the roots of cubic polynomial. Therefore, when $\nu_c < t_i(r_1 + r_2 + r_3)$, the optimum solution to (4.13) is given by $q_{i,1}$.

## B.5 Optimum Precoder for Differential STBC

### B.5.1 MISO Channel

The optimization problem involved in this case is similar to the water-filling problem in information theory, which has the optimal solution

$$p_i = \begin{cases} \frac{1}{\nu_d} - \frac{1}{t_i}, & \nu_d < t_i, \\ 0, & \text{otherwise}, \end{cases} \quad (B.7)$$

where the water-level $1/\nu_d$ is chosen to satisfy

$$\sum_{i=1}^{n_T} \max \left(0, \frac{1}{\nu_d} - \frac{1}{t_i} \right) = \frac{\pi \beta n_T}{8 + \beta}.$$

### B.5.2 $n_T \times 2$ MIMO Channel

The optimum $p_i$ for this case is

$$p_i = \begin{cases} A + \sqrt{K}, & \nu_d < t_i(r_1 + r_2); \\ 0, & \text{otherwise}, \end{cases}$$
where \( \nu \) is chosen to satisfy

\[
\sum_{i=1}^{n_T} \max \left( 0, A + \sqrt{K} \right) = \frac{7\beta n_T}{8 + \beta}
\]

with

\[
A = \frac{2r_1r_2t_i^2 - \nu dt_i(r_1 + r_2)}{2\nu d_1r_2t_i^2},
\]

and

\[
K = \frac{\nu^2 t_i^2(r_1 - r_2)^2 + 4r_1^2r_2^2t_i}{2\nu d_1r_2t_i^2}.
\]

**B.5.3 \( n_T \times 3 \) MIMO Channel**

For the case of \( n_T \) transmit antennas and \( n_R = 3 \) receive antennas, the optimum \( p_i \) is given by

\[
p_i = \begin{cases} 
-\frac{z_2}{3z_3} + Z, & \nu_d < t_i(r_1 + r_2 + r_3); \\
0, & \text{otherwise},
\end{cases}
\]

where \( \nu_d \) is chosen to satisfy

\[
\sum_{i=1}^{n_T} \max \left( 0, -\frac{z_2}{3z_3} + Z \right) = \frac{7\beta n_T}{8 + \beta}
\]

with

\[
Z = \left[ Z_2 + \sqrt{Z_3^2 + Z_2^2} \right]^{\frac{1}{2}} + \left[ Z_2 - \sqrt{Z_3^2 + Z_2^2} \right]^{\frac{1}{2}},
\]

\[
Z_1 = \frac{3z_1z_3 - z_2^2}{9z_3^2}, \quad Z_2 = \frac{9z_1z_2z_3 - 27z_0z_3^2 - 2z_3^3}{54z_3^3},
\]

\[
z_3 = \nu dt_1r_2r_3t_i^3, \quad z_2 = \nu dt_i^2(r_1r_2 + r_1r_3 + r_2r_3) - 3r_1r_2r_3t_i^3, \quad z_1 = \nu dt_i(r_1 + r_2 + r_3) - 2t_i^2(r_1r_2 + r_1r_3 + r_2r_3) \text{ and } z_0 = \nu_d - t_i(r_1 + r_2 + r_3).\n\]
Bibliography


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