Transmission problems for Dirac’s and Maxwell’s equations with Lipschitz interfaces

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A thesis submitted for the degree of Doctor of Philosophy of the Australian National University
Till minnet av min morfar Gunnard Rosén
som gav mig inspiration att ta steget
Declaration

The work in this thesis is my own except where otherwise stated.

Andreas Axelsson
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Erratum

 Listed below are the four papers that have been produced from this thesis.

Chapter 2:
Axelsson, A.
Transmission problems and boundary operator algebras.

Chapter 3:
Axelsson, A.
Oblique and normal transmission problems for Dirac operators with strongly Lipschitz interfaces.

Chapter 4:
Axelsson, A.; McIntosh, A.
Hodge decompositions on weakly Lipschitz domains.

Chapter 5:
Axelsson, A.
Transmission problems for Maxwell's equations with weakly Lipschitz interfaces.
(Submitted for publication)
Abstract

The aim of this thesis is to give a mathematical framework for scattering of electromagnetic waves by rough surfaces. We prove that the Maxwell transmission problem with a weakly Lipschitz interface, in finite energy norms, is well posed in Fredholm sense for real frequencies. Furthermore, we give precise conditions on the material constants $\epsilon$, $\mu$ and $\sigma$ and the frequency $\omega$ when this transmission problem is well posed.

To solve the Maxwell transmission problem, we embed Maxwell’s equations in an elliptic Dirac equation. We develop a new boundary integral method to solve the Dirac transmission problem. This method uses a boundary integral operator, the rotation operator, which factorises the double layer potential operator. We prove spectral estimates for this rotation operator in finite energy norms using Hodge decompositions on weakly Lipschitz domains.

To ensure that solutions to the Dirac transmission problem indeed solve Maxwell’s equations, we introduce an exterior/interior derivative operator acting in the trace space. By showing that this operator commutes with the two basic reflection operators, we are able to prove that the Maxwell transmission problem is well posed.

We also prove well-posedness for a class of oblique Dirac transmission problems with a strongly Lipschitz interface, in the $L_2$ space on the interface. This is shown by employing the Rellich technique, which gives angular spectral estimates on the rotation operator.

This thesis includes parts of the following papers.


(The main result given in Proposition 6.1.5.)


(A version included as Chapter 3.)


(A version included as Chapter 2.)
5.3 Spectral estimates ................................................. 93

6 Some complementary results ........................................ 103

   6.1 Duality in trace spaces ....................................... 103
   6.2 Hodge decomposition of trace spaces ....................... 106

A Pictures of spectra .................................................. 109

Bibliography .................................................................. 112
Chapter 0

Introduction

The motivation for this thesis comes from the study of the scattering of electromagnetic waves by a rough surface. We consider the following Maxwell transmission problem.

Let $\Sigma \subset \mathbb{R}^3$ be a surface/interface separating the bounded interior domain $\Omega^+$ and the exterior domain $\Omega^-$. Suppose that each of the domains $\Omega^+$ and $\Omega^-$ is composed of a linear, homogeneous, isotropic, possibly conducting material with permittivity $\varepsilon^\pm > 0$, permeability $\mu^\pm > 0$ and conductivity $\sigma^\pm \geq 0$. Define $\varepsilon^\pm := \varepsilon^\pm + i\sigma^\pm/\omega$, where $\omega$ is the frequency. In $\Omega^\pm$, monochromatic, i.e. time-harmonic, electric and magnetic fields $E(x,t) = E(x)e^{-i\omega t}$ and $B(x,t) = B(x)e^{-i\omega t}$ satisfy Maxwell’s equations

$$\nabla \cdot B = 0,$$
$$-i\omega B + \nabla \times E = 0,$$
$$-i\omega \varepsilon^\pm E - \nabla \times \left( \frac{1}{\mu^\pm} B \right) = 0,$$
$$\nabla \cdot (\varepsilon^\pm E) = 0.$$

Given an incoming field $\{E_0, B_0\}$ in $\Omega^-$, find a transmitted field $\{E^+, B^+\}$ in $\Omega^+$ and a reflected field $\{E^-, B^-\}$ in $\Omega^-$ such that $\{E^\pm, B^\pm\}$ satisfies Maxwell’s equations in $\Omega^\pm$, $\{E^-, B^-\}$ satisfies the Silver–Müller radiation condition at $\infty$ and on $\Sigma$ we have the jump relations

$$\nu \cdot (B_0 + B^- - B^+) = 0,$$
$$\nu \times (E_0 + E^- - E^+) = 0,$$
$$\nu \times \left( \frac{1}{\mu^-} (B_0 + B^-) - \frac{1}{\mu^+} B^+ \right) = 0,$$

$$\nu \cdot \left( \varepsilon^- (E_0 + E^-) - \varepsilon^+ E^+ \right) = 0,$$  \tag{1}

where $\nu$ denotes the outward pointing unit normal on $\Sigma$.

The main themes in this thesis are the following.

(i) Embedding of Maxwell’s equations in an elliptic Dirac equation.

(ii) A purely first order approach to Dirac’s and Maxwell’s equations. We develop methods for directly working with the first order Dirac equation, instead of reducing to a problem for the Laplace operator.
(iii) Splittings of function spaces. For example, two complementary projections \( A^\pm \) in a Hilbert space induce the splitting \( \mathcal{H} = A^+ \mathcal{H} \oplus A^- \mathcal{H} \). Also relevant to transmission problems are Hodge type splittings \( \mathcal{H} = R(\Gamma) \oplus (N(\Gamma) \cap N(\Gamma^*)) \oplus R(\Gamma^*) \) induced by a nilpotent operator \( \Gamma \), i.e. \( \Gamma^2 = 0 \).

(iv) Fredholm operator theory.

We now give an overview of our approach to solving the Maxwell transmission problem.

- A usual approach to Maxwell’s equations is to discard the two Gauss divergence equations as redundant. However, by only using the Faraday and Maxwell-Ampère equations the partial differential system is no longer elliptic. In this thesis we show in detail how to solve the Maxwell transmission problem (1) with elliptic methods, using the Gauss equations in a natural way. The basic idea is to rewrite Maxwell’s equations as an elliptic Dirac equation

\[
D_k F := e_1 \triangle \partial_1 F + e_2 \triangle \partial_2 F + e_3 \triangle \partial_3 F + k e_0 \triangle F = 0.
\]

Here \( F := -i/\sqrt{\varepsilon} e_0 \wedge E + 1/\sqrt{\mu} B \) is the full electromagnetic field, where we have replaced the magnetic field above with the corresponding Hodge dual field \( B = B_1 e_2 \wedge e_3 + B_2 e_3 \wedge e_1 + B_3 e_1 \wedge e_2 \), and where \( \{i e_0, e_1, e_2, e_3\} \) are the basis vectors for Minkowski space and \( k := \omega \sqrt{\varepsilon \mu} \) is the wave number. We write \( \wedge \) for the exterior product and \( \triangle \) for the Clifford product.

It is important to note that Maxwell’s equations combine to a Dirac equation, but also that the electromagnetic field \( F \) is a special solution of \( D_k F = 0 \) as it is a pure bivector field \( F : \Omega \to \wedge^2 \mathbb{R}^4 \). Surprisingly, it turns out that we can reformulate this constraint on the range of \( F \) as a divergence/curl free condition on \( F \), see Proposition 3.2.5.

- To solve the transmission problem for the Dirac operator \( D_k \), we develop a new boundary integral method. For simplicity, consider the “relativistic” case when \( e_+ \mu^+ = e^- \mu^- \). In this case, (1) can be written in the form

\[
\begin{cases}
    N^+ (\alpha^- f^+ - \alpha^+ f^-) = N^+ g & \text{on } \Sigma, \\
    N^- (\alpha^+ f^+ - \alpha^- f^-) = N^- g & \text{on } \Sigma, \\
    D_k F^\pm = 0 & \text{in } \Omega^\pm. 
\end{cases}
\]  

(2)

Here \( g \) is determined by \( \{E_0, B_0\} \), \( F^\pm \) is the electromagnetic field with components \( \{E^\pm, B^\pm\} \) and boundary trace \( f^\pm \), the jump parameters \( \alpha^\pm \) are determined by \( e^\pm \) and \( \mu^\pm \) and \( N^+ \) and \( N^- \) denote the tangential and normal projection operators respectively. In this Dirac transmission problem, \( F^\pm \) and \( g \) in general take values in the full exterior algebra \( \wedge = \wedge \mathbb{R}^{n+1} \).

Associated with the Dirac operator \( D_k \) is a Cauchy type singular integral operator \( E_k \). Denoting the corresponding Hardy type projection operator onto the space of
traces of monochromatic fields in $\Omega^\pm$ by $E_k^\pm$, we make the ansatz $f^\pm = E_k^\pm f$, where $f = f^+ + f^-$. We now introduce the reflection operators $E_k := E_k^+ - E_k^-$ and $N := N^+ - N^-$, where $E_k^2 = N^2 = I$, as well as the rotation operator $E_k N$. Just as one uses the double layer potential operator to solve the Dirichlet boundary value problem for the Laplace operator, so we use the rotation operator $E_k N$ to solve the Dirac transmission problem. Indeed, (2) is equivalent to the integral equation

$$ (\lambda - E_k N)f = 2E_k g, $$

where $\lambda := (\alpha^+ + \alpha^-)/(\alpha^+ - \alpha^-)$. A remark here is that the rotation operator essentially factorises the double layer potential type operator, just as $D_k$ squares to the Helmholtz operator $D_k^2 = \Delta + k^2$, see Proposition 2.1.2.

**Note** that a boundary value problem for the Dirac operator corresponds to the spectral points $\lambda = \pm 1$ in (2), and more generally a relativistic Maxwell transmission problem with non-conducting materials corresponds to real $\lambda$. Ideally, one would like to prove that the spectrum of $E_k N$ is close to the unit circle $|\lambda| = 1$, and bounded away from $\lambda = \pm 1$. In this thesis, we show how the local regularity of the interface $\Sigma$ influences the essential spectrum of the rotation operator. We consider two boundary function spaces.

(a) The space $L_2(\Sigma; \lambda)$ on a strongly Lipschitz interface $\Sigma$ together with a domain space $D(\Gamma; L_2) \subset L_2(\Sigma; \lambda)$ of mixed 0 and 1 regularity. In Chapter 3 we prove angular spectral estimates using the Rellich technique. In Section 3.1 we also prove estimates for more general oblique Dirac transmission problems in $L_2(\Sigma; \lambda)$.

(b) The energy trace space $\mathcal{E}(\Sigma; \lambda)$ on a weakly Lipschitz interface. This is a Hilbert space of mixed $\pm 1/2$ regularity which we define and use in Chapter 5 to solve Maxwell transmission problems for fields with locally finite energy. In Section 5.3, we prove estimates on $\sigma_{ess}(E_k N; \mathcal{E})$ using Hodge decompositions. In Chapter 4 we give a pure first order approach, free from Laplace operators, to Hodge decompositions on bounded, weakly Lipschitz domains.

**As a last step in solving the Maxwell transmission problem, we prove that a solution to the Dirac transmission problem (2) is actually a solution to (1). We here encounter Hodge type decompositions of Hilbert spaces again. Indeed, the Maxwell transmission problem is essentially a Dirac transmission problem restricted to the divergence/curl-free Hodge component of the trace space. We prove that there exists a nilpotent exterior/interior derivative operator $\Gamma_k$ in the boundary trace spaces such that**

- on normal fields, $\Gamma_k$ acts as a tangential divergence,
- on tangential fields, $\Gamma_k$ acts as a tangential curl, and
- $\Gamma_k$ commutes with both reflection operators $E_k$ and $N$. 

3
We now solve (1) as follows. A function $g$ in (2) coming from a Maxwell field $\{E_0, B_0\}$ satisfies $\Gamma_k g = 0$. Since $\Gamma_k E_k = E_k \Gamma_k$ and $\Gamma_k N = N \Gamma_k$, it follows that $\Gamma_k f^\pm = 0$. This essentially means that $F^\pm$ are Maxwell fields solving (1).

We now state our main result on the Maxwell transmission problem (1). This is a special case of Theorem 5.2.3 as explained in Example 5.2.4.

**Theorem 0.0.1.** Let $\Sigma \subset \mathbb{R}^3$ be a bounded, weakly Lipschitz surface, as in Definition 1.5.1, and consider the Maxwell transmission problem (1) in the finite energy trace space $\mathcal{E}$, defined in Section 5.1. Define jump parameters $\alpha := \sqrt{\varepsilon^+ / \varepsilon^-}$ and $\beta := \sqrt{\mu^- / \mu^+}$, and wave numbers $k^\pm = \omega \sqrt{\varepsilon^\pm \mu^\pm}$. Let $C_\Sigma$ and $C_{\Sigma, \Sigma}$ be the constants from Theorem 5.3.1 and Theorem 5.3.5. Then (1) is well posed in Fredholm sense if

$$\alpha \notin \{ix ; x \in \mathbb{R}, 1/C\Sigma \leq |x| \leq C_{\Sigma}\},$$

and it is well posed if $\text{Im} k^+ > 0$ and $\text{Im} k^- > 0$ and either

$$|\arg \alpha| + |\arg k^+ - \frac{\pi}{2}| + |\arg k^- - \frac{\pi}{2}| < \pi \quad \text{or}$$

$$\min(|\alpha - \beta|, |\frac{1}{\alpha} - \frac{1}{\beta}|) < 2/C_{\Sigma, \Sigma}.$$

The background to this thesis and in particular Chapter 3 is the development in harmonic analysis, the theory of Calderón–Zygmund operators, wavelet theory and boundary value problems on Lipschitz domains since the late 1970’s. The central result here is $L_2(\Sigma)$-boundedness of the Cauchy singular integral operator on strongly Lipschitz surfaces. In two dimensions this was first proved by Calderón [8] when the Lipschitz constant is small and by Coifman–McIntosh–Meyer [10] in the general case. There are by now many proofs and extensions of this celebrated result. The proof which seems most relevant to us in connection with transmission problems is Li–McIntosh–Semmes’ [33] higher dimensional extension of Coifman–Jones–Semmes’ first proof in [9], which we survey in Section 1.5.

A classical method for solving the Dirichlet and Neumann boundary value problems for the Laplace operator is to solve the associated boundary integral equation, an equation of the second kind involving the double layer potential operator or its adjoint. The $L_2(\Sigma)$-boundedness of the double layer potential operator follows from that of the Cauchy singular integral operator. As for the invertibility of the double layer potential equation on strongly Lipschitz surfaces, this was proved in the important work by Verchota [63] using Rellich estimates as a substitute for Fredholm theory. There is much related work on this topic by Dahlberg, Fabes, Jerison, Kenig and others. A good reference is the book [30] by Kenig.

The Rellich estimate technique for solving boundary value problems on strongly Lipschitz domain has also been applied with success to other partial differential systems such as the Lamé system of elasticity, the Stokes system of hydrostatics and Maxwell’s equations in electromagnetic theory. Relevant to us here is the latter, where Rellich estimates were adapted to the study of Maxwell’s equations by Mitrea–Torres–Wolland [51] and Mitrea [50] and to the Dirac equation by McIntosh–Mitrea [39] and McIntosh–Mitrea–Mitrea [38].
Compared with (a) above, the Maxwell transmission problem in energy norms in (b) does not require the techniques from harmonic analysis above, and a more complete spectral theory can be obtained. To show boundedness of the boundary integral operators, only Fourier theory on $\mathbb{R}^n$ is needed. For the classical double layer potential operator, early spectral estimates can be found in Kellogg’s classical book [29]. For the Dirac equation, the substitute for Rellich estimates is Hodge decompositions, and this theory goes through in the larger class of weakly Lipschitz domains. The theory presented in Chapter 4 and Chapter 5 has essentially been obtained independently here. However, the method employed to obtain Hodge decompositions on weakly Lipschitz domains has also been observed by Picard [52]. Also, Dorina and Marius Mitrea have investigated boundary value problems related to Chapter 5, on strongly Lipschitz domains in [43] and [44]. See the introduction to Chapter 5.

Finally, relevant to boundary value problems and transmission problems for Maxwell’s and Dirac’s equations, we would also like to mention the monographs Colton–Kress [11], Schwarz [57] and Jiang [27]. In the latter, it is pointed out that the elliptization procedure in (i) above is important for numerical stability and that the Gauss equations should not be neglected.
Chapter 1

Preliminaries

We start this chapter by describing the geometric algebra we use throughout this thesis. With geometric algebra we here mean the calculus with certain products $\wedge$, $\cdot$, $\wedge$ and $\triangle$ on an exterior algebra $\wedge E$. For a fixed real, euclidean $n$-dimensional space $E$, consider the full exterior algebra

$$\wedge E := \wedge^0 E \oplus \wedge^1 E \oplus \wedge^2 E \oplus \ldots \oplus \wedge^{n-1} E \oplus \wedge^n E.$$ 

Recall that the $\binom{n}{j}$ dimensional linear space $\wedge^j E$ consists of all $j$-vectors in $E$. In particular $\wedge^1 E$ are the vectors in $E$ and we identify $\wedge^0 E$ with the scalars $\mathbb{R}$. Using the euclidean structure of $E$ we always identify a $j$-vector with its dual (alternating) $j$-form. We call a general object $u = \sum_j u_j \in \wedge E$, with $u_j \in \wedge^j E$, a multivector.

Any given ON-basis $\{e_1, \ldots, e_n\}$ for $E \cong \wedge^1 E$ induces the basis $\{e_s\}_{s \subset \{1, \ldots, n\}}$ for $\wedge E$, where $e_s := e_{s_1} \wedge \ldots \wedge e_{s_j}$ if $s = \{s_1, \ldots, s_j\}$ and $1 \leq s_1 < \ldots < s_j \leq n$, and $\{e_s\}_{|s|=j}$ is a basis for $\wedge^j E$. Define the counting function

$$\sigma(s, t) := \# \{(s_i, t_j) ; s_i > t_j\},$$

where $s = \{s_i\}$, $t = \{t_j\} \subset \{1, \ldots, n\}$. We regard the following operations on $\wedge E$ as basic.

(i) The exterior product of two basis multivectors $e_s$ and $e_t$ is

$$e_s \wedge e_t = (-1)^{\sigma(s, t)} e_{s \cup t} \quad \text{if} \ s \cap t = \emptyset \ \text{and otherwise zero.}$$

(ii) The standard scalar product $(\cdot, \cdot)$ on $\wedge E$ is the one for which the induced basis $\{e_s\}_{s \subset \{1, \ldots, n\}}$ is an ON-basis.

(iii) The left (right) interior product $u \wedge v$ $(u \triangledown v)$ is the unique bilinear (non-associative) product for which $(u \wedge x, y) = (x, u \wedge y)$ and $(x \triangledown u, y) = (x, y \wedge u)$ respectively for all $u, x, y \in \wedge$. The action on two basis vectors $e_s$ and $e_t$ is

$$e_s \wedge e_t = (-1)^{\sigma(s, t \setminus s)} e_{t \setminus s}, \quad e_t \triangledown e_s = (-1)^{\sigma(t \setminus s, s)} e_{t \setminus s},$$

if $s \subset t$ and otherwise zero.
(iv) The reversion $\overline{u}$ and the involution $u^\top$ are the unique linear extensions of

$$
\overline{e_s} := \epsilon_{s_j} \wedge \ldots \wedge \epsilon_{s_1} = (-1)^{j(j-1)/2} e_s,
\epsilon_s^\top := (-\epsilon_{s_1}) \wedge \ldots \wedge (-\epsilon_{s_j}) = (-1)^j \epsilon_s.
$$

1.1 Clifford algebra

The Clifford product is the unique associative algebra product $\triangle$ on $\wedge E$ with identity $1 = e_0 \in \wedge^0 E$ which satisfies

(i) $a_1 \triangle \ldots \triangle a_k = a_1 \wedge \ldots \wedge a_k$ when $\{a_j\}$ are orthogonal vectors in $\wedge^1 E$, and

(ii) $a \triangle b = (a, b) + a \wedge b$ for all vectors $a, b \in \wedge^1 E$.

If $\triangle$ denotes the symmetric difference when acting on index sets, then we have

$$
e_s \triangle e_t = (-1)^{\sigma(s,t)} e_s \triangle_t.
$$

When there is no risk of confusion we will use the standard short-hand notation $uv := u \triangle v$ for the Clifford product.

One reason why the Clifford product is useful for us is its strong connection with reflection operators, see for example Example 1.1.6(iii) and Corollary 2.1.6. Also note that the inverse $a^{-1}$ of a non-zero vector $a \in \wedge^1 E$ with respect to the Clifford product is the inversion $a/|a|^2$ in the unit sphere. We now discuss the relation between Clifford algebra and the linear operators on the exterior algebra $\wedge E$. A source of inspiration has been the book [66] by Yu.

**Definition 1.1.1.** A real Clifford algebra $A$ of signature $(p, q)$ is a real associative algebra with unit $1$ and generators $\{a_i\}_{i=1}^k$ which satisfy the canonical anti commutation relation

$$
a_i a_j + a_j a_i = 2 \epsilon_i \delta_{ij} 1,
$$

where $\epsilon_i = +1$ for $i = 1, \ldots, p$ and $\epsilon_i = -1$ for $i = p + 1, \ldots, p + q = k$.

Recall the following basic linear independence lemma for Clifford algebras. For a proof, see Theorem 15.10 in Porteous [53].

**Lemma 1.1.2.** Let $A$ be a real Clifford algebra with signature $(p, q)$. Then $\dim A = 2^{p+q}$ if $p - q \neq 1(\text{mod } 4)$, and in case $p - q \equiv 1(\text{mod } 4)$ then either $\dim A = 2^{p+q}$ or $\dim A = 2^{p+q-1}$.

A basic example of a Clifford algebra is $L(\wedge E)$, the algebra of linear operators on the exterior algebra $\wedge E$.

**Proposition 1.1.3.** There exists two unique linear injective maps

$$
Q^\pm : \wedge E \longrightarrow L(\wedge E),
$$

with the following properties.
(i) For vectors $a, b \in \Lambda^1 E$, we have

$$a^\pm(\omega) := (Q^\pm(a))(\omega) = \pm a \wedge \omega + a \wedge \omega, \quad \omega \in \wedge E,$$

and the canonical anti commutation relation $a^\pm b^\pm + b^\pm a^\pm = \pm 2(a, b)I$ holds. The operators $a^\pm$ are self-adjoint and operators $b^- \wedge b^- a^\pm = 0$.

(ii) We have $Q^\pm(1) = I$ and if $\{a_j\}$ are orthogonal vectors, then

$$Q^\pm(a_1 \wedge \ldots \wedge a_k) = a_1^\pm \ldots a_k^\pm.$$

(iii) Define subspaces $\mathcal{L}^\pm_j(\wedge E) := Q^\pm(\wedge^j E)$. The full operator algebra $\mathcal{L}(\wedge E)$ contains the two subalgebras

$$\mathcal{L}^\pm(\wedge E) := Q^\pm(\wedge E) = \mathcal{L}_0^\pm(\wedge E) \oplus \mathcal{L}_1^\pm(\wedge E) \oplus \ldots \oplus \mathcal{L}_n^\pm(\wedge E),$$

the positive and negative Clifford subalgebra. The intersection is $\mathcal{L}^+(\wedge E) \cap \mathcal{L}^-(\wedge E) = \mathcal{L}_0^+(\wedge E) = \mathcal{L}_n^- = \text{span}(I)$ and $\mathcal{L}(\wedge E)$ is generated by $\mathcal{L}^{+}(\wedge E)$ and $\mathcal{L}^{-}(\wedge E)$.

The ON-basis $\{e_i\}_{i=1}^n$ gives linear bases $\{e_i^+\}_{i=1}^n$ for $\mathcal{L}_1^+(\wedge E)$ and $\{e_i^-\}_{i=1}^n$ for $\mathcal{L}_1^-(\wedge E)$. Furthermore we get an algebraic basis $\{e_i^+, e_i^-\}_{i=1}^n$ for $\mathcal{L}(\wedge E)$ and the corresponding linear basis $\{e_s^s e_i^t\}_{s,t \in \{1, \ldots , n\}}$.

The Clifford algebra $\mathcal{L}(\wedge E)$ contains two subalgebras $\mathcal{L}^\pm(\wedge E)$ which both look like $\wedge E$, since $Q^\pm : \wedge E \to \mathcal{L}^\pm(\wedge E)$ are linear isomorphisms. Using this we can define two algebra products on $\wedge E$. The positive/negative Clifford product $\Delta_{\pm}$ on $\wedge E$ is

$$u \Delta_{\pm} v := (Q^{\pm})^{-1}(Q^{\pm}(u)Q^{\pm}(v)), \quad u, v \in \wedge E.$$

The positive Clifford product $\Delta_{+}$ is the Clifford product $\Delta$ from above.

**Example 1.1.4.** When $n = 1$ and $n = 2$, the matrices for $e_1^+, e_1^-, (e_2^+)$ and $e_2^-$ in the bases $\{e_1, 1\} \subset \wedge \mathbb{R}$ and $\{e_1, e_2, 1, e_{12}\} \subset \wedge \mathbb{R}^2$ respectively are

$$e_1^+ \approx \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2^+ \approx \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$e_1^- \approx \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2^- \approx \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These are closely related to the classical Pauli’s and Dirac’s matrices in quantum mechanics.

**Proposition 1.1.5 (Universal property).** For any linear operator $T \in \mathcal{L}(E)$ on the vector space $E$, there exists a unique $\wedge$-homomorphism $T^\wedge \in \mathcal{L}(\wedge E)$ such that $T^\wedge|\wedge E = T$ and $T^\wedge(1) = 1$. The extension map $T \mapsto T^\wedge$ is an injective group homomorphism.
The matrix for $T^\vee$, which contains information on how $T$ acts on all subspaces of $E$, is block diagonal ($\wedge E$ is invariant under $T^\vee$) and contains the matrix for $T$ along with all its subdeterminants.

**Example 1.1.6.** (i) The operator $R_a := a^+a^-$, where $a \in S^{n-1} \subset \wedge^1 E$, is the $\wedge$-homomorphism on $\wedge E$ that extends the reflection in the hyperplane $\{a\}^\perp$ in the sense of Proposition 1.1.5. For example, the action of $R_i := R_{e_i}$ is

$$R_i(e_s) = \begin{cases} -e_s & i \in s, \\ e_s & i \notin s. \end{cases}$$

(ii) The symplectic operator $J := R_1 + \ldots + R_n$ acts like $J(e_s) = (n - 2|s|)e_s$ on the basis elements. Thus the spectrum is $\sigma(J) = \{n, n-2, \ldots, -(n-2), -n\}$, and the corresponding eigen spaces are $\{\wedge^0 E, \wedge^1 E, \ldots, \wedge^{n-1} E, \wedge^n E\}$. We can obtain any operator with these eigen spaces from functional calculus on $J$. In particular, the involution operator $u \mapsto u^\perp$ is the “volume element” $(-1)^{(n-J)/2} = R_1 \cdot \ldots \cdot R_n = e_1^+e_1^- \cdot \ldots \cdot e_n^+e_n^-.$

(iii) Using the involution operator, we can express $a^-$, where $a \in S^{n-1} \subset \wedge^1 E$, in terms of $\triangle_+$ since $a^\perp(\omega) = \omega^\perp \triangle_+ a$. Thus the action of $R_a$ is

$$R_a(\omega) = a^+a^-(\omega) = a \triangle \omega^\perp \triangle a = a\omega^\perp a = a \triangle (a \wedge \omega) - a \wedge (a \wedge \omega). \quad (1.1)$$

**Remark 1.1.7.** More generally, we see that all linear operators on $\wedge E$ can be expressed using the involution operator and (twosided multiplication with) the Clifford product $\triangle$.

We summarise basic geometric algebra identities, which will be used frequently in this thesis without explicit reference. There are analogous formulae for the right interior product, but we omit them here.

**Proposition 1.1.8.** For a vector $a \in \wedge^1 E$ and for general multivectors $u, v$ and $w \in \wedge E$ the following hold.

$$u \wedge (v \wedge w) = (v \wedge u) \wedge w, \quad u \wedge (v \wedge w) = (u \wedge v) \wedge w \quad (1.2)$$

$$a \triangle u = a \wedge u + a \wedge u \quad (1.3)$$

$$a \wedge u = -u^\perp \wedge a = \frac{1}{2}(a \wedge u - u^\perp \triangle a) \quad (1.4)$$

$$a \wedge u = u^\perp \wedge a = \frac{1}{2}(a \wedge u + u^\perp \triangle a) \quad (1.5)$$

$$a \wedge (u \wedge v) = (a \wedge u) \wedge v + u^\perp \wedge (a \wedge v) \quad (1.6)$$

$$a \wedge (u \wedge v) = (a \wedge u) \wedge v + u^\perp \wedge (a \wedge v) \quad (1.7)$$

$$a \wedge (u \wedge v) = (a \wedge u) \wedge v + u^\perp \wedge (a \wedge v) \quad (1.8)$$

$$(u \wedge v, w) = (v, \overline{w} \wedge u) = (u, w \wedge \overline{v}) \quad (1.9)$$

$$(\overline{u}, \overline{v}) = (u, v) = (u^\perp, v^\perp) \quad (1.10)$$

$$\overline{u \wedge v} = \overline{v} \wedge \overline{u}, \quad (u \wedge v)^\perp = u^\perp \wedge v^\perp \quad (1.11)$$

$$\overline{u \wedge v} = \overline{v} \wedge \overline{u}, \quad (u \wedge v)^\perp = u^\perp \wedge v^\perp \quad (1.12)$$

$$\overline{u \wedge v} = \overline{v} \wedge \overline{u}, \quad (u \wedge v)^\perp = u^\perp \wedge v^\perp \quad (1.13)$$

9
Here (1.2) are associativity properties of the interior products. The formulae (1.4) and (1.5), which are inverse to (1.3), are sometimes referred to as Riesz’ formulae, see Riesz [56]. An important special case of (1.4) is when $u = b \in \wedge^1 E$. Then $a \wedge b = b \wedge a = (a, b)$, so we have the canonical anti-commutation relation
\[
(a, b) = \frac{1}{2} (a \wedge b + b \wedge a),
\]
as in Definition 1.1.1. The formulae (1.6), (1.7) and (1.8) are derivation properties for the interior and exterior products. A classical example of (1.6) is when $a, b = u$ and $c = v$ are vectors in a three-dimensional space. Using the Hodge complement $u_\perp := u \wedge e_{123}$ and the vector product $b \times c = (b \wedge c)_\perp$, we get the well known identity
\[
-a \times (b \times c) = a \wedge (b \wedge c) = (a, b) c - (a, c) b.
\]

1.2 Calculus with $\wedge$

For the rest of this thesis, the euclidean space $E$ will be either the space $\mathbb{R}^n$ or spacetime $\mathbb{R}^{n+1}$ and for the analysis we will use the complexified exterior algebra $\wedge := \wedge C \mathbb{R}^n$ or $\wedge C \mathbb{R}^{n+1}$. The “algebraic” imaginary unit used in the complexification we denote by $i$ (as compared with the “geometric” imaginary unit $j := e_1 e_2 \in \wedge^2 \mathbb{R}^2$). When spacetime is considered, we add the vector $e_0$ to the standard ON-basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$. Here $e_0^2 = e_0 \wedge e_0 = 1$ and we interpret $\overline{e_0} := -ie_0$ as the forward time direction and $\overline{e_0} := ie_0$ as the backward time direction (thus the real Minkowski space is span$_{\mathbb{R}} \{\overline{e_0}, e_1, \ldots, e_n\}$).

The operations $\wedge$, $\wedge$, $\perp$, $\Delta$, $(\cdot, \cdot)$, $u^\perp$ and $\pi$ are extended to complex (bi-)linear operations and thus Proposition 1.1.8 remains valid.

Furthermore we use functions
\[
F : \Omega \longrightarrow \wedge,
\]
defined in some open set $\Omega \subset \mathbb{R}^n$ and which take values in the exterior algebra $\wedge$. On the boundary $\Sigma = \partial \Omega$ we denote boundary traces $F|_{\Sigma}$ with the corresponding small letter, e.g. $f$. Functions with range in $\wedge$ are referred to as (multivector-)fields.

Throughout this thesis we make use of the nabla symbol $\nabla = \sum_{j=1}^n e_j \partial_j$ and nabla symbol $\nabla_k = \nabla + k e_0$ with wave number $k$. We recall that the products $\wedge$, $\perp$ and $\Delta$ induce differential operators
\[
dF(x) := \nabla \wedge F(x) = \sum_{j=1}^n e_j \wedge (\partial_j F)(x),
\]
\[
\delta F(x) := \nabla \perp F(x) = \sum_{j=1}^n e_j \perp (\partial_j F)(x),
\]
\[
DF(x) := \nabla \Delta F(x) = \sum_{j=1}^n e_j \Delta (\partial_j F)(x) = dF(x) + \delta F(x).
\]

In the same spirit we also denote the full differential of $F$ by $\nabla \otimes F(x) = \sum e_j \otimes (\partial_j F)(x)$. Note that these operators do not involve any differentiation in time direction when $E =$
Here the formal adjoint of the exterior derivative operator $d$ is the negative of the interior derivative $\delta$; this differs from the standard convention. Sometimes we refer to $d$ as (generalised) curl and to $\delta$ as (generalised) divergence. The (elliptic) Dirac operator $D = d + \delta$ is formally skew-adjoint. Here $D$ is a square root of the Hodge–Laplace operator $\Delta = d\delta + \delta d$. Note that $D$ is a local differential operator as compared with the scalar pseudo differential square root $\sqrt{\Delta}$. On the other hand $\sqrt{\Delta}$ is a positive operator, whereas $D$ has two-sided unbounded spectrum.

The most important property of the differential operators $d$ and $\delta$ is that they commute with a change of variables if we change the direction of the field in an appropriate way.

**Definition 1.2.1.** Let $\rho : U \rightarrow V$ be a bijection between two open sets $U, V \subset \mathbb{R}^n$ such that both $\rho$ and $\rho^{-1}$ are smooth. Denote by $\rho_*|_{\wedge}$ the linearisation of $\rho$ at $x \in U$ and extend this map to an $\wedge$-isomorphism on $\wedge$ with Proposition 1.1.5. To a field $F : V \rightarrow \wedge$ we now associate the pullback and pushforward fields in $U$ as follows.

$$(\rho^* F)(x) := (\rho_*|_{\wedge})^*(F(\rho(x))), \quad (\rho_*^{-1} F)(x) := (\rho_*|_{\wedge})^{-1}(F(\rho(x))).$$

For convenience, we also define the reduced push forward $\tilde{\rho}_*^{-1} F(x) := J(\rho)(\rho_*^{-1} F)(x)$, where $J(\rho) \approx \rho|_{\wedge}$ denotes the Jacobian volume factor.

Note that $\rho_*$ and $\rho^*$ are adjoint linear operators pointwise whereas $\tilde{\rho}_*$ and $\rho^*$ are adjoint linear operators in integrated $L_2$-sense. Recall the following well known properties of these changes of variables.

**Proposition 1.2.2.** If $\rho$ and $F$ are as in Definition 1.2.1, then we have commutation properties

$$d(\rho^* F) = \rho^*(dF), \quad \delta(\tilde{\rho}_*^{-1} F) = \tilde{\rho}_*^{-1}(\delta F), \quad \text{(1.14)}$$

and homomorphism properties

$$\rho^* (F \wedge G) = \rho^* F \wedge \rho^* G, \quad \rho_*^{-1} (F \wedge G) = \rho_*^{-1} F \wedge \rho_*^{-1} G, \quad \text{(1.15)}$$

$$\rho^* (F \vee G) = \rho_*^{-1} F \vee \rho_*^{-1} G, \quad \rho_*^{-1} (F \triangledown G) = \rho^* F \triangledown \rho_*^{-1} G. \quad \text{(1.16)}$$

In particular, if $F^\perp := F \vee \varepsilon_{0 \ldots n}$ denotes the complement of $F$ (i.e. the Hodge star $\ast F$), then $\rho^*(F^\perp) = (\tilde{\rho}_*^{-1} F)^\perp$.

To study time harmonic solutions to the hyperbolic Dirac equation

$$(-\varepsilon_0 \frac{\partial}{\partial t} + D)F(t, x) = 0,$$

we need to introduce zero order perturbations

$$d_k F := \nabla_k \wedge F = (\nabla + \varepsilon_0) \wedge F = dF + \varepsilon_0 \wedge F;$$

$$\delta_k F := \nabla_k \vee F = (\nabla + \varepsilon_0) \vee F = \delta F + \varepsilon_0 \vee F;$$

$$D_k F := \nabla_k \triangle F = (\nabla + \varepsilon_0) \triangle F = D F + \varepsilon_0 \triangle F = d_k F + \delta_k F,$$
with wave number $k \in \mathbb{C}$, which satisfy the adjoint relation $d_k^* = -\delta_{-k}$ with respect to $\int (F, G^c)$, and $D_k = d_k + \delta_k$ which is skew adjoint if and only if $k \in i\mathbb{R}$. Note that $d_k$ and $\delta_k$ are nilpotent operators, i.e. $d_k^2 = \delta_k^2 = 0$ and that $D_k$ factorises the Helmholtz operator $D_k^2 = \Delta + k^2$. Fields satisfying $D_k F = 0$ we call ($k$-)monochromatic, and sometimes monogenic when $k = 0$.

The elliptic operator $D = D_0$ has a fundamental solution

$$E(x) = E_0(x) := \frac{x}{\sigma_{n-1}|x|^n},$$

where $\sigma_{n-1} = 2\pi^{n/2}(\Gamma(n/2))^{-1}$ is the area of the unit sphere in $\mathbb{R}^n$. For non-zero $k$, we use known expressions for fundamental solutions to the Helmholtz operator. Two references for Bessel functions used here are Watson [65] and Section 3.6 in Taylor [60]. Let $H^{(1)}_n(z)$ denote the $\alpha$ order Hankel function of the first kind. Then the Bessel potential

$$\Phi_k(x) = -\frac{i}{4} \left( \frac{k}{2\pi |x|} \right)^{n/2-1} H^{(1)}_{n/2-1}(k|x|)$$

is a radial fundamental solution to the Helmholtz operator, $(\Delta + k^2)\Phi_k = \delta$. In odd dimension, $\Phi_k$ can be expressed in elementary functions using the explicit expression

$$H^{(1)}_{1/2}(z) = -i (\frac{2}{\pi z})^{1/2} e^{iz}$$

and the recursion formula $H^{(1)}_{\alpha+1}(z) = (\frac{\alpha}{z} - \frac{d}{dz}) H^{(1)}_{\alpha}(z)$, for $\alpha \in \mathbb{R}$. For example, $\Phi_k(x) = -e^{ik|x|/(4\pi |x|)}$ when $n = 3$. In odd dimensions $k \mapsto \Phi_k$ is analytic in $\mathbb{C}$ while in even dimensions it is analytic on the Riemann surface for $\log z$ since $H^{(1)}_{n/2-1}$ contains a logarithm. However, with abuse of notation we will still write $k \in \mathbb{C}$ in the even dimensional case.

**Definition 1.2.3.** Factorise $\Delta + k^2 = (D + ik \vec{e}_0)^2 = (D + ik \vec{e}_0)^2$. Define two families of fundamental solutions

$$\vec{E}_k(x) := (D + ik \vec{e}_0)\Phi_k(x) = (\hat{\rho} \frac{\partial}{\partial \hat{\rho}} + ke_0)\Phi_k(r),$$

$$\vec{E}_{-k}(x) := (D + ik \vec{e}_0)\Phi_k(x) = (\hat{\rho} \frac{\partial}{\partial \hat{\rho}} - ke_0)\Phi_k(r).$$

Here $D_k \vec{E}_k(\hat{x}) = \vec{E}_k(\hat{x})D_k = \delta_0(x)$ and $D_{-k} \vec{E}_{-k}(\hat{x}) = \vec{E}_{-k}(\hat{x})D_{-k} = \delta_0(x)$, where $\hat{x}$ denotes the variable which the differential operator acts on. With a little abuse of notation we write $\mathcal{F}D_k := \sum_j (\theta_j F) \triangle e_j + kF \triangle e_0$.

**Lemma 1.2.4.** We have the three identities

$$\vec{E}_k(x) = -\vec{E}_{-k}(-x) = e_0 \vec{E}_k(-x)e_0 = (\vec{E}_k(x))^c.$$

In odd dimension $n = 2d + 1$, the fundamental solution $\vec{E}_k(x)$ explicitly is

$$\vec{E}_k(x) = \left( \frac{\hat{\rho}}{\sigma_{n-1} r^{n-1}} \right) \delta + \sum_{j=1}^{d-1} \frac{\xi_j (-2ik)^j}{\sigma_{n-1} r^{n-1-j}} \frac{(2d-j)\hat{\rho} + j \vec{e}_0}{2(d-j)} + \left( \frac{-ik}{2\pi} \right) \frac{d\hat{\rho} + \vec{e}_0}{2r^d} e^{ikr},$$

where $\xi_j := \frac{(d-1) \cdots (d-j)}{(d-1) \cdots (2d-j)}$. In particular for $n = 3$ we have

$$\vec{E}_k(x) = \left( \frac{\hat{\rho}}{r^2} - \frac{ik}{r}(\hat{\rho} + \vec{e}_0) \right) e^{ikr},$$

as in McIntosh–Mitrea [39]. In even dimension, the fundamental solution have similar asymptotics. From Watson [65] and Definition 1.2.3 we obtain the following.
**Proposition 1.2.5.** For even dimension $n = 2b + 2$, $b = 0, 1, 2, \ldots$, there exists entire analytic $\wedge^1$-valued functions $f_b(z)$ and $g_b(z)$ such that

$$
\vec{E}_k(x) = \frac{1}{\sigma_{n-1}} \sum_{j=0}^{b} \frac{(b-j)!}{b!} \left( \frac{k}{2} \right)^{2j} \frac{1}{r^{2(b-j)}} \left( \frac{\hat{r}}{r} - \frac{ke_0}{2(b-j)} \right)^n \left( \log \left( \frac{k}{2r} \right) f_b(kr) + g_b(kr) \right).
$$

In any dimension $n \geq 3$ we have asymptotics

$$
\vec{E}_k(x) - E(x) - k e_0 \Phi(x) = O(r^{-(n-3)}), \quad r \to 0,
$$

$$
\nabla \otimes (\vec{E}_k(x) - E(x)) = O(r^{-(n-1)}), \quad r \to 0,
$$

where $\Phi(x) = -1/((n-2)\sigma_{n-1}r^{n-2})$. On the other hand, the asymptotics $r \to \infty$, when $-\pi < \arg(k) < 2\pi$, are

$$
\vec{E}_k(x) e^{-i kr} - \frac{1}{2} e^{-i \frac{n-1}{2} \hat{r}} \left( \frac{k}{2\pi} \right)^{(n-1)/2} \hat{r} + \hat{e}_0 \frac{e^{-\frac{1}{2}\hat{r}}}{\hat{r}^{(n-1)/2}} = O(r^{-(n+1)/2}), \quad r \to \infty.
$$

Note that the factor $\hat{r} + \hat{e}_0$ appearing in asymptotic expression for $r \to \infty$ is directed along the future null cone in spacetime and thus is a non-invertible element in the Clifford algebra. As in McIntosh–Mitrea [39] we note that this can be interpreted as a generalised Silver–Müller radiation condition. For more on radiation conditions, see Section 5.1. Corresponding statements for $\vec{E}_k$ are true as well. The asymptotics at $\infty$ for $\vec{E}_k$ involves $\hat{r} + \hat{e}_0$ which is directed along the past null cone.

**Lemma 1.2.6.** Consider the Cauchy convolution/balayage operator (also called Teodorescu transform)

$$
B_k F(x) = \int \vec{E}_k(x - y) F(y) \, dy,
$$

acting on compactly supported distributions $F \in \mathcal{E}(\mathbb{R}^n; \wedge)$. When $\text{Im} \, k \geq 0$, this is the Fourier (Clifford) multiplier $\hat{F}(\xi) \mapsto (i\xi + ke_0)^{-1} \hat{F}(\xi)$. If $U$ and $V$ are two bounded sets and $k \in \mathbb{C}$, then $B_k : L_2(U) \to L_2(V)$ is a compact operator. We have anti commutation relations

$$
d_k B_k + B_k d_k = I, \quad (1.17)
$$

$$
\delta_k B_k + B_k \delta_k = I, \quad (1.18)
$$

and there exists a universal constant $C$ such the essential norm $\|B_k d_k\|_{L_2(U) \to L_2(V), \text{ess}} \leq C$.

**Proof.** That $B_k : L_2(U) \to L_2(V)$ is compact follows from Schur’s lemma. To estimate $B_k d_k$, write

$$
B_k d_k = B_0 d + B_0 (ke_0 \wedge) + (B_k - B_0) d_k.
$$

The first term is a Fourier multiplier with the bounded symbol $(i\xi)^{-1}i\xi \wedge$ whereas the other two terms are compact operators due to the kernel estimates in Proposition 1.2.5.
To prove the first anti commutation relation for example we calculate
\[
d_k B_k F(x) = \nabla_k \wedge \int \vec{E}_k(x - y) F(y) \\
= \int (\nabla_k, \vec{E}_k)(x - y) F(y) - \vec{E}_k(x - y)(\nabla_k \wedge F)(y) \\
= F(x) - \int \vec{E}_k(x - y)((-\nabla + k\epsilon_0) \wedge F)(y) \\
= F(x) - \int \vec{E}_k(x - y)(\nabla_k \wedge F)(y) = (I - B_k d_k) F(x),
\]
where we used the derivation property (1.8) on the second line and that \((\nabla_k, \vec{E}_k)(x) = (\Delta + k^2)\Phi_k(x) = \delta_0(x)\) on the third line.

We now construct the basic reproducing formulae for the Dirac operators \(D_k\). In order to treat Stokes’ type theorems in a unified way, we record the following theorem, here referred to as the boundary theorem.

**Theorem 1.2.7.** Let \(\Omega \subset \mathbb{R}^n\) be a smooth bounded open set and let \(V\) be a finite dimensional linear space. Then for a function \(F : \overline{\Omega} \to V\), smooth up to \(\partial \Omega\), with boundary trace \(f := F|_\Sigma\) we have
\[
\int_{\partial \Omega} \nu(y) \otimes f(y) \, d\sigma(y) = \int_{\Omega} \nabla \otimes F(\hat{x}) \, dx,
\]
where the integrand is \(\mathbb{R}^n \otimes V\) valued, \(\nu\) is the outward pointing normal and \(d\sigma\) is the scalar surface measure.

Recall that this theorem is universal in the sense that for any given finite dimensional linear space \(W\) and bilinear form \(L : \mathbb{R}^n \times V \to W\), \(L\) can be lifted to a linear map \(L : \mathbb{R}^n \otimes V \to W\). Applying this to the formula in the boundary theorem gives the special case \(\int_{\partial \Omega} L(\nu(y), f(y)) \, d\sigma(y) = \int_{\Omega} L(\nabla, F(\hat{x})) \, dx\).

We will frequently use the boundary theorem. Let us here record two important examples.

1. Let \(V = \wedge \otimes \wedge\) and \(L : \mathbb{R}^n \otimes (\wedge \otimes \wedge) \to \mathbf{C} : a \otimes (G \otimes F) \mapsto (G, a \wedge F)\). Applying the boundary theorem and Leibniz’ formula gives
\[
\int_{\partial \Omega} (g(y), \nu(y) \wedge f(y)) \, d\sigma(y) = \int_{\Omega} ((G, dF) + (\delta G, F)) \, dx.
\]

2. Let \(V = \wedge \otimes \wedge\) and \(L : \mathbb{R}^n \otimes (\wedge \otimes \wedge) \to \wedge : a \otimes (G \otimes F) \mapsto G \wedge a \wedge F\). Here we get
\[
\int_{\partial \Omega} g(y) \nu(y) f(y) \, d\sigma(y) = \int_{\Omega} ((G(\hat{x})D) F(x) + G(x)(DF(\hat{x}))) \, dx.
\]

In particular, for fixed \(x_0 \in \Omega\) let \(G(y) = \vec{E}_{-k}(y - x_0) = -\vec{E}_k(x_0 - y)\) and assume \(F\) is a \(k\)-monochromatic field in \(\Omega\). Then the \(\Omega\)-integrand becomes \((\vec{E}_{-k}k\epsilon_0 + \delta_{x_0}) F + \vec{E}_{-k}(-k\epsilon_0 F) = \delta_{x_0} F\) and we obtain the reproducing formula
\[
F(x_0) = \int_{\partial \Omega} \vec{E}_{-k}(y - x_0) \nu(y) f(y) \, d\sigma(y) = -\int_{\partial \Omega} \vec{E}_k(x_0 - y) \nu(y) f(y) \, d\sigma(y). \tag{1.20}
\]

Note that for arbitrary \(f\) on \(\partial \Omega\) the last integral produces a \(k\)-monochromatic field in \(\Omega\) since the Clifford product is associative. We now make the following formal definition.
**Definition 1.2.8.** Write \( \Omega^+ = \Omega \) and let \( \Omega^- := \mathbb{R}^n \setminus \overline{\Omega}^+ \) be the exterior domain and let \( \nu \) denote the outward (into \( \Omega^- \)) pointing normal on the hypersurface \( \Sigma = \partial \Omega^+ = \partial \Omega^- \).

For a field \( f : \Sigma \to \wedge \), let the **Cauchy reflection operator** \( E_k \) with wave number \( k \) be

\[
E_k f(x) := 2 \text{p.v.} \int_{\partial \Omega} \frac{E_{-k}(y - x)\nu(y)f(y)}{y(x)} \, d\sigma(y), \quad x \in \Sigma.
\]

The **Cauchy extension operators** \( C_k^\pm \) with wave number \( k \) are

\[
C_k^\pm f(x) := \int_\Sigma \left( E_{k}(y - x) \pm (\pm \nu)(y)f(y) \right) \, d\sigma(y), \quad x \in \Omega^\pm,
\]

and the interior/exterior **Hardy projection operator** \( E_k^\pm \) with wave number \( k \) is the boundary trace

\[
E_k^\pm f(x) := \lim_{\Omega^\pm \ni z \to x} \frac{z}{\kappa} \left( C_k^\pm f(z) - \frac{1}{2} (I \pm E_k) f(x) \right), \quad x \in \Sigma.
\]

When \( k = 0 \), we omit the subscript \( k \).

We finish this section with a simple but important observation, which we will see later on tells us that on Lipschitz type interfaces \( \Sigma \) it is natural to define the curl of a tangential field and the divergence of a normal field, but not vice versa.

**Proposition 1.2.9.** Let \( \sigma \in \mathcal{D}'(\mathbb{R}^n; \wedge^0) \) denote the surface measure on a smooth surface \( \Sigma \) and let \( S = \nu \sigma = -\nabla \chi_\Omega \in \mathcal{D}'(\mathbb{R}^n; \wedge^1) \) denote the weighted surface measure, where \( \chi_\Omega \) denotes the characteristic function for \( \Omega \). Then in distribution sense we have

\[
\nabla_\wedge S = 0, \quad \nabla_\j S = \frac{\partial \sigma}{\partial \nu} - H \sigma,
\]

where \( H \) is the mean curvature of \( \Sigma \) in the direction of \( \nu \).

### 1.3 Three classical equations

#### 1.3.1 The \( \overline{\partial} \)-equation

We first demonstrate how the Cauchy integral in Definition 1.2.8 generalises the classical Cauchy integral in \( \mathbb{C} \). Let \( k = 0 \) and consider the real exterior algebra

\[
\wedge \mathbb{R}^2 = \wedge^0 \oplus \wedge^1 \oplus \wedge^2 = \wedge^1 \oplus \mathbb{C},
\]

using the “geometric characterisation” \( \mathbb{C} = \wedge^0 \oplus \wedge^2 \). Fix a basis \( \{e_1, e_2\} \) for \( \wedge^1 \) and make the identification \( \wedge^1 \approx \mathbb{C} \) so that

\[
x = x_1 e_1 + x_2 e_2 \iff z = e_1 x = x_1 + x_2 j,
\]

\[
y = y_1 e_1 + y_2 e_2 \iff w = e_1 y = y_1 + y_2 j,
\]

where \( j = e_1 e_2 \in \wedge^2 \subset \mathbb{C} \) is the “geometric imaginary unit”. In this way we see that the Dirac equation in two dimensions is essentially Cauchy–Riemann’s equations since

\[
e_1 DF(x) = (\partial_1 + j \partial_2) F(x) = 2 \overline{\partial} F(x), \quad F : \mathbb{C} \approx \mathbb{R}^2 \to \mathbb{C} \approx \wedge^0 \oplus \wedge^2.
\]
To put the Cauchy integral from Definition 1.2.8 in a more familiar form, note that the two weighted measures \( \nu da \) and \( dw \) are related by \( j e_1(\nu da) = dw \); first identify the normal \( \nu \) with the complex number \( e_1 \nu \) and then rotate this to the tangent \( j e_1 \nu \). For \( f : \Sigma \to \mathbb{C} \), we now calculate

\[
\int_{\Sigma} E(y - x)\nu(y)f(y)\,d\sigma(y) = \frac{1}{2\pi} \int_{\Sigma} \frac{e_1(w - z)}{|w - z|^2} (e_1 \frac{1}{2} dw)f(w)
\]

\[
= \frac{1}{2\pi} \int_{\Sigma} \frac{w - z}{|w - z|^2} (\frac{1}{2} dw)f(w) = \frac{1}{2\pi} \int_{\Sigma} \frac{f(w)}{w - z} \, dw,
\]

using that \( \mathbb{C} \) is a commutative algebra. In particular, the logarithmic double layer potential operator is

\[
\int_{\Sigma} (E(y - x), n(y))u(y)\,d\sigma(y) = \frac{1}{2\pi} \int_{\Sigma} \text{Im} \left( \frac{dw}{w - z} \right) u(w),
\]

where \( u : \Sigma \to \mathbb{R} \approx \wedge^0 \). For more about double layer potential operators, see Chapter 2.

1.3.2 The Laplace equation

We here view the Laplace equation \( \Delta U = 0 \) as a special case of the Dirac equation by considering the monogenic vector field

\[
F := \nabla U : \Omega \longrightarrow \wedge^1
\]

instead of the harmonic potential \( U \). Since \( \Delta U = D^2U = DF = 0 \), \( F \) is monogenic if and only if \( U \) is harmonic. Note that since \( F \) is a vector field, we also have \( dF = 0 \) and \( \delta F = 0 \) in this case. The trace \( f := F|_{\Sigma} \) is

\[
f = N^+ f + N^- f = \nu \lhd \nu \lhd f + \nu \lhd (\nu \rhd f) = \Gamma u + \frac{\partial u}{\partial \nu}\nu,
\]

where \( N^\pm \) denotes the tangential/normal projection operators and \( \Gamma u = \nabla_{\text{tan}} u \) denotes the tangential gradient of \( u \). For \( x \in \Omega \), we have Cauchy extensions

\[
C^+((\partial u/\partial \nu)\nu)(x) = \int_{\Sigma} E(y - x)\frac{\partial u}{\partial \nu}(x),
\]

\[
C^+(\Gamma u)(x) = \nabla x \wedge \int_{\Sigma} E(y - x)\nu(y)u(y) = \nabla x K u(x),
\]

where \( \Phi \) and \( K \) are the classical single and double layer potential operators as defined in Section 2.2.2. More details on the last calculation are found in Proposition 3.2.10 in Chapter 3. Thus if \( K \) denotes the principal value double layer potential we formally get the commutative diagrams

\[
\begin{array}{ccc}
\mathcal{X}_0(\Sigma; \wedge^0) & \xrightarrow{\nu(\cdot)} & \mathcal{X}_0(\Sigma; \wedge^1) \\
\downarrow K^* & & \downarrow N^- E \\
\mathcal{X}_0(\Sigma; \wedge^0) & \xrightarrow{\nu(\cdot)} & \mathcal{X}_0(\Sigma; \wedge^1),
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{X}_1(\Sigma; \wedge^0) & \xrightarrow{\Gamma} & \mathcal{X}_1(\Sigma; \wedge^1) \\
\downarrow K & & \downarrow N^+ E \\
\mathcal{X}_1(\Sigma; \wedge^0) & \xrightarrow{\Gamma} & \mathcal{X}_1(\Sigma; \wedge^1),
\end{array}
\]
where the function space $\mathcal{X}_1$ have one order higher regularity than $\mathcal{X}_0$. Note that in the second diagram, the range of $\Gamma$ is essentially the half of $N^+\mathcal{X}_0(\Sigma; \wedge^1)$ consisting of tangential vector fields with vanishing tangential curl.

We also see that

$$C^+ f = C^+ (\Gamma u) + C^+ \left( \frac{\partial}{\partial t} \nu \right) = \nabla(Ku - \Phi(\frac{\partial}{\partial t})),$$

i.e. the reproducing formula $F = C^+ f$ is “the gradient of Green’s second formula”.

### 1.3.3 Maxwell’s equations

We now show that the electromagnetic field satisfying Maxwell’s equations is a monochromatic field, i.e. it satisfies a Dirac equation $D_k F = 0$. Indeed, it is this example which motivates the terminology “monochromatic”, a word borrowed from optics. The starting point is the classical four equations for the electric and magnetic fields. Replacing divergence and curl with the interior and exterior derivatives in $\wedge \mathbb{R}^3$, the equations (in suitable units) read as follows.

\[
\nabla \wedge \mathbf{B} = 0, \\
\partial_0 \mathbf{B} + \nabla \wedge \mathbf{E} = 0, \\
\partial_0 \mathbf{D} + \nabla \cdot \mathbf{H} = -\mathbf{J}, \\
\nabla \cdot \mathbf{D} = \rho,
\]

(1.22)

where the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are

\[
\mathbf{E}(t, x) = E_1(t, x)e_1 + E_2(t, x)e_2 + E_3(t, x)e_3 \in \wedge^1 \mathbb{R}^3 \\
\mathbf{B}(t, x) = B_1(t, x)e_23 + B_2(t, x)e_31 + B_3(t, x)e_{12} \in \wedge^2 \mathbb{R}^3,
\]

(the classical magnetic field $\mathbf{B}^\perp$ is the Hodge complement $\mathbf{B}^\perp = B_1e_1 + B_2e_2 + B_3e_3 \in \wedge^1 \mathbb{R}^3$) and $\mathbf{D} \in \wedge^1 \mathbb{R}^3$ is the dielectric displacement and $\mathbf{H} \in \wedge^2 \mathbb{R}^3$ is the magnetic intensity depending on the material properties. Moreover, $\mathbf{J} \in \wedge^1 \mathbb{R}^3$ is the (free) electric current and $\rho$ is the (free) electric charge density. From bottom to top the equations are Gauss’ law, Ampère–Maxwell’s law, Faraday’s induction law and the magnetic Gauss’ law and they take values in $\wedge^0$, $\wedge^1$, $\wedge^2$ and $\wedge^3$ respectively.

We will here consider time harmonic fields ($\partial_0 = -i\omega$) in a linear, homogeneous, isotropic, possibly conducting material, i.e. we have scalar material constants $\varepsilon > 0$ and $\mu > 0$ (permittivity and permeability) and conductivity $\sigma \geq 0$ such that $\mathbf{D} = \varepsilon \mathbf{E}$, $\mathbf{H} = \mu^{-1} \mathbf{B}$, $\mathbf{J} = \sigma \mathbf{E}$, $\rho = 0$. In this situation, the reduced Maxwell’s equations becomes

\[
\nabla \wedge \tilde{\mathbf{B}} = 0, \\
-i k \tilde{\mathbf{B}} + \nabla \wedge \tilde{\mathbf{E}} = 0, \\
-i k \tilde{\mathbf{E}} + \nabla \cdot \tilde{\mathbf{B}} = 0, \\
\nabla \cdot \tilde{\mathbf{E}} = 0,
\]

(1.23)
where $\hat{E} := \sqrt{\varepsilon} E$, $\hat{B} := \frac{1}{\sqrt{\mu}} B$ with wave number $k := \omega/c$, propagation speed $c := (\varepsilon_0 \mu)^{-1/2}$ and complex permittivity $\varepsilon_* := \varepsilon + i \sigma/\omega$. We here assume that $\omega \in \mathbb{C} \setminus \{0\}$.

We now translate to spacetime notation and embed $\wedge \mathbb{R}^3 \subset \wedge \mathbb{R}^4$. A physically natural quantity is the full electromagnetic field

$$F(x) := \sqrt{\varepsilon_*} e^0 \wedge E(x) + \frac{1}{\sqrt{\mu}} B(x) : \mathbb{R}^3 \rightarrow \bigwedge^2 \mathbb{C} \mathbb{R}^4,$$

(1.24)

since $|F|^2$ is the energy density of the electromagnetic field. Conversely, we recover $E$ and $B$ from $F$ as the time/space parts

$$\sqrt{\varepsilon_*} e^0 E(x) = T^- F(x) := e^0 \wedge (e^0 \wedge F(x)),
\frac{1}{\sqrt{\mu}} B(x) = T^+ F(x) := e^0 \wedge (e^0 \wedge F(x)).$$

The reduced Maxwell’s equations are now equivalent with the Dirac equation

$$D_k F(x) = (D + ik \gamma_0) F(x) = 0.$$ (1.25)

Indeed, we check that the time-like $\wedge^1$ part of this Dirac equation is Gauss’ law. Likewise the space-like $\wedge^1$ part is Ampère–Maxwell’s law, the time-like $\wedge^3$ part is Faraday’s law and the space-like $\wedge^3$ part is the magnetic Gauss’ law. For more details on this, see Janecewicz [26]. It is important to note that since $F : \Omega \rightarrow \wedge^2$, we in fact have $d_k F = \delta_k F = 0$. In other words, $F$ is a $k$-monochromatic field which satisfies the extra constraint $d_k F = 0$. In particular $\|F\|_{D(d_k L^2(\Omega))} = \|F\|_{L^2}$ is the energy of $F$.

We have the reproducing formula $F(x) = C_k^+ f(x) = \int_{\partial U} \hat{E}_k(x - y) \nu(y) f(y) d\sigma(y)$ for $F$. Expanding this in terms of the electric and magnetic fields using Proposition 1.1.8, this formula contains the two classical Stratton–Chu formulae, which in classical vector calculus notation reads

$$E(x) = \nabla \times \int_{\partial U} \Phi_k(x - y) \nu(y) \times E(y) d\sigma(y)
- \nabla \int_{\partial U} \Phi_k(x - y) \nu(y) \cdot E(y) d\sigma(y) + i k c \int_{\partial U} \Phi_k(x - y) \nu(y) \times B^\perp(y) d\sigma(y),$$

$$B^\perp(x) = \nabla \times \int_{\partial U} \Phi_k(x - y) \nu(y) \times B^\perp(y) d\sigma(y)
- \nabla \int_{\partial U} \Phi_k(x - y) \nu(y) \cdot B^\perp(y) d\sigma(y) - ik/c \int_{\partial U} \Phi_k(x - y) \nu(y) \times E(y) d\sigma(y),$$

Remark 1.3.1. We have seen that the $\bar{\jmath}$-equation, the Laplace equation and (time-harmonic) Maxwell’s equations all can be seen as a Dirac equation. However, the Laplace equation and Maxwell’s equations are more than a Dirac equation. For the Laplace equation, we require that $F$ is a pure vector field, and for Maxwell’s equations we require that $F$ is a pure bivector field. In Chapters 3 and 5 we will see this implies that also the boundary trace $f$ will satisfy a divergence/curl-free constraint. This explains why for example the $\wedge^3$ component of $C^+(\Gamma u)$ vanishes for the Laplace equation above, and why the $\wedge^0$ and $\wedge^4$ components of $C^+_k f$ vanish for Maxwell’s equations above. That the $\wedge^1$ part of $F$ is required to be zero for the $\bar{\jmath}$-equation does not matter since $\wedge^0 \oplus \wedge^2$ is a subalgebra of $\wedge \mathbb{R}^2$.  

18
1.4 Operator theory

In this section we collect basic operator theoretic results which we will use throughout this thesis. A good reference is Kato [28]. We denote Hilbert spaces by \( \mathcal{H} \), \( \mathcal{K} \), etc and Banach spaces by \( \mathcal{X} \), \( \mathcal{Y} \), etc. With \( x \lesssim y \) we mean that \( x \leq Cy \) for some constant \( C < \infty \).

**Definition 1.4.1.** A closed, densely defined operator \( T : \mathcal{X} \to \mathcal{Y} \) is a semi-Fredholm operator if its range \( \text{R}(T) \subset \mathcal{Y} \) is closed and its null space \( \text{N}(T) \subset \mathcal{X} \) is finite dimensional. The index of \( T \) is \( i(T) := \alpha(T) - \beta(T) := \text{dim} \text{N}(T) - \text{dim}(\mathcal{Y}/\text{R}(T)) \). We say that \( T \) is a Fredholm operator if \( \text{dim}(\mathcal{Y}/\text{R}(T)) < \infty \).

**Remark 1.4.2.** For convenience we do not call \( T \) for which \( \text{dim}(\mathcal{Y}/\text{R}(T)) < \infty \) (and thus \( \text{R}(T) \) is closed by Proposition 1.4.4) and \( \text{dim}(\mathcal{N}(T)) = \infty \) semi-Fredholm here.

Recall that if \( \mathcal{X} \) and \( \mathcal{Y} \) above are Hilbert spaces, then \( T \) is a Fredholm operator if and only if it has a Fredholm inverse, i.e. an inverse modulo compact operators.

**Proposition 1.4.3 (Uniqueness, a priori estimates).** Let \( T : \mathcal{X} \to \mathcal{Y} \) be a bounded linear operator. Then \( T \) is a semi-Fredholm operator if and only if it has a priori estimates, i.e. there exists a compact operator \( K : \mathcal{X} \to \mathcal{Z} \) such that

\[
\|f\|_{\mathcal{X}} \lesssim \|Tf\|_{\mathcal{Y}} + \|Kf\|_{\mathcal{Z}}, \quad f \in \mathcal{X}.
\]

Furthermore, \( T \) is an injective semi-Fredholm operator if and only if it has exact a priori estimates, i.e. a priori estimates with \( K = 0 \).

Another way to prove that \( \text{R}(T) \) is closed uses the open mapping theorem.

**Proposition 1.4.4.** If \( T : \mathcal{X} \to \mathcal{Y} \) is a closed operator and if the cokernel \( \mathcal{Y}/\text{R}(T) \) is finite dimensional, then \( \text{R}(T) \) is closed.

**Theorem 1.4.5 (Existence, method of continuity).** For \( \lambda \in [0,1] \), let \( T_{\lambda} : \mathcal{X} \to \mathcal{Y} \) be a bounded semi-Fredholm operator and assume \( \lambda \mapsto T_{\lambda} \) is continuous. Then \( i(T_{0}) = i(T_{1}) \).

**Theorem 1.4.6 (Analytic Fredholm theory).** Let \( D \) be a connected open subset of \( \mathbb{C} \). Assume that to each \( \lambda \in D \), there is associated a compact operator \( K_{\lambda} : \mathcal{X} \to \mathcal{X} \), such that \( \lambda \mapsto K_{\lambda} \) is an analytic map.

If \( I - K_{\lambda_{0}} \) is invertible for some \( \lambda_{0} \in D \), then \( I - K_{\lambda} \) is invertible for all \( \lambda \) except on a discrete subset \( D_{0} \subset D \).

**Definition 1.4.7.** A bi-linear or sesqui-linear pairing \( \langle \mathcal{H}', \mathcal{H} \rangle \) between two Hilbert spaces \( \mathcal{H}' \) and \( \mathcal{H} \) is called a duality if the estimates

\[
|\langle f', f \rangle| \lesssim \|f'\| \|f\|, \quad \|f'\| \lesssim \sup_{\|f\|=1} |\langle f', f \rangle| \quad \text{and} \quad \|f\| \lesssim \sup_{\|f'\|=1} |\langle f', f \rangle|
\]

hold. By Riesz theorem, there exists a unique isomorphism \( A : \mathcal{H} \to \mathcal{H}' \) such that \( \langle f', f \rangle = \langle f', Af \rangle_{\mathcal{H}} = (A^* f', f)_{\mathcal{Y}} \). The annihilator of a set \( M \subset \mathcal{H} \) with respect to this duality is \( M^{\perp} := \{ f' \in \mathcal{H}' : \langle f', f \rangle = 0 \text{ for all } f \in M \} = A(M)^{\perp} \) and similarly for \( M \subset \mathcal{H}' \).
Let \( \langle K', K \rangle \) be a second duality. Two closed, densely defined operators \( T : \mathcal{H} \to K \) and \( T' : K' \to \mathcal{H}' \) are said to be adjoint/dual if \( G(T) := \{ (x, Tx) \mid x \in D(T) \} \) and \( IG(T') := \{ (-T'y', y') \mid y' \in D(T') \} \), where \( I(x, y) := (-y, x) \), are each others annihilators with respect to the graph duality \( \langle \mathcal{H}' + K', \mathcal{H} + K \rangle \).

**Theorem 1.4.8.** Let \( T \) and \( T' \) be dual Hilbert space operators as in Definition 1.4.7. Then \( R(T) \) is closed if and only if \( R(T') \) is closed. Also, \( R(T)^a = N(T') \) and \( R(T')^a = N(T) \).

These theorems are the tools we need for proving (Fredholm) invertibility of the boundary integral operators under consideration. We also recall that the basic tool for proving continuity of Hilbert space operators is Schur’s lemma, i.e. interpolation between \( L_1 \) and \( L_\infty \) using Cauchy–Schwarz’ inequality. One version of Schur’s lemma is used for Lemma 3.1.8 below. Note that since this technique only uses size estimates on the kernel, one typically obtains not only a bounded map, but in fact a compact map.

### 1.5 Lipschitz interfaces

Unless otherwise stated, \( \Omega^+ \subset \mathbb{R}^n \) denotes a bounded open set, separated from the exterior domain \( \Omega^- = \mathbb{R}^n \setminus \overline{\Omega^+} \) by an interface \( \Sigma = \partial \Omega^+ = \partial \Omega^- \). The regularity assumption on \( \Sigma \) depends on which boundary function space we want to use for the transmission problem.

**Definition 1.5.1.** Let \( \Omega^\pm := \rho(\mathbb{R}^n) \) and \( \Sigma_\rho := \rho(\mathbb{R}^{n-1}) \) when \( \rho : \mathbb{R}^n \to \mathbb{R}^n \) is a global homeomorphism.

(i) If \( \rho \) is a bi-Lipschitz map, then we say that \( \Sigma_\rho \) is a special weakly Lipschitz surface/interior.

(ii) Let \( \rho \) be a bi-Lipschitz map which is smooth off \( \mathbb{R}^{n-1} \) such that

\[
\int_Q \int_{0 < |x_n| < \ell(Q)} |(\nabla \otimes)^2 \rho(x)|^2 |x_n| dx_n dx \leq C |Q|, \tag{1.26}
\]

for a fixed \( C < \infty \) and all cubes \( Q \subset \mathbb{R}^{n-1} \) with side length \( \ell(Q) \). In words, we require \( d\mu = |(\nabla \otimes)^2 \rho(x)|^2 x_n dx \) to be a Carleson measure in both \( \mathbb{R}^n_+ \) and \( \mathbb{R}^n_- \). Then we say that \( \Sigma_\rho \) is a special Carleson–Lipschitz surface/interior.

(iii) Assume there exists a ON-basis with coordinates \( \{ y_1, \ldots, y_n \} \) and a Lipschitz function \( \phi : \mathbb{R}^{n-1} \to \mathbb{R} \) such that \( \Sigma_\rho = \{ y \in \mathbb{R}^n ; y_n = \phi(y_1, \ldots, y_{n-1}) \} \). Then we say that \( \Sigma_\rho \) is a special strongly Lipschitz surface/interior.

Now consider the bounded domain \( \Omega^+ \). Assume that for all \( y \in \Sigma \), there exists a neighbourhood \( V_y \ni y \) and \( \Sigma_{\rho}, \rho = \rho_y \), such that \( \Omega^+ \cap V_y = \Omega^+ \cap V_y, \Omega^- \cap V_y = \Omega^- \cap V_y \) and \( \Sigma \cap V_y = \Sigma \cap V_y \). If for all \( y \in \Sigma \), \( \Sigma_{\rho} \) can be chosen to be special weakly/Carleson/strongly Lipschitz interfaces then we say that \( \Sigma \) is a bounded weakly/Carleson/strongly Lipschitz interface.

We will also use the terminology (special/bounded) weakly/Carleson/strongly Lipschitz domain for \( \Omega_{\rho}^\pm \) and \( \Omega^\pm \) respectively.
Recall that by Rademacher’s theorem, any weakly Lipschitz interface \( \Sigma \) has a tangent plane and an outward (into \( \Omega^- \)) pointing unit normal \( \nu(y) \) at almost every \( y \in \Sigma \). Also, note that any Carleson–Lipschitz surface by definition is a weakly Lipschitz surface. Less trivial is the following.

**Proposition 1.5.2.** If \( \Sigma \) is a strongly Lipschitz surface, then it is a Carleson–Lipschitz surface.

This proposition is due to Dahlberg. Note that it suffices to construct a Carleson–Lipschitz parametrisation \( \rho : \mathbb{R}^n \to \mathbb{R}^n \) which extends \( \mathbb{R}^{n-1} \ni x' \mapsto x' + \phi(x')e_n \) in (iii), where \( \phi \) is a given Lipschitz function. An explicit formula for such \( \rho \) is

\[
\rho(x) = x + (\eta_{x_n} * \phi)(x')e_n,
\]
due to Kenig and Stein. Here \( \eta : \mathbb{R}^{n-1} \to \mathbb{R} \) is a bump function supported in a sufficiently small neighbourhood of 0 and \( \int \eta = 1 \). To verify the Carleson condition (1.26), one calculate

\[
\partial_i \partial_j \rho_n = \frac{1}{x_n} (\partial_j \eta)_{x_n} * (\partial_i \phi), \quad 1 \leq i, j \leq n - 1,
\]

\[
\partial_n \partial_i \rho_n = -\frac{1}{x_n} \sum_{j=1}^{n-1} (\partial_j (x_j \eta))_{x_n} * (\partial_i \phi), \quad 1 \leq i \leq n - 1,
\]

\[
\partial^2_n \rho_n = \frac{1}{x_n} \sum_{i,j=1}^{n-1} (\partial_j (x_j x_i \eta))_{x_n} * (\partial_i \phi),
\]

and applies the following lemma.

**Lemma 1.5.3.** Let \( \varphi \in C^\infty_0(\mathbb{R}^{n-1}) \) be such that \( \int \varphi = 0 \). Then for all \( b \in BMO(\mathbb{R}^{n-1}) \)

\[
\frac{1}{|Q|} \int_Q \int_{0 < x_n < \ell(Q)} |(\varphi_{x_n} * b)(x')|^2 \frac{dx_n}{x_n} dx' \lesssim \|b\|^2_{BMO}.
\]

In words, the continuous wavelet transform (in the sense of Daubechies [13]) of a BMO function is a Carleson measure.

The most important property of a strongly Lipschitz surface for us is the following global result, which will be used in Chapters 2 and 3.

**Proposition 1.5.4.** Let \( \Sigma \) be a bounded strongly Lipschitz interface. Then there exists a smooth transversal vector field \( \theta \) for \( \Sigma \), i.e. \( \theta \in C^\infty_0(\mathbb{R}^n, \mathbb{R}^n) \) such that \( (\theta(y), \nu(y)) \geq c_0 > 0 \) a.e. on \( \Sigma \).

**Proof.** By compactness, we have \( \Sigma \subset \bigcup_{j=1}^N V_j \). Let \( \{\eta_j\} \) be a \( C^\infty \) partition of unity subordinate to \( \{V_j\} \) for a neighbourhood of \( \Sigma \), and define the “vertical” constant unit vector field \( \theta_j = -\nabla y_n \) in the chart \( V_j \) and the global smooth vector field \( \theta(y) := \sum \eta_j(y) \theta_j \). Then for a.a. \( y \in \Sigma \)

\[
(\theta(y), \nu(y)) = \sum \eta_j(y)(\theta_j, \nu(y)) \geq \sum \eta_j c = c,
\]
since \( (\theta_j, \nu(y)) \geq c \) on \( \text{supp} \eta_j \subset V_j \).

\[\square\]
Definition 1.5.5. Let us introduce quantities

\[ L_\theta := \inf\{ L : |\theta \wedge \nu| \leq L(\theta, \nu) \text{ a.e. on } \Sigma \}, \]
\[ L_\Sigma := \inf\{ L_\theta : \theta \text{ a smooth transversal vector field for } \Sigma \}. \]

We call \( L_\Sigma \) the local Lipschitz constant for \( \Sigma \).

We note that every \( C^1 \) surface has local Lipschitz constant 0. It is easy to see that any bounded open convex set is a bounded strongly Lipschitz domain. However, a general strongly Lipschitz domain may be “non-convex on all scales”.

Example 1.5.6. We now give two examples of Carleson–Lipschitz domains (and in particular weakly Lipschitz domains) which are not strongly Lipschitz.

(i) Let \( \rho_0 : S^{n-1} \to S^{n-1} \) be a bilipschitz homeomorphism of the unit sphere, which is such that \( |(\nabla_{\text{tang}})^2 \rho_0(x)|^2 |x_n| \) is a Carleson measure in \( S^n \) and \( S^n_\omega \). Consider the conical surface

\[ \Sigma := \{ x \in \mathbb{R}^n \setminus \{ 0 \} : x/|x| \in \rho_0(S^{n-1} \cap \mathbb{R}^{n-1}) \} \cup \{ 0 \}. \]

The natural parametrisation for \( \Sigma \) is

\[ \rho(r\omega) := r\rho_0(\omega), \quad r \geq 0, \omega \in S^{n-1}. \]

Using the identity \( |r\omega - r'\omega'|^2 = |r - r'|^2 + rr'|\omega - \omega'|^2 \), it is straightforward to show that \( \rho : \mathbb{R}^n \to \mathbb{R}^n \) is a bilipschitz map. One can also verify that \( \rho \) satisfies the Carleson condition (1.26).

An important special case is the “two brick” domain, defined as the interior of

\[ \{ (x, y, z) \in \mathbb{R}^3 : y \leq 0, z \leq 0 \} \cup \{ (x, y, z) \in \mathbb{R}^3 : x \leq 0, z \geq 0 \}. \]

Indeed, the intersection with the unit sphere \( S^2 \) is a two dimensional strongly Lipschitz domain. Nevertheless, the three dimensional cone is not a strongly Lipschitz domain.

(ii) Let \( a_1 > 0 \) and \( e^{-2\pi} a_2 < a_1 < a_2 \) and consider the logarithmic spiral

\[ \Omega := \{ re^{i\theta} : r > 0, \theta \in \mathbb{R}, a_1 e^{-\theta} < r < a_2 e^{-\theta} \} \subset \mathbb{R}^2. \]

To see that \( \Omega \) is a special Carleson–Lipschitz domain, define the maps

\[ \rho_s(x, y) := (x \cos(s \ln r) - y \sin(s \ln r), x \sin(s \ln r) + y \cos(s \ln r)), \]

where \( r^2 = x^2 + y^2 \). We see that \( \rho_s \circ \rho_t = \rho_{s+t}, s, t \in \mathbb{R} \) and that

\[ |\nabla \rho_s| \leq C, \quad |(\nabla \rho_s)^2| \leq C/r. \]

Furthermore, \( \rho_{-1} \) maps the sector \( \ln a_1 < \theta < \ln a_2 \) onto \( \Omega \). As \( 1/r \) is a Carleson measure in \( \mathbb{R}^2_+ \), this proves that \( \Omega \) is a special Carleson–Lipschitz domain.
We remark that the Rellich technique for solving boundary value problems for the Laplace equation has been extended in Verchota [64] to the cones in (i) above when $\Sigma \cap S^{n-1}$ is a strongly Lipschitz domain. See also Grisvard [22] for another two dimensional weakly Lipschitz domain which is not strongly Lipschitz.

**Example 1.5.7.** There are few interfaces $\Sigma$ for which one explicitly can calculate the spectrum of the double layer potential operator. It is well known, see Fabes–Jodeit–Lewis [19] and [18], that this can be done with Mellin transform techniques for certain conical interfaces, using the group structure on these interfaces. We here extend these techniques to the two dimensional logarithmic spirals from (ii) above.

Let $\alpha_1 < \alpha_2 < \alpha_1 + 2\pi$, $\omega \in \mathbb{R}$ and consider the two parametrisations

$$\gamma_i : \mathbb{R} \rightarrow \gamma_i(\mathbb{R}) : t \mapsto e^{t+i(\omega t + \alpha_i)},$$

and the double layer potential operator $D$ from (1.21) on the Carleson–Lipschitz curve $\gamma_1 \cup \gamma_2$, where $\gamma_1$ is oriented towards 0 and $\gamma_2$ towards $\infty$. We claim that $D$ is isomorphic to the convolution operator $D_0$ on $L_2(\mathbb{R}; \mathbb{C}^2)$ with kernel $K(t-s)$ having Fourier transform

$$\hat{K}(-2\xi) = \frac{1}{2\pi} \left[ \frac{\cot \zeta_1 + \cot \zeta_2}{\sin \zeta_1} e^{\frac{i\alpha_1 \zeta_1}{\sin \zeta_1}} - \frac{e^{-i\alpha_2 \zeta_2}}{\sin \zeta_2} \cot \zeta_1 - \cot \zeta_2 \right],$$

where $\zeta_1 := \frac{\pi + i \xi}{2(1+i\omega)}$, $\zeta_2 := \frac{\pi + i \xi}{2(1-i\omega)}$ and $\alpha := \frac{\alpha_2 - \alpha_1}{\pi} - 1$. To see this, use the map

$$\gamma^* f(t) := e^{t/2} \left( \frac{f(\gamma_1(t))}{f(\gamma_2(t))} \right),$$

which is an isomorphism $\gamma^* : L_2(\gamma_1 \cup \gamma_2) \rightarrow L_2(\mathbb{R}; \mathbb{C}^2)$ and intertwines $D_0 \gamma^* = \gamma^* D$. The expression for $\hat{K}$ follows from residue calculus using that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 + i\omega}{1 - e^{i\beta t + (1+i\omega)t}} e^{(1+i\xi)t/2} dt = \begin{cases} e^{-i(\beta-1)\zeta_1} / \sin \zeta_1 & 0 < \beta < 2\pi \\ \cot \zeta_1 & \beta = 0 \text{ (p.v.)} \\ e^{-i(\beta+1)\zeta_1} / \sin \zeta_1 & -2\pi < \beta < 0. \end{cases}$$

We now obtain, for fixed $\alpha_1 < \alpha_2 < \alpha_1 + 2\pi$ and $\omega \in \mathbb{R}$, the spectrum

$$\sigma(D; L_2(\gamma_1 \cup \gamma_2)) = \left\{ \pm \frac{1}{2} \sqrt{\left( \frac{\cos \zeta_1}{\sin \zeta_1} - \frac{\cos \zeta_2}{\sin \zeta_2} \right) \left( \frac{\sin \zeta_1}{\sin \zeta_2} - \frac{\sin \zeta_2}{\sin \zeta_1} \right) - \left( \cot \zeta_1 - \cot \zeta_2 \right)^2 ; \xi \in \mathbb{R} \right\}.$$

For some examples of how these spectra looks like, see Figure 2.1-2.9 in the Appendix. In particular, we can verify that the spectral radius for the double layer potential operator is less that 1 in this example. Note that the special case of a cone with opening angle $(\alpha + 1)\pi$ is when $\omega = 0$, i.e. $\zeta_1 = \zeta_2$, in which case $\zeta = \frac{\pi}{2}(1 + i\xi)$ and $\sigma(D; L_2(\gamma_1 \cup \gamma_2)) = \left\{ \frac{\sin \alpha \zeta_1}{\sin \zeta_1} \right\}$.

When investigating transmission problems on Lipschitz type domains one need to understand the connection between the function spaces of monochromatic fields in $\Omega^\pm$ and the corresponding trace space on $\Sigma$. In Chapters 4 and 5, we use the energy norm $L_2(\Omega; \Lambda)$ and construct an associated trace space $E$ on a weakly Lipschitz interface $\Sigma$. In Chapter 3 we use the trace space $L_2(\Sigma; \Lambda)$ on a strongly Lipschitz interface $\Sigma$. We finish this chapter with a discussion of the function spaces in $\Omega$ associated with $L_2(\Sigma; \Lambda)$ on a Carleson–Lipschitz interface $\Sigma$. 

23
Definition 1.5.8. Let $\Omega \subset \mathbb{R}^n$ be an open set with distance function $\lambda(x) \approx \text{dist}(x, \partial \Omega)$, $x \in \Omega$. Then

$$
\|F\|_{W_{2,1}^2(\Omega)}^2 := \int_{\Omega} |\nabla \otimes F(x)|^2 \lambda(x) \, dx
$$

$$
\|F\|_{W_{2,2}^{1/2}(\Omega)}^2 := \int_{\Omega} \int_{\Omega} \frac{|F(y) - F(x)|^2}{|y - x|^n} \, dx \, dy.
$$

The fundamental theorem for us in Chapter 3 is the following higher dimensional version of the famous theorem on the $L_2$ boundedness of the Cauchy integral, first proved for general strongly Lipschitz curves by Coifman–McIntosh–Meyer [10]. We finish this section with a survey of the proof of this theorem from Li–McIntosh–Sermes [33], and point out that the proof extends to Carleson–Lipschitz surfaces. Note that one can prove that singular integral operators like $E_k$ are $L_2$-bounded on more general surfaces than Carleson–Lipschitz surfaces. The appropriate class is the Ahlfors regular surfaces, see for example David [15]. However, this class is not appropriate for invertibility of double layer type operators.

Theorem 1.5.9. Let $\Sigma$ be a bounded Carleson–Lipschitz interface. Then the Cauchy reflection operator $E_k$ from Definition 1.2.8 is well defined as a bounded singular integral operator on $L_2(\Sigma ; \wedge)$. The space $L_2(\Sigma ; \wedge)$ splits topologically

$$
L_2(\Sigma ; \wedge) = E_k^+ L_2 \oplus E_k^- L_2
$$

into two Hardy type subspaces, where $E_k^\pm = \frac{1}{2}(I \pm E_k)$ are the projection operators corresponding to $E_k$ as in Chapter 2. The Cauchy extensions $C_k^\pm$ from Definition 1.2.8 give a one to one $(f = F|_{\Sigma})$ correspondence $E_k^\pm L_2 \ni f^\pm \leftrightarrow F^\pm \in C_k^\pm L_2$, where $C_k^\pm L_2 := R(C_k^\pm)$. If $\Omega \supset \Sigma$ denotes any bounded neighbourhood of $\Sigma$, then

$$
\|f\|_{L_2(\Sigma)} \approx \|F^\pm\|_{W_{2,1}^2(\Omega^\pm \cap \Omega)} + \|F^\pm\|_{L_2(\Omega^\pm \cap \Omega)} \approx \|F^\pm\|_{W_{2,2}^{1/2}(\Omega^\pm \cap \Omega)} + \|F^\pm\|_{L_2(\Omega^\pm \cap \Omega)}.
$$

If $\rho : \mathbb{R}^n \to \mathbb{R}^n$ denotes a special Carleson–Lipschitz parametrisation of a part of $\Sigma$ and $\rho_t(x') := \rho(x' + te_n)$, then for all $F \in C^\pm L_2$ we locally have convergence $F \circ \rho_t \to F \circ \rho_0$, $t \to 0$, both in $L_2(\mathbb{R}^{n-1})$ and pointwise almost everywhere. Moreover, we have control of the non-tangential maximal function of $F \circ \rho$ as $\|\mathcal{N}(F \circ \rho)\|_{L_2(\mathbb{R}^{n-1})} \lesssim \|f\|_{L_2}$.

The terms $\|F^\pm\|_{L_2(\Omega^\pm \cap \Omega)}$, which arises from a partition of unity argument, are compact terms and are of minor importance. We will hence only consider the case when $k = 0$ and a special Carleson–Lipschitz interface $\Sigma$, in which case the lower order terms $\|F^\pm\|_{L_2(\Omega^\pm \cap \Omega)}$ can be omitted.

Theorem 1.5.10. (i) Let $\Omega \subset \mathbb{R}^n$ be open and $c > 0$ small enough. Then for any monogenic $F : \Omega \to \wedge$ we have

$$
\int_{\Omega} |\nabla \otimes F(x)|^2 \lambda(x) \, dx \lesssim \int_{\Omega} \int_{y \in B(x, c\lambda(x))} \frac{|F(y) - F(x)|^2}{|y - x|^n} \, dx \, dy.
$$
(ii) Let $\Omega \subset \mathbb{R}^n$ be a special weakly Lipschitz domain. Then for any $F : \Omega \to \mathbb{R}$ we have

$$\int_{\Omega} \int_{\Omega} \frac{|F(y) - F(x)|^2}{|y - x|} \, dx \, dy \leq \int_{\Omega} |\nabla \otimes F(x)|^2 \lambda(x) \, dx.$$

This theorem can be found in Fabes [17], but the proof contains a false “Besov–Sobolev result”. As pointed out to the author by I. Mitrea, a correction has appeared in Mitrea–Mitrea–Pipher [45]. The correction below was found independently and is different. For completeness we include it here.

Proof. (i) As in [17], this estimate is obtained by pointwise estimating $|\nabla \otimes F(x)|^2$, using the reproducing formula (1.20) on spheres with radii $r \leq \lambda(x)$.

(ii) Since all norms are equivalent under bilipschitz parametrisations, we may assume that $\Sigma = \mathbb{R}^{n-1}$. In order to obtain the estimate $\|F\|_{\dot{W}^{1/2}_2(\mathbb{R}^n)} \lesssim \|F\|_{\dot{W}^{1/2}_{2,\lambda}(\mathbb{R}^n)}$, we consider the three regions

$$D_1 := \{(x, y) ; x, y \in \mathbb{R}^n, |y' - x'| < x_n, |y_n - x_n| < x_n/2\},$$
$$D_2 := \{(x, y) ; x, y \in \mathbb{R}^n, |y' - x'| < x_n, 0 < y_n < x_n/2\},$$
$$D_3 := \{(x, y) ; x, y \in \mathbb{R}^n, |y' - x'| > x_n, 0 < x_n < x_n\}.$$

It suffices to bound the double integral over these three regions. For $D_1$, we estimate $|F(y) - F(x)|$ with the integral of $|\nabla \otimes F|$ on the radial line from $x$ to $y$ as in [17].

For $D_2$, we write

$$|F(y) - F(x)| \leq |F(y', y_n) - F(y', x_n)| + |F(y', x_n) - F(x', x_n)|,$$

use Hardy inequality on the half-axis $y' + t e_n$, $t > 0$, for the first term and Parseval’s formula in $x_n e_n + \mathbb{R}^{n-1}$ for the second term as in [17].

For $D_3$, we need to modify the argument in [17] as follows: instead of following the euclidean straight line from $x$ to $y$, we need to follow the hyperbolic geodesic of the Poincaré half space. We do this by writing

$$|F(y) - F(x)| \leq |F(y', y_n) - F(y', y' - x')| + |F(y', y' - x') - F(x', y' - x')| + |F(x', y' - x') - F(x', x_n)|.$$
where on the third line we used polar coordinates $y' = x' + r\omega$, and on the fourth that $x_n(r - x_n)^2 \leq r^3$ and Hardy's inequality. As for the second term, we get

$$II = \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}^{n-1}} dy' \int_0^{y' - x'} dy_n \int_0^{x_n} dy_n \frac{|F(y', |y' - x'|) - F(x', |y' - x'|)|^2}{|y' - x'|^{n+1}}$$

$$\approx \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}^{n-1}} dy' \int_0^{x_n} dh \frac{|F(y', |y' - x'|) - F(x', |y' - x'|)|^2}{|y' - x'|^{n-1}}$$

$$= \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}^{n-1}} dh \frac{d\xi}{|h|^{n-1}} |F(x' + h, |h|) - F(x', |h|)|^2$$

$$\approx \int_{\mathbb{R}^{n-1}} d\xi \int_0^\infty dr \frac{r^2 |\hat{F}(\xi, r)|^2}{r^{2n}} \approx \int_{\mathbb{R}^{n-1}} |\nabla_{\mathbb{R}^{n-1}} \otimes F(x)|^2 x_n \, dx,$$

where we on the last two lines have used Parseval formula and where $\hat{F}$ denotes the horizontal partial Fourier transform.

To prove Theorem 1.5.9 for a special Carleson–Lipschitz interface $\Sigma = \Sigma_\rho$, we will use the following lemma. This lemma is an operator theoretic version of the method from Coifman–Jones–Semmes [9] to deduce the $L_2$-boundedness of the Cauchy integral from Theorem 1.5.15 below. For a version of this lemma in the language of functional calculus, see McIntosh–Qian [40].

**Lemma 1.5.11.** Let $\mathcal{H}$ and $\mathcal{K}^\pm$ be Hilbert spaces ($\mathcal{K}^\pm$ being “wavelet transform spaces” for $\mathcal{H}$) and assume the following.

1. We have densely defined “synthesis maps” $S_0^\pm : \mathcal{K}^\pm \rightarrow \mathcal{H}$.

2. There exists a self-adjoint isometry $J : \mathcal{H} \rightarrow \mathcal{H}$ such that $R(J S_0^\pm) \subset D(S_0^\pm) \cap D(S_0^\pm)$.

3. The subspace $R(S_0^+) + R(S_0^-)$ is dense in $\mathcal{H}$.

4. We have a continuous pairing

$$|\langle S_0^+ F, J S_0^- G \rangle| \leq C_2 \|G\| \|F\|,$$

Then we have a topological splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm = N(S_0^\pm)^*$, and there exists a unique bounded extension $\mathcal{S}^\pm : \mathcal{K}^\pm \rightarrow \mathcal{H}$ of $S_0^\pm$ and $S_0^\pm = S_0^\pm^*$.

If we let $\mathcal{K}_0^\pm := R(S_0^\pm)$ be the range of the “analysing map” $S_0^\pm$, then $\mathcal{K}_0^\pm$ are closed and $S_0^\pm : \mathcal{K}^\pm \rightarrow \mathcal{K}_0^\pm$ are isomorphisms. Thus there exists unique bounded $\gamma_\pm : \mathcal{K}_0^\pm \rightarrow \mathcal{H}$ such that $P_\pm := \gamma_\pm S_0^\pm$ is the projection onto $\mathcal{H}^\pm$ along $\mathcal{H}^\mp$ and we have $\|P_\pm\| \leq C_2^2 C_2$.

Furthermore $R(S_0^\pm) = (\mathcal{H}^\pm)^\perp = J \mathcal{H}^\pm$ is closed, $S_0^\pm : \mathcal{K}_0^\pm \rightarrow J \mathcal{H}^\pm$ are isomorphisms and $(S_0^+)^\perp, (S_0^-)^\perp$ is a duality.

**Proof.** If $F \in D(S_0^+)$, then (2) and (4) give

$$\|S_0^+ F\| = \|J S_0^+ F\| \leq C_1 \|S_0^+ J S_0^+ F\| = C_1 \sup_{G \in D(S_0^-)} \|G\| \leq C_2 \|F\|.$$
Thus $S_0^\pm$ extends to a bounded operator $S^\pm$, and $\mathcal{D}(S_0^{\pm\ast}) = \mathcal{H}$ and $S^\pm = S_0^{\pm\ast}$.

If $f^+ + f^- \in R(JS_0^-) + R(JS_0^+)$, then the a priori estimate in (2) gives

$$
\|f^\pm\| \leq C_1 \|S_0^{\pm\ast} f^\pm\| = C_1 \|S_0^{\pm\ast}(f^+ + f^-)\| \leq C_1^2 C_2 \|f^+ + f^-\|.
$$

Thus $f^+ + f^- \cong f^+ + f^-$ and from (3) we get $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $\mathcal{H}^\pm := R(S_0^\pm)$. It follows from (2) that $\mathcal{N}(S^{\pm\ast}) = \mathcal{H}^\mp$ and $S^{\pm\ast} : \mathcal{H}^\pm \to K_0^{\pm}$ are isomorphisms. Now Theorem 1.4.8 shows that $R(S^\mp)$ is closed and that $R(S^\mp) = \mathcal{N}(S^{\pm\ast})$ = $\mathcal{H}^\pm$. Finally we see that $\mathcal{JH}^\pm = R(S_0^\pm) = R(S^\mp)$.

The idea behind Lemma 1.5.11 is that in order to prove that an essentially self-adjoint (modulo the factor $J$) singular integral operator like $E_k$ from Definition 1.2.8 is $L^2$-bounded, it suffices to prove a “reverse square function estimate” like $\|f\| \lesssim \|S_0^{\pm\ast} f\|$ in (2). The “square function estimate” $\|S_0^{\pm\ast} f\| \lesssim \|f\|$ then follows by duality (4). For more details, see Albrecht–Duong–McIntosh [1].

**Example 1.5.12.** Using a distance function $\lambda(x) \approx \text{dist } (x, \Sigma)$ we let

$$
\mathcal{H} = L_2(\Sigma) \otimes \Lambda, \quad K^{\pm} = L_2(\Omega^{\pm}; \frac{dx}{\lambda(x)}) \otimes \mathbb{R}^n \otimes \Lambda, \quad J f(y) = \nu(y) f(y),
$$

$$
S_0^{\pm} \mathcal{H}(y) = \nu(y) \int_{\Omega^{\pm}} \partial_i E(y - x) H_i(x) dx, \quad \mathcal{H} = \{H_i \in \mathcal{D}(S_0^\pm) \equiv C_0(\Omega^{\pm}; \mathbb{R}^n \otimes \Lambda),
$$

$$
S_0^{\pm\ast} f(x) = \lambda(x) \nabla \otimes C^\pm f(x) = \lambda(x) e_i \otimes \int_{\Sigma} \partial_i E(y - x) \nu(y) f(y) d\sigma(y).
$$

To verify (3) and (4) one uses Lemma 1.5.13 and Lemma 1.5.14 below respectively. Moreover, the a priori estimate in (2) follows from the trace Theorem 1.5.15. Thus the $L^2$-boundedness of the Cauchy integral $E = P^+ - P^-$ follows from Lemma 1.5.11.

**Lemma 1.5.13.** Let $\rho$ be a weakly Lipschitz parametrisation of $\Sigma$ and define $\eta_{t,x'} : \Sigma \to \lambda^1 : y \mapsto E(y - \rho(x', \epsilon)) - E(y - \rho(x', -\epsilon))$, $\epsilon > 0$, $x' \in \mathbb{R}^{n-1}$.

Then $|\eta_{t,x'}(y)| \lesssim \epsilon / (|y - \rho(x', 0)| + \epsilon)$ and

$$
\int_{\Sigma} \eta_{t,x'}(y) \nu(y) d\sigma(y) = \int_{\Sigma} \nu(y) \eta_{t,x'}(y) d\sigma(y) = 1.
$$

**Lemma 1.5.14.** Let $\Sigma$ be a special weakly Lipschitz interface. If $F \in L_2(\Omega^+; \frac{dx}{\lambda(x)}) \otimes \mathbb{R}^n \otimes \Lambda$ and $G \in L_2(\Omega^-; \frac{dx}{\lambda(y)}) \otimes \mathbb{R}^n \otimes \Lambda$ then

$$
\left| \int_{\Omega^+} dx \int_{\Omega^-} dy \left( F_j(x), \partial_j \partial_i E(x - y) G_i(y) \right) \right|^2 \lesssim \int_{\Omega^+} \frac{dx}{\lambda(x)} |F(x)|^2 \int_{\Omega^-} \frac{dy}{\lambda(y)} |G(y)|^2.
$$

The proofs of these two lemmata are straightforward, using the boundary theorem 1.2.7 and Cauchy–Schwarz inequality respectively.

**Theorem 1.5.15.** If $\Sigma$ is a special Carleson–Lipschitz interface and if $f \in R(JS_0^\mp) \subset E^\pm L_2(\Sigma)$ and $F := C^\pm f$, then

$$
\int_{\Sigma} |f|^2 \lesssim \int_{\Sigma} |\nabla \otimes F(x)|^2 \lambda(x) dx.
$$
This trace theorem is due to Dahlberg. Let us recall the proof from Li–McIntosh–Semmes [33]. The main idea is that since \(DF = 0\) in \(\Omega^\pm\), one can estimate the normal derivative with the tangential derivatives. More precisely, we make the change of variables \(G(x', x_n) := F(\rho(x', x_n))\). Then using the expression \(\nabla = \sum e_i^* \partial_{e_i}\), where \(e_{is} := \rho_s(e_i)\) and \(e_i^* := (\rho^{-1})^* e_i\) are dual bases, we obtain from Sobolev trace theorem \(\tilde{W}^1_2(\mathbb{R}_+^n) \to \tilde{W}^{1/2}_2(\mathbb{R}^{n-1})\) that

\[
\int_{\Omega} |f|^2 \approx \int_{\mathbb{R}^{n-1}} |g|^2 - 2 \int_{\mathbb{R}_+^n} (G, \partial_n G) = 2 \sum_{i=1}^{n-1} \int_{\mathbb{R}_+^n} (G, (e_i^*)^{-1} e_i \partial_i G)
\]

\[
= \sum_{i=1}^{n-1} \int_0^\infty dx_n \int_{\mathbb{R}^{n-1}} dx' \left(|\nabla x'|^{1/2}(\alpha_i G(x', x_n)), |\nabla x'|^{-1/2} \partial_i G(x', x_n)\right)
\]

\[
\lesssim \sum_{i=1}^{n-1} \left(\int_{\mathbb{R}_+^n} |\nabla \otimes G|^2 x_n dx + \int_{\mathbb{R}_+^n} |G|^2 (|\nabla \otimes a_i|^2 x_n dx)\right)^{1/2} \left(\int_{\mathbb{R}_+^n} |\nabla \otimes G|^2 x_n dx\right)^{1/2},
\]

where \(a_i := 2(e_i^*)^{-1} e_i^* : \mathbb{R}_+^n \to \Lambda^0 \otimes \Lambda^2\). Using (1.26), we can now apply Carleson’s lemma 1.5.16 for the Carleson measure \(d\mu := |\nabla \otimes a_i|^2 x_n dx\) to get

\[
\int_{\mathbb{R}_+^n} |G|^2 d\mu \lesssim \left(\sup_Q \frac{1}{|Q|} \int_{Q \times (0, t(Q))} d\mu\right) \int_{\mathbb{R}_+^n} |\nabla G|^2 \lesssim \int_{\mathbb{R}_+^n} |G|^2 \lesssim \int_{\mathbb{R}_+^n} |x|^2.
\]

Here we have estimated the non-tangential maximal function \(NG\) of \(G\) with the Hardy–Littlewood maximal function \(Mg\) of \(g\) by applying Lemma 1.5.13 to the identity

\[
G(x) = F(\rho(x)) = \int_{\mathbb{R}_+^n} \eta_{x, x'}(\rho(y)) \nu(\rho(y)) g(y) d\sigma(y), \quad x \in \mathbb{R}_+^n.
\]

From this we get the estimate \(\int_{\mathbb{R}_+^n} |x|^2 \lesssim \int_{\mathbb{R}_+^n} |\nabla \otimes G|^2 x_n dx\), which proves Theorem 1.5.15.

**Lemma 1.5.16 (Carleson).** Let \(F : \mathbb{R}_+^n \to \mathbb{R}\) be a function and \(\mu\) be a measure in \(\mathbb{R}_+^n\) and define the non-tangential maximal function

\[
NF(x') := \sup\{|F(y', y_n)| : |y' - x'| < y_n\}, \quad x' \in \mathbb{R}^{n-1}.
\]

Then

\[
\left|\int_{\mathbb{R}_+^n} F d\mu\right| \lesssim \left(\sup_Q \frac{1}{|Q|} \int_{Q \times (0, t(Q))} d\mu\right) \int_{\mathbb{R}_+^n} NF,
\]

where the supremum is over all cubes \(Q \subset \mathbb{R}^{n-1}\).

The following example shows that we indeed need to eliminate the normal derivative of \(F\) and that the Sobolev trace map fails on all \(W^{1/2}_2\).

**Example 1.5.17.** For \(r > 0\), consider the function \(f_R : \mathbb{R}_+^n \to \mathbb{R}\)

\[
f_R(x) := \begin{cases} 
1 & x < e^{-R}, \\
-\ln(x)/R & e^{-R} < x < 1, \\
0 & x > 1.
\end{cases}
\]

Then \(f_R(0) = 1\) for all \(R\), although \(\int_{0}^{\infty} |f_R'(x)|^2 x \, dx \to 0\).
In Chapter 6 we will need the following lemma.

**Lemma 1.5.18.** Let \( \Sigma_\rho = \rho(R^{n-1}) \) be a special Carleson-Lipschitz interface, let \( \rho_t(x') := \rho(x' + te_n) \) and assume \( \Phi \in C^\infty_0(R^n; \Lambda) \). Then \( \rho_0^* \Phi \) and \( \rho_0^{-1} (\nu \perp \phi) \), defined pointwise a.e., belongs to \( D(d\rho_{R^n-1}) \) and \( D(\delta \rho_{R^n-1}) \) respectively and

\[
d_{R^n-1}(\rho_0^* \Phi) = \rho_0^*(d\Phi),
\]
\[
\delta_{R^n-1}(\rho_0^{-1}(\nu \perp \phi)) = -\rho_0^{-1}(\nu \perp (\delta \Phi_{|\Sigma})).
\]

Moreover, we have \( L^2 \) convergence \( \rho_t^* \Phi \to \rho_0^* \Phi \) as \( t \to 0 \) and uniform bounds

\[
\int_{\mathbb{R}_n} |\rho_t^* \Phi|^2 \lesssim \int_{\Omega^+} |\nabla \otimes \Phi|^2 + \int_{\mathbb{R}_n} |\nabla (\Phi \circ \rho)|^2,
\]

\( t \geq 0 \).

The corresponding convergence and bounds are also valid for \( \delta \).

**Proof.** (i) First consider \( \Psi \in C^\infty_0(R^n; \Lambda) \). As for Theorem 1.5.15, using the Sobolev trace map \( \tilde{W}^2_n(\mathbb{R}_+^n) \to \tilde{W}_{2/2}^1(\mathbb{R}^{n-1}) \), we get

\[
\int_{\mathbb{R}^n} \left| e_n \wedge \psi \right|^2 = 2 \int_{\mathbb{R}^n} (e_n \wedge \Psi, e_n \wedge \partial_n \Psi) = 2 \int_{\mathbb{R}^n} \left( e_n \wedge \Psi, d\Psi - \sum_{i=1}^{n-1} e_i \wedge \partial_i \Psi \right)
\]
\[
\lesssim \int_{\mathbb{R}^n} |\nabla \otimes \Psi|^2 x_n dx + \int_{\mathbb{R}^n} |\Psi|^2 dx + \int_{\mathbb{R}^n} |\nabla \wedge \Psi|^2 dx.
\]

Consider the pullback \( G := \rho^* \Phi \) in \( \mathbb{R}^n_+ \). Using Definition 1.2.1, the chain rule, (1.26) and Carleson’s lemma 1.5.16, we see that

\[
\int_{\mathbb{R}^n_+} |\nabla \otimes G|^2 x_n \lesssim \int_{\Omega^+} |\nabla \otimes \Phi|^2 + \int_{\mathbb{R}^n} |\nabla (\Phi \circ \rho)|^2 < \infty.
\]

(ii) Consider the pullbacks \( g_t := \rho_t^* \Phi = e_n \perp (e_n \wedge G)|_{[te_n+R]} \) in \( \mathbb{R}^n \). Using \( \Psi(x) := G(x + te_n) - G(x + se_n) \), \( t, s > 0 \), it follows from (i) that \( \{ g_t \}_{t \geq 0} \) is \( L^2 \)-Cauchy and we obtain that \( g_t \to g_0 = \rho_0^* \Phi \) in \( L^2 \). To see that the limit is indeed \( g_0 \), one can for example pick a subsequence converging weakly in \( L^2 \) to \( g_0 \). Moreover, for \( t > 0 \), Lemma 1.2.2 proves that \( d_{R^n-1} g_t \to \rho_t(d\Phi) \) and so \( d_{R^n-1} g_t \to \rho_0(d \Phi) \) in \( L^2 \). Since \( d_{R^n-1} \) is a closed operator, we obtain that \( g_0 \in D(d_{R^n-1}) \) and \( d_{R^n-1} g_0 = \rho_0^*(d \Phi) \).

(iii) We now show how to derive the corresponding result for \( \delta \) from (i) using Hodge complements. Writing \( \Phi = \Phi \perp = \Phi_1 \perp V \), where \( V := e_{12...n} \), it follows that \( \delta \Phi = (d \Phi_{V}) \perp \perp \). Using Proposition 1.1.8 and Lemma 1.2.2 we calculate

\[
\rho_{0}^{-1}(\nu \perp \phi) = \rho_{0}^{-1}(\nu \perp (\Phi_1 \perp V)) = (\rho_0^*(\Phi_1)) \perp \rho_{0}^{-1}(\nu \perp V) \perp (\rho_0^*(\Phi_1) \perp (\nu \perp V)),
\]
\[
\delta_{R^n-1}(\rho_{0}^{-1}(\nu \perp \phi)) = \left( d_{R^n-1} - \rho_0^*(\Phi_1) \right) \perp (\rho_0^*(\Phi_1) \perp V) = -\rho_0^{-1}(\nu \perp (\delta \Phi_{|\Sigma})).
\]
Chapter 2

Transmission problems and boundary operator algebras

A classical method for solving boundary value problems such as the Dirichlet and Neumann problems for the Laplace operator is to solve the associated boundary integral equation, an equation of the second kind involving the double layer potential operator or its adjoint, see e.g. Kress [31] and Maz’ya [34]. However, there has been little discussion in the literature about other alternative boundary integral operators, besides double layer type operators, that are relevant for solving boundary value and transmission problems. On the other hand, operator algebras generated by two reflection operators are used in Toeplitz operator theory, see for example Böttcher–Karlovich [7] and Böttcher–Silbermann [6]. In this chapter we show that there is indeed a strong connection between transmission problems and operator algebras generated by two reflection operators. For example, to solve the Dirichlet problem one uses the double layer potential operator

\[ Ku(x) = \text{p.v.} \int_{\Sigma} \frac{(\nabla \Phi)(y - x) \cdot \nu(y) u(y)}{y - x} \, d\sigma(y), \quad x \in \Sigma, \]

on the surface \( \Sigma \). To understand the operator-theoretic structure of \( K \), it helps to introduce a “boundary data reflection operator” \( A \) and a “boundary trace reflection operator” \( B \), which generate a Banach algebra \( \mathcal{A} \). These are reflection operators, i.e. \( A^2 = B^2 = I \), and they act on the “full trace space” which, since \( \Delta \) is a second order operator, will be a boundary function space of \( C^2 \)-valued functions. Some ideas in this direction can be found in for example Kress–Roach [32]. In Example 2.2.2 we show that \( 2K \) is “half” of the anti-commutator, or cosine operator \( \frac{1}{2}(AB + BA) \). More precisely \( 2K \) is the compression of \( B \) to the 1-eigenspace of \( A \).

On the other hand, when the elliptic partial differential equation is first order, for example the Hilbert and Riemann problems for \( \overline{\Omega} \), we show that the transmission problem is naturally formulated as a resolvent equation \( (\lambda - BA)f = g \). Thus it is here the rotation operator \( BA \) which is naturally associated with the transmission problem. From a practical point of view, this rotation operator is superior to the double layer type operator in the first order case since it “acts in the whole boundary function space” as explained in Example 2.2.1.
We will also demonstrate how the decoupled transmission problem, i.e. the boundary value problem, can be solved with Toeplitz type operators. Indeed, these are operators of the same structure as double layer type operators. In Proposition 2.1.2 we investigate the connection between the cosine operator $\frac{1}{2}(AB + BA)$, which is useful for the Laplace transmission problem, and the rotation operator $BA$ which is useful for Dirac and $\overline{\Omega}$ transmission problems.

In Section 2.2 we show how to apply the abstract techniques in Section 2.1 to transmission problems for $\overline{\Omega}$ and $\Delta$. Two main results we obtain are the following.

- Consider the following Hilbert problem. Let $\Omega^+$ be a bounded strongly Lipschitz domain in the complex plane with $\Sigma := \partial \Omega^+$ and outward pointing normal $\nu$ and let $a = a_1 + a_2 j$ be a measurable unit vector field on $\Sigma$. Given a function $\alpha \in L_2(\Sigma; \mathbb{R})$, find a complex valued function $F$ in $\Omega^+$ with non-tangential maximal function $\mathcal{N}F \in L_2(\Sigma)$, such that

$$\begin{cases} \mathcal{T}F = 0 & \text{in } \Omega^+, \\ a_1 f_1 + a_2 f_2 = \alpha & \text{on } \Sigma, \end{cases}$$

where $f = f_1 + f_2 j = F|_{\Sigma}$. If $\nu a^2$ is locally accretive, then this boundary value problem is well-posed in Fredholm sense.

- Let $\Omega^+$ be a bounded strongly Lipschitz domain in $\mathbb{R}^n$ with boundary $\Sigma$ and let $L_\Sigma$ be the local Lipschitz constant for $\Sigma$. For any $0 \leq s \leq 1$, the essential spectrum of the double layer potential operator is contained in the hyperbolic region

$$\sigma_{\text{ess}}(2K; W^s_2(\Sigma)) \subset \{ \lambda = \lambda_1 + i \lambda_2 ; \lambda_1^2 \leq L^2_\Sigma \lambda_2^2 + L^2_\Sigma/(L^2_\Sigma + 1) \}.$$  

Both these results are proved by using the underlying rotation operator for the transmission problem. When applying Rellich estimates directly to the double layer potential operator, usually only real spectral parameters are considered. By applying a “complexified” Rellich estimate here on the rotation operator, we directly obtain a priori estimates in an optimal double sector around the real axis. The corresponding spectral hyperbola for the double layer potential have also been obtained by Spencer [58].

### 2.1 Banach algebras generated by two reflections

Consider an elliptic partial differential operator $D$ and two complementary domains $\Omega^+$ and $\Omega^-$ in $\mathbb{R}^n$ separated by a surface $\Sigma = \partial \Omega^+ = \partial \Omega^-$ (with appropriate regularity). The formal structure of a transmission problem across $\Sigma$ for $D$ is as follows. Given some data $g$ (a function on $\Sigma$), we are looking for a pair of functions $(F^+, F^-)$ satisfying $DF^\pm = 0$ in $\Omega^\pm$ respectively (and some decay/radiation condition for $F^-$ at $\infty$), such that the boundary traces $f^\pm := F^\pm|_{\Sigma}$ are related in a way prescribed by $g$. The function space $\mathcal{X}$ on $\Sigma$ splits into two Hardy type subspaces $B^\pm \mathcal{X}$ consisting of all traces $f^\pm$ as above, where $B^\pm$ are the two complementary Hardy type projections. Here “complementary
projections" means \((B^\pm)^2 = B^\pm, B^+B^- = B^-B^+ = 0\) and \(B^+ + B^- = I\). The relation between \(f^\pm\) we consider here is given by the two jump conditions

\[
\begin{align*}
A^+(\alpha^-f^- - \alpha^+f^-) &= A^+g, \\
A^-(\alpha^+f^+ - \alpha^-f^-) &= A^-g,
\end{align*}
\]

where \(A^\pm\) are two given complementary projections and \(\alpha^\pm \in \mathbb{C}\) are given jump parameters, which we without loss of generality assume satisfy \(\alpha^+ \neq \pm \alpha^-\). In this section we study the operator algebra generated by the two reflection operators associated with this transmission problem.

**Definition 2.1.1.** Let \(\mathcal{X}\) be a Banach space and let \(\mathcal{A}\) be the Banach sub-algebra of \(\mathcal{B}(\mathcal{X})\) which is generated by two bounded reflection operators \(A\) and \(B\), i.e. \(A^2 = B^2 = I\). Define the following operators in \(\mathcal{A}\).

- The **\(A\)-projections** \(A^\pm = \frac{1}{2}(I \pm A)\) and the **\(B\)-projections** \(B^\pm = \frac{1}{2}(I \pm B)\),
- the **rotation operator** \(BA\),
- the **cosine operator** \(C := \frac{1}{2}(AB + BA)\) and the **sine operator** \(S := \frac{1}{2i}(AB - BA)\),
- the **double layer type operators** \(A^\pm BA^\pm : A^\pm \mathcal{X} \to A^\pm \mathcal{X}\) and
  the **single layer type operators** \(A^\mp BA^\pm : A^\mp \mathcal{X} \to A^\mp \mathcal{X}\),
- the **Toeplitz type operators** \(B^\mp AB^\pm : B^\mp \mathcal{X} \to B^\mp \mathcal{X}\) and
  the **Hankel type operators** \(B^\pm AB^\pm : B^\pm \mathcal{X} \to B^\pm \mathcal{X}\).

The **even sub-algebra** of \(\mathcal{A}\), denoted \(\mathcal{A}^+\), is the Banach sub-algebra generated by \(AB\) and \(BA\). We say that \(A\) and \(B\) are **transversal** if \(\sigma(C) \cap \{1, -1\} = \emptyset\).

The even sub-algebra \(\mathcal{A}^+\) is a commutative Banach algebra. The operator \(C\) is central in \(\mathcal{A}\), i.e. \(C\) commutes with all \(T \in \mathcal{A}\), while \(S\) anti-commutes with \(A\) and \(B\). Basic properties of these operators are \(A^+ - A^- = I, A^+-A^- = A, B^+ + B^- = I, B^+ - B^- = B, (A^\pm)^2 = A^\pm, (B^\pm)^2 = B^\pm, (BA)^{-1} = AB\). Note also the identities

\[
\begin{align*}
I - BA &= 2(B^+A^- + B^-A^+), \\
I + BA &= 2(B^+A^+ + B^-A^-), \\
BA &= B^+A^+ + B^-A^- - B^+A^- - B^-A^+, \\
C &= A^+BA^+ - A^-BA^- = B^+AB^+ - B^-AB^-, \\
iS &= A^+BA^- - A^-BA^+ = B^-AB^+ - B^+AB^-, \\
I &= C^2 + S^2,
\end{align*}
\]

where (2.3) shows that \(BA\) “acts in all \(\mathcal{X}\)”, while (2.4) and (2.5) show that \(C\) and \(S\) decompose as direct sums of operators acting in the subspaces.

The simplest case is when \(\mathcal{A}\) is a \(C^*\)-algebra and \(A\) and \(B\) are self-adjoint reflections acting on a Hilbert space of functions on \(\Sigma\), see for example Halmos [23] and Davis [16].
In this case, since $(BA)^*(BA) = ABBA = I = BAAB = (BA)(BA)^*$ and $0 \leq (A \pm B)^2 = 2(I \pm C)$, we see that the rotation operator is unitary and $-1 \leq C \leq 1$.

The heuristics are that the $C^*$ case corresponds to a flat or spherical surface $\Sigma$ and as long as $\Sigma$ has enough smoothness (for example $C^{1+\epsilon}$), then the reflection operator $B$ is self-adjoint modulo compact operators. In the examples we consider, $A$ is self-adjoint (in the standard norm). Note that by changing to an equivalent norm on the Hilbert space we can always make one of the reflection operators self adjoint, but in general not both simultaneously.

**Proposition 2.1.2.** Define spectral parameters $\lambda := (\alpha^+ + \alpha^-)/(\alpha^+ - \alpha^-)$ and

$$c := \frac{1}{2}(\lambda + 1/\lambda) = ((\alpha^+)^2 + (\alpha^-)^2)/((\alpha^+)^2 - (\alpha^-)^2),$$

and consider the following four equations/systems.

\[
\begin{align*}
A^+(\alpha^- B^+ f - \alpha^+ B^- f) &= A^+ g \\
A^-(\alpha^+ B^+ f - \alpha^- B^- f) &= A^- g \\
(\lambda I - BA)f &= \frac{2}{\alpha^+ - \alpha^-}Bg \\
(cI - C)f &= \left(\frac{B}{\alpha^+ - \alpha^-} - \frac{A}{\alpha^+ + \alpha^-}\right)g
\end{align*}
\]

(2.7)

(2.8)

(2.9)

(2.10)

Then (2.7) and (2.8) are equivalent and so are (2.9) and (2.10). Furthermore (2.7) and (2.8) each imply (2.9) and (2.10). If $\lambda \notin \sigma(BA)$ then all equations/systems (2.7)-(2.10) are equivalent and uniquely solvable.

The conformal map

$$\sigma(BA) \ni \lambda \overset{\Psi}{\mapsto} c = \frac{1}{2}(\lambda + 1/\lambda) \in \sigma(C)$$

yields a $2 \to 1$ correspondence between the two spectra and

$$\sigma(C) = \sigma(A^+ B; A^\perp \mathcal{X}) \cup \sigma(A^- B; A^- \mathcal{X}) = \sigma(B^+ A; B^\perp \mathcal{X}) \cup \sigma(B^- A; B^- \mathcal{X}).$$

From Proposition 2.1.2 we see that understanding the transmission problem (2.7) means understanding the rotation spectrum $\sigma(BA)$ or equivalently the cosine spectrum $\sigma(C)$. These spectra contain information about the geometry between the two splittings of the function space $\mathcal{X}$ induced by the projections $\{A^\pm\}$ and $\{B^\pm\}$.

**Proof.** Adding both equations in (2.7) gives (2.8). But applying $A$ to the equations (2.7) changes sign on the second equation, so also subtracting the equations will give (2.8). Multiplying (2.8) by $I - \frac{1}{\lambda}AB$ and using the identity

$$2(cI - C) = (I - \frac{1}{\lambda}AB)(\lambda I - BA)$$

(2.11)

gives (2.9). From (2.4) we see that splitting (2.9) with $A^\pm$ gives the equivalent system (2.10).

The properties of $\psi$ follows from the spectral mapping theorem since $C = \frac{1}{2}(BA + (BA)^{-1})$, or more directly from the identities (2.11) and $\lambda - BA = A(\lambda - AB)A$. 

\[ \square \]
We now consider boundary value problems (abbreviated BVP). These are special cases of a transmission problem, as seen by letting $\alpha^+ = 0$ or $\alpha^- = 0$ in the transmission problem, or equivalently $\lambda = \pm 1$ or $c = \pm 1$. In this case the two equations in (2.7) are decoupled and the rotation equation (2.8) and the cosine equation (2.9) solve two boundary value problems simultaneously: the case $\alpha^+ = 0$ solves an interior $A^+ \text{ BVP}$ and an exterior $A^- \text{ BVP}$, and the case $\alpha^- = 0$ solves an interior $A^- \text{ BVP}$ and an exterior $A^+ \text{ BVP}$.

Note that if $A$ and $B$ are transversal, then Proposition 2.1.2 implies that the eight restricted projections

\[
A^+: B^X \rightarrow A^+ X, \quad A^-: B^X \rightarrow A^- X, \\
B^+: A^X \rightarrow B^+ X, \quad B^-: A^X \rightarrow B^- X,
\]

are all isomorphisms. More generally, if $A$ and $B$ are essentially transversal, i.e. if $\pm I - BA$ are Fredholm operators, then we can deduce from (2.1) and (2.2) that all the restricted projections are Fredholm operators and their indices satisfy

\[
\text{Ind}(I - BA) = \text{Ind}(B^+: A^- X \rightarrow B^+ X) + \text{Ind}(B^-: A^+ X \rightarrow B^- X), \\
\text{Ind}(I + BA) = \text{Ind}(B^+: A^+ X \rightarrow B^+ X) + \text{Ind}(B^-: A^- X \rightarrow B^- X).
\]

Even if both $\text{Ind}(\pm I - BA) = 0$, the restricted projections may have non-zero indices. This is the case for example when $X$ is finite dimensional and the four subspaces have different dimension. However, see Proposition 6.1.5 for one method to prove the the restricted projection have index zero.

**Example 2.1.3.** Consider the interior $A^+ \text{ BVP}$: let $\alpha^+ = 0$, $\alpha^- = 1$ and $g \in A^+ X$ and assume that $A$ and $B$ are transversal. Then the four equations reduce to

\[
A^+ f = g, \quad f \in B^+ X, \\
(I + BA) f = 2Bg, \\
(I + C) f = 2B^+ g, \\
(A^- B^-)(A^- f) = A^- B^+ g, \quad A^+ f = g.
\]

Note that indeed the last three equations imply $f \in B^+ X$ if $g \in A^+ X$ and $A$ and $B$ are transversal. For a BVP, the double layer and Toeplitz type operators are more efficient than the cosine and rotation operators. We demonstrate three ways to solve the BVP.

- Solve the last equation above: invert $A^- B^-$ on $A^- X$ which give us the complementary (conjugate) function $A^- f$ to $g = A^+ f$ and obtain $f = g + A^- f$.
- The standard way to solve this BVP with the double layer operators is different: from the ansatz $f = B^+ h$, $h \in A^+ X$, we get the double layer equation $A^+ B^+ h = g$. Solving for $h = (A^+ B^+ |_{A^+ X})^{-1} g$ then gives the solution $f = B^+ h$. 

34
• A way to solve this BVP which is well suited for a first order partial differential equation, is the \textit{Toeplitz type operator equation}

\[ B^+ A^+ f = B^+ g, \quad f \in B^+ \mathcal{X}, \]  

which follows immediately by applying \( B^+ \) to the BVP.

\textbf{Example 2.1.4.} We now consider some typical estimates on \( \sigma(BA) \) and \( \sigma(C) \). Below, (i) and (ii) are examples of spectral estimates which implies that \( A \) and \( B \) are transversal (and thus the boundary value problems are well posed). On the other hand (iii) and (iv) are, roughly speaking, estimates on how far from the self-adjoint case \( (A^* = A \text{ and } B^* = B) \) we are. We give corresponding estimates for the jump parameter \( \alpha \), the spectral parameter \( \lambda \) for the rotation operator \( BA \) and the spectral parameter \( c \) for the cosine operator \( C \).

(i) An angular estimate \( |\text{Re}\lambda| \leq L|\text{Im}\lambda| \) corresponds to the hyperbolic estimate

\[
\left( \frac{\text{Re}c}{L/\sqrt{L^2 + 1}} \right)^2 - \left( \frac{\text{Im}c}{1/\sqrt{L^2 + 1}} \right)^2 \leq 1.
\]

See Figure 1.1 and 1.2 in the Appendix.

(ii) An estimate \( |\text{Im} \alpha| > 1/L \) and \( |\text{Im}(1/\alpha)| > 1/L \) corresponds to

\[ \sigma(BA) \subset (\overline{D}(1 + iL, L) \cap \overline{D}(-1 + iL, L)) \cup (\overline{D}(1 - iL, L) \cap \overline{D}(-1 - iL, L)). \]

See Figure 1.3 (as well as 1.4 for the corresponding region for the cosine operator) in the Appendix.

(iii) A radial estimate \( 1/R \leq |\lambda| \leq R \) corresponds to the elliptic estimate

\[
\left( \frac{\text{Re}c}{\frac{1}{2}(R + 1/R)} \right)^2 + \left( \frac{\text{Im}c}{\frac{1}{2}(R - 1/R)} \right)^2 \leq 1.
\]

(iv) An estimate \( |\text{Re}\alpha| \leq \eta|\text{Im}\alpha| \) corresponds to \( \sigma(BA) \subset (\overline{\sigma}_{\eta} \cup \overline{\sigma}_{-\eta}) \setminus (\sigma_{\eta} \cap \sigma_{-\eta}) \), where \( \sigma_{\eta} \) is the open disc with centre at \( i\eta \) which touches \( \pm 1 \). For the cosine operator this estimate reads \( \sigma(C) \subset \overline{\sigma}_{(1/\eta-\eta)/2} \cap \overline{\sigma}_{(\eta-1/\eta)/2} \) if \( \eta \leq 1 \) and \( \sigma(C) \subset \overline{\sigma}_{(1/\eta-\eta)/2} \cup \overline{\sigma}_{(\eta-1/\eta)/2} \) if \( \eta > 1 \). See Figure 1.5 and 1.6 in the Appendix.

The rest of this section discusses the structure of the Banach algebra \( \mathcal{A} \). The results here will not be used in the next section. However, we will calculate with the bi-complex numbers \( \mathcal{A}_2^+ \) in Example 2.2.1.

Besides the operators above, it is natural to ask if and how the spectrum of a general operator \( T \in \mathcal{A} \) can be determined from that of \( BA \) or \( C \). This has been investigated in Toeplitz operator theory, see e.g. Böttcher–Karlovich [7]. It uses the following general result by G.R. Allan and R.G. Douglas about the Gelfand transform for non-commutative Banach algebras. For a proof, see e.g. Theorem 1.34 in Böttcher–Silbermann [6].
Theorem 2.1.5. Let $A$ be a Banach algebra with identity $e \in Z$, where $Z$ is a central sub-algebra of $A$. Let $M(Z)$ be the maximal ideal space of $Z$ and consider the inclusion

$$M(Z) \ni \omega \subset J_\omega := \omega A \subset A$$

of $\omega$ in the smallest two sided closed $A$-ideal containing $\omega$. Then $a \in A$ is invertible if and only if $a + J_\omega$ is invertible in $A/J_\omega$ for all $\omega \in M(Z)$. (By definition $A$ is invertible in $A/A$.)

A proof of Corollary 2.1.6 below can be found in [7]. We here give an alternative proof from the point of view of Clifford algebras. The idea is that $A$ can be considered as an operator version of the complex Clifford algebra for two dimensional euclidean space, with the reflection operators corresponding to (complex) unit vectors. Recall that if $\{e_1, e_2\}$ denotes a given ON basis for the euclidean plane, then this complex Clifford algebra $A_2 = A^0_2 \oplus A^2_2$ is four dimensional and spanned by $\{1, e_1, e_2, e_1e_2\}$. The algebraic basis $\{e_1, e_2\}$ satisfies the canonical anti-commutation relation

$$e_i e_j + e_j e_i = 2\delta_{ij}.$$ 

The even sub-algebra $A^1_2 := A^0_2 \oplus A^2_2 = \text{span}_C \{1, e_1e_2\}$ is commutative and is isomorphic to the bi-complex numbers. On the other hand, as in Example 1.1.4, the full algebra $A$ is isomorphic to the matrix algebra $C^{2 \times 2}$ with

$$e_1 \approx \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \approx \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$ 

Corollary 2.1.6. Consider the operator algebra $A$ in Definition 2.1.1 and let $Z$ in Theorem 2.1.5 be the central sub-algebra generated by the cosine operator $C$. Then the commutative Gelfand transform $\hat{C} : M(Z) \to \sigma_Z(C) \subset C$ is a bijection, and $J_\omega \neq A$ if and only if $\hat{C}(\omega) \in \sigma_A(C) \subset \sigma_Z(C)$.

When $c = \hat{C}(\omega) \in \sigma_A(C) \setminus \{\pm 1\}$, the quotient space is $A/J_\omega \cong A_2 \cong C^{2 \times 2}$ and the quotient map $A \to A/J_\omega$ corresponds to the homomorphism generated by

$$C \mapsto c \in A^0_2, \quad A \mapsto e_1 \in A^1_2, \quad B \mapsto e_1 + se_2 \in A^1_2, \quad iS \mapsto se_1e_2 \in A^2_2,$$

where $s$ is such that $c^2 + s^2 = 1$.

Assuming that $A$ and $B$ are transversal, the $A$-Gelfand transform of $T \in A$ is by definition the induced function $\hat{T} : \sigma_A(C) \to A_2$. Then $T$ is invertible in $A$ if and only if $\hat{T}(c)$ is invertible in $A_2$ for all $c \in \sigma_A(C)$.

Proof. The properties of the commutative Gelfand transform $\hat{C} : M(Z) \to \sigma_Z(C)$ follow as in Chapter 8 in [7]. We here construct an algebraic Clifford basis $\{e_1, e_2\}$ for $A/J_\omega$. For fixed $c = \hat{C}(\omega) \in \sigma_A(C) \setminus \{\pm 1\}$, define the following elements of $A/J_\omega$,

$$1 := \pi(I) \neq 0, \quad e_1 := \pi(A), \quad e_{12} := \frac{1}{s}\pi(iS), \quad e_2 := e_1e_{12},$$

36
where $c^2 + s^2 = 1$. We verify that $\pi(A) = e_1$ and $\pi(B) = \pi(CA + iAS) = ce_1 + se_2$ and using (2.6) and that $AS = -SA$ we get

\[
\begin{align*}
\epsilon_1^2 &= \pi(A^2) = 1, \\
\epsilon_2^2 &= (\frac{i}{\sqrt{2}}\pi(AS))^2 = \frac{1}{2\sqrt{2}} \pi(-A^2S^2) = \frac{1}{4\sqrt{2}} \pi(S^2) = \frac{1}{4\sqrt{2}} \pi(I - C^2) = 1, \\
\epsilon_1\epsilon_2 + e_2\epsilon_1 &= \epsilon_{12} + \epsilon_1\epsilon_{12} - 1 = \frac{i}{\sqrt{2}} \pi(S + ASA) = 0.
\end{align*}
\]

Therefore the canonical anti-commutation relation $\epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 2\delta_{ij}$ holds, so the basic linear independence lemma for Clifford algebras, see Theorem 15.10 in Porteous [53], shows that $\{1, \epsilon_1, \epsilon_2, \epsilon_{12}\}$ are linear independent. Moreover, since $\pi(A)\pi(B) + \pi(B)\pi(A) = 2c$ is scalar, it follows that $\{1, \epsilon_1, \epsilon_2, \epsilon_{12}\}$ spans $A/J_\omega$. \[\square\]

In the degenerate case, e.g. when $c = 1$, there are four different possibilities. As before we know that $\{1, a, b, ab\}$ span $A/J_\omega$, where $a := \pi(A), b := \pi(B)$ and $a^2 = b^2 = 1 \neq 0$ and $ab + ba = 2$, but we may also have non-trivial relations $a = b = 1$, $a = b$ and $a(ab - ba) = ab - ba$. It can be shown that the quotient algebra can be realised as one of the algebras $C$, the one dimensional Clifford algebra $A_1 \cong C \oplus C$, the algebra of upper triangular complex $2 \times 2$ matrices, where $a \approx \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $b \approx \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, or the sub-algebra of $C^{2 \times 2} \oplus C^{2 \times 2}$ generated by $a \approx \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ \oplus \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and

\[
b \approx \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.
\]

2.2 Two basic transmission problems

2.2.1 The $\overline{\partial}$-equation

The theory from Section 2.1 will be used in Chapter 3 to investigate transmission problems for the Dirac operator. However, this requires more knowledge about non-commutative exterior and Clifford algebras. But for the two dimensional Dirac operator, which is essentially the classical $\overline{\partial}$ operator, the algebra is commutative and easier to study. The spectral estimates we obtain in Theorem 2.2.2 are sharper than those obtained in Chapter 3 for $\mathbb{R}^n, n \geq 3$. An application of the main Theorem 2.2.2 yields the following theorem on the classical Hilbert problem.

**Theorem 2.2.1.** Consider the following Hilbert problem. Let $\Omega^+$ be a bounded strongly Lipschitz domain in the complex plane with $\Sigma := \partial\Omega^+$ and outward pointing normal $\nu$ and let $a = a_1 + a_2 j$ be a measurable unit vector field on $\Sigma$. Given a function $\alpha \in L_2(\Sigma; \mathbb{R})$, find a complex valued function $F$ in $\Omega^+$ with non-tangential maximal function $NF \in L_2(\Sigma)$, such that

\[
\begin{align*}
\overline{\partial} F &= 0 \quad \text{in } \Omega^+, \\
a_1 f_1 + a_2 f_2 &= \alpha \quad \text{on } \Sigma,
\end{align*}
\]

37
where \( f = f_1 + f_2 j = F|_\Sigma \). If \( \nu a^2 \) is locally accretive, as defined in Theorem 2.2.2, then this boundary value problem is well-posed in Fredholm sense. In particular, there exists \( \epsilon = \epsilon(\Sigma) > 0 \) such that \( \nu a^2 \) is locally accretive if \( \sup_{z,w \in \Sigma} |a(z) - a(w)| < \epsilon \).

Before giving the proof, we describe how to treat transmission problems for \( \overline{\partial} \) with the formalism from Section 2.1. Consider the operator \( \overline{\partial} = \frac{1}{2}(\partial_1 + j \partial_2) \) in the complex plane with fundamental solution 1/\( \pi z \). Since our reflection operator \( A \) will be complex anti-linear with respect to the complex structure \( j \), we need to expand the two-dimensional real algebra \( \text{span}_R\{1, j\} \) to the bi-complex numbers \( \mathbb{A}_2^+ = \text{span}_\mathbb{C}\{1, j\} \), where \( \mathbb{C} = \text{span}_R\{1, i\} \). Thus we have both a geometric imaginary unit \( j (= e_2 e_2 \in \mathbb{A}_2^2 \) in Section 2.1) and an auxiliary algebraic imaginary unit \( i \). They both square to \(-1\), and \( ij = ji \). If \( z = z_1 + z_2 j = (x_1 + iy_1) + (x_2 + iy_2) j \in \mathbb{A}_2^+ \) then

\[
\overline{\partial} z = z_1 - z_2 j, \quad z^c = z_1^c + z_2^c j, \\
(z)_0 := z_1, \quad (z)_2 := z_2 j, \quad \Re z = \Re z_1 + \Re z_2 j, \quad \Im z = \Im z_1 + \Im z_2 j,
\]

where \( z_k^c := x_k - iy_k \), \( \Re z_k := x_k \) and \( \Im z_k := y_k \). We identify \( \mathbb{R}^2 \ni x_1 e_1 + x_2 e_2 \approx x_1 + x_2 j \in \Re \mathbb{A}_2^+ \).

The reflection operators for an oblique \( \overline{\partial} \) transmission problem are

\[
Bf(z) := \frac{1}{\pi j} \text{p.v.} \int_\Sigma \frac{f(w)}{w - z} dw, \quad Af(z) := -a^2(z) f(z), \quad z \in \Sigma,
\]

where \( \Sigma = \partial \Omega^+ \) and \( \Omega^+ \) is a bounded strongly Lipschitz domain in \( \mathbb{R}^2 \). Here \( a = a_1 + a_2 j \) is a real (i.e. \( a_k \in \mathbb{R} \)) unit vector field on \( \Sigma \), and \( z, w \in \mathbb{R}^2 = \Re \mathbb{A}_2^+ = \text{span}_R\{1, j\} \).

In this example we consider the complex Hilbert space \( L^2(\Sigma; \mathbb{A}_2^+) \), in which case \( B \) is a well defined, bounded \( j \)-linear (and \( j \)-linear) operator, as in Section 1.5. Note that if we define Cauchy extensions

\[
C^\pm f(z) := \pm \frac{1}{2\pi j} \int_\Sigma \frac{f(w)}{w - z} dw, \quad z \in \Omega^\pm,
\]

then the Plenell jump formulae shows that the Hardy projections are

\[
B^\pm f(z) = \frac{1}{2}(I \pm B)f(z) = \lim_{\Omega^\pm \ni \zeta \to z} C^\pm f(\zeta), \quad z \in \Sigma.
\]

On the other hand, the definition of \( A \) means that \( A^+ f = a(f\overline{a})_2 \) is the part of \( f \) orthogonal to \( a \) and \( A^- f = a(f\overline{a})_0 \) is the part of \( f \) parallel to \( a \). The reflection operator \( A \) is a bounded \( j \)-linear operator, but \( j \)-anti-linear.

The transmission problem (2.7) becomes

\[
\begin{align*}
a(z)((a^- f^+ (z) - a^+ f^- (z))\overline{a(z)})_2 &= a(z)(g(z)\overline{a(z)})_2, \\
a(z)((a^+ f^+ (z) - a^- f^- (z))\overline{a(z)})_0 &= a(z)(g(z)\overline{a(z)})_0, \quad z \in \Sigma,
\end{align*}
\]

(2.13)

where \( F^\pm \) is analytic in \( \Omega^\pm \) and \( f^\pm = F^\pm|_\Sigma \) (and \( F^- \) vanishes at infinity). Note that the boundary value problems \( a^+ = 0 \) or \( a^- = 0 \) are exactly the classical Hilbert problems

38
for analytic functions, here extended to non-smooth domains. On the other hand, the
transmission problem (2.13) with \( \alpha^\pm \in \mathbb{R} \) can be thought of as a non \( j \)-linear version of
the classical Riemann problem for analytic functions.

We here prefer to solve the transmission problem with an operator acting in the whole
space \( L^2(\Sigma; A^\pm_2) \) like the rotation operator, rather than with the double layer type oper-
ators which act in \( A^\pm L^2 \). Proposition 2.1.2 shows that solving the transmission problem
(2.13) is equivalent to solving the singular integral equation
\[
\lambda f(z) + \frac{1}{\pi j} \text{p.v.} \int_{\Sigma} \frac{a^2(w)\overline{f(w)}}{w-z} \, dw = \frac{2}{\alpha^+ - \alpha^-} \frac{1}{\pi j} \text{p.v.} \int_{\Sigma} \frac{g(w)}{w-z} \, dw, \quad z \in \Sigma.
\]

Note that the operators \( B^\pm AB^\pm (B^\mp AB^\pm) \), except for the conjugation \( \overline{f(w)} \), are Toeplitz
(Hankel) operators with symbol \( -\alpha^2 \), which motivates this terminology in Definition 2.1.1.

**Proof of Theorem 2.2.1.** In operator notation the Hilbert problem is \( A^- f = \alpha a, f \in B^+ L^2 \). Note that since \( \alpha(z) \in \mathbb{R} \), we have \( \alpha a \in A^- L^2 \). Under the hypothesis that \( \nu a^2 \) is
locally accretive it follows from Theorem 2.2.2 below that \( I - BA \) is a Fredholm operator.
As discussed in Section 2.1 this shows that
\[
A^- : B^+ L^2 \to A^- L^2
\]
also is a Fredholm operator, which proves the theorem.

To solve the Hilbert problem in Theorem 2.2.1, note that we can use the Toeplitz type
operator \( B^+ A^- B^+ \). Applying \( B^+ \) shows that \( B^+ A^- f = B^+(\alpha a) \), or equivalently
\[
f - B^+ Af = 2B^+(\alpha a).
\]
Writing this out as an integral equation, it becomes
\[
F(z) + \frac{1}{2\pi j} \int_{\Sigma} \frac{a^2(w)\overline{f(w)}}{w-z} \, dw = \frac{1}{\pi j} \int_{\Sigma} \frac{a(w)a(w)}{w-z} \, dw, \quad z \in \Omega^+.
\]
We now use a Rellich type argument to estimate \( \sigma_{\text{ess}}(BA) \). Here \( \nu = \nu_1 + \nu_2 j \) is the
outward pointing normal vector.

**Theorem 2.2.2.** Assume that \( \nu a^2 \) is locally accretive, i.e. there exists a real smooth
vector field \( \theta = \theta_1 + \theta_2 j \in C^\infty_0(\mathbb{R}^2; \text{Re} A^\pm_2) \) and \( L < \infty, c > 0 \) such that \( (\nu a^2 \overline{\theta})_0 \geq c \)
and \( |(\nu a^2 \overline{\theta})_0| \leq L(\nu a^2 \overline{\theta})_0 \) on \( \Sigma \). Then the rotation operator \( BA \) has the essential angular spectral estimate
\[
\sigma_{\text{ess}}(BA; L^2(\Sigma; A^\pm_2)) \subset \{ \lambda = \lambda_1 + i\lambda_2 ; |\lambda_1| \leq L|\lambda_2| \}.
\]

**Proof.** Consider the variables
\[
g = (\lambda - BA)h, \quad f^\pm := B^\pm Ah = \frac{1}{2}(Ah \pm (\lambda h - g)),
\]

39
where the two auxiliary functions $f^\pm$ belong to the Hardy spaces, and define the sesqui linear $A^+_2$-valued form
\[
\langle g, f \rangle_\theta = \int_\Sigma \nu(z)g(z)\overline{f^c(z)\overline{\theta(z)}}
\]
ds(z).

On one hand, the boundary theorem 1.2.7 shows that
\[
\pm\langle f^\pm, f^\pm \rangle_\theta = \int_{\Omega^\pm} 2\overline{\nu(F^\pm F^\pm^c \overline{\theta})} = 2\int_{\Omega^\pm} F^\pm F^\pm^c \overline{\theta},
\]
where $F^\pm = C^\pm(f^\pm)$. This implies the estimate
\[
|\langle f^\pm, f^\pm \rangle_\theta| \lesssim \|\partial_z \theta\|_{L^\infty(\Omega^\pm)} \int_{\text{supp } \theta \cap \Omega^\pm} |F^\pm|^2 dx.
\]
On the other hand $\langle f^+, f^+ \rangle_\theta - \langle f^-, f^- \rangle_\theta = \text{Re}(\lambda^c \langle Ah, h \rangle_\theta - \langle Ah, g \rangle_\theta)$. Here the integrand of $\langle Ah, h \rangle_\theta$ is $\nu(-a \overline{\theta})h^c \overline{\theta} = -(\nu a \overline{\theta})(|h|^2 + i \text{Im}(\overline{\theta}h^c)_2)$, and so we get the estimate
\[
|\text{Re}(\lambda^c \langle Ah, h \rangle_\theta)| \geq |\text{Re}(\lambda^c \langle Ah, h \rangle_\theta)| = \left| \int_{\Sigma} \text{Re}(\lambda^c \nu a^2 \overline{\theta} h^c \overline{\theta})_0 \right|
\]
\[
\geq \int_{\Sigma} (|\lambda_1|(|\nu a^2 \overline{\theta})_0 - |\lambda_2||(|\nu a^2 \overline{\theta})_2)|h|^2.
\]
The hypothesis now shows that $|\text{Re}(\lambda^c \langle Ah, h \rangle_\theta)| \gtrsim \|h\|^2$ if $|\lambda_1| > L|\lambda_2|$. Since in this case $\|h\|^2 \lesssim \|h\|\|g\| + \|F\|_{L^2(\text{supp } \theta)}^2$ and $L_2(\Sigma) \ni h \mapsto F^\pm \in L^2(\text{supp } \theta \cap \Omega^\pm)$ are compact maps, it follows that $\lambda - BA$ is semi-Fredholm. By the method of continuity 1.4.5, the index of $\lambda - BA$ must be 0 since $BA$ is a bounded operator and the double sector $|\lambda_1| > L|\lambda_2|$ is connected to $\infty$. This completes the proof. \qed

Two important special cases when $\nu a^2$ is locally accretive are the following.

- If $a = 1$, then the quantity $L_\Sigma := \inf_{\Sigma} \sup_{\Sigma}(\|\nu \overline{\theta}\|^2/\|\nu \theta\|^2)_0$ is called the local Lipschitz constant for $\Sigma$. In this case Theorem 2.2.2 shows that the transmission problem
\[
(\alpha^- f^+(z) - \alpha^+ f^-(z))_2 = (g(z))_2,
\]
\[
(\alpha^+ f^+(z) - \alpha^- f^-(z))_0 = (g(z))_0, \quad z \in \Sigma,
\]
is well-posed in the Fredholm sense when $|\lambda_1| > L_\Sigma|\lambda_2|.$

- If both $\nu$ and $a$ are continuous, then the optimal $L$ is 0, so $\sigma_{\text{ess}}(BA) \subset i\mathbb{R}$. Note that in this case $a$ can be allowed to have any direction. The algebraic reason for this is that the algebra $A^+_2$ is commutative. In higher dimensions, the corresponding oblique boundary value problems for the Dirac operator are not well posed in the tangential case.

### 2.2.2 The Laplace equation

Next we demonstrate how the classical Dirichlet and Neumann problems for the Helmholtz equation fit into the framework in Section 2.1. Consider the elliptic operator $(\Delta + k^2)U = 0$
in $\mathbb{R}^n$, $n \geq 3$, and assume $\text{Im} k > 0$ for simplicity. Recall that the Bessel potential $\Phi_k(x)$ with $\Phi_k(\xi) = 1/(-|\xi|^2 + k^2)$ is a fundamental solution to the Helmholtz operator and in particular in $\mathbb{R}^3$ we have $\Phi_k(x) = -ie^{ik|x|}/4\pi|x|$.

We start by, on a formal level, investigating the operator algebra behind the Helmholtz transmission problem. Motivated by the Green reproducing formula for $\Delta + k^2$, we define the Green extension

$$G^\pm_k(\pi)(x) := \pm \int_\Sigma \left((\nabla \Phi_k)(y - x) \cdot \nu(y)u_1(y) - \Phi_k(y - x)u_2(y)\right) d\sigma(y), \quad x \in \Omega^\pm,$$

of a function $\pi = (u_1, u_2): \Sigma \to \mathbb{C}^2$. Taking boundary traces we define the Calderón projections

$$B^\pm_k(\pi)(x) := \lim_{\Omega^\pm_{\pi z \to x}} (G^\pm_k(\pi)(z), \nu(x) \cdot \nabla G^\pm_k(\pi)(z)), \quad x \in \Sigma,$$

acting in a function space $\mathcal{X}$ of functions $\pi: \Sigma \to \mathbb{C}^2$. Note that Green’s second formula shows that if $(\Delta + k^2)U = 0$ in $\Omega^\pm$ (with decay at $\infty$ for $\Omega^-$) then $B^\pm U = \pi$, where $\pi = (U|_{\Sigma}, \nu \cdot \nabla U|_{\Sigma})$.

Next we recall the following basic operators acting on scalar functions $u: \Sigma \to \mathbb{C}$.

- The single layer potential operator $\Phi_k u(x) := \int_\Sigma \Phi_k(y - x)u(y) \, d\sigma(y), \quad x \in \Sigma$.
- The double layer potential operator $K_k u(x) := \text{p.v.} \int_\Sigma (\nabla \Phi_k)(y - x) \cdot \nu(y)u(y) \, d\sigma(y), \quad x \in \Sigma$.

We recall the trace/jump formula

$$\lim_{\Omega^\pm_{\pi z \to x}} \int_\Sigma (\nabla \Phi_k)(y - x) \cdot \nu(y)u(y) \, d\sigma(y) = (\pm \frac{1}{2} + K_k)u(x), \quad x \in \Sigma.$$

- The adjoint of the double layer potential

$$(K_{k'})^* u(x) := \nu(x) \cdot \text{p.v.} \int_\Sigma (\nabla \Phi_k)(x - y)u(y) \, d\sigma(y), \quad x \in \Sigma,$$

Here $k' := -k^c$ is the reflection in the imaginary axis. We recall the trace/jump formula $\nu(x) \cdot \lim_{\Omega^\pm_{\pi z \to x}} \nabla \int_\Sigma \Phi_k(y - x)u(y) \, d\sigma(y) = (\mp \frac{1}{2} + (K_{k'})^*)u(x), \quad x \in \Sigma$.

- The Neumann operators $N^\pm_k(U|_{\Sigma}) := \nu \cdot \nabla U|_{\Sigma}$, where $U$ satisfies $(\Delta + k^2)U = 0$ in $\Omega^\pm$ (and decays at $\infty$ for $\Omega^-$). These operators are sometimes called Poincaré–Steklov operators.

**Proposition 2.2.3.** The basic relations between these operators are

$$(\Phi_k)^* = \Phi_{k'}$$  \hspace{1cm} (2.15)

$$(N_{k'}^*)^* = N_k^+$$  \hspace{1cm} (2.16)

$$N_k^\pm \Phi_k = \mp \frac{1}{2} + (K_{k'})^*$$  \hspace{1cm} (2.17)

$$\Phi_k N_k^\pm = \mp \frac{1}{2} + K_k$$  \hspace{1cm} (2.18)

$$K_k \Phi_k = \Phi_k (K_{k'})^*$$  \hspace{1cm} (2.19)

$$N^\pm K_k = (K_{k'})^* N_k^\pm.$$  \hspace{1cm} (2.20)

Moreover, if $\text{Re} k = 0$ then $\Phi_k \leq 0$, $N_k^+ \geq 0$ and $N_k^- \leq 0$.  

41
The adjointness (2.15) is immediate from the definition. From Parseval’s formula we get
\[ (\Phi_k u, u) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |u\overline{d\sigma}(\xi)|^2 \frac{d\xi}{|\xi|^2 + k^2}, \]
from which \( \Phi_k \leq 0 \) follows if \( \text{Re} \ k = 0 \). Next Green’s first formula shows that
\[ (N^\pm_k u, v) = \pm \int_{\Omega^\pm} (-k^2 U(x))V^c(x) + \nabla U(x) \cdot \nabla V^c(x) \, dx = (u, N^\pm_k v), \]
which gives (2.16) and that \( \pm N^\pm_k \geq 0 \) when \( \text{Re} \ k = 0 \).

The factorisation formula (2.17) is exactly the jump/trace formula for the adjoint of the double layer potential from above. Formulas (2.18), (2.19) and (2.20) follow from (2.15), (2.16) and (2.17).

Using these scalar operators, we can decompose the Calderón projections \( B^\pm \) in (2.14) as
\[ B^+ = \begin{pmatrix} \frac{1}{2} + K_k & -\Phi_k \\ N^+_k(\frac{1}{2} + K_k) & -N^+_k \Phi_k \end{pmatrix}, \quad B^- = \begin{pmatrix} \frac{1}{2} - K_k & \Phi_k \\ N^-_k(\frac{1}{2} - K_k) & N^-_k \Phi_k \end{pmatrix}. \]

Therefore the two basic reflection operators are
\[ B = \begin{pmatrix} 2K_k & -2\Phi_k \\ N^+_k(I + 2K_k) & -2(K_k')^* \end{pmatrix}, \quad A := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \]
Indeed, Proposition 2.2.3 shows that \( B^2 = I \). For the (2,1) element in \( B \), note the identity \( N^+_k(\frac{1}{2} + K_k) = N^-_k(-\frac{1}{2} + K_k) \), i.e. the well-known fact that the normal derivative of the double layer potential does not jump across \( \Sigma \), which follows from (2.18). Moreover, it follows that
\[ \frac{1}{2}(AB + BA) = \begin{pmatrix} 2K_k & 0 \\ 0 & -2(K_k')^* \end{pmatrix}, \quad BA = \begin{pmatrix} 2K_k & 2\Phi_k \\ N^+_k(I + 2K_k) & 2(K_k')^* \end{pmatrix}. \]

Comparing with (2.4), we see that \( A^+BA^+ = 2K_k \) and \( A^-BA^- = 2(K_k')^* \) and also \( A^+BA^- = -2\Phi_k \). Note that \( N^+_k = A^-(A^+|_{B^+ \mathcal{X}})^{-1} \) and that for \( k = 0 \) the integral kernel for \( G^+(A^+|_{B^+ \mathcal{X}})^{-1} \) is the harmonic measure for \( \Omega^\pm \).

Just as we solved the \( \mathcal{S} \) transmission problem in Example 2.2.1 with the rotation operator since the full trace space \( L^2_2(\Sigma; A^+_2) \) was natural, we here prefer to solve the Helmholtz transmission problem with the double layer potentials since the full trace space splits naturally into the scalar spaces \( A^\pm \mathcal{X} \).

Next we justify the formal calculations above using \( L^2_2 \) based Sobolev spaces. We here assume that \( \Sigma \) is a bounded, strongly Lipschitz surface in \( \mathbb{R}^n \), and use the Sobolev spaces \( W^s_2(\Sigma) \), \(|s| \leq 1 \). It is known that if \(|s| \leq 1/2 \), then we have bounded operators
\[ \Phi_k : W^{s-1/2}_2(\Sigma) \longrightarrow W^{s+1/2}_2(\Sigma), \]
\[ K_k : W^{s+1/2}_2(\Sigma) \longrightarrow W^{s+1/2}_2(\Sigma), \]
\[ K^*_k : W^{s-1/2}_2(\Sigma) \longrightarrow W^{s-1/2}_2(\Sigma), \]
\[ N^\pm_k : W^{s+1/2}_2(\Sigma) \longrightarrow W^{s-1/2}_2(\Sigma). \]
Moreover $\Phi_k : W_2^{s-1/2}(\Sigma) \rightarrow W_2^{s+1/2}(\Sigma)$ and $N_k^\pm : W_2^{s+1/2}(\Sigma) \rightarrow W_2^{s-1/2}(\Sigma)$ (and therefore $\pm \frac{1}{2} + K_k : W_2^{s+1/2}(\Sigma) \rightarrow W_2^{s+1/2}(\Sigma)$) are invertible operators. In particular, the boundary value problems are well posed. These results follow from the boundedness of the higher dimensional Cauchy integral Li–McIntosh–Semmes [33], the Rellich estimates in Verchota [63], duality and interpolation. More on the single and double layer potential operators can be found in for example Costabel [12] and Torres–Welland [62]. Note that when $\Sigma$ is smooth, then $K_k$ is compact, so by (2.17) and (2.18) $\Phi_k$ is essentially an inverse to $N_k^\pm$.

The reflection operators $A$ and $B$ here act in the full trace space

$$\mathcal{X}^s := W_2^{s+1/2}(\Sigma) \oplus W_2^{s-1/2}(\Sigma), \quad |s| \leq 1/2.$$ 

We now demonstrate how to use these reflection operators to obtain spectral estimates on the double layer potential. Both the Rellich form in the next lemma and the form in Theorem 2.2.2 can be seen as special cases of the generalised Rellich form used in Chapter 3.

**Lemma 2.2.4.** The Rellich form for $\mathcal{X}^{1/2}$ with respect to a smooth vector field $\theta \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$ in $\mathbb{R}^n$ is the indefinite sesqui linear symmetric form

$$\langle \pi, \pi \rangle_\theta := \int_\Sigma \left( \left( (\nabla_T u_1, \nabla_T v_1) - u_2 \bar{v}_2 \right) (\nu, \theta) - \left( (\nabla_T u_1, v_2 \theta) - (u_2 \theta, \nabla_T v_1) \right) \right) d\sigma,$$

where $\pi = (u_1, u_2), \bar{\pi} = (v_1, v_2) \in \mathcal{X}^{1/2}$ and $\nabla_T$ denotes the tangential gradient operator. If $\pi \in A^+ \mathcal{X}^{1/2}$ then $\langle \pi, \pi \rangle_\theta = \int_\Sigma |\nabla_T u_1|^2 (\nu, \theta)$ and if $\pi \in A^- \mathcal{X}^{1/2}$ then $\langle \pi, \pi \rangle_\theta = -\int_\Sigma |u_2|^2 (\nu, \theta)$. Moreover, if $\pi \in B^\pm \mathcal{X}^{1/2}$ then we have the estimate

$$|\langle \pi, \pi \rangle_\theta| \lesssim \left( \|k\|^2 \theta \|_\infty + \|\nabla \theta \|_\infty \right) \|G_k^\pm \pi\|_{W_2^1(\text{supp} \theta \cap \Omega^\pm)}^2,$$

and the Green extension $G_k^\pm : \mathcal{X}^{1/2} \rightarrow W_2^1(\text{supp} \theta \cap \Omega^\pm)$ is compact.

This lemma is well known, although we have formulated it slightly different here to emphasize the full trace space $\mathcal{X}^{1/2}$. For completeness we include a proof.

**Proof.** To prove the estimate, assume $\pi \in B^\pm \mathcal{X}^{1/2}$ and write $U := G_k^\pm \pi, F := \nabla U$ and $f = \nabla_T u_1 + u_2 \nu \in L_2(\Sigma; \mathbb{C}^n)$. Using the general Stokes’ theorem, Helmholtz equation $\text{div } F = -k^2 U$ and nilpotence of the the exterior derivative $\nabla \wedge F = 0$, we get

$$\langle \pi, \pi \rangle_\theta = \int_\Sigma |f|^2 (\nu, \theta) - 2 \text{Re} \left( (u_2 \nu, u_2 \theta) + (\nabla_T u_1, u_2 \theta) \right)$$

$$= \int_\Sigma |f|^2 (\nu, \theta) - 2 \text{Re} \left( (f, \theta)(\nu, f) \right)$$

$$= \pm \int_{\Omega^\pm} |F|^2 \text{div } \theta + 2 \text{Re} (\theta, \nabla)(\hat{F}, F)$$

$$- 2 \text{Re} ((F, \theta)(-k^2 U^c) + (F, \hat{\theta})(\nabla, F) + (\hat{F}, \theta)(\nabla, F))$$

$$= \pm \int_{\Omega^\pm} |F|^2 \text{div } \theta - 2 \text{Re} ((F, \theta)(-k^2 U^c) + (F, \hat{\theta})(\nabla, F)),$$

43
where $\hat{x}$ shows where the differential operator is acting. From this calculation the estimate follows. Note that the last equality is a consequence of the identity $0 = (\nabla \wedge \hat{F}, \theta \wedge F) = (\nabla, \theta)(\hat{F}, F) - (\nabla, F)(\hat{F}, \theta)$. The compactness result follows from Schur estimates and the factorisation (2.18).

We now use this Rellich form to prove spectral estimates on the double layer potential. Here we use a smooth vector field $\theta \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ such that $(\theta, \nu) \geq c$ and $|\theta \wedge \nu| \leq L(\theta, \nu)$ on $\Sigma$ for some constants $L < \infty$ and $c > 0$, as in Definition 1.5.5.

**Theorem 2.2.5.** Let $\Omega^+$ be a bounded strongly Lipschitz domain in $\mathbb{R}^n$ with boundary $\Sigma$ and let $L_\Sigma$ be the local Lipschitz constant for $\Sigma$. Then for any $0 \leq s \leq 1$ we have the hyperbolic essential spectral estimate

$$
\sigma_{\text{ess}}(2K_k; \mathcal{W}_2^s(\Sigma)) \subset \{ \lambda = \lambda_1 + i\lambda_2 \mid \lambda_1^2 \leq L_\Sigma^2 \lambda_2^2 + L_\Sigma^2/(L_\Sigma^2 + 1) \}.
$$

**Proof.** As in Theorem 2.2.2 we introduce variables

$$
\overline{\theta} = (\lambda - BA)\overline{h},
\overline{u}^\pm := B^\pm A\overline{h} = \frac{1}{2}(A\overline{h} \pm (\lambda\overline{h} - \overline{\theta})),
$$

and $U^\pm := G_k^\pm A\overline{h}$. On one hand, Lemma 2.2.4 shows that

$$
|\langle \overline{u}^\pm, \overline{u}^\pm \rangle_\theta| \lesssim (|k|\|\theta\|_{\infty} + \|\nabla\theta\|_{\infty})\|U^\pm\|_{W^2_2(\text{supp}\theta; \Omega^+)}^2.
$$

On the other hand $\langle \overline{u}^+, \overline{u}^+ \rangle_\theta - \langle \overline{u}^-, \overline{u}^- \rangle_\theta = \text{Re}(\lambda^c \langle A\overline{h}, \overline{h} \rangle_\theta - \langle A\overline{h}, \overline{\theta} \rangle_\theta).$ Here the integrand of $\langle A\overline{h}, \overline{h} \rangle_\theta$ is $|\nabla \tau h_1|^2 + |h_2|^2)(\nu, \theta) + 2\text{Im}(h_2\theta, \nabla \tau h_1)$ which gives the coercivity estimate

$$
|\text{Re}\lambda^c \langle A\overline{h}, \overline{h} \rangle_\theta| \geq \int_{\Sigma} (|\lambda_1| - L|\lambda_2|)c(|\nabla \tau h_1|^2 + |h_2|^2) \gtrsim \|\overline{h}\|_{X^{1/2}}^2 - \text{compact}
$$

if $|\lambda_1| > L|\lambda_2|$, using that the embedding $W^2_2(\Sigma) \hookrightarrow \mathcal{L}_2(\Sigma)$ is compact. Therefore we have the a priori estimate $\|\overline{h}\|_{X^{1/2}}^2 \lesssim \|\overline{h}\|_{X^{1/2}}^2 \|\overline{h}\|_{X^{1/2}} + \text{compact in the double sector $|\lambda_1| > L|\lambda_2|$.}$ As in Theorem 2.2.2, this implies angular spectral estimates as in Example 1.4 on $\sigma_{\text{ess}}(BA; \lambda^{1/2})$. By Proposition 2.1.2 this translates to the corresponding hyperbolic estimates on $\sigma_{\text{ess}}(K_k; \mathcal{W}^s_2(\Sigma))$ and $\sigma_{\text{ess}}((K_k^*)^*; \mathcal{L}_2(\Sigma))$. Using duality and interpolation now completes the proof.

We remark that when $\text{Im} k > 0$, one can also get a similar hyperbolic estimate on the full spectrum $\sigma(K_k; \mathcal{W}_2^s(\Sigma))$, but with a constant larger than $L_\Sigma$, depending on $k$, as in Chapter 3.

We finish with some spectral estimates in the “energy trace space” $\mathcal{X}^0$. The techniques used below here have been used in Steinbach–Wendland [59]. However the methods were rediscovered independently here, and for completeness we include a short discussion. The basic observation here is that if $\text{Im} k > 0$ then

$$
\|u\|_{W^{1/2}_2(\Sigma)}^2 \approx |(N_k^\pm u, u)| \quad \text{and} \quad \|u\|_{W^{-1/2,2}_2(\Sigma)}^2 \approx |(N_k^\pm)^{-1} u, u|.
$$

(2.21)
To prove the first equivalence, note that on one hand

\[
|(N_k^\pm u, u)| \lesssim \|N_k^\pm u\|_{W_2^{1/2}(\Sigma)} \|u\|_{W_2^{1/2}(\Sigma)} \lesssim \|u\|_{W_2^{1/2}(\Sigma)}^2.
\]

On the other hand Green’s first formula shows that

\[
\|u\|^2_{W_2^{1/2}(\Sigma)} \lesssim \|U\|^2_{W_2^{1/2}(\Omega^\pm)} \approx \int_{\Omega^\pm} |\nabla U|^2 - k^2|U|^2 = |(N_k^\pm u, u)|
\]

if \( U = G^\pm(u, N_k^\pm u) \). The second equivalence in (2.21) follows from the first and that \( N_k^\pm : W_2^{-1/2}(\Sigma) \rightarrow W_2^{-1/2}(\Sigma) \) is an isomorphism.

The best case is when \( \text{Re} \, k = 0 \). Then, using the equivalent \( W_2^{-1/2}(\Sigma) \) scalar products \((u, v)^\pm_{W_2^{-1/2}(\Sigma)} := ((\pm N_k^\pm)^{-1}u, v)\), we get from (2.17) that

\[
((-\frac{1}{2} + K_k^*)u, u)^+_{W_2^{-1/2}(\Sigma)} = (\Phi ku, u) \leq 0,
\]

\[
((-\frac{1}{2} + K_k^*)u, u)^-_{W_2^{-1/2}(\Sigma)} = -(\Phi ku, u) \geq 0.
\]

This proves that \( \sigma(K_k^*; W_2^{-1/2}(\Sigma)) \), \( \sigma(K_k^*; W_2^{-1/2}(\Sigma)) \subset \left(-\frac{1}{2}, \frac{1}{2}\right) \subset \mathbb{R} \). More generally we have the following.

**Proposition 2.2.6.** Assume \( \text{Im} \, k > 0 \), let \( \eta := |\text{Re} \, k|/\text{Im} \, k \) and let \( \sigma_k \) be the open disc centred at \( i \xi \) and which touches \( \pm 1 \). Then we have the spectral estimates

\[
\sigma(2K_k^*; W_2^{-1/2}(\Sigma)), \sigma(2K_k^*; W_2^{-1/2}(\Sigma)) \subset \left\{ \frac{\pi}{\eta} \left\lfloor \frac{\eta}{\eta-1} \right\rfloor /2 \cap \left\{ \frac{\pi}{\eta} \left\lfloor \frac{\eta-1}{\eta-1} \right\rfloor /2 \right\} \{\pm 1\} \right\} \{\pm 1\} \eta \leq 1,
\]

\[
\left\{ \frac{\pi}{\eta} \left\lfloor \frac{\eta}{\eta-1} \right\rfloor /2 \cup \left\{ \frac{\pi}{\eta} \left\lfloor \frac{\eta-1}{\eta-1} \right\rfloor /2 \right\} \{\pm 1\} \right\} \eta > 1,
\]

as in Figure 1.6 in the Appendix.

**Proof.** Using the formulation (2.7) of the transmission problem, assume that

\[
\begin{cases}
\alpha^- U^+|_\Sigma - \alpha^+ U^-|_\Sigma = g_1, \\
\alpha^+ \nu \cdot \nabla U^+|_\Sigma - \alpha^- \nu \cdot \nabla U^-|_\Sigma = g_2,
\end{cases}
\]

and let \( \Gamma^\pm := (U^\pm|_\Sigma, \nu \cdot \nabla U^\pm|_\Sigma) \in B^\pm, \mathcal{X}^0, \Gamma := \Gamma^+ + \Gamma^- \) and \( \gamma = (g_1, g_2) \in \mathcal{X}^0 \). Using these jump conditions we see from Green’s first formula that

\[
\int_{\Omega^+} |\nabla U^+|^2 - k^2|U^+|^2 = (N_k^+ u^+, u^+) = \frac{\alpha^+}{\alpha^-} \left( \frac{\alpha^-}{\alpha^+} \right) c (N_k^- u^-, u^-) + \ldots
\]

\[
= -\frac{(\alpha^+/\alpha^-)^2}{|\alpha^+/\alpha^-|^2} \int_{\Omega^-} |\nabla U^-|^2 - k^2|U^-|^2 + \ldots
\]

where \( \ldots \) are terms that can be estimated by \( \|\Gamma\|_{\mathcal{X}^0} \||\Gamma|\|_{\mathcal{X}^0} + \|\Gamma\|^2_{\mathcal{X}^0} \). Since the integrands take values in the sector between \( \mathbb{R}_+ \) and \(-k^2\mathbb{R}_+\), this gives the estimate \( |(N_k^+ u^+, u^+)| + |(N_k^- u^-, u^-)| \leq \|\Gamma\|_{\mathcal{X}^0} \|\Gamma\|_{\mathcal{X}^0} + \|\Gamma\|^2_{\mathcal{X}^0} \), provided \( 2|\text{arctan} \, \eta| + |\text{arg}(\alpha^+/\alpha^-)| < \pi \). Using (2.21), this implies the a priori estimate \( \|\Gamma\|_{\mathcal{X}^0} \lesssim \|\eta\|_{\mathcal{X}^0} \). By Proposition 2.1.2 the estimate \( 2|\text{arctan} \, \eta| + |\text{arg}(\alpha^+/\alpha^-)| \leq \pi \) on the spectrum translates to the spectral estimates above. Note that we already know that there exists neighbourhoods \( U_\pm \) of \( \pm 1 \) such that \( (U_+ \cup U_-) \cap \sigma(2K_k^*; W_2^{1/2}(\Sigma)) = \emptyset \). \( \square \)
Chapter 3

Oblique and normal transmission problems for Dirac operators with strongly Lipschitz interfaces

Boundary value problems on Lipschitz domains for the Dirac operator have been investigated by McIntosh-Mitrea [39] and McIntosh-Mitrea-Mitrea [38], where Rellich estimates were generalised and adapted to the Dirac equation. In Axelsson-Grognard-Hogan-McIntosh [4], we showed how boundary value problems for the Dirac operator are related to the geometry of two pairs of complementary projections. In Chapter 2, we investigated the appropriate boundary integral operator, the rotation operator, for solving transmission problems for first order elliptic systems such as the Dirac equation. In this chapter we apply Rellich estimates in a new way to obtain angular spectral estimates for this operator.

The motivation for this chapter comes from the following boundary value problem for generalised Maxwell’s equations. Let \( \Omega^+ \) be a bounded strongly Lipschitz domain with \( \partial \Omega^+ =: \Sigma \) and outward pointing unit normal \( \nu \), and let \( k \in \mathbb{C} \) be a fixed wave number. Given a normal \( j \)-form \( g \in L_2(\Sigma; \wedge^j) \) with vanishing tangential divergence \( \Gamma_k g = 0 \), find \( F : \Omega^+ \to \wedge^j \) with non-tangential maximal function \( N F \in L_2(\Sigma) \) such that

\[
\begin{align*}
  d^*_k F &= 0 & \text{in } \Omega^+, \\
  \delta_k F &= 0 & \text{in } \Omega^+, \\
  \nu \perp f &= \nu \perp g & \text{on } \Sigma.
\end{align*}
\]

(3.1)

The last condition means that the normal part of \( f := F|_{\Sigma} \) is prescribed to be \( g \), and \( d^*_k = d + ke_0 \wedge \) and \( \delta_k = \delta + ke_0 \perp \) are zero-order perturbations of the exterior and interior derivative operators.

Using ideas from McIntosh-Mitrea [39] and Axelsson-Grognard-Hogan-McIntosh [4], we solve this boundary value problem by embedding Maxwell’s equations \( d^*_k F = \delta_k F = 0 \) in an elliptic Dirac equation \( D^*_k F = d^*_k F + \delta_k F = 0 \). From this point of view we thus have “Maxwell’s equations = Dirac equation + the constraint \( d^*_k F = 0 \)”.

In Section 3.1 we consider more general oblique transmission problems for the elliptic
Dirac operator, like the following. Let \( \Omega^\pm := \mathbb{R}^n \setminus \overline{\Sigma^\pm} \), let \( \alpha^\pm \in \mathbb{C} \) be two jump parameters and let \( v \) be a measurable unit vector field on \( \Sigma \). Given \( g \in L^2(\Sigma; \wedge) \), find Hardy type functions \( F^\pm : \Omega^\pm \to \wedge \) with non-tangential maximal functions \( \mathcal{M} F^\pm \in L^2(\Sigma) \) with \( F^- \) satisfying a radiation condition at infinity, such that

\[
\begin{aligned}
\text{D}_k F^\pm &= 0 \quad \text{in } \Omega^\pm, \\
v \wedge (\alpha^- F^+|_\Sigma - \alpha^+ F^-|_\Sigma) &= v \wedge g \quad \text{on } \Sigma, \\
v \perp (\alpha^+ F^+|_\Sigma - \alpha^- F^-|_\Sigma) &= v \perp g \quad \text{on } \Sigma.
\end{aligned}
\tag{3.2}
\]

We express this transmission problem in terms of two reflection operators \( E_k \) and \( V \), i.e. \( E_k^2 = V^2 = I \), acting boundedly in \( L^2(\Sigma; \wedge) \). Here

\[
E_k(F^+|_\Sigma + F^-|_\Sigma) = F^+|_\Sigma - F^-|_\Sigma,
\]

\[
V(v \perp (v \wedge g) + v \wedge (v \perp g)) = v \perp (v \wedge g) - v \wedge (v \perp g),
\]

with associated projection operators \( E_k^\pm := \frac{1}{2}(I \pm E_k) \) and \( V^\pm := \frac{1}{2}(I \pm V) \). In terms of these reflection operators, the transmission problem (3.2) is equivalent to the equation

\[
(\lambda - E_k V)f = 2E_k g,
\]

where \( f = F^+|_\Sigma + F^-|_\Sigma \) and \( \lambda = (\alpha^+ + \alpha^-)/(\alpha^+ - \alpha^-) \). The main result in Section 3.1 is Theorem 3.1.3, where we show that, under a geometric condition on \( v \), there exists a constant \( C = C(v) \) such that the essential spectrum of the rotation operator \( E_k V \) satisfies

\[
\sigma_{\text{ess}}(E_k V; L^2(\Sigma; \wedge)) \subset \{ \lambda : |\text{Re} \lambda| \leq C|\text{Im} \lambda| \}.
\]

This spectral estimate means that the transmission problem (3.2) is well posed in Fredholm sense when \( |\text{Re} \lambda| > C|\text{Im} \lambda| \). For the normal transmission problem \( v = \nu \), this constant \( C \) equals the local Lipschitz constant \( L_\Sigma \) for \( \Sigma \).

In Section 3.2 we return to generalised Maxwell’s equations (3.1) and investigate the “homogeneous” constraint \( d_k F = 0 \). We prove in Theorem 3.2.1 that the Maxwell boundary value problem (3.1) is well posed in Fredholm sense. The idea is that both the divergence free condition on the normal part \( g \) and the constraint \( d_k F = 0 \) on the Hardy field \( F \) express the fact that \( f \) and \( g \) are in the null space of an operator \( \Gamma_k \). Parts of this operator have been investigated in Mitrea–Mitrea–Taylor [46] in connection with boundary value problems for the Hodge Laplacian. We prove in Theorem 3.2.3 that for each wave number \( k \), there exists a closed, nilpotent exterior/interior derivative operator \( \Gamma_k \) acting in \( L^2(\Sigma; \wedge) \) with the following properties.

- On normal fields, \( \Gamma_k \) acts as a tangential interior derivative.
- On tangential fields, \( \Gamma_k \) acts as a tangential exterior derivative.
- On Hardy fields, \( \Gamma_k \) acts as both an exterior and interior derivative since these coincides.
• In the sense of unbounded operators, $\Gamma_k$ commutes with both the Hardy reflection operator $E_k$ and the normal reflection operator $N$.

Here $N$ is the special case of $V$ when $v = \nu$. After constructing $\Gamma_k$, we investigate the action of $E_k$ and $N$ in the domain $D(\Gamma)$ and the null space $N(\Gamma_k)$ and prove spectral estimates on the rotation operator $E_kN$ in Theorem 3.2.15 and regularity results, from $L_2(\Sigma; \Lambda)$ to the space $D(\Gamma_k)$ of mixed 0 and 1 regularity, with Proposition 3.2.16.

3.1 Spectral estimates for oblique Dirac transmission problems

A transmission problem, as explained in Chapter 2, involves two reflection operators. Here we consider the Hardy type reflection operator $E_k$ from Definition 1.2.8 together with a bi-oblique reflection operator $V$.

**Definition 3.1.1.** The bi-oblique boundary data reflection operator $V$ determined by the real unit vector fields $v_1, v_2 \in L_\infty(\Sigma; \Lambda^1)$ on $\Sigma$ is the operator

\[
(Vf)(x) := v_1(x)f(x)^{-}v_2(x), \quad x \in \Sigma.
\]

Indeed, $V$ is a reflection operator, i.e. $V^2 = I$, and its spectral projections are $V^\pm := \frac{1}{2}(I \pm V)$. When both $v_1$ and $v_2$ are the unit normal vector field $\nu$, we denote this reflection operator by $\nu$, with spectral projections $\mu^\pm := \frac{1}{2}(I \pm \nu)$.

Note that $V$ preserves the homogeneous exterior powers $\Lambda^j$ if and only if $v_1 = \pm v_2$.

In the case $v_1 = v_2 =: v$, (1.4) and (1.5) show that $V^+f = v^\perp(v \wedge f)$ is the part of $f$ “orthogonal” to $v$ and $V^-f = v \wedge (v \perp f)$ is the part of $f$ “parallel” to $v$. In particular, $N^+f$ is the tangential part of $f$ and $N^-f$ is the normal part of $f$. We note that if $v_1 = v_2$, then $L_2(\Sigma; \Lambda^0) \subset V^+L_2$ and $L_2(\Sigma; \Lambda^{n+1}) \subset V^-L_2$.

The bi-oblique transmission problem we consider in this section is

\[
\begin{align*}
V^+(\alpha^- E_k^+ f - \alpha^+ E_k^- f) &= V^+g, \\
V^-(\alpha^+ E_k^+ f - \alpha^- E_k^- f) &= V^-g,
\end{align*}
\]

(3.3)

with jump parameters $\alpha^\pm \in \mathbb{C}$. Note that this reduces to (2.3) when $v_1 = v_2 =: v$. We here generalise to a bi-oblique transmission problem $v_1 \neq v_2$ since this is a natural setting for the Rellich estimates below. Before stating the main Theorem 3.1.3, we give some of its applications to the transmission problem (3.3).

**Example 3.1.2.** Consider the case $v_1 = v_2 = v$. By Proposition 1.5.4 there exists a smooth transversal vector field $\theta$ for $\Sigma$. Assume that

\[
\gamma := \sup_{\Sigma} (\angle v \nu + \angle v \theta) < \frac{\pi}{2},
\]

(3.4)

where $\angle ab$ denotes the angle between the two vectors $a$ and $b$. Then the transmission problem (3.3) is well posed in Fredholm sense when $|\text{Re} \lambda| > \tan \gamma |\text{Im} \lambda|$, where $\lambda := (\alpha^+ + \alpha^-)/(\alpha^+ - \alpha^-)$.

Special cases when the condition (3.4) holds are the following.
• There exists \( \epsilon = \epsilon(\theta) > 0 \) such that every \( \nu \) which satisfies \( |v - \nu| < \epsilon \) a.e. on \( \Sigma \) gives (3.4).

• Any \( \nu \) directed along \( t\theta + (1 - t)\nu \), where \( 0 \leq t = t(y) \leq 1 \) for all \( y \in \Sigma \), gives (3.4).

• When \( \Sigma \) as well as the vector field \( \nu \) are smooth, we can choose a smooth transversal vector field \( \theta \) such that \( \theta|_\Sigma = \nu \), which gives \( \angle v\theta = 0 \). Therefore (3.4) holds provided that \( v \) is non-tangential.

We see from (2.8) that the transmission problem (3.3) is equivalent to the rotation operator equation

\[
(\lambda I - E_k V)f = \frac{2}{\alpha^+ - \alpha^-}E_k g.
\]

The transmission problem thus reduces to the problem of estimating the spectrum of the rotation operator \( E_k V \). We now state the main theorem in this section.

**Theorem 3.1.3.** Let \( \Sigma \) be a bounded strongly Lipschitz surface and let \( v_1 \) and \( v_2 \) be real, measurable unit vector fields on \( \Sigma \) as in Definition 3.1.1. Suppose that we can find a smooth vector field \( \theta \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^d) \) such that \( c_0 := \inf_{\Sigma}|\theta| > 0 \) and constants \( L_\nu, L_\theta < \infty \) such that \( |v_1 \wedge \nu| \leq L_\nu(v_1, \nu) \) and \( |v_2 \wedge \theta| \leq L_\theta(v_2, \theta) \) on \( \Sigma \), and

\[
L_\nu L_\theta < 1.
\]

Then we have the angular estimate of the essential spectrum

\[
\sigma_{ess}(E_k V; L_2(\Sigma; \lambda)) \subset \left\{ \lambda = \lambda_1 + i\lambda_2 : |\lambda_1| \leq \frac{L_\nu + L_\theta}{1 - L_\nu L_\theta} |\lambda_2| \right\},
\]

(3.5)

In particular, if there is a smooth vector field \( \theta \) such that \( L_\nu L_\theta < 1 \), then \( E_k \) and \( V \) are transversal in the Calkin algebra, i.e. the boundary value points \( \pm 1 \) are not in \( \sigma_{ess}(E_k V) \).

Note that the geometric meaning of the condition \( L_\nu L_\theta < 1 \) is that the sum of the angles \( \sup_\Sigma \angle v_1 \nu \) and \( \sup_\Sigma \angle v_2 \theta \) should be less than \( \frac{\pi}{2} \) on \( \Sigma \). In fact, by examining the proof of Theorem 3.1.3, this condition can be relaxed to \( \sup_\Sigma (\angle v_1 \nu + \angle v_2 \theta) < \frac{\pi}{2} \) as noted in Example 3.1.2.

We now consider oblique boundary value problems (BVP) for the Dirac operator. Note that the special cases \( \alpha^+ = 0 \) or \( \alpha^- = 0 \) in (3.3) are each two independent boundary value problems, one for \( \Omega^+ \) and one for \( \Omega^- \). Our result on the oblique BVP is the following.

**Theorem 3.1.4.** Consider the eight restricted projections

\[
V^+ : E^+_{k}L_2 \rightarrow V^+L_2, \quad V^- : E^+_{k}L_2 \rightarrow V^-L_2,
\]

\[
E^+_{k} : V^\pm L_2 \rightarrow E^+_{k}L_2, \quad E^-_{k} : V^\pm L_2 \rightarrow E^-_{k}L_2.
\]

Under the hypothesis of Theorem 3.1.3, these are all Fredholm operators with index zero for all \( k \in \mathbb{C} \). Moreover, there exists a discrete set \( S_V \subset \mathbb{C} \) of resonances contained in the region

\[
\Im k \leq (|\Re k| \|\theta\|_\infty + \frac{1}{2} \sum_j \|\partial_j \theta\|_\infty) \frac{L_\nu(1 + E^2_\theta)}{c_0(1 - L_\nu L_\theta)},
\]

such that if \( k \notin S_V \) then all the restricted projections are isomorphisms. In particular, if \( f \in E^+_{k}L_2 \) and \( k \notin S_V \) then we have the classical Rellich equivalence \( \|V^+ f\| \approx \|V^- f\| \).
**Example 3.1.5.** Consider the following interior oblique Dirac BVP. Given a $V$-tangential field $g \in V^+L_2(\Sigma; \wedge)$ find a Hardy field $F \in C^+_kL_2$ with non-tangential maximal function $\mathcal{NF} \in L_2(\Sigma)$ such that

$$\begin{cases}
    \mathbf{D}_kF = 0 & \text{in } \Omega^+, \\
    v \wedge f = v \wedge g & \text{on } \Sigma.
\end{cases}$$

In operator notation, this reads $V^+f = g$, $f \in E^+_kL_2$. Since $V^+: E^+_kL_2 \to V^+L_2$ is an isomorphism except for a discrete set of resonances according to Theorem 3.1.4, the BVP is well posed for these $k$.

The proofs of these two theorems use the following four lemmata.

**Lemma 3.1.6.** The Rellich form with respect to a smooth vector field $\theta \in C^\infty_0(\mathbb{R}^n; \mathbb{R}^n)$ in $\mathbb{R}^n$ is the indefinite sesqui-linear symmetric form

$$\langle g, f \rangle_\theta := \int_\Sigma (\nu g^\perp, f^\psi \theta), \quad g, f \in L_2(\Sigma; \wedge).$$

If $g \in N^\pm L_2$, then $\langle f, f \rangle_\theta = \pm \int_\Sigma |f|^2(\nu, \theta)$, and if $f \in E^\pm_kL_2$ then we have the estimate

$$|\langle f, f \rangle_\theta| \leq \left\{ 2|\text{Re } k| \|\theta\|_\infty + \sum_j \|\partial_j \theta\|_\infty \right\} \int_{\Omega^\pm \cap \supp \theta} |F|^2.$$  (3.6)

**Proof.** The proof of the first statement is omitted since we will not use this later. To prove (3.6) for $f \in E^\pm_kL_2$, apply the boundary theorem 1.2.7 and Leibniz’ formula to get

$$\pm \langle f, f \rangle_\theta = \int_{\Omega^\pm} (-(-ke_0F)^\perp, F^\psi \theta) + (F^\perp, (-ke_0F)^\psi \theta) + (F^\perp, \mathbf{D} F^\psi \theta(\hat{x}))$$

$$= \int_{\Omega^\pm} -2\text{Re } k (e_0 F^\perp, F^\psi \theta) + \sum_j (F^\perp, e_j F^\psi (\partial_j \theta)).$$

From this identity the estimate (3.6) follows. Here we have used the fact that the Cauchy extension $F = C^\pm_kf$ satisfies $\mathbf{D}F = -ke_0F$ in $\Omega^\pm$. \hfill \Box

**Lemma 3.1.7.** For $f = F|_{\Sigma} \in E^+_kL_2$ we have the following integral identities.

$$\int_{\Sigma} (\overline{e}_0 f, \nu f^\psi) = 2\text{Im } k \int_{\Omega^+} |F|^2,$$  (3.7)

$$\int_{\Sigma} (\overline{e}_0 f, \nu \wedge f^\psi) = \int_{\Omega^+} \text{Im } k |F|^2 + i \text{Re } k (F, TF^\psi) + 2i \text{Im } (d_k F, \overline{e}_0 F^\psi),$$  (3.8)

$$\int_{\Sigma} (\Gamma_k f, \nu \wedge f^\psi) = \int_{\Omega^+} |d_k F|^2 - 2\text{Re } k (d_k F, e_0 \wedge F^\psi).$$  (3.9)

Here $Tf := e_0 f^\perp e_0$ denotes the time reversion operator and $\Gamma_k f := (d_k F)|_{\Sigma}$ as in Section 3.2. In the last two equations we assume that $d_k F \in C^+_kL_2$.

The same holds true for $f \in E^-_kL_2$ when $\text{Im } k > 0$, provided $\int_{\Omega^+}$ is replaced by $-\int_{\Omega^-}$.
\textbf{Proof.} From the boundary theorem 1.2.7 and the identities $\mathbf{D}F = -k\varepsilon_0 F$, $-\delta F = dF + ke_0 F$ and $d_k F = -\delta_k F$ respectively, we obtain

\begin{align*}
\int_{\Sigma} (\bar{\zeta} e_0 f, \nu f^c) &= \int_{\Omega^+} (\bar{\zeta} e_0 f, (ke_0 F)^c) - (\bar{\zeta} e_0 (ke_0 F), F^c) = 2\text{Im} \ i \int_{\Omega^+} |F|^2, \\
\int_{\Sigma} (\bar{\zeta} e_0 f, \nu \wedge f^c) &= \int_{\Omega^+} (\bar{\zeta} e_0 F, dF^c) + (\bar{\zeta} e_0 (\delta_k F), F^c) \\
&= -i \int_{\Omega^+} (e_0 (dF + ke_0 F), F^c) \\
&= -i \int_{\Omega^+} (e_0 F, (d_k F - ke_0 \wedge F)^c) + (d_k F - ke_0 \wedge F, e_0 F^c) \\
&= -i \int_{\Omega^+} 2\text{Re} (d_k F, e_0 F^c) - k^c (F, T^+ F^c) + k (T^- F, F^c) \\
&= \int_{\Omega^+} \text{Im} \ k |F|^2 + i \text{Re} \ k (F, T F^c) + 2i \text{Im} \ (d_k F, \bar{\zeta} e_0 F^c), \\
\int_{\Sigma} (\Gamma_k f, \nu \wedge f^c) &= \int_{\Sigma} ((d_k F)|_{\Sigma} \nu \wedge f^c) = \int_{\Omega^+} (d_k F, dF^c) + (\delta d_k F, F^c) \\
&= \int_{\Omega^+} (d_k F, (d_k F - ke_0 \wedge F)^c) - ((\delta_k - ke_0 \wedge) \delta_k F, F^c) \\
&= \int_{\Omega^+} |d_k F|^2 - 2\text{Re} \ k (d_k F, e_0 \wedge F^c). & \Box
\end{align*}

\begin{lemma}
Let $U$ be a bounded neighbourhood of $\Sigma \subset \mathbb{R}^n$. Consider an integral operator $L_2(\Sigma) \ni f \mapsto T f \in L_2(U \setminus \Sigma)$ of the form

$$T f(x) := \int_{\Sigma} K(x, y) f(y) \, d\sigma(y), \quad x \in U \setminus \Sigma,$$

where the kernel satisfies the estimate $|K(x, y)| \lesssim |x - y|^{-\gamma}$ for some $\gamma < n - \frac{1}{2}$. Then $T$ is a compact operator.

\textbf{Proof.} We use a modified Schur estimate as follows. Pick $\alpha, \beta > 0$ such that $\alpha + \beta = 1$, $2\alpha \gamma < n - 1$ and $2\beta \gamma < n$. Using Cauchy–Schwarz’ inequality we obtain

$$|T f(x)|^2 \leq \left( \int_{\Sigma} |K(x, y)|^\alpha |K(x, y)|^\beta |f(y)| \, d\sigma(y) \right)^2 \leq \left( \sup_{x \in U \setminus \Sigma} \int_{\Sigma} |K(x, y)|^2 \, d\sigma(y) \right) \int_{\Sigma} |K(x, y)|^2 |f(y)|^2 \, d\sigma(y).$$

Integrating $x \in U \setminus \Sigma$ gives

$$\|T f\|_{L_2(U \setminus \Sigma)}^2 \leq \sup_{x \in U \setminus \Sigma} \int_{\Sigma} |K(x, y)|^2 \, d\sigma(y) \sup_{y \in \Sigma} \oint_{U \setminus \Sigma} |K(x, y)|^2 \, dx \|f\|_{L_2(\Sigma)}^2,$$

which shows that $T$ is bounded. Replacing the kernel $K$ with $K_\varepsilon(x, y) := K(x, y) \chi_{\{|x - y| < \varepsilon\}}$ and performing the same estimates for the corresponding operators, it follows that $T$ in fact can be uniformly approximated with Hilbert–Schmidt operators. Therefore $T$ is compact. \hfill \Box

51
Proof of Theorem 3.1.3. Let $h \in L_2(\Sigma; \wedge)$ and $g := (\lambda - E_k V)h$. Introduce auxiliary Hardy fields

$$F^\pm|_\Sigma = f^\pm := E_k^\pm V h = \frac{1}{2}(Vh \pm (\lambda h - g)).$$

Note that Lemma 3.1.8 shows that $L_2(\Sigma; \wedge) \ni h \mapsto F^\pm \in L_2(\text{supp } \theta \cap \Omega^\pm; \wedge)$ is a compact map. Thus by Proposition 1.4.3 and Theorem 1.4.5 it is enough to prove an a priori estimate

$$\|h\|_{L_2(\Sigma)} \lesssim \|g\|_{L_2(\Sigma)} + \|F\|_{L_2(\text{supp } \theta)},$$

where $F = F^+ + F^-$, for all $\lambda$ such that $|\lambda_1|(1 - L_\nu L_\theta) > |\lambda_2|(L_\nu + L_\theta)$. Note that

$$\langle f^+, f^+ \rangle_\theta - \langle f^-, f^- \rangle_\theta = \Re(\lambda^c \langle Vh, h \rangle_\theta - \langle Vh, g \rangle_\theta)$$

and that Lemma 3.1.6 gives

$$|\langle f^+, f^+ \rangle_\theta| + |\langle f^-, f^- \rangle_\theta| \leq \left(2|\Re k|\|\theta\|_\infty + \sum_j \|\partial_j \theta\|_\infty \right) \int_{\text{supp } \theta} |F|^2.$$

Thus the needed a priori estimate follows if we can prove the coercivity estimate

$$\Re(\lambda^c \langle Vh, h \rangle_\theta) \gtrsim \|h\|_2^2.$$

The integrand of $\langle Vh, h \rangle_\theta$ is

$$(\nu v_1 h v_2, h^c \theta) = \langle \nu v_1 h, h^c \theta v_2 \rangle = (\langle \nu v_1, v_1 \rangle h, h^c ((\theta, v_2) + \theta \wedge v_2))$$

$$= \Re (\langle \nu v_1, v_2 \rangle |h|^2 + (\nu \wedge v_1) h, h^c (\theta \wedge v_2))$$

$$+ i \Im (\langle \nu v_1, h, h^c (\theta \wedge v_2) \rangle + (\nu \wedge v_1) h, h^c (\theta, v_2)).$$

Therefore

$$\sgn \lambda_1 \Re(\lambda^c \langle \nu v_1 h v_2, h^c \theta \rangle) \geq \lambda_1 \langle \nu v_1, (\theta, v_2) \rangle |h|^2 - \lambda_1 (|\nu \wedge v_1| |\theta \wedge v_2| |h|^2$$

$$- |\lambda_2 (|\nu v_1| |\theta \wedge v_2| |h|^2 - |\lambda_2 (|\nu \wedge v_1| |\theta, v_2| |h|^2)$$

$$\geq c_\theta \frac{\lambda_1 (1 - L_\nu L_\theta) - \lambda_2 (L_\nu + L_\theta) |h|^2}{\sqrt{(1 + L_\nu^2)(1 + L_\theta^2)},$$

which completes the proof. \hfill \Box

Note that in $\mathbb{R}^2$ we can get better estimates due to commutativity, as shown in Section 2.2.1, where the space $L_2(\Sigma; \wedge^{\text{even}})$ on a bounded strongly Lipschitz curve $\Sigma$ in $\mathbb{R}^2$ is considered. In this case, the even sub-algebra $\wedge^{\text{even}} := \wedge^0 \oplus \wedge^2$ is commutative and can be identified with the bi-complex numbers. Here the reflection operator $E = E_0$ reduces to the classical Cauchy integral operator in complex analysis and $V f = v_1 f^\wedge v_2 = (v_1 v_2) f = -a^\wedge f$ is the oblique boundary data reflection operator used in Section 2.2.1.

From Proposition 2.1.2 we also get the following spectral hyperbolas for double layer type operators. In the special case of the normal (Dirichlet and Neumann) transmission problem for the Laplace equation these estimates have been proved in Spencer [58].
Corollary 3.1.9. Under the hypothesis of Theorem 3.1.3, the essential spectra in $L_2(\Sigma; \lambda)$ of the cosine operator $\frac{1}{2}(E_kV + V E_k)$, the double layer type operators $V^\pm E_k V^\pm : V^\pm L_2 \to V^\pm L_2$ and the Toeplitz type operators $E_k^\pm V E_k^\pm : E_k^\pm L_2 \to E_k^\pm L_2$ are all contained in the hyperbolic region

$$\lambda_1^2 \leq L^2 \lambda_2^2 + L^2/(L^2 + 1),$$

where $L := (\lambda_1 + \lambda_2)/(1 - \lambda_1 \lambda_2)$ and $\lambda =: \lambda_1 + i \lambda_2$.

We now show that if $\text{Im} \, k > 0$, then we can use Lemma 3.1.7 to control the compact term $\|F\|_{L_2(\text{supp} \, \theta)}$ in the proof of Theorem 3.1.3 and show that $\lambda - E_k V$ is an isomorphism in a slightly smaller double sector $|\text{Re} \, \lambda| > C|\text{Im} \, \lambda|$.

Theorem 3.1.10. Assume $\text{Im} \, k > 0$, let $v_1$, $v_2$ and $\theta$ be as in Theorem 3.1.3 and define

$$C_{\theta, k} := \frac{(1 + L_\theta^2)^{1/2}(2|\text{Re} \, k| \|\theta\|_\infty + \sum_j \|\partial_j \theta\|_\infty)}{2c_\theta \text{Im} \, k}.$$ 

If $L_\nu(L_\theta + C_{\theta, k}) < 1$, then we have the angular estimate of the spectrum

$$\sigma(E_k V; L_2(\Sigma; \lambda)) \subset \left\{ \lambda = \lambda_1 + i \lambda_2 : |\lambda_1| \leq \frac{L_\nu + L_\theta + C_{\theta, k}}{1 - L_\nu(L_\theta + C_{\theta, k})} |\lambda_2| \right\}. \quad (3.10)$$

Thus if there is a smooth vector field $\theta$ such that $L_\nu(L_\theta + C_{\theta, k}) < 1$, then $E_k$ and $V$ are transversal, i.e. the boundary value points $\pm 1$ are not in $\sigma(E_k V)$.

Proof. Using the same notation as in the proof of Theorem 3.1.3 and in Definition 6.1.1, we get from (3.7) in Lemma 3.1.7 that

$$|\text{Re}(\lambda^c \langle V \theta, h \rangle \theta - \langle V h, g \rangle \theta)| \leq \frac{c_\theta C_{\theta, k}}{(1 + L_\theta^2)^{1/2}} ((f^+, f^{+c}) \overline{e}_0 - (f^-, f^{-c}) \overline{e}_0)
= \frac{c_\theta C_{\theta, k}}{(1 + L_\theta^2)^{1/2}} \text{Re}(\lambda^c \langle V \theta, h^c \rangle \overline{e}_0 - \langle V h, g^c \rangle \overline{e}_0). \quad (3.11)$$

Estimating the integrand of $\text{Re}(\lambda^c \langle V \theta, h^c \rangle \overline{e}_0)$, we have

$$(\overline{e}_0 v_1 h^\nu v_2, v h^c) = -(v_1, \nu)(\overline{e}_0 h^\nu v_2, h^c) - (\overline{e}_0 h^\nu v_2, (v_1 \wedge \nu) h^c)
= -(v_1, \nu) i \text{Im} \, (\overline{e}_0 h^\nu v_2, h^c) - \text{Re} \, (\overline{e}_0 h^\nu v_2, (v_1 \wedge \nu) h^c),$$

and therefore

$$|\text{Re}(\lambda^c (\overline{e}_0 v_1 h^\nu v_2, v h^c))| \leq |\lambda_2(v_1, \nu)| (\overline{e}_0 h^\nu v_2, h^c) + |\lambda_1(\overline{e}_0 h^\nu v_2, (v_1 \wedge \nu) h^c)|
\leq (|\lambda_2| + L_\nu |\lambda_1|) |\nu, v_1| \|h\|^2.$$ 

On the quadratic term in (3.11) we now get the coercivity estimate

$$|\text{Re}(\lambda^c \langle V \theta, h \rangle \theta)| - \frac{c_\theta C_{\theta, k} |\text{Re}(\lambda^c \langle V h, h^c \rangle \overline{e}_0)|}{\sqrt{1 + L_\theta^2}}
\geq c_\theta \frac{|\lambda_1| (1 - L_\nu (L_\theta + C_{\theta, k})) - |\lambda_2| (L_\nu + L_\theta + C_{\theta, k})}{\sqrt{1 + L_\theta^2 (1 + L_\theta^2)}} \|h\|^2.$$ 

So for $\lambda$ not satisfying the right hand side estimate in (3.10) the operator $\lambda - E_k V$ has exact a priori estimates. Now Theorem 1.4.5 proves (3.10).
Proposition 3.1.11. Under the hypothesis of Theorem 3.1.3 and for fixed \( \lambda \) with \( \| \text{Re} \lambda \| > \frac{L_0 + L_\theta}{L_0}, \) \( \text{Im} \lambda \| , \) there exists a discrete set \( S_{V, \lambda} \subset \mathbb{C} \) such that if \( k \notin S_{V, \lambda} \) then \( \lambda - E_k V \) is invertible.

Proof. By Proposition 1.2.5, \( E_k \) depends analytically on \( k \) and \( E_k - E_{k'} \) is a compact operator for any \( k \) and \( k' \in \mathbb{C} \). Note that \( C_{\theta, k} \to 0 \) as \( k \to \infty \) along \( i\mathbb{R}^+ \). Therefore Theorem 3.1.10 shows that \( \lambda - E_k V \) is invertible for large imaginary \( k \). We can now apply analytic Fredholm theory from Theorem 1.4.6, which proves the proposition. \( \square \)

Proof of Theorem 3.1.4. By Theorem 3.1.3, \( I \pm E_k V \) is a Fredholm operator and Proposition 3.1.11 with \( \lambda = \pm 1 \) shows that except on the discrete set \( S_V := S_{V,1} \cup S_{V,-1}, \) \( I \pm E_k V \) are isomorphisms, and thus also all the restricted projections. Solving for \( k \) in \( L_\theta (L_\theta + C_{\theta, k}) \geq 1 \) in Theorem 3.1.10 gives the stated estimate on \( S_V \). What is left to prove is that the restricted projections have index zero for all \( k \). We know from Theorem 3.1.10 that \( \pm I - E_k V \) is Fredholm with index zero. This implies that the restricted projections are Fredholm but not in general that the indices are zero, as noted in Chapter 2. For the normal transmission problem one can use duality relations with respect to the Dirac form from Proposition 6.1.4 and Proposition 6.1.5 in the Chapter 6. For the bi-oblique transmission problem, we use here perturbation theory. It suffices to demonstrate that \( \text{Ind} (V^+ : E_k^+ L_2 \to V^+ L_2) \) is a locally constant function of \( k \). This is a non-standard perturbation problem, since it is the domain and not the operator that is variable. Fix \( k_0 \in \mathbb{C} \). We observe that

\[
E_k + \frac{1}{2} (E_k - E_k) = E_k^+ E_k^- + E_k^+ E_k^-.
\]

Thus \( E_k^+ : E_k^+ L_2 \to E_k^+ L_2 \) is invertible for \( |k - k_0| < \epsilon \), and we can factorise

\[
V^+|_{E_k^+ L_2} = (V^+ (\frac{1}{2} (E_k + E_k))^{-1}) E_k^+ = A_k B_k,
\]

where \( B_k : E_k^+ L_2 \to E_k^+ L_2 \) is an isomorphism and \( A_k : E_k^+ L_2 \to V^+ L_2 \) acts between fixed spaces and depends continuously on \( k \). Standard perturbation theory implies that \( \text{Ind} (V^+ : E_k^+ L_2 \to V^+ L_2) = \text{Ind} A_k = \text{Ind} A_k = \text{Ind} (V^+ : E_k^+ L_2 \to V^+ L_2) \). \( \square \)

As we have seen above in Theorem 3.1.3 and Theorem 3.1.10, angular estimates on the spectrum of \( E_k V \) can be obtained with techniques based on the boundary theorem 1.2.7. On the other hand, estimating the spectral radius \( r(E_k V) \) is much harder. Only the crude estimate \( r(E_k V) \leq \| E_k V \| = \| E_k \| < \infty \) is known. It is shown in David [14] that the \( L_2 \) norm of the Cauchy integral can be arbitrarily large for a general strongly Lipschitz domain. There is no reason to believe that \( r(E_k V; L_2) \) should be small for general \( V \) and \( \Sigma \), but one can ask whether the (essential) spectral radius \( r(E_k N; L_2) \) is uniformly bounded for all strongly Lipschitz domains. Note that a special case of this is the spectral radius conjecture for the classical double layer potential, see open problem 3.2.10 in Kenig [30]. See also Fabes–Sand–Seo [20] for the special case of a convex domain.
3.2 The normal transmission problem and the exterior interior derivative operator

In this section we specialise to the normal reflection operator $N$. The aim here is to develop a theory, based on the rotation operator $E_k N$, for solving the following Maxwell type boundary value problems.

- Given a normal $j$-vector field $g \in N^- L_2(\Sigma; \wedge^j)$ with vanishing tangential divergence $\Gamma^N_k g = 0$, as in Definition 3.2.7(ii) below, find $F : \Omega^+ \rightarrow \wedge^j$ with non-tangential maximal function $N F \in L_2(\Sigma)$ such that

$$
\begin{align*}
&d_k F = 0 \quad \text{in } \Omega^+, \\
&\delta_k F = 0 \quad \text{in } \Omega^+, \\
&\nu \bot f = \nu \bot g \quad \text{on } \Sigma,
\end{align*}
$$

(3.12)

- Given a tangential $j$-vector field $g \in N^+ L_2(\Sigma; \wedge^j)$ with vanishing tangential curl $\Gamma^N_k g = 0$, as in Definition 3.2.7(i) below, find $F : \Omega^+ \rightarrow \wedge^j$ with non-tangential maximal function $N F \in L_2(\Sigma)$ such that

$$
\begin{align*}
&d_k F = 0 \quad \text{in } \Omega^+, \\
&\delta_k F = 0 \quad \text{in } \Omega^+, \\
&\nu \wedge f = \nu \wedge g \quad \text{on } \Sigma.
\end{align*}
$$

(3.13)

As in Theorem 3.1.4, we will investigate the BVP’s (3.12) and (3.13) using the restricted projections

$$
\begin{align*}
N^- &= E^-_2 (I - E_k N ) : E^+_k L_2 \rightarrow N^- L_2 : f \mapsto g, \\
N^+ &= E^+_2 (I + E_k N ) : E^+_k L_2 \rightarrow N^+ L_2 : f \mapsto g,
\end{align*}
$$

respectively. The first main result in this section is the following.

**Theorem 3.2.1.** For any wave number $k$, the boundary value problems (3.12) and (3.13) are well posed in Fredholm sense, i.e. there exists $\{F_1, \ldots, F_{m_1}\} \subset C^+_k L_2$, $F_i : \Omega^+ \rightarrow \wedge^j$, and $\{g_1, \ldots, g_{m_2}\} \subset N^- L_2(\Sigma; \wedge^j)$, $\Gamma^N_k g_i = 0$, where $m_i = m_i(k)$, such that if $g$ is orthogonal to $\{g_1, \ldots, g_{m_2}\}$, then there exists a solution $F$ to (3.12) which is unique modulo $\{F_1, \ldots, F_{m_1}\}$. Similarly for (3.13). Moreover:

(i) There exists a discrete set $S_{\Omega^+} \subset \mathbb{R}$ such that if $k \notin S_{\Omega^+}$ then $m_1 = m_2 = 0$, i.e. the boundary value problems (3.12) and (3.13) have a unique solution.

(ii) For any $k$, we have the following regularity result. If $F \in C^+_k L_2$ is such that $N^- f \in D(\Gamma^N_k)$, then $d_k F \in C^+_k L_2$ and $N^- (d_k F |_{\Sigma}) = \Gamma^N_k (N^- f)$. In particular if $k \notin S_{\Omega^+}$, then $N^- f \in N(\Gamma^N_k)$ implies $d_k F = 0$. The same is true if $N^-$ is replaced by $N^+$.

(iii) Even though $0 \in S_{\Omega^+}$, we do have that $d F = 0$ if $f \in E^+_k L_2$ and either $N^- f \in N(\Gamma^N_k)$ or $N^+ f \in N(\Gamma^N_k)$. 

55
This theorem extends the results on boundary value problems for Maxwell’s equations with \( j = 2 \) in \( \mathbb{R}^3 \) obtained in Axelsson–Grognaard–Hogan–McIntosh [4]. The Maxwell boundary value problems (3.12) and (3.13) have been investigated with different methods in Mitrea–Mitrea–Taylor [46] in connection with boundary value problems for the Hodge–Laplace operator, and in Mitrea [49] in manifold setting.

The methods in this section also give results for the following Maxwell type transmission problem. Given a \( j \)-vector field \( g \in L^2(\Sigma; \wedge^j) \) with vanishing tangential divergence/curl \( \Gamma_k g = 0 \) as in Theorem 3.2.3 below, find \( F^- : \Omega^- \rightarrow \wedge^j \) with non-tangential maximal functions \( N F^- \in L^2(\Sigma) \) with \( F^- \) satisfying a radiation condition at infinity, such that

\[
\begin{cases}
d_k F^- = \delta_k F^- = 0 & \text{in } \Omega^-, \\
\nu \wedge (\alpha^+ f^+ - \alpha^- f^-) = \nu \wedge g & \text{on } \Sigma, \\
\nu \perp (\alpha^+ f^+ - \alpha^- f^-) = \nu \perp g & \text{on } \Sigma,
\end{cases}
\]

(3.14)

with jump parameters \( \alpha^\pm \in \mathbb{C} \) and \( f^\pm := F^\pm |_{\Sigma} \). We will investigate this transmission problem using the operator

\[
\frac{\alpha^+ - \alpha^-}{2} E_k (\lambda - E_k N) : L^2(\Sigma; \wedge) \rightarrow L^2(\Sigma; \wedge) : f \mapsto g,
\]

where \( \lambda := (\alpha^+ + \alpha^-) / (\alpha^+ - \alpha^-) \) and \( f = f^+ + f^- \). The second main result in this section is the following.

**Theorem 3.2.2.** Let \( L^2(\Sigma) \) be the local Lipschitz constant for \( \Sigma \) as in Definition 1.5.5 and assume that \( \lambda \) belongs to the open double sector \( |\text{Re } \lambda| > L^2(\Sigma)|\text{Im } \lambda| \). Then the transmission problem (3.14) is Fredholm with index zero, i.e. there exists \( \{f_1, \ldots, f_m\} \subset L^2(\Sigma; \wedge^j) \), \( C^I_k f_i : \Omega^\pm \rightarrow \wedge^j \), and \( \{g_1, \ldots, g_m\} \subset L^2(\Sigma; \wedge^j) \), \( \Gamma_k g_i = 0 \), where \( m = m(k, \lambda) \), such that if \( g \) is orthogonal to \( \{g_1, \ldots, g_m\} \), then there exists a solution \( f \) to (3.14), where \( F^\pm := C^I_k f : \Omega^\pm \rightarrow \wedge^j \), which is unique modulo \( \{f_1, \ldots, f_m\} \). Moreover:

(i) **There exists a discrete set \( S_\lambda \subset \mathbb{C} \) such that if \( k \notin S_\lambda \) then \( m = 0 \), i.e. the transmission problem (3.14) has a unique solution.**

(ii) **For any \( k \), we have the following regularity result. If \( F^\pm \in C^I_k L^2 \) is such that \( g \in D(\Gamma_k) \), where \( g \) is the jump data for \( F^\pm \), then \( d_k F^\pm \in C^I_k L^2 \) and the jump data for \( d_k F^\pm \) is \( \Gamma_k g \). In particular, if \( k \notin S_\lambda \) then \( \Gamma_k g = 0 \) implies \( d_k F^\pm = 0 \).**

The first step in the proof of Theorem 3.2.1 and Theorem 3.2.2 is to investigate the constraint \( d_k F = 0 \) in the elliptic Dirac equation \((d_k + \delta_k) F = 0\). To handle this constraint we introduce, besides the two reflection operators \( E_k \) and \( N \), an **exterior/interior derivative operator** \( \Gamma_k \) acting in \( L^2(\Sigma; \wedge) \).

**Theorem 3.2.3.** For each wave number \( k \), there exists a unique closed, densely defined, nilpotent operator \( \Gamma_k \) acting in \( L^2(\Sigma; \wedge) \), which restricts to the operators \( \Gamma^N_k \) and \( \Gamma^{E_k}_k \).
introduced in Definition 3.2.7 below, in the four subspaces $N^±L_2$ and $E^±_k L_2$. The domain of $\Gamma_k$ splits

$$D(\Gamma_k) = N^+D(\Gamma_k) \oplus N^-D(\Gamma_k) = E^+_k D(\Gamma_k) \oplus E^-_k D(\Gamma_k)$$

$$= D(\Gamma^{N^+}_k) \oplus D(\Gamma^{N^-}_k) = D(\Gamma^{E^+_k}_k) \oplus D(\Gamma^{E^-_k}_k).$$

The domain $D(\Gamma_k)$ is independent of $k$, and we write $D(\Gamma) := D(\Gamma_k)$ for this Hilbert space, which is dense in $L_2(\Sigma; \wedge)$. In the sense of unbounded operators we have intertwining relations

$$\Gamma_k E_k = E_k \Gamma_k \quad \text{and} \quad \Gamma_k N = N \Gamma_k,$$

and $N$ and $E_k$ act boundedly in the Hilbert space $D(\Gamma)$. Since $\Gamma_k$ is nilpotent, it also acts boundedly in $D(\Gamma)$, and it leaves the four subspaces $E^±_k D(\Gamma)$ and $N^± D(\Gamma)$ invariant.

For example, on scalar fields $L_2(\Sigma; \wedge^0) \subset N^+L_2$, $\Gamma_k$ acts like tangential $k$-gradient; on tangential vector fields $N^+L_2(\Sigma; \wedge^1)$, $\Gamma_k$ acts like tangential $k$-curl; on normal vector fields $N^-L_2(\Sigma; \wedge^1)$, $\Gamma_k = 0$; on normal bivector fields $N^-L_2(\Sigma; \wedge^2) = \nu \wedge N^+L_2(\Sigma; \wedge^1)$, $\Gamma_k$ acts like tangential $k$-divergence. More generally, considering $d$ as generalised curl and $\delta$ as generalised divergence, the null space $N(\Gamma^{N^+}_k)$ consists of “$k$-curl free” tangential fields, the null space $N(\Gamma^{N^-}_k)$ consists of “$k$-divergence free” normal fields and by Proposition 3.2.5 below, the null space $N(\Gamma^{E^±}_k)$ consists of “decoupled $k$-monochromatic” Hardy fields. On the other hand, the auxiliary space $D(\Gamma)$ is useful since the operators $E_k$, $N$ and $\Gamma_k$ act boundedly in it for all $k \in \mathbb{C}$.

**Definition 3.2.4.** Let $N(\Gamma_k, \wedge^j) := N(\Gamma_k) \cap L_2(\Sigma; \wedge^j)$ be the space of $k$-divergence/curl free $j$-vector fields.

**Proof of Theorem 3.2.2.** From Theorem 3.2.15 and 3.2.3 it follows that $\lambda - E_k N$, and thus $a^{±}a^{−}E_k(\lambda - E_k N)$ is a Fredholm operator with index zero on all four spaces

$$L_2(\Sigma; \wedge) \supset D(\Gamma) \supset N(\Gamma_k) \supset N(\Gamma_k, \wedge^j),$$

when $|\Re \lambda| > L_2|\Im \lambda|$. Thus (observing Theorem 3.2.3, Proposition 3.2.5 and Definition 3.2.7), given jump data $g \in N(\Gamma_k, \wedge^j)$, the transmission problem (3.14) have, in Fredholm sense, a twosided monochromatic solution $f = f^+ + f^− \in N(\Gamma_k, \wedge^j)$. This proves the well posedness.

(i) Let $S_\chi := S_{N, \chi}$ be the discrete set from Proposition 3.1.11. Then $\lambda - E_k N$ is injective, and thus surjective since the index is zero, on all four spaces above. In particular, $a^{±}a^{−}E_k(\lambda - E_k N) : N(\Gamma_k, \wedge^j) \rightarrow N(\Gamma_k, \wedge^j)$ is an isomorphism.

(ii) For the regularity statement, we consider the action of $a^{±}a^{−}E_k(\lambda - E_k N)$ on $L_2(\Sigma; \wedge) \supset D(\Gamma)$. Using that this inclusion is dense, we deduce that if $g \in D(\Gamma)$, then by Theorem 3.2.3 $(\lambda - E_k N)f \in D(\Gamma)$, and by Corollary 3.2.17 $f \in D(\Gamma)$. Moreover, from Theorem 3.2.3 we get

$$a^{±}a^{−}E_k(\lambda - E_k N)(\Gamma_k f) = \Gamma_k g.$$

If $k \notin S_\chi$, then by injectivity $\Gamma_k g = 0$ implies $\Gamma_k f = 0$. \qed
Proof of Theorem 3.2.1. We only consider the normal BVP (3.12). The tangential BVP (3.13) is treated similarly.

Essentially, this is the special case \( \lambda = 1 \) in the previous proof. However, there are some differences since we use only half the operator \( I - E_k N \), acting in the space \( N(\Gamma_k, \Lambda^j) \) from Definition 3.2.4, for the interior BVP. From Theorem 3.2.15 it follows that

\[
I - E_k N : \mathcal{N}(\Gamma_k, \Lambda^j) \longrightarrow \mathcal{N}(\Gamma_k, \Lambda^j)
\]  
(3.15)

is a Fredholm operator with index zero, and so Lemma 3.2.13 and the observation that

\[
E_k^\pm (I - E_k N) = N^- E_k^+ - N^+ E_k^-
\]

give that

\[
N^- : E_k^+ \mathcal{N}(\Gamma_k, \Lambda^j) \longrightarrow N^\pm \mathcal{N}(\Gamma_k, \Lambda^j)
\]  
(3.16)

is a Fredholm operator. Indeed, we see that \( \mathcal{N}(N^- E_k^+ - N^+ E_k^-) = \mathcal{N}(N^- E_k^+) \oplus \mathcal{N}(N^+ E_k^-) \) and \( \mathcal{R}(N^- E_k^+ - N^+ E_k^-) = \mathcal{R}(N^- E_k^+) \oplus \mathcal{R}(N^+ E_k^-) \). This proves that the BVP (3.12) is well posed in Fredholm sense.

(i) Let \( S_1 = S_{\lambda, 1} \subset \{ k \mid \text{Im} k \leq 0 \} \) be the (minimal) discrete set from Proposition 3.1.11. It follows that (3.15) is an isomorphism when \( k \notin S_1 \), and so is (3.16). We will now show that (3.16) is an isomorphism also when \( \text{Im} k < 0 \). Thus the discrete set \( S_{1+} := S_1 \cap \mathbb{R} \) will have the desired properties. Using perturbation theory as in Theorem 3.1.4, it follows that for any \( k \),

\[
N^- : E_k^+ \mathcal{D}(\Gamma) \longrightarrow N^\pm \mathcal{D}(\Gamma)
\]  
(3.17)

is a Fredholm operator with index zero. We claim that if \( \text{Im} k \neq 0 \), then (3.17) is injective and thus an isomorphism. To see this, assume \( N^- f = 0 \), \( f \in E_k^+ \mathcal{D}(\Gamma) \) and take real part of (3.8) to get

\[
\text{Im} k \int_{\Omega^+} |F|^2 = \text{Re} \int_{\Sigma} (\overline{\nu} N^- f, \nu N^+ f^c) = 0,
\]

and therefore \( f = 0 \). In particular (3.16) is injective when \( \text{Im} k < 0 \). To show that (3.16) is also surjective when \( \text{Im} k < 0 \), let \( g \in N^- \mathcal{N}(\Gamma_k, \Lambda^j) \subset N^\pm \mathcal{D}(\Gamma) \). Then since (3.17) is an isomorphism, there exists \( f \in E_k^+ \mathcal{D}(\Gamma) \) such that \( N^- f = g \), and thus by Theorem 3.2.3 we have \( N^- (\Gamma_k f) = \Gamma_k g = 0 \). Since (3.17) is injective, \( \Gamma_k f = 0 \). The j-vector part \( f_j \) of \( f \) obviously solves \( N^- f_j = g \) and Proposition 3.2.5 shows that \( f_j \in E_k^+ \mathcal{N}(\Gamma_k, \Lambda^j) \).

(ii) As in the proof of Theorem 3.2.2, we deduce from Corollary 3.2.17 and Theorem 3.2.3 that \( f \in \mathcal{D}(\Gamma) \) if \( N^- f = E_k^\pm (I - E_k N) f \in \mathcal{D}(\Gamma) \).

(iii) For \( k = 0 \), we see that \( N^- f = g \), \( f \in E^+ L_2 \) and \( \Gamma g = 0 \) implies \( N^- (\Gamma f) = 0 \). We need to show that \( \mathcal{N}(N^-) \cap \mathcal{R}(E^+) = \{ 0 \} \). For this, use (3.9) in Lemma 3.1.7 to get

\[
\int_{\Omega^+} |dF|^2 = \int_{\Sigma} (\Gamma g, \nu \wedge f^c) = 0,
\]

which implies \( dF = 0 \), i.e. \( \Gamma f = 0 \).  

We now investigate the parts \( \Gamma_k^{N^\pm} \) and \( \Gamma_k^{E^\pm} \) of the exterior/interior derivative operator acting in the four subspaces \( N^\pm L_2 \) and \( E_k^\pm L_2 \) respectively. An important observation is
that the reflection operator $N$ is naturally associated with the differential operators $d$ and $\delta$. To see this, let $\chi_+$ be (multiplication with) the characteristic function of $\Omega^+$ and let

$$\iota_* : L^2(\Sigma; \lambda) \longrightarrow \mathcal{D}'(\mathbb{R}^n; \lambda)$$

be the push forward map, i.e. $(\iota_* f, \phi) = \int f_\Sigma f, \phi \in C_0^\infty(\mathbb{R}^n; \lambda)$. In fact $\iota_*$ maps $L^2(\Sigma; \lambda)$ onto a Hilbert space $\mathcal{R}(\iota_*)$ which is isomorphically embedded in the Besov space $B^0_{2,\infty}(\mathbb{R}^n; \lambda)$. Indeed the range $\mathcal{R}(\iota_*)$ is exactly the distributions in $B^{-1/2}_{2,\infty}(\mathbb{R}^n; \lambda)$ which are supported on $\Sigma$. This is proved in Theorem 7.1.26-27 in Hörmander [24] for $C^1$ surfaces, but the proof goes through for strongly Lipschitz surfaces. Now if for example $F \in C^\infty(\mathbb{R}^n; \lambda)$, then the commutators

$$-\nu \perp \iota_*^{-1}([d, \chi_+] F) = N^+(F|_\Sigma) \in N^+L_2,$$

$$-\nu \wedge \iota_*^{-1}([\delta, \chi_+] F) = N^-(F|_\Sigma) \in N^-L_2,$$

are the tangential and normal traces of $F$ on $\Sigma$. To see the first identity for example, note that $[d, \chi_+] F = d(\chi_+ F) - \chi_+(dF)$. Here $d(\chi_+ F)$ is the measure $-\nu \wedge \mu \sigma$ on $\Sigma$ plus the function $dF$ in $\Omega^+$, which the second commutator term cancels. Note that $[d, \chi_+] = [d, \chi_+]$ and $[\delta, \chi_+] = [\delta, \chi_+]$.

On the other hand, $E_k$ is also related to the differential operators $d$ and $\delta$ since $D_k = d_k + \delta_k$. The following proposition interprets the constraint $d_k F = \delta_k F = 0$ for $k$-monochromatic fields.

**Proposition 3.2.5.** Let $F$ be a $k$-monochromatic field in an open set $\Omega \subset \mathbb{R}^n$ and let $F_j : \Omega \rightarrow \Lambda^j$ be its homogeneous components, $F = \sum F_j$. Then the even part $F_{\text{even}} := \sum F_{2j}$ and the odd part $F_{\text{odd}} := \sum F_{2j+1}$ are both $k$-monochromatic. Furthermore, the following are equivalent.

(i) All the homogeneous components $F_j$ are $k$-monochromatic in $\Omega$.

(ii) $F$ is two-sided $k$-monochromatic in $\Omega$, i.e. $D_k F(\tilde{x}) = F(\tilde{x}) D_k = 0$.

(iii) $F$ satisfies the constraint $d_k F = \delta_k F = 0$ in $\Omega$.

The properties (i)-(iii) express the fact that the homogeneous components of $F$ are decoupled.

**Proof.** That $D_{k}F_{\text{even}} = D_{k}F_{\text{odd}} = 0$ follows from the fact that $D_k$ switches $\Lambda^{\text{even}} := \oplus \Lambda^{2j}$ and $\Lambda^{\text{odd}} := \oplus \Lambda^{2j+1}$. For the equivalences, observe that since $\perp$ is a $\triangle$-automorphism, $F D_k = 0$ if and only if

$$0 = F^\perp D_k = F^\perp (d_k + \delta_k) = (d_k - \delta_k) F.$$

Here we have used $F d_k := \sum_j (\partial_j F) \wedge e_j + kF \wedge e_0$ and $F \delta_k := \sum_j (\partial_j F) \wedge e_j + kF \wedge e_0$. Since $D_k F = (d_k + \delta_k) F$, this proves the equivalence of (ii) and (iii).

On the other hand, from the mapping properties $d_k : \Lambda^j \rightarrow \Lambda^{j+1}$ and $\delta_k : \Lambda^j \rightarrow \Lambda^{j-1}$, it follows that both (i) and (iii) are equivalent with $d_k F_j = 0 = \delta_k F_j$ for all $j$. \hfill \Box
We now define closed, nilpotent (i.e. \( R(T) \subset N(T) \)) operators in the four subspaces \( N^\pm L_2 \) and \( E^\pm_k L_2 \) of \( L_2(\Sigma; \wedge) \). For this, recall the following basic construction.

**Lemma 3.2.6.** If \( \mathcal{X} \) is a Banach space embedded in a Hausdorff topological linear space \( \mathcal{Y} \) and if \( T : \mathcal{X} \to \mathcal{Y} \) is a linear continuous operator defined on the whole of \( \mathcal{X} \), then the restriction of \( T \) with natural domain

\[
D(T) := \{ x \in \mathcal{X} : Tx \in \mathcal{X} \}
\]

is a closed operator in \( \mathcal{X} \).

Consider the maps

\[
\begin{align*}
E^\pm_k L_2 & \overset{C^\pm_k}{\longrightarrow} C^\pm_k L_2 \overset{i}{\longrightarrow} \mathcal{D}'(\Omega^\pm; \wedge), \\
N^\pm L_2 & \overset{\iota_*}{\longrightarrow} \mathcal{D}'(\mathbf{R}^n; \wedge).
\end{align*}
\]

Here \( C^\pm_k \) and \( \iota_* \) are Hilbert space isomorphisms and the natural inclusions \( i \) play the role of the embedding in Lemma 3.2.6. More explicitly, the maps \( C^\pm_k \) are as in Definition 1.2.8 and \( \iota_* \) are

\[
\begin{align*}
N^+ L_2 \ni f & \mapsto \iota_*(\nu \wedge f) \in \iota_*(\nu \wedge L_2), \\
N^- L_2 \ni f & \mapsto \iota_*(\nu \perp f) \in \iota_*(\nu \perp L_2).
\end{align*}
\]

We here intertwine with \( \nu \) in order to define the tangential \( d \) and \( \delta \) operators in \( N^\pm L_2 \) using the full \( d \) and \( \delta \) in \( \mathbf{R}^n \) acting in the distribution sense.

**Definition 3.2.7.** Using Lemma 3.2.6 we define the following four closed, nilpotent operators.

(i) The continuous operator \( d_k : \iota_*(\nu \wedge L_2) \to \mathcal{D}'(\mathbf{R}^n; \wedge) \) induces a closed operator in \( \iota_*(\nu \wedge L_2) \). In \( N^+ L_2 \) this corresponds to an operator \( \Gamma^N_k \) for which

\[
-\iota_*(\nu \wedge \Gamma^N_k(f)) = d_k(\iota_*(\nu \wedge f)), \quad f \in \mathcal{D}(\Gamma^N_k) \subset N^+ L_2.
\]

(ii) The continuous operator \( \delta_k : \iota_*(\nu \perp L_2) \to \mathcal{D}'(\mathbf{R}^n; \wedge) \) induces a closed operator in \( \iota_*(\nu \perp L_2) \). In \( N^- L_2 \) this corresponds to an operator \( \Gamma^-_k \) for which

\[
-\iota_*(\nu \perp \Gamma^-_k(f)) = -\delta_k(\iota_*(\nu \perp f)), \quad f \in \mathcal{D}(\Gamma^-_k) \subset N^- L_2.
\]

(iii) The continuous operator \( d_k = -\delta_k : C^\pm_k L_2 \to \mathcal{D}'(\Omega^\pm; \wedge) \) induces a closed operator in \( C^\pm_k L_2 \). In \( E^\pm_k L_2 \) this corresponds to an operator \( \Gamma^E_k \) for which

\[
C^\pm_k(\Gamma^E_k f) = d_k(C^\pm_k f) = -\delta_k(C^\pm_k f), \quad f \in \mathcal{D}(\Gamma^E_k) \subset E^\pm_k L_2.
\]

For an intrinsic characterisation of the operators \( \Gamma^{N^\pm}_k \), see Proposition 6.2.5.
Remark 3.2.8. Consider the weighted surface measure distribution $S = \nu \sigma = -\nabla \chi_{\Omega^+} \in \mathcal{D}'(\mathbb{R}^n; \Lambda^1)$, where $\chi_{\Omega^+}$ is the characteristic function of $\Omega^+$ and $\sigma$ is the scalar surface measure on $\Sigma$. Since $\nabla \wedge S = 0$, we get for any $\phi \in C^\infty(\mathbb{R}^n; \Lambda)$ that

\[
\nabla_k \wedge (\dot{\mathcal{S}} \wedge \dot{\phi}) = -S \wedge (\nabla_k \wedge \dot{\phi}) = -(S \wedge \nabla_k) \wedge \dot{\phi},
\]

\[
\nabla_k \wedge (\dot{\mathcal{S}} \wedge \dot{\phi}) = -S \wedge (\nabla_k \wedge \dot{\phi}) = +(S \wedge \nabla_k) \wedge \dot{\phi},
\]

using the first associativity formulae in (1.2). Here we formally have $S \wedge \nabla_k = S \Delta \nabla_{k,\text{tan}}$. This is why (i) and (ii) above make sense as a tangential exterior (− interior) derivative.

To see that (iii) is a reasonable definition, i.e. the operator can be expected to be densely defined, note that $\mathbf{D}_k F = 0$ in $\Omega^\pm$ implies $\mathbf{D}_k d_k F = \delta_k \mathbf{D}_k F = 0$.

Since $\mu \mapsto \epsilon_0 \wedge \mu : \iota_* (\nu \wedge L_2) \rightarrow \iota_* (\nu \wedge L_2)$ and $\mu \mapsto \epsilon_0 \wedge \mu : \iota_* (\nu \wedge L_2) \rightarrow \iota_* (\nu \wedge L_2)$ are bounded operators, the domains $\mathbf{D}(\Gamma_k^{N^\pm})$ are actually independent of $k$.

Proposition 3.2.9. The domain $\mathbf{D}(\Gamma_k^{N^\pm})$ is dense in $N^\pm L_2$.

Proof. By using a mollifier as in Theorem 7.1.27 in Hörmander [24], it is straightforward to show that $[d, \chi_+]C^\infty(\mathbb{R}^n; \Lambda)$ is dense in $\iota_* (\nu \wedge L_2)$ in the $L_2(\Sigma)$-topology. But $d_k$ acts in $[d, \chi_+]C^\infty(\mathbb{R}^n; \Lambda)$ since

\[
d_k[d, \chi_+]\phi = -[d, \chi_+]d_k \phi.
\]

Similarly $[\delta, \chi_+]C^\infty(\mathbb{R}^n; \Lambda) \subset \mathbf{D}(\delta_k; \iota_* (\nu \wedge L_2))$ is dense. \hfill $\Box$

We now investigate the relation between the four Hilbert spaces $\mathbf{D}(\Gamma_k^{N^\pm})$ and $\mathbf{D}(\Gamma_k^{E^\pm})$.

Proposition 3.2.10. We have continuous embeddings

\[
\mathbf{D}(\Gamma_k^{E^+}) \hookrightarrow \mathbf{D}(\Gamma_k^{N^+}) \oplus \mathbf{D}(\Gamma_k^{N^-}), \quad \mathbf{D}(\Gamma_k^{N^\pm}) \hookrightarrow \mathbf{D}(\Gamma_k^{E^+}) \oplus \mathbf{D}(\Gamma_k^{E^-}),
\]

and moreover we have intertwining relations

\[
\Gamma_k^{N^+} (N^+ f) + \Gamma_k^{N^-} (N^- f) = \Gamma_k^{E^+} f, \quad f \in \mathbf{D}(\Gamma_k^{E^+}),
\]

\[
\Gamma_k^{E^+} (E^+ f) + \Gamma_k^{E^-} (E^- f) = \Gamma_k^{N^\pm} f, \quad f \in \mathbf{D}(\Gamma_k^{N^\pm}).
\]

Proof. (1) Assume $F \in C^\infty_k L_2$ and $d_k F \in C^\infty_k L_2$ and that $F|_{\Sigma} = f = N^+ f + N^- f$ on $\Sigma$. We need to show that $d_k \iota_* (\nu \wedge f) \in \iota_* (\nu \wedge L_2)$. For arbitrary $\phi \in C^\infty(\mathbb{R}^n; \Lambda)$, applying the special case (1.19) of the boundary theorem twice shows that

\[
(d_k \iota_* (\nu \wedge f), \phi) = \int_{\Sigma} (\nu \wedge f, -\delta \phi + k \epsilon_0 \wedge \phi)
\]

\[
= \pm \int_{\Omega^\pm} (dF, -\delta \phi + k \epsilon_0 \wedge \phi) + (F, 0 - k \epsilon_0 \wedge \delta \phi)
\]

\[
= \pm \int_{\Omega^\pm} (0 + k \epsilon_0 \wedge dF, \phi) - (dF + k \epsilon_0 \wedge F, \delta \phi)
\]

\[
= \int_{\Sigma} (-\nu \wedge (\Gamma_k^{E^w} f), \phi) = (\iota_* (\nu \wedge (\Gamma_k^{E^w} f)), \phi).
\]

61
Therefore $N^+ f \in \mathcal{D}(\Gamma_k^{N^+})$ and $\Gamma_k^{N^+}(N^+ f) = N^+ \Gamma_k^{E^+_k} f$. Similarly we show $N^- f \in \mathcal{D}(\Gamma_k^{N^-})$ and $\Gamma_k^{N^-}(N^- f) = N^- \Gamma_k^{E^-_k} f$.

(2) Assume $\mathcal{D}(\Gamma_k^{N^+}) \ni f = E^+_k f + E^-_k f$ with $E^+_k f = F^\pm|_\Sigma$ and $F^\pm \in C^+_k L_2$. We have $d_k\iota_*(\nu \wedge f) \in \iota_*(\nu \wedge L_2)$, and if $x \in \Omega^\pm$ then

$$F^\pm(x) = \pm \int_{\Sigma} \langle \overrightarrow{E}_k(y - x), (\nu \wedge f(y)) \rangle \sigma(y),$$

$$d_k \langle \overrightarrow{E}_k(y - x), (\nu \wedge f(y)) \rangle = - \langle \overrightarrow{E}_k(y - x), ((\nabla + k e_0) \wedge \nu \wedge f(y)) \rangle,$$

where we have used the derivation property (1.8) and that $(\nabla + k e_0) \perp \overrightarrow{E}_k(y - x) = \delta_k \overrightarrow{E}_k(y - x) = (\Delta + k^2)\Phi_k(x - y) = 0$ for $y \neq x$. We need to show that $d_k F^\pm \in C^\pm L_2$. For arbitrary fixed $w \in \Lambda$ and $x \in \Omega^\pm$ we get

$$\pm(w, d_k F^\pm(x)) = - \left( w, \int_{\Sigma} \langle \overrightarrow{E}_k(y - x), ((\nabla + k e_0) \wedge \nu \wedge f(y)) \rangle \sigma(y) \right),$$

where we have on the second last line paired a function, $C^\infty$ in a neighbourhood of $\Sigma$, with a $\Sigma$-supported distribution. This shows that $f^\pm = E^\pm_k f \in \mathcal{D}(\Gamma_k^{E^\pm_k})$ and $\Gamma_k^{E^\pm_k} E^\pm_k f = E^\pm_k \Gamma_k^{N^\pm} f$. The proof for $f \in \mathcal{D}(\Gamma_k^{N^-})$ is similar.

We now want to piece together these four Hilbert spaces and operators. For this, we use the following abstract lemma. The proof is straightforward and we omit it.

**Lemma 3.2.11.** Let $X^+_1, X^+_2 \hookrightarrow V$ be two pairs of Banach spaces with $X^+_1 \cap X^-_1 = \{0\}$; embedded in a vector space $\mathcal{V}$. If we have continuous embeddings $X^+_1 \hookrightarrow X^+_1 + X^-_2$ and $X^+_2 \hookrightarrow X^+_1 + X^-_1$, then the Banach space direct sums $X^+_1 + X^-_1$ and $X^+_1 + X^-_2$ are equal both as sets and as Banach spaces with equivalent norms.

**Proof of Theorem 3.2.3.** Define the direct sum $\Gamma_k := \Gamma_k^{N^+_k} \oplus \Gamma_k^{N^-}$. Since $\mathcal{D}(\Gamma_k) = \mathcal{D}(\Gamma_k^{N^+_k}) \oplus \mathcal{D}(\Gamma_k^{N^-})$ and $N = I \oplus (-I)$ in this splitting, we get $\Gamma_k N = N \Gamma_k$. Since $\mathcal{D}(\Gamma_k^{N^+_k}) = \mathcal{D}(\Gamma_k^{N^-})$, it follows that $\mathcal{D}(\Gamma_k) = \mathcal{D}(\Gamma_{k'})$. Proposition 3.2.9 shows that $\mathcal{D}(\Gamma)$ is dense in $L_2(\Sigma; \Lambda)$. Proposition 3.2.10 and Lemma 3.2.11 show that $\mathcal{D}(\Gamma_k) = \mathcal{D}(\Gamma_k^{E^+_k}) \oplus \mathcal{D}(\Gamma_k^{E^-_k})$ and that $\Gamma_k(f^+ + f^-) = \Gamma_k^{E^+_k} f^+ + \Gamma_k^{E^-_k} f^-$ if $f^\pm \in E^\pm_k L_2$. In the same way $\Gamma_k E_k = E_k \Gamma_k$ now follows.

Thus, replacing the space $L_2(\Sigma; \Lambda)$ with $\mathcal{D}(\Gamma)$, we can use the reflection operators $E_k$ and $N$ to solve the transmission problem in this space using the formalism from Chapter 2.
However, before discussing the spectral estimates in this space, we point out an important
difference. Even though both $E_k$ and $E_{k'}$ act boundedly in $D(\Gamma_k) = D(\Gamma_{k'})$, due to the
mixed 0 and 1 regularity of the space $D(\Gamma)$, the difference $E_k - E_{k'}$ will no longer be a
compact operator when $k \neq k'$ as the following example shows.

**Example 3.2.12.** We consider $\Sigma = \mathbb{R}^2 \subset \mathbb{R}^3$ and the operator $\Gamma N^+(E_k - E)N^-$, where
$\Gamma := \Gamma_0$, acting on $\nu \phi \in \nu \cap L_2(\Sigma; \Lambda^0) \subset \mathcal{N}(\Gamma_k^-)$. The last inclusion holds since $\delta \mid_{\nu \phi} = 0$.
Proposition 1.2.5 shows that

$$E_k(x) - E(x) = k e_0 \Phi(x) + O(1),$$

and we have

$$N^+ L_2 \ni (E_k - E)(\nu \phi)(x) = 2k e_0 \Phi(\phi)(x) + \ldots$$

$$\Gamma N^+(E_k - E)(\nu \phi) = -2k e_0 \nabla_{\text{tan}} \Phi(\phi)(x) + \ldots$$

So if $E_k - E$ where compact on $D(\Gamma)$, this would imply that $\nabla_{\text{tan}} \Phi$ is compact $L_2(\Sigma; \Lambda^0) \to L_2(\Sigma; \Lambda)$, which is not true. Hence $E_k - E$ is not compact on $D(\Gamma)$.

We have seen that $E_k$ and $N$ act in $D(\Gamma_k)$ and in $\mathcal{N}(\Gamma_k)$. Moreover, we have the following.

**Lemma 3.2.13.** The operator $N$ preserves $L_2(\Sigma; \Lambda^j)$ for any $j$, while

$$E_k L_2(\Sigma; \Lambda^j) \subset L_2(\Sigma; \Lambda^{j-2} \oplus \Lambda^j \oplus \Lambda^{j+2}).$$

However, if $f \in \mathcal{N}(\Gamma_k, \Lambda^j)$ from Definition 3.2.4, then

$$E_k f(x) = 2p.v. \int_{\Sigma} E_k(y - x) \perp (\nu \wedge f(y)) + E_k(y - x) \wedge (\nu \perp f(y)) \in \mathcal{N}(\Gamma_k, \Lambda^j).$$

**Proof.** It follows from Definitions 1.2.3 and 3.2.7 that if $f \in N^+ \mathcal{N}(\Gamma_k, \Lambda^j)$, then

$$\pm (C_k^\pm f(x))(j+2) = \int_{\Sigma} \nabla_k (\Phi_k(x - \hat{y})) \wedge (\nu \wedge f)(y)$$

$$= - \int_{\Sigma} \Phi_k(x - y) \nabla_k \wedge (\nu \wedge f)(\hat{y}) = \int_{\Sigma} \Phi_k(x - y) (\nu \wedge \Gamma_k f)(y) = 0.$$  

Taking the boundary traces of these two identities and the corresponding calculation for $f \in N^- \mathcal{N}(\Gamma_k, \Lambda^j)$ now proves the lemma. \hfill $\square$

**Lemma 3.2.14.** Let $A_i$ be a closed operator in a Banach space $\mathcal{X}_i$, $i = 1, 2$, and assume
that the bounded operator $T : \mathcal{X}_1 \to \mathcal{X}_2$ is such that

$$T A_1 \subset A_2 T.$$

Then $T : D(A_1) \to D(A_2)$ is bounded and if $T : \mathcal{X}_1 \to \mathcal{X}_2$ has (exact) a priori estimates, then the restriction $T : D(A_1) \to D(A_2)$ also has (exact) a priori estimates.

63
Proof. Let the a priori estimates be \( \|f\|_{X_1} \lesssim \|Tf\|_{X_2} + \|Kf\|_{Y} \), where \( K : X_1 \to Y \) is compact. Then the operator \( \widetilde{K} : D(A_1) \to Y \oplus Y \) defined by \( \widetilde{K} f := (Kf, KA_1 f) \) is compact and we have the a priori estimate

\[
\|f\|_{D(A_1)} = \|f\|_{X_1} + \|A_1 f\|_{X_1} \lesssim \|Tf\|_{X_2} + \|Kf\|_{Y} + \|TA_1 f\|_{X_2} + \|KA_1 f\|_{Y} \\
= \|Tf\|_{X_2} + \|A_2 T f\|_{X_2} + \|\widetilde{K} f\|_{Y \oplus Y} = \|Tf\|_{D(A_2)} + \|\widetilde{K} f\|_{Y \oplus Y}.
\]

Furthermore \( K = 0 \) implies \( \widetilde{K} = 0 \). \qed

**Theorem 3.2.15.** The rotation operator \( E_k N \) acts boundedly in \( L_2(\Sigma; \wedge) \), in \( D(\Gamma) \), in \( N(\Gamma_k) \) and in \( M(\Gamma_k, \wedge^j) \). Let \( \mathcal{X} \) be either \( L_2(\Sigma; \wedge) \), \( D(\Gamma) \), \( N(\Gamma_k) \) or \( M(\Gamma_k, \wedge^j) \). Then we have

\[
\sigma_{ess}(E_k N; \mathcal{X}) \subset \{ \lambda = \lambda_1 + i \lambda_2 ; | \lambda_1 | \leq L_{\Sigma} | \lambda_2 | \}, \\
\sigma(E_k N; \mathcal{X}) \subset \{ \lambda = \lambda_1 + i \lambda_2 ; | \lambda_1 | \leq (L_{\Sigma} + C_{\Sigma,k}) | \lambda_2 | \}, \quad \text{Im} k > 0,
\]

where \( L_{\Sigma} < \infty \) is the local Lipschitz constant of \( \Sigma \) as in Definition 1.5.5 and

\[
C_{\Sigma,k} := (\text{Im} k)^{-1} \inf_{\theta} \{(1 + L_\theta^2)^{1/2} (| \text{Re} k | \| \theta \|_{\infty} + \frac{1}{2} \sum \| \theta_j \|_{\infty}) / e_\theta, \}
\]

where the infimum is taken over all smooth transversal vector fields \( \theta \) for \( \Sigma \).

If \( \text{Im} k > 0 \), then \( \lambda - E_k N \) is injective on \( D(\Gamma) \), \( N(\Gamma_k) \) and \( N(\Gamma_k, \wedge^j) \) if \( \lambda \notin (\overline{D}_\eta \cup D_{-\eta}) \setminus (D_\eta \cap D_{-\eta}) \), where \( \eta := | \text{Re} k | / \text{Im} k \) and \( D_\eta \) is the open disc \( D(\eta, \sqrt{1 + \eta^2}) \) centred at \( \eta \) and which touches \( \pm 1 \).

**Proof.** To prove the spectral estimates, note that the a priori estimates follow from Theorem 3.1.3, Theorem 3.1.10 and Lemma 3.2.14. Then apply Theorem 1.4.5.

To prove the estimate on the eigenvalues, assume \( (\lambda - E_k N) f = 0 \) where \( f \in D(\Gamma) \).

We may assume that \( f \in N(\Gamma_k) \), because Theorem 3.2.3 shows that \( (\lambda - E_k N) \Gamma_k f = 0 \) and hence \( \Gamma_k f \in N(\Gamma_k) \) by nilpotence. From Proposition 2.1.2 we see that \( (\lambda - E_k N) f = 0 \) is equivalent with

\[
\begin{cases}
N^+(f^+ - \alpha f^-) = 0, \\
N^-(\alpha f^+ - f^-) = 0,
\end{cases}
\]

where \( \lambda = (\alpha + 1)/(\alpha - 1) \) and \( f^\pm := E_k^\pm f \). Using these jump conditions in the Dirac form from Definition 6.1.1, it follows that \( \langle f^+, N^+ f^+ \rangle \varphi_0 = \langle N^- f^+, N^+ f^+ \rangle \varphi_0 = \frac{q^2}{\alpha} \langle N^- f^+, N^+ f^- \rangle \varphi_0 = \frac{q^2}{\alpha} \langle f^-, N^+ f^+ \rangle \varphi_0 \) and (3.8) in Lemma 3.1.7 implies that

\[
\int_{\Omega^+} k_2 |F^+|^2 + i k_1 (F^+, T F^+) + \frac{|k|^2}{\alpha} \int_{\Omega^-} k_2 |F^-|^2 + i k_1 (F^-, T F^-) = 0,
\]

where \( k =: k_1 + i k_2 \). Since the integrands take values in the sector \( | \text{arg} z | \leq \arctan \eta \) we must have \( F^+ = F^- = 0 \) if \( 2 \arctan \eta + \text{arg}(\alpha^2) < \pi \). This translates to the stated eigenvalue estimate for \( \lambda = (\alpha + 1)/(\alpha - 1) \). \qed

Theorem 3.2.15 shows that in the double sector \( |\text{Re} \lambda| > L_{\Sigma} |\text{Im} \lambda| \), the operator \( \lambda - E_k N \) is Fredholm with index zero on both spaces \( D(\Gamma) \subset L_2(\Sigma; \wedge) \). The following algebraic result shows that this implies regularity.
Proposition 3.2.16. Assume we have Banach spaces $\tilde{X} \hookrightarrow X$ and $\tilde{Y} \hookrightarrow Y$, where the inclusion $\tilde{Y} \hookrightarrow Y$ is dense. Further assume that $T : X \rightarrow Y$ is a bounded Fredholm operator which restricts to $\tilde{T} : \tilde{X} \rightarrow \tilde{Y}$ which is also a bounded Fredholm operator. Denote the dimensions of the null spaces by $\alpha(T)$ and $\alpha(\tilde{T})$, the dimensions of the co-kernels by $\beta(T)$ and $\beta(\tilde{T})$ and the indices by $\iota(T) := \alpha(T) - \beta(T)$ and $\iota(\tilde{T}) := \alpha(\tilde{T}) - \beta(\tilde{T})$. Then $\Delta := \iota(T) - \iota(\tilde{T}) \geq 0$,
\[
\alpha(T) - \Delta \leq \alpha(\tilde{T}) \leq \alpha(T), \quad \beta(T) \leq \beta(\tilde{T}) \leq \beta(T) + \Delta,
\]
$\mathcal{N}(\tilde{T}) \subset \mathcal{N}(T)$ and $T(\tilde{X}) \subset T(X) \cap \tilde{Y}$. Furthermore, the following are equivalent.

(i) $\Delta = 0$, i.e. $\iota(\tilde{T}) = \iota(T)$
(ii) $\alpha(\tilde{T}) = \alpha(T)$ and $\beta(\tilde{T}) = \beta(T)$
(iii) $\mathcal{N}(\tilde{T}) = \mathcal{N}(T)$ and $T(\tilde{X}) = T(X) \cap \tilde{Y}$
(iv) Regularity: $T(x) \in \tilde{Y}$ if and only if $x \in \tilde{X}$

A proof of this theorem can be found in McIntosh [37]. However, the theorem was found independently here and for completeness we include a proof.

Proof. Obviously $\mathcal{N}(\tilde{T}) \subset \mathcal{N}(T)$, thus $\alpha(\tilde{T}) \leq \alpha(T)$ and
\[
\beta(\tilde{T}) = \alpha(\tilde{T}) - \iota(\tilde{T}) \leq \alpha(T) - \iota(T) = \beta(T) + \Delta.
\]
Obviously $T(\tilde{X}) \subset T(X) \cap \tilde{Y}$. For the rest of the proof, let $\{y_j + T(X)\}_{j=1}^k$, where $k = \beta(T)$, be a basis for $Y/T(X)$. Since $\tilde{Y}$ is dense in $Y$, we may assume $y_j \in \tilde{Y}$. Now $T(\tilde{X}) \subset T(X)$ shows that $\{y_j + T(\tilde{X})\}_{j=1}^k$ are linear independent in $\tilde{Y}/T(\tilde{X})$, so $\beta(T) \leq \beta(\tilde{T})$ and
\[
\alpha(\tilde{T}) = \beta(\tilde{T}) + \iota(\tilde{T}) \geq \beta(T) + \iota(T) = \alpha(T) - \Delta
\]
and $\Delta \geq 0$.

For the last part, it follows from above that (i) and (ii) are equivalent.

Assume (ii). It follows that $\mathcal{N}(\tilde{T}) = \mathcal{N}(T)$ and that $\{y_j + T(X)\}_{j=1}^k$ is a basis for $\tilde{Y}/T(\tilde{X})$. Let $y \in T(\tilde{X}) \cap \tilde{Y}$. Since $y \in \tilde{Y}$ we may write $y = y' + \tilde{y}$, where $y'$ is in the span of $\{y_j\}$ and $\tilde{y} \in T(\tilde{X}) \subset T(X)$. Since $y' = y - \tilde{y} \in T(X)$ and $\{y_j + T(X)\}_{j=1}^k$ are linear independent in $\tilde{Y}/T(\tilde{X})$, it follows that $y' = 0$ and $y = \tilde{y} \in T(\tilde{X})$. Thus $T(\tilde{X}) = T(X) \cap \tilde{Y}$, so (iii) holds.

Now assume (iii). If $T(x) \in \tilde{Y}$, then we have $T(x) \in T(X) \cap \tilde{Y} = T(\tilde{X})$ and there exists $\tilde{x} \in \tilde{X}$ such that $x = \tilde{x} + N(T)$. But since $\mathcal{N}(T) = \mathcal{N}(\tilde{T}) \subset \tilde{X}$, this gives $x \in \tilde{X}$ and so (iv) holds.

Finally assume (iv). Then obviously $\alpha(T) = \alpha(\tilde{T})$. Take $y \in \tilde{Y} \subset Y$. Then $y = \sum \lambda_j y_j + T(x)$. But $T(x) \in \tilde{Y}$, so $x \in \tilde{X}$. Thus $\{y_j + T(\tilde{X})\}_{j=1}^k$ is a basis for $\tilde{Y}/T(\tilde{X})$ and $\beta(\tilde{T}) = \beta(T)$ follows, so (ii) holds.

We can now apply this theorem with $\tilde{X} = \tilde{Y} = D(\Gamma)$, $X = Y = L_2(\Sigma; \Lambda)$ and $T = \lambda - E_k N$ since this is a Fredholm operator with index zero on both spaces $L_2(\Sigma; \Lambda)$ and $D(\Gamma)$ when $|\text{Re} \lambda| > L_2|\text{Im} \lambda|$ according to Theorem 3.2.15.
Corollary 3.2.17. Assume that $|\text{Re} \lambda| > L_\Sigma \| \text{Im} \lambda \|$ and that

$$(\lambda - E_k N)f = g,$$

where $f \in L_2(\Sigma; \Lambda)$ and $g \in D(\Gamma)$. Then in fact $f \in D(\Gamma)$. 
Chapter 4

Hodge decompositions on weakly Lipschitz domains

In this chapter we discuss Hodge type decompositions of Hilbert spaces. It can be seen as part of the preliminaries since the results here essentially are well known. Two early references are Hodge [25] and Gaffney [21]; for further references see for example Mitrea–Mitrea [43]. However, some novelty is introduced by approaching the subject from a pure first order, operator theoretic point of view here. By first order we mean that the focus is on nilpotent operators (see Definition 4.1.1 below) like the exterior derivative $d$ and not on the Hodge–Laplace operator $\Delta$.

The main properties of a nilpotent operator $\Gamma$ are summarised in Proposition 4.1.7 and Proposition 4.2.3, where we discuss when and how to deduce whether $\Gamma$ induces a Hodge type splitting

$$\mathcal{H} = R(\Gamma) \oplus (N(\Gamma) \cap N(\Gamma^*)) \oplus R(\Gamma^*)$$

of the Hilbert space $\mathcal{H}$. Parallel with this general discussion we introduce and investigate the nilpotent $d$ and $\delta$ operators with and without $1/2$ boundary conditions. We use Proposition 4.2.3 to prove Hodge decompositions for these operators on a general bounded, weakly Lipschitz domain in the main Theorem 4.2.5, which will be used in Chapter 5. The important “reduction to the smooth case” technique in Lemma 4.2.4 was found independently here, but has also been observed by Picard [52]. We survey three different ways to deduce Hodge decomposition in this reduced case.

(i) The classical a priori estimate technique due to Gaffney, which gives optimal $W^1_2$ regularity in the smooth case.

(ii) The boundary integral method, which gives optimal regularity $W^{1/2}_2$ in the class of strongly Lipschitz domains by using Rellich estimates.

(iii) A path integral method for a star shaped domain. This method, which is based on the classical Poincare’s lemma, seems new. Although it does not give optimal regularity, it has the advantage of being entirely explicit.

I would like to thank Alan Mcintosh for teaching me Definition 4.1.2 and Proposition 4.1.7.
4.1 Nilpotent operators

**Definition 4.1.1.** A closed, densely defined operator $\Gamma$ in a Hilbert space $\mathcal{H}$ is nilpotent if $\mathcal{R}(\Gamma) \subset \mathcal{N}(\Gamma)$.

Standard examples of nilpotent operators are $\Gamma_1 = d_{\mathcal{H}}$ and $\Gamma_2 = d_{\Omega}$ with adjoints $\Gamma_1^* = -\delta_{\Omega}$, $\Gamma_2^* = -\delta_{\mathcal{H}}$, as defined below. Unless otherwise stated, we assume in this chapter that $\Omega = \Omega^+$ is a bounded, weakly Lipschitz domain.

We also use the bounded nilpotent operator $\Gamma_0 = \varepsilon_0 \wedge (\cdot)$ with adjoint $\Gamma_0^* = \varepsilon_0 \vee (\cdot)$, which give zero order perturbations $\Gamma_1 + k\Gamma_0 = d_{\mathcal{H}}$ and $\Gamma_2 + k\Gamma_0 = d_{\Omega}$, where $k \in \mathbb{C}$. Note that these are nilpotent as well, since $\Gamma_i \Gamma_0 + \Gamma_0 \Gamma_i = 0$.

**Definition 4.1.2.** (i) Let $d_{\Omega}$ and $\delta_{\Omega}$ be the closed, nilpotent $d$ and $\delta$ operators (without boundary conditions) in $L_2(\Omega; \wedge)$ with natural domains, i.e.
\[ D(d_{\Omega}) := \{ F \in L_2(\Omega; \wedge) \mid dF \in L_2(\Omega; \wedge) \}, \]
and similarly for $\delta_{\Omega}$.

(ii) Let $d_{\mathcal{H}}$ ($d$ with normal boundary conditions) and $\delta_{\mathcal{H}}$ ($\delta$ with tangential boundary conditions) be the closed, nilpotent $d$ and $\delta$ operators in $\Omega$ with domains
\[ D(d_{\mathcal{H}}) := \{ F \in L_2(\Omega; \wedge) \mid d(F_0) \in L_2(R^n; \wedge) \}, \]
\[ D(\delta_{\mathcal{H}}) := \{ F \in L_2(\Omega; \wedge) \mid \delta(F_0) \in L_2(R^n; \wedge) \}, \]
where $F_0 \in L_2(R^n; \wedge)$ denotes the zero-extension of $F$ to $R^n$.

Formally we have $d(F_0) = dF|_\Omega - (\nu \wedge f)\sigma$ and $\delta(F_0) = \delta F|_\Omega - (\nu \vee f)\sigma$, where $\sigma$ denotes the surface measure on $\Sigma$. Note that the boundary conditions for $d_{\mathcal{H}}$ and $\delta_{\mathcal{H}}$ are half boundary conditions, since only the normal/tangential part respectively of $F$ is required to vanish on $\Sigma$. Obviously we have
\[ d_{\mathcal{H}} \subset d_{\Omega}, \quad \delta_{\mathcal{H}} \subset \delta_{\Omega}. \]

To see that $d_{\mathcal{H}}$ is a closed operator, consider the closed, unbounded operator $d_{R^n}$ on $R^n$ and identify $L_2(\Omega; \wedge)$ with the closed subspace $\{ F \in L_2(R^n; \wedge) \mid F|_{\Omega^-} = 0 \}$. Then we have a closed graph
\[ \mathcal{G}(d_{\mathcal{H}}) = \mathcal{G}(d_{R^n}) \cap (L_2(\Omega; \wedge) \oplus L_2(\Omega; \wedge)). \]

**Proposition 4.1.3.** The operators $d_{\Omega}$ and $d_{\mathcal{H}}$ have cores (i.e. a subset of the domain which is dense in graph norm)
\[ C_0^\infty(R^n; \wedge)|_\Omega \subset D(d_{\Omega}), \quad C_0^\infty(\Omega; \wedge) \subset D(d_{\mathcal{H}}) \]
respectively. In particular, the inclusions $d(C_0^\infty(R^n; \wedge)|_\Omega) \subset \mathcal{R}(d_{\Omega})$ and $d(C_0^\infty(\Omega; \wedge)) \subset \mathcal{R}(d_{\mathcal{H}})$ are dense. We also have dense subspaces
\[ \{ F|_\Omega \mid F \in C_0^\infty(R^n; \wedge), \supp dF \subset \Omega^- \} \subset \mathcal{M}(d_{\Omega}), \]
\[ \{ F \in C_0^\infty(\Omega; \wedge) \mid dF = 0 \} \subset \mathcal{M}(d_{\mathcal{H}}). \]
The same holds true when $d$ is replaced by $\delta$. 

68
Before giving the proof, we note the following important corollary.

**Corollary 4.1.4.** The two operator \( d_{\Omega} \) and \( -\delta_{\Omega} \) are adjoint in the sense of unbounded operators (see Definition 1.4.7) and so are \( d_{\Omega} \) and \( -\delta_{\Omega} \).

**Proof.** Consider the first pair. Assume \((U, F)^t \in \mathcal{G}(d_{\Omega}^\times)\). Then in particular
\[
\int_{\Omega} (U, d\phi) = \int_{\Omega} (F, \phi), \quad \text{for all } \phi \in C_0^\infty(\Omega) \subset D(d_{\Omega}).
\]

Thus \(-\delta_{\Omega} U = F \in L_2(\Omega)\), which proves \( d_{\Omega}^\times \subset -\delta_{\Omega} \). Conversely, assume \((U, -\delta_{\Omega} U)^t \in \mathcal{G}(-\delta_{\Omega})\). By Proposition 4.1.3 and the fact that the operators are closed, it is enough to consider \( U \in C_0^\infty(\mathbb{R}^n; \Lambda) \). Then for any \( G \in D(d_{\Omega}) \subset D(d_{\mathbb{R}^n})\)
\[
\int_{\Omega} (U, dG) = \int_{\mathbb{R}^n} (U, dG_0) = \int_{\mathbb{R}^n} (-\delta U, G_0) = \int_{\Omega} (-\delta_{\Omega} U, G),
\]

so \((U, -\delta_{\Omega} U)^t \in \mathcal{G}(d_{\Omega}^\times)\). \(\square\)

To prove Proposition 4.1.3, we use Lie flows \( t \mapsto \alpha_t^* \) and \( t \mapsto \tilde{\alpha}_t^{-1} \) constructed as follows.

**Lemma 4.1.5.** There exists a family \( \alpha_t : \mathbb{R}^n \to \mathbb{R}^n \) of bilipschitz maps (all being identity outside some ball), \( |t| < T = T(\Omega) \), with the following properties.

\[
\begin{align*}
\alpha_t(\Omega) &\subset \Omega, \quad 0 < t < T, \\
\alpha_t(\Omega) &\supset \overline{\Omega}, \quad -T < t < 0,
\end{align*}
\]

\[
\| \alpha_t^* F - F \|_{L_2(\mathbb{R}^n)} \to 0, \quad t \to 0, \quad F \in L_2(\mathbb{R}^n),
\]

\[
\| \tilde{\alpha}_t^{-1} F - F \|_{L_2(\mathbb{R}^n)} \to 0, \quad t \to 0, \quad F \in L_2(\mathbb{R}^n).
\]

**Proof.** We use \( N \) local parametrisations \( \rho_j : U_j \to V_j \) and smaller open sets \( U_j^0 \subset U_j \) such that \( \Sigma \subset \bigcup_{j=1}^N V_j^0 \), \( V_j^0 := \rho_j(U_j^0) \). Let \( \beta^j_t : U_j \to U_j \) be a smooth invertible map such that
\[
\beta^j_t = \begin{cases} 
\text{translation with } t e_n \text{ on } U_j^0, \\
\text{identity outside a compact subset of } U_j
\end{cases}
\]

and such that \( \beta^j_t(U_j \cap \mathbb{R}^n_+) \subset \mathbb{R}^n_+ \) depending on the sign of \( t \). Now define \( \alpha_t^j := \rho_t \circ \beta_t^j \circ \rho_j^{-1} \), extend to identity outside \( V_j \) and let \( \alpha_t := \alpha_t^1 \circ \cdots \circ \alpha_t^N \). Obviously the required mapping properties for \( \alpha_t \) holds.

For the Lie pullback flow, if \( \alpha_t^* F = (\alpha_t^j)^* G \) where \( G := (\alpha_t^2)^* \cdots (\alpha_t^N)^* F \) then
\[
\| \alpha_t^j F - F \|_{L_2(\mathbb{R}^n)} \leq \| (\alpha_t^j)^* G - F \|_{L_2(V_j)} + \| (\alpha_t^j)^* G - F \|_{L_2(V_j)}
\]
\[
\leq \| (\alpha_t^j)^* G - G \|_{L_2(V_j)} + \| G - F \|_{L_2(V_j)} + \| G - F \|_{L_2(V_j)}.
\]

Thus it suffices to show that \( \| (\alpha_t^j)^* F - F \|_{L_2(V_j)} \to 0 \). But (essentially) since translation is \( L_2 \) continuous, this follows from
\[
\| (\alpha_t^j)^* F - F \|_{L_2(V_j)} \lesssim \| (\beta_t^j)^* (\rho_j^* F) - (\rho_j^* F) \|_{L_2(V_j)} \to 0.
\]

The proof of the \( L_2 \)-continuity of the reduced push forward flow is similar. \(\square\)
Note that if $\Sigma$ is a strongly Lipschitz interface, then we may use the flow of the Rellich field from Proposition 1.5.4 as $\alpha_t$.

Proof of Proposition 4.1.3. Let $\eta_s(x) := s^{-n} \eta(x/s)$ be a mollifier. For $F \in \mathcal{D}(d_\Omega)$, let $0 < s << t$ and define approximating fields

$$F_{s,t} := \eta_s \ast (\alpha_t F_0).$$

Then $\|F_{s,t} - F\|_{\mathcal{D}(d_\Omega)} \approx \|F_{s,t} - F\|_{L^2} + \|(dF)_{s,t} - dF\|_{L^2} \to 0$ as $s, t \to 0$. Furthermore, if $d_\Omega F = 0$ and $0 < s << t$, then $dF_{s,t} = 0$ in a neighbourhood of $\overline{\Omega}$, due to Proposition 1.2.2.

On the other hand for $F \in \mathcal{D}(d_{\Omega'})$, let $0 < s << -t$ and define approximating fields $F_{s,t}$ as above. Then $F_{s,t} \in C_0^\infty(\Omega)$ if $0 < s << -t$ and $\|F_{s,t} - F\|_{\mathcal{D}(d_{\Omega'})} \to 0$ as $s, t \to 0$. Furthermore, if $d_{\Omega'} F = 0$ then $d_{\Omega'} F_{s,t} = 0$.

Example 4.1.6. Let $a \in \Lambda^1$ be a constant vector, and consider the vector field

$$F_a(x) := \begin{cases} a & x_n > 0, \\ 0 & x_n < 0, \end{cases}$$

locally around $x = (x', x_n) = 0$. Then $F_a \in \mathcal{D}(\delta)$ if and only if $a$ is normal to $\mathbb{R}^{n-1}$, and on the other hand $F_a \in \mathcal{D}(\delta)$ if and only if $a$ is tangential to $\mathbb{R}^{n-1}$.

In particular, the normal part of the trace $F|_{\mathbb{R}^{n-1}}$ is in general not well defined for $F \in \mathcal{D}(\delta)$, and the tangential part of the trace is in general not well defined for $F \in \mathcal{D}(\delta)$.

Also note that we cannot have full local regularity for $F \in \mathcal{N}(d)$ or $\mathcal{N}(\delta)$ since we have extension maps by Proposition 4.1.9 which (essentially) commute with $d$ or $\delta$ respectively.

The basic property of the four $d$ and $\delta$ operators, is that they are “half-elliptic”. In fact, any nilpotent operator $\Gamma$ is “at most half elliptic”, or more precisely $\Gamma : \mathcal{N}(\Gamma) \perp \to \mathcal{N}(\Gamma)$ is injective. Therefore it is convenient to also consider the sum $\Gamma + \Gamma^*$. 

Proposition 4.1.7. Let $\Gamma$ be a nilpotent operator in a Hilbert space $\mathcal{H}$. Then

$$R(\Gamma) \subset \overline{R(\Gamma)} \subset \mathcal{N}(\Gamma) \subset D(\Gamma).$$

Fixing a scalar product on $\mathcal{H}$, we have an adjoint operator $\Gamma^*$ with the same properties as $\Gamma$ above. Associated with the operator $\Gamma$ are the splittings

$$\mathcal{H} = \overline{R(\Gamma)} \oplus \mathcal{N}(\Gamma^*) = \mathcal{N}(\Gamma) \oplus \overline{R(\Gamma^*)} = \overline{R(\Gamma)} \oplus (\mathcal{N}(\Gamma) \cap \mathcal{N}(\Gamma^*)) \oplus \overline{R(\Gamma^*)}$$

(4.1)

of $\mathcal{H}$ into closed orthogonal subspaces.

Since $\Gamma$ is nilpotent, it preserves its domain and acts boundedly $\Gamma : D(\Gamma) \to D(\Gamma)$. Similarly $\Gamma^* : D(\Gamma^*) \to D(\Gamma^*)$ is bounded. The domains split

$$D(\Gamma) = \overline{R(\Gamma)} \oplus (\mathcal{N}(\Gamma) \cap \mathcal{N}(\Gamma^*)) \oplus (D(\Gamma) \cap \overline{R(\Gamma^*)}),$$

$$D(\Gamma^*) = (D(\Gamma^*) \cap \overline{R(\Gamma)}) \oplus (\mathcal{N}(\Gamma) \cap \mathcal{N}(\Gamma^*)) \oplus \overline{R(\Gamma^*)}.$$
For each $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, the Dirac type operator $\Pi_\alpha$ is

$$
\Pi_\alpha := \Gamma + \alpha \Gamma^* = \begin{bmatrix} 0 & 0 & \Gamma \\ 0 & 0 & 0 \\ \alpha \Gamma^* & 0 & 0 \end{bmatrix}
$$

with domain

$$
D(\Gamma^*) \cap D(\Gamma) = \left( D(\Gamma^*) \cap \overline{R(\Gamma)} \right) \oplus \left( N(\Gamma) \cap N(\Gamma^*) \right) \oplus \left( D(\Gamma) \cap \overline{R(\Gamma^*)} \right).
$$

This Dirac type operator is a closed, densely defined, unbounded operator in $\mathcal{H}$ and it is normal, in fact $(\Pi_\alpha)^* = \alpha \Pi_\alpha$, and has null space $N(\Pi_\alpha) = N(\Gamma) \cap N(\Gamma^*)$ and range $R(\Pi_\alpha) = R(\Gamma) \oplus R(\Gamma^*)$. In particular $\Pi_{\pm 1}$ are self/skew adjoint (in the sense of unbounded operators).

**Proof.** We focus here on the splitting of $D(\Gamma)$. Surely the inclusion $\subset$ holds. For the opposite inclusion, decompose $x \in D(\Gamma)$ with (4.1), which follows from Theorem 1.4.8, as $x = x_1 + x_0 + x_2$. Since $x_1 \in \overline{R(\Gamma)} \subset D(\Gamma)$ and $x_0 \in N(\Gamma) \cap N(\Gamma^*) \subset D(\Gamma)$ we deduce that $x_2 = x - x_1 - x_0$ also belongs to $D(\Gamma)$. Similarly for $D(\Gamma^*)$. It follows from these decompositions that $D(\Gamma) \cap N(\Gamma) = D(\Gamma^*) \cap N(\Gamma^*)$ and $D(\Pi_\alpha) \subset \mathcal{H}$ are dense inclusions. That $\Pi_\alpha$ is closed and that the duality $(\Pi_\alpha)^* = \alpha \Pi_\alpha$ hold is now a consequence of the decomposition (4.2). \[\square\]

**Example 4.1.8.** (i) If $\Gamma = \frac{d}{\partial \eta}$, then the Dirac operator on $\Omega$ with normal boundary conditions is

$$
D_{\Omega^\perp} := \delta_\Omega + \delta_\Omega.
$$

On the other hand, if $\Gamma = \frac{d}{\partial \eta}$, then the Dirac operator on $\Omega$ with tangential boundary conditions is

$$
D_{\Omega^\parallel} := \delta_\Omega + \delta_\Omega.
$$

Note that $D_{\Omega^\perp} = \frac{d}{\partial \eta} \delta_\Omega + \delta_\Omega \frac{d}{\partial \eta}$ is the Hodge–Laplace operator with relative (generalised Dirichlet) boundary conditions. Its domain is $D(D_{\Omega^\perp}) = \{ F \in D(d_{\partial \eta}) \cap D(\delta_\Omega) : d_{\partial \eta} F \in D(\delta_\Omega), \delta_\Omega F \in D(d_{\partial \eta}) \}$ and since $D_{\Omega^\perp}$ is skew-adjoint, Theorem 1.4.8 shows that $N(D_{\Omega^\perp}) = \overline{D(D_{\Omega^\perp})}$. Also, $D_{\Omega^\parallel} = d_\Omega \delta_\Omega + \delta_\Omega d_\Omega$ is the Hodge–Laplace operator with absolute (generalised Neumann) boundary conditions. Its domain is $D(D_{\Omega^\parallel}) = \{ F \in D(d_\Omega) \cap D(\delta_\Omega) ; d_\Omega F \in D(\delta_\Omega), \delta_\Omega F \in D(d_\Omega) \}$. 

We end this section with constructing extension operators for $d_\Omega$ and $\delta_\Omega$. These will be used in Chapter 5.

**Proposition 4.1.9.** The restriction operator $\chi_{d_\Omega} : D(d_{\mathbb{R}^n}) \to D(d_\Omega)$ has a bounded right inverse, an extension operator $\chi^{-1}_{d_\Omega} : D(d_\Omega) \to D(d_{\mathbb{R}^n})$, which essentially commutes with $d$ in the sense that $[d, \chi^{-1}_{d_\Omega}] : L^2(\Omega) \to L^2(\mathbb{R}^n)$ is bounded. If we consider the exterior algebra $\Lambda \mathbb{R}^{n+1}$ for spacetime, we can choose $\chi^{-1}_{d_\Omega}$ so that $\chi^{-1}_{d_\Omega}(f_1 + e_0 \wedge f_2) = \chi^{-1}_{d_\Omega} f_1 + e_0 \wedge \chi^{-1}_{d_\Omega} f_2$.

The same holds true when $d$ is replaced by $\delta$.
Proof. Let $\overline{\Omega} \subset \bigcup_{j=0}^N V_j$, supp $\eta_j \subset V_j$, $\sum_{j=0}^N \eta_j = 1$ and let $\rho_j : U_j \rightarrow V_j, j = 1 \ldots N$, be the local bilipschitz parametrisations from Definition 1.5.1 and $V_0 \subset\subset \Omega$ be contained in the interior.

(i) We first note that it suffices to construct an extension map $\chi^{-1} : D(d_{\Omega}) \rightarrow D(d_{\mathbb{R}^n})$ acting on fields supported in $\rho_j^{-1}\text{supp } \eta_j \cap \overline{\mathbb{R}^n_+}$. Indeed, from Proposition 1.2.2 we see that this gives local extension maps $\chi_j^{-1} : (\rho_j^{-1})^* D(d_{\bar{\Omega}}) \rightarrow D(d_{\bar{V}_j})$, extending fields supported in $\text{supp } \eta_j \cap \overline{\Omega}$ to fields compactly supported in $V_j$. Then we can construct $\chi^{-1}_{d\Omega}$ as

$$\chi^{-1}_{d\Omega}F := \eta_0 F + \sum_{j=1}^N \chi_j^{-1}(\eta_j F). \quad (4.3)$$

Moreover, from the construction of $\chi^{-1}$ below, $d$ commutes with $\chi_j^{-1}$ and thus

$$[d, \chi^{-1}_{d\Omega}]F = (d\eta_0) \wedge F + \sum_{j=1}^N \chi_j^{-1}((d\eta_j) \wedge F). \quad (4.4)$$

Clearly, both (4.3) and (4.4) define $L_2$-bounded operators.

(ii) To construct the extension map $\chi^{-1}$, consider the stretched reflections

$$r_k : (x', -x_n) \mapsto (x', kx_n).$$

By Proposition 4.1.3, it suffices to consider $G \in C_0^\infty(\mathbb{R}^n; \wedge)|_{\mathbb{R}^n_+}$. If we decompose $G(x) = G_1(x) + e_n \wedge G_2(x)$, $e_n \wedge G_i = 0$, into parts tangential and normal to $\mathbb{R}^{n-1}$, then the pullbacks are given by $r_k^* G(x', -x_n) = G_1(x', kx_n) - k e_n \wedge G_2(x', kx_n)$, and we see that both tangential and normal parts of the field

$$\tilde{G} := \begin{cases} G & x_n > 0, \\ 3r_1^* G - 2r_2^* G & x_n < 0, \end{cases}$$

are continuous across $\Sigma$. We can assume that $\text{supp } \eta_{ij}$ is small enough so that $\text{supp } \tilde{G} \subset U_j$ if $\text{supp } G \subset \rho_j^{-1}\text{supp } \eta_j$. Now define $\chi^{-1} := 3r_1^* - 2r_2^*$.

The proof for $\delta$ is analogous. We here use the reduced pushforwards $(r_k^{-1})F = -k F_1 + e_n \wedge F_2$. \hfill \square

Remark 4.1.10. (i) We see that in a natural way $D(d_{\Omega}) \subset D(d_{\mathbb{R}^n})$ and

$$D(d_{\Omega}) = D(d_{\mathbb{R}^n})/N(\chi_{d\Omega}).$$

Proposition 4.1.9 shows that $\chi_{d\Omega} : D(d_{\mathbb{R}^n}) \rightarrow D(d_{\Omega})$ is surjective and that $\chi^{-1}_{d\Omega} : D(d_{\Omega}) \rightarrow D(d_{\mathbb{R}^n})$ embeds $D(d_{\Omega})$ as a topological complement of $N(\chi_{d\Omega})$ in $D(d_{\mathbb{R}^n})$.

72
(ii) From expressions (4.3) and (4.4) we obtain norm estimates
\[
\|\varepsilon\|_{L_2(\Omega;\Lambda)} \lesssim 1 + \sum_{j=1}^N \left( \sup_{x \in \rho_j V_j} \frac{\|\rho_j(x)\|_{\mathcal{O}_p}}{\sqrt{\mathcal{J}(\rho_j)(x)}} \right) \left( \sup_{y \in V_j} \frac{\|\rho_j^{-1}(y)\|_{\mathcal{O}_p}}{\sqrt{\mathcal{J}(\rho_j^{-1})(y)}} \right),
\]
\[
\|d, \varepsilon\|_{L_2(\Omega;\Lambda)} \lesssim \|\nabla \eta\|_{\infty} + \sum_{j=1}^N \left( \sup_{x \in \rho_j V_j} \frac{\|\rho_j(x)\|_{\mathcal{O}_p}}{\sqrt{\mathcal{J}(\rho_j)(x)}} \right) \left( \sup_{y \in V_j} \frac{\|\rho_j^{-1}(y)\|_{\mathcal{O}_p}}{\sqrt{\mathcal{J}(\rho_j^{-1})(y)}} \right) \|\eta_j\|_{\infty},
\]
where \(\rho_j(x) : \Lambda \to \Lambda\) denotes the \(\Lambda\)-homomorphism from Proposition 1.1.5 and \(\mathcal{J}(\rho_j)\) is the Jacobian function.

### 4.2 Hodge decompositions

We now investigate when a nilpotent operator is maximal in the sense that it is “half elliptic”. More precisely we make the following definition.

**Definition 4.2.1.** Let \(\Gamma\) be a nilpotent operator in a Hilbert space \(\mathcal{H}\). We say that \(\Gamma\) is a **Fredholm-nilpotent operator** if the reduced operator
\[
\tilde{\Gamma} : \mathcal{H}/N(\Gamma) \to N(\Gamma)
\]
with domain \(D(\tilde{\Gamma}) := D(\Gamma)/N(\Gamma)\) is a Fredholm operator, i.e. the range \(R(\tilde{\Gamma}) = R(\Gamma)\) is a closed subspace of finite codimension in \(N(\Gamma)\).

**Definition 4.2.2.** An unbounded Fredholm operator \(T : \mathcal{H} \to \mathcal{K}\) is said to be **diffuse** if its Fredholm inverse (parametrix) \(T^{-1} : \mathcal{K} \to \mathcal{H}\) is compact, or equivalently if the embedding \(D(T) \subset \mathcal{H}\) is compact.

A Fredholm-nilpotent operator \(\Gamma\) is said to be diffuse if \(\tilde{\Gamma} : \mathcal{H}/N(\Gamma) \to N(\Gamma)\) is a diffuse Fredholm operator.

**Proposition 4.2.3.** Let \(\Gamma\) and \(\Pi_\alpha\) be as in Proposition 4.1.7. Then the following are equivalent.

(i) \(\Gamma\) is a Fredholm-nilpotent operator.

(i') \(\Gamma^*\) is a Fredholm-nilpotent operator.

(ii) \(\Pi_\alpha\) is a Fredholm operator.

When this holds, \(\Gamma\) induces a Hodge type decomposition (splitting) of \(\mathcal{H}\), i.e.
\[
\mathcal{H} = R(\Gamma) \oplus (N(\Gamma) \cap N(\Gamma^*)) \oplus R(\Gamma^*),
\]
where the ranges \(R(\Gamma)\) and \(R(\Gamma^*)\) are closed and \(N(\Gamma) \cap N(\Gamma^*)\) is finite dimensional. If in addition \(N(\Gamma) \cap N(\Gamma^*) = \{0\}\), then the splitting is said to be exact.

The equivalence of (i), (i') and (ii) remains true if “Fredholm(-nilpotent) operator” is replaced with “diffuse Fredholm(-nilpotent) operator”. In this case, we also have the following.
(iii) The spectrum $\sigma(\Pi_\alpha)$ is discrete.

(iv) If $\Gamma_0$ is a bounded, nilpotent operator such that $\Gamma \Gamma_0 + \Gamma_0 \Gamma = 0$, then the perturbed operator $\Gamma + \Gamma_0$ is also a diffuse Fredholm-nilpotent operator.

**Proof.** Split $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \oplus \mathcal{H}_2$, where

\[
\mathcal{H}_1 := \overline{R(\Gamma)} = N(\Gamma^*) = \mathcal{H}/N(\Gamma^*),
\]

\[
\mathcal{H}_0 := N(\Gamma) \cap N(\Gamma^*) \approx N(\Gamma)/R(\Gamma) \approx N(\Gamma^*)/\overline{R(\Gamma^*)},
\]

\[
\mathcal{H}_2 := \overline{R(\Gamma^*)} = N(\Gamma^*) \approx \mathcal{H}/N(\Gamma).
\]

Assume that (i) holds. Then $R(\Gamma) = \mathcal{H}_1$ is closed and $\mathcal{H}_0$ is finite dimensional. Theorem 1.4.8 shows that $R(\Gamma^*)$ is closed, and since $N(\Gamma^*)/R(\Gamma^*) \approx \mathcal{H}_0$ is finite dimensional, (i') follows. By symmetry (i) and (i') are equivalent. Moreover, since $\Gamma : \mathcal{H}_2 \to \mathcal{H}_1$ and $\Gamma^* : \mathcal{H}_1 \to \mathcal{H}_2$ are dual operators, $\Gamma^*$ is a diffuse Fredholm-nilpotent operator if and only if $\Gamma$ is so.

Since $R(\Gamma)$ and $R(\Gamma^*)$ are orthogonal, they are both closed if and only if $R(\Pi_\alpha) = R(\Gamma) \oplus R(\Gamma^*)$ is closed. Since $N(\Pi_\alpha) = \mathcal{H}_0 \approx \mathcal{H}/\overline{R(\Pi_\alpha)}$ it follows that (i) and (i') are equivalent with (ii). Moreover, since $D(\Pi_\alpha) = D(\Gamma) \oplus \Pi_\alpha \oplus D(\Gamma^*)$, $\Pi_\alpha$ is a diffuse Fredholm operator if and only if both $\Gamma$ and $\Gamma^*$ are diffuse Fredholm-nilpotent operators. \qed

An important observation here is that the statement (i) is independent of which Hilbert norm on $\mathcal{H}$ we are using (as long as it induces the same topology), whereas in (i') and (ii), the adjoint operator $\Gamma^*$ and $\Pi_\alpha$ depends on which scalar product we are using. We use this below to choose a “good” adjoint $\Gamma^*$.

**Lemma 4.2.4.** Let $\Gamma_i$ be nilpotent operators in Hilbert spaces $\mathcal{H}_i$ and assume there exists an isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$ which intertwines $\Gamma_1$ and $\Gamma_2$, i.e. $\Gamma_2 T = T \Gamma_1$ in the sense of unbounded operators. Assume that $\Pi_i$ is a Dirac type operator associated with $\Gamma_i$ as in Proposition 4.1.7. Then $\Pi_1$ is a (diffuse) Fredholm operator if and only if $\Pi_2$ is a (diffuse) Fredholm operator.

**Proof.** That $\Gamma_i$ are (diffuse) Fredholm-nilpotent operators simultaneously is obvious as we may consider $\Gamma_1$ and $\Gamma_2$ to be the same operator and $\mathcal{H}_1$ and $\mathcal{H}_2$ as the same Hilbert space with different but equivalent norms. The lemma now follows from the equivalence of (i) and (ii) in Proposition 4.2.3. \qed

Next we consider the situation for the concrete $\delta$ and $\delta$ operators. As is well known, the null spaces $N(d_\Omega^\perp) \cap N(\delta_\Omega)$ and $N(d_\Omega) \cap N(\delta_\Omega^\perp)$ of the Dirac operators $D_{\Omega^\perp}$ and $D_{\Omega^\perp}$ from Example 4.1.8 can be identified with the de Rham cohomology spaces of $\Omega$ with normal (relative) and tangential (absolute) boundary conditions, and are thus determined by the global topology of $\Omega$. However, we are here mainly interested in how the local regularity of the boundary $\Sigma$ influences the Fredholm properties of the Dirac operators $D_{\Omega^\perp}$ and $D_{\Omega^\perp}$. 74
Theorem 4.2.5. Let \( \Omega \) be a bounded weakly Lipschitz domain. Then the four operators \( d_{T}, d_{\Omega}, \delta_{T} \) and \( \delta_{\Omega} \) are diffuse Fredholm-nilpotent operators in \( L_{2}(\Omega; \wedge) \) or equivalently \( D_{\Omega_{+}} \) and \( D_{\Omega_{\|}} \) are diffuse Fredholm operators.

Proof. We here only give the proof for \( D_{\Omega_{+}} \), since that for \( D_{\Omega_{\|}} \) is similar. Using Definition 1.5.1 we see that there exist bilipschitz maps \( \rho_{j} : B \to \Omega_{j}, j = 1, \ldots, N, \) where \( B \) denotes the open unit ball in \( \mathbb{R}^{n} \), such that \( \Omega = \bigcup_{j=1}^{N} \Omega_{j} \). Furthermore we may assume that \( \rho_{j} \) extends to a bilipschitz maps between slightly larger open sets. Choose a smooth partition of unity \( \{ \eta_{j} \} \) such that \( \text{supp} \eta_{j} \cap \Omega \subset \Omega_{j} \) and \( \sum \eta_{j}^2 = 1 \) on \( \Omega \).

(i) We see that it suffices to prove that the Dirac operators \( D_{\Omega_{j}} \) are diffuse Fredholm operators. Indeed, if \( T_{j} \) are compact Fredholm inverses to \( D_{\Omega_{j}} \), respectively, then a compact Fredholm inverse to \( D_{\Omega_{+}} \) is

\[
T(F) := \sum_{j} \eta_{j} T_{j}(\eta_{j} F).
\]

(ii) Next we claim that it suffices to prove that the Dirac operator \( D_{B_{+}} \) is a diffuse Fredholm operator, where \( B \) is the unit ball in \( \mathbb{R}^{n} \). Using Proposition 1.2.2, this is a consequence of Lemma 4.2.4 with \( \Gamma_{1} = d_{T}, \Gamma_{2} = d_{\Omega_{j}}, \) and \( T = \rho_{j}^{*} \).

(iii) That \( D_{B_{+}} \) is a diffuse Fredholm operator now follows from either Theorem 4.2.9, Theorem 4.2.12 or Theorem 4.2.16 below.

We now consider the zero-order perturbations \( \Gamma_{1} + k\Gamma_{0} = d_{kT} \) and \( \Gamma_{2} + k\Gamma_{0} = d_{k\Omega} \), where \( k \in C \), and give a sufficient condition for when a zero order perturbation “kills the cohomology”.

Proposition 4.2.6. Let \( \Gamma \) be a diffuse Fredholm-nilpotent operator and let \( \Gamma_{0} \) be a bounded, nilpotent operator such that \( \Gamma\Gamma_{0} + \Gamma_{0}\Gamma = 0 \). Then for any \( k \in C \), \( \Gamma + k\Gamma_{0} \) is a diffuse Fredholm-nilpotent operator. Furthermore, if \( \Gamma_{0} \) give an exact Hodge splitting of \( \mathcal{H} \) and \( \Gamma^{*}\Gamma_{0} + \Gamma_{0}\Gamma^{*} = 0 \), then \( \Gamma + k\Gamma_{0} \) give an exact Hodge splitting of \( \mathcal{H} \) if \( k \neq 0 \).

Proof. To prove that \( \Gamma + k\Gamma_{0} \) give an exact Hodge splitting, assume that \( (\Gamma + k\Gamma_{0})f = (\Gamma^{*} + k\Gamma_{0}^{*})f = 0 \). This implies \( f = 0 \), since

\[
-k\|f\|^{2} \approx (-k\Gamma_{0}f, \Gamma_{0}f) + (\Gamma_{0}^{*}f, -k\Gamma_{0}^{*}f) = (\Gamma f, \Gamma_{0}f) + (\Gamma_{0}^{*}f, \Gamma^{*}f) = (f, (\Gamma^{*}\Gamma_{0} + \Gamma_{0}\Gamma^{*})f) = 0.
\]

\( \square \)

Corollary 4.2.7. Let \( \Omega \subset \mathbb{R}^{n} \) is a bounded, weakly Lipschitz domain. Then for any \( k \in C \), all operators \( d_{kT}, \delta_{kT}, d_{k\Omega} \) and \( \delta_{k\Omega} \) are diffuse Fredholm-nilpotent operators. Moreover, if \( k \neq 0 \), then we have exact Hodge splittings

\[
L_{2}(\Omega; \wedge) = N(d_{\Omega} + k\epsilon_{0}\wedge) \oplus N(-\delta_{\Omega} + k\epsilon_{0}\wedge) = N(d_{\Omega} + k\epsilon_{0}\wedge) \oplus N(-\delta_{\Omega} + k\epsilon_{0}\wedge).
\]

75
We end this section with a discussion of various ways to prove that the Dirac operators $D_{\Omega^\perp}$ and $D_{\Omega^\parallel}$ are diffuse Fredholm operators under certain additional regularity and topological assumptions on $\Sigma$. First we recall the standard proof in the smooth case. Both here and in Theorem 4.2.12 we use the following observation.

**Lemma 4.2.8.** Let $\Pi_\alpha$ be a Dirac type operator as in Proposition 4.1.7. If the embedding $D(\Pi_\alpha) \hookrightarrow H$ is compact, then $\Pi_\alpha$ is a diffuse Fredholm operator with index zero.

**Proof.** Consider the operators

$$\lambda I - \Pi_1 : D(\Pi_\alpha) \to H.$$  

(5.5)

Since $\Pi_1$ is a self adjoint operator by Proposition 4.1.7, (5.5) is an isomorphism when $\lambda \notin \mathbb{R}$. Now observe that $\lambda I : D(\Pi_\alpha) \to H$ is a compact operator. Thus perturbation theory shows that $\Pi_1 : D(\Pi_\alpha) \to H$ is a Fredholm operator with index zero. □

**Theorem 4.2.9 (Gaffney).** Assume that $\Omega$ is a bounded open set with $C^2$-regular boundary $\Sigma$. Then

$$D(D_{\Omega^\perp}) = D(d_{\Omega}) \cap D(\delta_{\Omega}) = W^1_\Omega,$$

$$D(D_{\Omega^\parallel}) = D(d_\Omega) \cap D(\delta_{\Omega}) = W^1_\Omega,$$

In particular, Rellich’s compactness theorem shows that $D(d_{\Omega}) \cap D(\delta_{\Omega})$ are compactly embedded in $L^2(\Omega; \Lambda)$.

Moreover, if $\{v_1, \ldots, v_{n-1}\}$ is an ON-frame on $\Sigma$ of directions of principal (inward) curvatures $\kappa_i$, then we have the Weitzenböck formulas

$$\int_\Omega |\nabla \times F|^2 = \int_\Omega |dF(x)|^2 + |\delta F(x)|^2 - \sum_{i=1}^{n-1} \int_\Sigma \kappa_i(y) |v_i(y) \times f(y)|^2, \quad F \in D(D_{\Omega^\perp}).$$  

(6.4)

**Remark 4.2.10.** (i) Note that when $\Sigma$ is convex, but not necessarily $C^2$, then $\kappa_i \geq 0$ and we obtain the inequality $\int_\Omega |\nabla \times F(x)|^2 \leq \int_\Omega |dF(x)|^2 + |\delta F(x)|^2$ if either $\nu \times f = 0$ or $\nu \perp f = 0$. See Mitrea [48] for generalisations of this result.

(ii) Consider also the special case of the Laplace equation as explained in Section 1.3.2.

If $U$ is in the domain $D(\Delta_D)$ of the Dirichlet Laplacian in $\Omega$, then the gradient $F := \nabla U \in D(D_{\Omega^\perp})$. The Weitzenböck formula now reads

$$\int_\Omega |\nabla \times \nabla U(x)|^2 = \int_\Omega |\Delta U|^2 - (n-1) \int_\Sigma H(y) |\partial_{\nu^\perp}(y)|^2,$$

where $H$ is the (inward) mean curvature of $\Sigma$, since for normal vector fields $|v_i \times f| = |v_i||f| = |f|$. This formula is known as Kadlec’s formula, see p.341 in Taylor [60].

**Proof.** We here only give the proof for $D_{\Omega^\perp}$, since that for $D_{\Omega^\parallel}$ is similar.

(i) Assume that $F \in W^1_\Omega \subset D(D_{\Omega^\perp})$. The boundary theorem 1.2.7 shows that

$$\int_\Omega |\nabla \times F|^2 = (F, \Delta F) = \int_\Sigma (f, (\nu, \nabla) \hat{F})|_{\Sigma},$$

$$\int_\Omega |dF|^2 + (F, \delta dF) = \int_\Sigma (f, \nu \cdot dF)|_{\Sigma} = 0,$$

$$\int_\Omega |\delta F|^2 + (F, \delta dF) = \int_\Sigma (f, \nu \cdot \delta F)|_{\Sigma}.$$
Thus, subtracting the last two equations from the first gives
\[ \int_\Omega \left| \nabla \otimes F \right|^2 = \int_\Omega |dF|^2 + |\delta F|^2 - \int_\Sigma (f, \nu \wedge \delta F |_{\Sigma}) + (f, (\nu, \nabla) \hat{F} |_{\Sigma}). \]

Using the derivation property (1.6) and that \( \nu \wedge f = 0 \) and \( \partial_\nu \nu = \kappa_i v_i \) we rewrite the boundary integral as
\[ \int_\Sigma (f, \nu \wedge \delta F |_{\Sigma}) - (f, (\nu, \nabla) \hat{F} |_{\Sigma}) = - \sum_{i=1}^{n-1} \int_\Sigma (v_i \wedge f, \nu \wedge \partial_\nu f) = \sum_{i=1}^{n-1} \int_\Sigma (v_i \wedge f, (\partial_\nu \nu) \wedge f) = \sum_{i=1}^{n-1} \kappa_i |v_i \wedge f|^2. \]

Since \( \Sigma \) is of regularity \( C^2 \), \( \kappa_i \) are continuous on \( \Sigma \) and thus the Sobolev trace theorem shows that the inclusion \( i_{\Omega} : W^1_\Omega(\Omega, \Lambda) \hookrightarrow D(D_{\Omega_\perp}) \) is a bounded semi-Fredholm map.

(ii) What is left to prove is that the inclusion is surjective. Note that since \( \varepsilon_0 D_{\Omega_\perp} \) is a self-adjoint operator, we have that \( D_{\Omega\perp} + i\varepsilon_0 : D(D_{\Omega\perp}) \to L_2 \) is an isomorphism (for any weakly Lipschitz domain). Thus it suffices to prove that \( D_{\Omega\perp} + i\varepsilon_0 : W^1_\perp(\Omega, \Lambda) \to L_2(\Omega_\perp, \Lambda) \) is surjective.

One way to prove this is to use a domain in a manifold with an isometric double as in Taylor [60], e.g. the upper half \( T^n : = \{ x \in \mathbb{R}^n \mid 0 < x_n < 1 \}/(2\mathbb{Z} + 1)^n \) of the flat \( n \)-torus \( T^n : = \mathbb{R}^n/(2\mathbb{Z} + 1)^n \) and the method of continuity 1.4.5. Since the problem is local, it suffices to prove that if \( \rho_t : \Omega = \Omega_0 \to \Omega_t \) is a continuous family of \( C^2 \) diffeomorphisms, where \( \Omega_t \) is a \( C^2 \) domain in \( T^n \) for \( t \in [0,1] \) and \( \Omega_1 = T^n \), then \( D_{\Omega_0} + i\varepsilon_0 : W^1_\perp(\Omega_0, \Lambda) \to L_2(\Omega_0, \Lambda) \) is an isomorphism. From (i) we have a continuous family of semi-Fredholm maps
\[ \rho_t^* (D_{\Omega_t} + i\varepsilon_0)(\rho_t^{-1})^* = d_{\Omega_t} + \rho_t^* \delta_{\Omega_t}(\rho_t^{-1})^* + i\varepsilon_0 : W^1_\perp(\Omega_0, \Lambda) \to L_2(\Omega_0, \Lambda), \]
since pullbacks preserves normal boundary conditions, and we see that \( [\rho_t^* \delta] : W^1 \to L_2 \) depends continuously on \( t \). By Theorem 1.4.5 it thus suffices to prove that \( D_{(T^n_\perp)}^+ + i\varepsilon_0 : W^1_\perp(T^n_\perp, \Lambda) \to L_2(T^n_\perp, \Lambda) \) is surjective. Note that \( D_{T^n} + i\varepsilon_0 : W^1(T^n, \Lambda) \to L_2(T^n, \Lambda) \) is an isomorphism. We see that, given any \( G \in L_2(T^n, \Lambda) \) with \( \text{supp} G \subset T^n_\perp \), there exists \( F \in W^1(T^n, \Lambda) \) such that \( (D_{T^n} + i\varepsilon_0)F = G \). Now the anti symmetrised field \( F - r^*F \), where \( r : T^n_\perp \to T^n_\parallel \) is the isometric reflection, belongs to \( W^1_\perp(T^n_\perp, \Lambda) \) and \( (d + \delta + i\varepsilon_0)(F - r^*F) = G - r^*G = G \) in \( T^n_\parallel \) since \( r^* = r^{-1} \). This finishes the proof. \( \square \)

For non-smooth \( \Sigma \), not only the source function \( F := D_{\Omega_\perp}U \) influences the regularity of \( U \in D(D_{\Omega_\perp}) \), but also \( \Sigma \). A standard example, see e.g. Grisvard [22], is the following.

**Example 4.2.11.** Consider a bounded domain \( \Omega \subset \mathbb{R}^2 \) whose boundary \( \Sigma \) is smooth except at \( 0 \) where it coincides with \( \mathbb{R}_+ \cup e^{i\alpha} \mathbb{R}_+ \). Let \( U : \mathbb{R}^2 \to \Lambda^0 \) be a scalar function in \( \Omega \) such that \( U(x) = r^\alpha \sin(\underline{\alpha} \theta) \) around \( 0 \) and \( U|_{\Sigma} = 0 \). Define
\[ F(x) := \nabla U(x) = \underline{\alpha} r \alpha^{-1} (\sin(\underline{\alpha} \theta)\hat{r} + \cos(\underline{\alpha} \theta)\hat{\theta}), \quad x \approx 0, \]

\[ 77 \]
where \( \hat{r} \) and \( \hat{\theta} \) denotes the radial and angular unit vector fields. Then the estimate \( |F| \lesssim r^{(\frac{\alpha - 1}{2})} \) shows that \( F \in \text{D}(d_{(\pi)}^*) \cap \text{D}(d_{(\pi)}) \), whereas the estimate \( |dF|/dr | \lesssim r^{(\frac{\alpha - 2}{2})} \) shows that
\[
\|F\|^2_{W^2_2(\alpha)} \gtrsim \int_{\Omega} |\frac{\partial F}{\partial r}|^2 \gtrsim \int_{0}^{1} r^{2(\frac{\alpha - 2}{2})} r \, dr.
\]
But in the non convex case \( \alpha > \pi \) the right hand side is infinite so that \( F \notin W^1_2(\Omega; \land) \). However, one can verify that \( \|F\|_{W^{1/2}} < \infty \) for any \( 0 < \alpha < 2\pi \).

For a strongly Lipschitz domain, we obtain the following theorem from the Rellich estimates in Chapter 3.

**Theorem 4.2.12.** Assume that \( \Omega \) is a bounded, strongly Lipschitz domain. Then we have continuous inclusions
\[
\text{D}(\text{D}_{\Omega}) \subset \text{W}^{1/2}(\Omega; \land).
\]
In particular, Rellich’s compactness theorem shows that \( \text{D}(d_{\Omega}) \cap \text{D}(\delta_{\Omega}) \) and \( \text{D}(d_{\Omega}) \cap \text{D}(\delta_{\Omega}) \) are compactly embedded in \( L_2(\Omega; \land) \).

**Proof.** Consider the map \( \text{D}_{\Omega} + i\epsilon_0 : \text{D}(\text{D}_{\Omega}) \to L_2(\Omega; \land) \), which is an isomorphism since \( \epsilon_0 \text{D}_{\Omega} \) is self-adjoint, and the dense subset
\[
S := \{ F \in \text{D}(\text{D}_{\Omega}) : (\text{D}_{\Omega} + i\epsilon_0)F \in C^0(\Omega; \land) \} \subset \text{D}(\text{D}_{\Omega}).
\]
It suffices to show that we have a continuous inclusion \( S \hookrightarrow \text{W}^{1/2}(\Omega; \land) \). Given \( G = (\text{D}_{\Omega} + i\epsilon_0)F \in C^0(\Omega; \land) \), let \( F_0 := B_1G \in \text{C}^0(\Omega; \land) \), where \( B_1 \) is the Cauchy convolution (balayage) operator from Lemma 1.2.6. It follows that the tangential trace \( N^+f_0 \in N^+\text{D}(\Gamma; L_2(\Sigma)) \), with notation as in Chapter 3. Theorem 3.2.15 shows the existence of \( f_1 \in E^+_i \text{D}(\Gamma; L_2(\Sigma)) \) such that \( N^+f_1 = N^+f_0 \). Now let \( F' := F_0 - F_1 \). We see that
\[
\|F_0\|_{\text{W}^{1/2}(\Omega)} \lesssim \|F_0\|_{\text{W}^{1}(\Omega)} \lesssim \|G\|_{L_2(\Omega)} \approx \|F\|_{\text{D}(\text{D}_{\Omega})}
\]
\[
\|F_1\|_{\text{W}^{1/2}(\Omega)} \approx \|f_1\|_{L_2(\Sigma)} \approx \|N^+f_0\|_{L_2(\Sigma)} \lesssim \|F_0\|_{\text{W}^{1}(\Omega)} \lesssim \|F\|_{\text{D}(\text{D}_{\Omega})},
\]
using Theorem 1.5.9. Moreover \( dF_0, \delta F_0 \in C^0(\mathbb{R}^n; \land) \), \( dF_1 = -\delta F_1 \in \text{W}^{1/2}(\Omega; \land) \subset L_2(\Omega; \land) \) and \( \nu \cdot f' = \nu \cdot f_0 - \nu \cdot f_1 = 0 \). Thus \( F' \in \text{D}(\text{D}_{\Omega}) \cap \text{W}^{1/2}(\Omega; \land) \) and \( (\text{D}_{\Omega} + i\epsilon_0)F = G = (\text{D}_{\Omega} + i\epsilon_0)F \). Thus \( F = F' \in \text{W}^{1/2}(\Omega; \land) \). \( \square \)

Note that Theorem 4.2.12 is more constructive than Theorem 4.2.9 in the sense that \( F' \) is found by solving the boundary equation \( N^+f_1 = N^+f_0 \). However, if one is only interested in solving for example \( d_3U = F \), where \( F \in \text{N}(d_3) \), with a “good inverse” in the sense that \( F \mapsto U \) is an \( L_2 \) compact map, then this can be done much more explicit using path integrals as we now explain.

**Lemma 4.2.13.** Let \( \Gamma \) be a nilpotent operator in a Hilbert space \( \mathcal{H} \). Assume there exists a bounded (compact) operator \( T : \text{N}(\Gamma) \to \mathcal{H} \) such that
\[
\Gamma T = I_{\text{M}(\Gamma)} + K,
\]
where \( K : \text{N}(\Gamma) \to \text{N}(\Gamma) \) is compact. Then \( \Gamma \) is a (diffuse) Fredholm-nilpotent operator.
Proof. Let $\pi : \mathcal{H} \to \mathcal{H}/\mathbb{N}(\Gamma) \cong \mathbb{N}(\Gamma)\bot$ be the projection. Since $\pi$ is bounded, it follows that $\pi T$ is a bounded (compact) Fredholm inverse to $\tilde{T} : \mathcal{H}/\mathbb{N}(\Gamma) \to \mathbb{N}(\Gamma)$. \hfill \Box

**Lemma 4.2.14.** Let $\Omega \subset \mathbb{R}^n$ be an open set with a smooth retraction $\mathcal{F}_t : \Omega \to \mathcal{F}_t(\Omega) \subset \Omega$ to $p \in \Omega$ such that $\mathcal{F}_t = I$, $\mathcal{F}_t \mathcal{F}_s = \mathcal{F}_{t+s}$ for $0 \leq t, s \leq 1$ and $\mathcal{F}_0 = p$. If $\theta = d\mathcal{F}_t/dt|_{t=1}$ is the vector field with flow $\mathcal{F}_t$, then for smooth fields $F$ in $\Omega$ we have the path integral formulae

\[
F(x) = \nabla \wedge \left( \int_0^1 \theta(x) \cdot \mathcal{F}_t^* F(x) \frac{dt}{t} \right), \quad \text{if } \nabla \wedge F = 0 \text{ and } F|_{\wedge 0} = 0,
\]

\[
F(x) = \nabla \cdot \left( \int_0^1 \theta(x) \wedge \mathcal{F}_t^{-1} F(x) \frac{dt}{t} \right), \quad \text{if } \nabla \cdot F = 0 \text{ and } F|_{\wedge n} = 0.
\]

One proves this lemma by using Cartan’s formula

\[L_0 F = \frac{d}{dt}(\mathcal{F}_t^* F(x))|_{t=1} = \nabla \wedge (\theta \cdot F) + \theta \cdot (\nabla \wedge F)\]

for the Lie derivative of the differential form $F$ and using the homomorphism formulae from Proposition 1.2.2. For details, see for example Taylor [60].

**Example 4.2.15.** Let $\Omega$ be star shaped with respect to 0 and $\mathcal{F}_t(x) := tx$. Then for a smooth $j$-vector field $F : \Omega \to \wedge^j$ we have the path integrals

\[
F(x) = \nabla \wedge \left( \int_0^1 t^{j-1} x \cdot F(tx) \, dt \right), \quad \text{if } \nabla \wedge F = 0 \text{ and } j \geq 1,
\]

\[
F(x) = \nabla \cdot \left( \int_0^1 t^n-j-1 x \wedge F(tx) \, dt \right), \quad \text{if } \nabla \cdot F = 0 \text{ and } j \leq n-1.
\]

In particular, a curl free vector field $F$ has a scalar potential $U(x) = \int_0^1 (x, F(tx)) \, dt$ and a divergence free vector field $F$ has a bivector potential $U(x) := \int_0^1 t^{n-2} x \wedge F(tx) \, dt$. In classical notation in $\mathbb{R}^3$, the latter translates to $F = -\nabla \times U$ if $U(x) := \int_0^1 tx \times F(tx) \, dt$.

The third way to prove that the Dirac operators are diffuse Fredholm operators uses an $L_2$ version of the classical Poincaré lemma.

**Theorem 4.2.16.** Let $\Omega \subset \mathbb{R}^n$ be an open set for which there exists $0 < \epsilon < R < \infty$ such that $B(0, \epsilon) \subset \Omega \subset B(0, R)$ and that $\Omega$ is star shaped with respect to each $p \in B(0, \epsilon)$ (in particular, $\Omega$ is a strongly Lipschitz domain). For $1 \leq j \leq n$, let $T_j$ denote the integral operator

\[
T_j F(x) := \int_\Omega (x-y) \cdot F(y) k_j(x,y) dy, \quad F \in L_2(\Omega; \wedge^j),
\]

where $k_j$ denotes the kernel

\[
k_j(x,y) := \int_0^1 \eta \left( \frac{y-tx}{1-t} \right) \frac{t^{j-1} dt}{(1-t)^{n+1}}
\]

for some fixed $\eta \in C_0^\infty(B(0, \epsilon))$ with $\int \eta = 1$. In particular $\text{supp} k_j \subset \{(x,y) \ ; \ y \in \text{conv}(B(0,\epsilon), x)\}$, $k_j$ is smooth off the diagonal $\{x = y\}$ with estimates

\[
|k_j(x,y)| \leq \frac{1}{n} \|\eta\|_\infty (R + \epsilon)^n \frac{1}{|x-y|^n}
\]

79
and $T_j$ defines a compact operator $L_2(\Omega; \wedge^j) \to L_2(\Omega; \wedge^{j-1})$.

Then the restriction of $T := 0 \oplus T_1 \oplus \ldots \oplus T_n$ to $\mathcal{N}(d_\Omega) = \text{span}(1) \oplus \mathcal{N}(d_\Omega|_{\wedge^1}) \oplus \ldots \oplus \mathcal{N}(d_\Omega|_{\wedge^{n-1}}) \oplus L_2(\Omega; \wedge^n)$ is an $L_2$-compact left Fredholm inverse to $d_\Omega$ as in Lemma 4.2.13.

The corresponding result for $\delta_\Omega$ holds true as well.

More precisely, it can be shown that $T_j$ is a bounded operator $L_2(\Omega; \wedge^j) \to W^s_2(\Omega; \wedge^{j-1})$ for $s < 1/2$.

**Proof.** Assume that $F \in C^\infty_0(\mathbb{R}^n; \wedge^j)$ and $\text{supp} (\nabla \wedge F) \cap \Omega = \emptyset$ as in Proposition 4.1.3. We have

$$T_j F(x) = \int \eta(p) dp \left( \int_0^1 t^{j-1}(x - p) \cdot F(p + t(x - p)) dt \right),$$

which by using Fubini’s theorem and the change of variables $y = p + t(x - p)$ becomes (4.7). Since $\int \eta = 1$, it follows from Example 4.2.11 that $d_\Omega T_j F = F$.

Since the full kernel for $T_j$ has the estimate $\lesssim 1/|x - y|^{n-1}$, Schur’s lemma, c.f. Lemma 3.1.8, shows that $T_j$ defines a compact operator $L_2(\Omega; \wedge^j) \to L_2(\Omega; \wedge^{j-1})$. ∎
Chapter 5

Transmission problems for Dirac’s and Maxwell’s equations with weakly Lipschitz interfaces

This is the main chapter of this thesis. We here investigate “non-relativistic” transmission problems for Dirac operators in energy norms, with emphasis on applications to Maxwell’s equations.

In Section 5.1 we construct a “boundary function space” $\mathcal{E} = \mathcal{E}(\Sigma; \wedge)$ on a weakly Lipschitz interface $\Sigma$. On a smooth interface $\Sigma$ this space $\mathcal{E}$ is the space of fields such that $f$, $\nabla_{\text{tan}} \wedge (N^+ f)$, and $\nabla_{\text{tan}} \cdot (N^- f)$ all belong to the Sobolev space $W^{1/2}_2(\Sigma; \wedge)$, and is thus a function space of mixed $\pm 1/2$ regularity. On a Lipschitz interface however, the smoothness of $f$ is measured “relative to the orientation of $\Sigma$” and is not a component wise condition on $f$. We will not be concerned here with describing $\mathcal{E}$ in terms of Sobolev spaces, since we do not need this in order to understand the transmission problem. Instead we use an “extrinsic and distributional” approach where $\mathcal{E} \subset (\nu C_0^\infty(\mathbb{R}^n; \wedge)|_{\Sigma})^*$ and

$$\nu \mathcal{E} = \{ \nu \triangle f ; f \in \mathcal{E} \} \subset \{ \mu \in \mathcal{D}'(\mathbb{R}^n; \wedge) ; \text{supp } \mu \subset \Sigma \}.$$

In Section 5.2, the formalism from Chapter 2 is generalised in order to treat transmission problems with different speeds of propagation, and thus different wave numbers, in the two regions $\Omega^\pm$. The main theorem in this chapter is Theorem 5.2.3 which in particular shows that the Maxwell transmission problem, as stated in Example 5.2.4, is well posed in Fredholm sense in energy norms, for all bounded weakly Lipschitz interfaces. This theorem is proved by (1) removing the constraint $F \in N(\Gamma_k)$ and using the elliptic Dirac equation, (2) establishing a priori estimates and (3) applying an ansatz and the method of continuity. The main part here is (2) which is the topic of Section 5.3.

The tangential and normal parts $\mathcal{A}_{\text{nor}}^2 = \nu \wedge \mathcal{E}$ and $\mathcal{A}_{\text{tan}}^2 = \nu \cdot \mathcal{E}$ of the full space $\nu \mathcal{E}$ on a strongly Lipschitz interface were introduced and used by Mitrea–Mitrea in [43], where invertibility of double layer type operators were proved. Spectral estimates in Corollary 5.3.3 have appeared in Mitrea–Mitrea [44] where also Proposition 5.2.7 was proved.
5.1 The energy trace space \( \mathcal{E} \)

We start by defining spaces of monochromatic fields in the domains \( \Omega^\pm \).

**Definition 5.1.1.** Let \( \mathcal{O} \supset \Sigma \) be a bounded smooth neighbourhood of \( \Sigma \). In \( \Omega^+ \), we define the Hilbert space

\[
C_k^+ \mathcal{E} := \mathcal{N}(d_{k,\Omega^+} + \delta_{k,\Omega^+})
\]

with norm \( \| F \|_{\mathcal{D}(d_{k,\Omega^+ + \partial \mathcal{O}})} = (\| F \|^2_{L^2(\Omega^{\mathcal{O}+})} + \| d_k F \|^2_{L^2(\Omega^{\mathcal{O}+})})^{1/2} \). In \( \Omega^- \), we define the Hilbert space \( C_k^- \mathcal{E} \) to be the set of \( k \)-monochromatic fields \( F : \Omega^- \rightarrow \Lambda \) for which \( F|_{\Omega^- \cap \partial \mathcal{O}} \in \mathcal{D}(d_{k,\Omega^+ - \partial \mathcal{O}}) \) and which satisfies the following radiation condition: if \( x \in \Omega^- \setminus \overline{\mathcal{O}} \) then

\[
F(x) = \int_{\partial \Omega \cap \Omega^-} \hat{E}_k(x - y)(-\nu_{\partial \mathcal{O}}(y))F(y) \, d\sigma(y),
\]

(5.1)

The norm on \( C_k^- \mathcal{E} \) is \( \| F \|_{\mathcal{D}(d_{k,\Omega^+ - \partial \mathcal{O}})} \).

**Remark 5.1.2.** (i) The radiation condition (5.1) does not depend on \( \mathcal{O} \), as is seen using the reproducing formula (1.20). Moreover, if \( D_k F = 0 \) in \( \Omega^- \) and \( F \) satisfies (5.1), then so does the twosided \( k \)-monochromatic field \( d_k F \). To see this, for \( x \in \Omega^- \setminus \overline{\mathcal{O}} \), we calculate

\[
d_k F(x) = \nabla_k \cdot \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nu \cdot F + \nu \cdot F)(y)
\]

\[
= \nabla_k \cdot \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nu \cdot F)(y) - \nabla_k \cdot \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nu \cdot F)(y)
\]

\[
= - \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nabla \cdot (\nu \cdot F))(y) + \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nabla \cdot (\nu \cdot F))(y)
\]

\[
= - \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nabla \cdot (\nu \cdot F))(y) + \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nabla \cdot (\nu \cdot F))(y)
\]

\[
= \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nu \cdot d_k F)(y) - \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nu \cdot d_k F)(y)
\]

\[
= \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_k(x - y)(\nu \cdot d_k F)(y),
\]

using derivation properties (1.7) and (1.8) and Proposition 1.2.9.

(ii) Different \( \mathcal{O} \) give rise to equivalent norms on \( C_k^\pm \mathcal{E} \). This follows from the reproducing formula for \( k \)-monochromatic fields in \( \Omega^+ \) and from the radiation condition (5.1) in \( \Omega^- \).

(iii) The space \( C_k^+ \mathcal{E} \) is complete since \( \mathcal{N}(d_{k,\Omega^+} + \delta_{k,\Omega^+}) = \pi_1(\mathcal{G}(d_{k,\Omega^+}) \cap \mathcal{G}(-\delta_{k,\Omega^+})) \). To see that \( C_k^- \mathcal{E} \) is complete, considered it for example as a subspace of \( \mathcal{N}(d_{k,\Omega^- \cap \partial \mathcal{O}} + \delta_{k,\Omega^- \cap \partial \mathcal{O}}) \) and let \( C_k^- \mathcal{E} \ni F_n \rightarrow F \in \mathcal{N}(d_{k,\Omega^- \cap \partial \mathcal{O}} + \delta_{k,\Omega^- \cap \partial \mathcal{O}}) \). Then \( F_n \) and \( d_k F_n \) converges uniformly on compact subsets of \( \Omega^- \). Hence both sides in (5.1) for \( F_n \) converges and proves that \( F \) extends to \( \lim F_n \in C_k^- \mathcal{E} \).

(iv) When \( \operatorname{Im} k \geq 0 \), we obtain more concrete characterisations of the radiation condition (5.1) as follows. Let \( \mathcal{O} \subset B(0, R), x \in \Omega^- \cap B(0, R) \setminus \mathcal{O} \) and \( D_k F = 0 \) in \( \Omega^- \). Then the reproducing formula gives us

\[
F(x) = \int_{\Omega^- \cap \partial \mathcal{O}} \hat{E}_{-k}(y - x)(-\nu)(y)F(y) \, d\sigma(y) + \int_{\partial B(0, R)} \hat{E}_{-k}(y - x)(\nu)F(y) \, d\sigma(y),
\]

(5.2)
Using the asymptotics of \( \tilde{E}_{-k}(x) = -\tilde{E}_{k}(-x) \) from Proposition 1.2.5 in (5.1) for a necessary condition and in the last integral in (5.2) for a sufficient condition, we get the following.

- If \( \text{Im} k > 0 \), then a sufficient condition for (5.1) is \( F = o(r^{-\frac{n-1}{2}}e^{(\text{Im} k)r}) \), and a necessary condition is \( F = O(r^{-\frac{n-1}{2}}e^{-\text{Im} k r}) \) as \( r \to \infty \).

- If \( k \in \mathbb{R} \setminus \{0\} \), then a sufficient condition for (5.1) is \( (\tilde{r} + k_{0})F = o(r^{-\frac{n-1}{2}}) \) and \( (\tilde{r} + k_{0})F = o(r^{-\frac{n-3}{2}}) \), and a necessary condition is \( (\tilde{r} + k_{0})F = O(r^{-\frac{n+1}{2}}) \) and \( (\tilde{r} + k_{0})F = O(r^{-\frac{n-3}{2}}) \).

- If \( k = 0 \), then a sufficient condition for (5.1) is \( F = o(1) \), and a necessary condition is \( F = O(r^{-\frac{n-1}{2}}) \) as \( r \to \infty \).

Note that the necessary conditions seem stronger than the sufficient ones, although this is just apparent.

We are now ready to state the main result in this section.

**Theorem 5.1.3.** On a bounded weakly Lipschitz surface \( \Sigma \), there exists a unique boundary function space \( \mathcal{E} \subset (\mathcal{D}_{0}^{\infty}(\mathbb{R}^{n}; \Lambda)|\Sigma)^{\ast} \) and a space of distributions \( \nu \mathcal{E} \subset \{ \mu \in \mathcal{D}'(\mathbb{R}^{n}; \Lambda) ; \supp \mu \subset \Sigma \} \) with the following properties. The two Hilbert spaces \( \mathcal{E} \) and \( \nu \mathcal{E} \) split into subspaces

\[
\begin{array}{c|c|c|c|c}
D(d\mathbb{R}^{n}) & D(\delta\mathbb{R}^{n}) & C_{k}^{+} \mathcal{E} & C_{k}^{-} \mathcal{E} \\
\gamma_{d} & \gamma_{\delta} & \gamma_{+} & \gamma_{-} \\
\nu \mathcal{E} = \nu N^{+} \mathcal{E} & \nu N^{-} \mathcal{E} = \nu E_{k}^{+} \mathcal{E} & \nu E_{k}^{-} \mathcal{E} \\
\nu & \nu & \nu & \nu & E_{k}^{+} \mathcal{E} & E_{k}^{-} \mathcal{E} \\
\mathcal{E} = N^{+} \mathcal{E} & N^{-} \mathcal{E} & E_{k}^{+} \mathcal{E} & E_{k}^{-} \mathcal{E} \\
\end{array}
\]

The trace maps in the diagram are

\[
\begin{align*}
\gamma_{d} & : F \mapsto \nu \wedge f := [-d, \chi]F = -d(\chi F) + \chi(dF) = [-d_{k}, \chi]F, \quad F \in D(d\mathbb{R}^{n}), \\
\gamma_{\delta} & : F \mapsto \nu \wedge f := [-\delta, \chi]F = -\delta(\chi F) + \chi(\delta F) = [-\delta_{k}, \chi]F, \quad F \in D(\delta\mathbb{R}^{n}), \\
\gamma_{\pm} & : F \mapsto \nu f := \mp D_{k}(F_{0}), \quad F \in C_{k}^{\pm} \mathcal{E},
\end{align*}
\]

where \( \chi \) denotes (multiplication with) the characteristic function for \( \Omega^{+} \) and \( F_{0} \) denotes the zero-extension of \( F \) in \( \Omega^{\pm} \) to \( \Omega^{\mp} \). The four subspaces of \( \nu \mathcal{E} \) are defined as the range of the corresponding trace map.

The Hilbert space \( \mathcal{E} \) is mainly introduced for notational convenience. It is identified with the Hilbert space \( \nu \mathcal{E} \) by requiring that \( \nu \triangle (-) : \mathcal{E} \to \nu \mathcal{E} \) is an isomorphism.

The topology of \( \nu \mathcal{E} \) is the one induced by either of the equivalent norms \( \| \cdot \|_{E^{,1}, \mathcal{O}} \), \( \| \cdot \|_{N^{,1}, \mathcal{O}} \) and \( \| \cdot \|_{N^{,1}, \mathcal{O}} \) from Definition 5.1.6 below. Indeed, if \( d(\Sigma, \partial \mathcal{O}) \) denotes the distance from \( \Sigma \) to the boundary of the bounded smooth neighbourhood \( \mathcal{O} \) and \( L^{w} \) are the
constants from Definition 5.1.7 below, then

\[(1 + d(\Sigma, \partial \Omega)^{-1})^{-1} \| \nu f \|_{E_k, \Omega} \leq \| \nu f \|_{N, \Omega} \leq L^u(\Omega^+, \Omega^-) \| \nu f \|_{E_k, \Omega},\]

\[(1 + d(\Sigma, \partial \Omega)^{-1})^{-1} \| \nu f \|_{E_k, \Omega} \leq \| \nu f \|_{N, \Omega} \leq L^u(\Omega^-, \Omega^+) \| \nu f \|_{E_k, \Omega},\]

where \( \lesssim \) here means up to universal constants and compact terms. Moreover, we have Hilbert space isomorphisms

\[\gamma_d : N^+(\Omega^+) := D(d_{\Omega^+})/D(d_{\Omega^+}) \to \nu N^+ \mathcal{E} = \nu \wedge \mathcal{E},\]

\[\gamma_\delta : N^-(\Omega^-) := D(d_{\Omega^-})/D(d_{\Omega^-}) \to \nu N^- \mathcal{E} = \nu \perp \mathcal{E},\]

\[\gamma_\pm : C^+_k \mathcal{E} \to \nu E^+ \mathcal{E}.\]

We also use the norms \( \| \cdot \|_{E_k, \Omega} \), \( \| \cdot \|_{N, \Omega^+} \) and \( \| \cdot \|_{N, \Omega^-} \) in \( \mathcal{E} \) under the identification \( \mathcal{E} \ni f \leftrightarrow \nu f \in \nu \mathcal{E} \).

**Definition 5.1.4.** Let the tangential/normal projection operators \( N^\pm \) be the projections associated with the splitting \( \mathcal{E} = N^+ \mathcal{E} \oplus N^- \mathcal{E} \), and let the interior/exterior Hardy type projection operators \( E_k^\pm \) with wave number \( k \) be the projections associated with the splitting \( \mathcal{E} = E_k^+ \mathcal{E} \oplus E_k^- \mathcal{E} \). Furthermore, define the reflection operators \( N := N^+ - N^- \) and \( E_k := E_k^+ - E_k^- \).

We note that the normal/tangential reflection operator \( N \) is self-adjoint in the norms \( \| \cdot \|_{N, \Omega^\pm} \) whereas the Hardy reflection operator \( E_k \) is self-adjoint in the norms \( \| \cdot \|_{E_k, \Omega} \).

**Corollary 5.1.5.** The essential spectral radius of the rotation operator \( E_k N \) in \( \mathcal{E} \) can be estimated by

\[r_{ess}(E_k N; \mathcal{E}) \leq C(1 + d(\Sigma, \partial \Omega)^{-1}) \min(L^u(\Omega^+, \Omega^-), L^u(\Omega^-, \Omega^+)).\]

Note that this estimate is far from being sharp. Indeed, we will show in Section 5.3 that \( \sigma_{ess}(E_k N; \mathcal{E}) \subset \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) for all bounded, weakly Lipschitz interfaces \( \Sigma \) and all wave numbers \( k \in \mathbb{C} \).

**Definition 5.1.6.** Let \( \Omega \supset \Sigma \) denote a bounded smooth neighbourhood of \( \Sigma \) and write \( \Omega^\pm := \Omega^\pm \cap \Omega \). On \( \nu \mathcal{E} \) we define the norms

\[\| \nu f \|_{N, \Omega^\pm} := \| \nu \wedge f \|_{N, \Omega^\pm}^2 + \| \nu \perp f \|_{N, \Omega^\pm}^2,\]

\[\| \nu f \|_{N, \Omega^-} := \| \nu \wedge f \|_{N, \Omega^-}^2 + \| \nu \perp f \|_{N, \Omega^-}^2,\]

\[\| \nu f \|_{E, \Omega^\pm} := \| \nu f^\dagger \|_{E, \Omega^\pm}^2 + \| \nu f^\dagger \|_{E, \Omega^\pm}^2,\]

where in the four subspaces we have

\[\| \nu \wedge f \|_{N, \Omega^\pm} := \inf_{\nu \wedge g = \nu \wedge f} \| G \|_{D(d_k, \Omega^\pm)}, \quad F \in D(d_{\mathbb{R}^n}),\]

\[\| \nu \perp f \|_{N, \Omega^\pm} := \inf_{\nu \perp g = \nu \perp f} \| G \|_{D(\delta_k, \Omega^\pm)}, \quad F \in D(\delta_{\mathbb{R}^n}),\]

\[\| \nu f^\perp \|_{E, \Omega^\pm} := \| F^\perp \|_{D(d_k, \Omega^\pm)}, \quad F^\perp \in C^+_k \mathcal{E}.\]
We here introduce “local Lipschitz constants” for a weakly Lipschitz interface which replace those from Definition 1.5.5.

**Definition 5.1.7.** Let $U \subseteq V \subset \mathbb{R}^n$ be open sets such that the boundary $\overline{U} \cap \overline{V \setminus U}$ of $U$ relative to $V$ is a bounded, weakly Lipschitz interface. As in Proposition 4.1.9, we can construct an extension map $\chi^{-1} : \mathcal{D}(dU) \to \mathcal{D}(dV)$ such that $\text{supp}(\chi^{-1} F) \cap (V \setminus U)$ is compact in $V$, uniformly for $F \in \mathcal{D}(dV)$, and where $\chi^{-1}$ and $[d, \chi^{-1}]$ are $L_2(U) \to L_2(V)$ bounded and $\chi^{-1}(f_1 + \varepsilon_0 \wedge f_2) = \chi^{-1}f_1 + \varepsilon_0 \wedge \chi^{-1}f_2$. We define quantities

$$
L_0^w(U, V \setminus \overline{U}) := \inf_{\chi^{-1}} \| \chi^{-1} \|_{L_2(U) \to L_2(V \setminus \overline{U})},
$$

$$
L_1^w(U, V \setminus \overline{U}) := \inf_{\chi^{-1}} \| [d, \chi^{-1}] \|_{L_2(U) \to L_2(V \setminus \overline{U})},
$$

$$
L^w(U, V \setminus \overline{U}) := L_0^w(U, V \setminus \overline{U}) + L_1^w(U, V \setminus \overline{U}),
$$

where the infima are taken over extension maps satisfying the properties above.

Note that by Hodge duality, the same quantities are obtained if we take the corresponding infima over extension maps for $\delta$. We also always have $L_0^w(U, V \setminus \overline{U}) \geq 1$.

**Proof of Theorem 5.1.3.** (i) We first show that, as sets of distributions supported on $\Sigma$, $\nu N^+\mathcal{E} \cap \nu N^-\mathcal{E} = \{0\}$. Assume $F \in \mathcal{D}(d\mathbb{R}^n)$, $G \in \mathcal{D}(\delta\mathbb{R}^n)$ and $[-d, \chi]F = [-\delta, \chi]G$, or equivalently

$$
d(\chi F) - \delta(\chi G) = \chi(dF - \delta G) \in L_2(\mathbb{R}^n).
$$

Since the ranges of $d$ and $\delta$ are orthogonal on $\mathbb{R}^n$, we must have $d(\chi F), \delta(\chi G) \in L_2(\mathbb{R}^n; \wedge)$, so $[-d, \chi]F, [-\delta, \chi]G \in L_2(\mathbb{R}^n; \wedge)$. But since the traces are supported on $\Sigma$, they must vanish.

Also note that the null space of $\gamma_d : \mathcal{D}(d\mathbb{R}^n) \to \nu N^+\mathcal{E}$ is $\mathcal{D}(d\mathbb{R}^n)$.

(ii) Next, we show that, as sets of distributions supported on $\Sigma$, $\nu E^+_k\mathcal{E} \cap \nu E^-_k\mathcal{E} = \{0\}$. Assume that $F^\pm \in C^\pm_k \mathcal{E}$ and $\gamma_+ F^+ = \gamma_- F^-$. It follows that the field $F = F^+ + F^-$ is $k$-monochromatic on $\mathbb{R}^n$. But since $D_k F = 0$ in $\mathcal{O} \cup \Omega^+$, the right hand side for the radiation condition (5.1) must vanish. Thus $F = 0$ in $\Omega^- \setminus \mathcal{O}$ and analyticity shows that $F = 0$ in all $\mathbb{R}^n$.

Note that we have also shown that the maps $\gamma_\pm$ are injective.

(iii) We now have the four Hilbert spaces $N^+\mathcal{E}$ and $E^\pm_k\mathcal{E}$ well defined and $N^+\mathcal{E} \cap N^-\mathcal{E} = \{0\}$ and $E^+_k\mathcal{E} \cap E^-_k\mathcal{E} = \{0\}$. From Lemma 3.2.11 and Lemma 5.1.8 below we deduce that the boundary trace space

$$
\mathcal{E} = N^+\mathcal{E} \oplus N^-\mathcal{E} = E^+_k\mathcal{E} \oplus E^-_k\mathcal{E}
$$

is well defined. To estimate the distance between the norms $\| f \|_\cdot := \| f \|_{N,k; \mathcal{O}^-}$
\[ \|f\|_{E,k,O} =: \|f\|, \text{ use Lemma 5.1.8 to get} \]

\[
\|f\| \approx \|E_k^+ f\| + \|E_k^- f\| \\
\leq (\|E_k^+ N^+ f\| + \|E_k^- N^+ f\|) + (\|E_k^+ N^- f\| + \|E_k^- N^- f\|) \\
\lesssim (1 + d(\Sigma, \partial O)^{-1}) (\|N^+ f\|_{-} + \|N^- f\|_{-}) \approx (1 + d(\Sigma, \partial O)^{-1}) \|f\|_{-}, \\
\|f\|_{-} \approx \|N^+ f\|_{-} + \|N^- f\|_{-} \\
\leq (\|N^+ E_k^+ f\|_{-} + \|N^- E_k^+ f\|_{-}) + (\|N^+ E_k^- f\|_{-} + \|N^- E_k^- f\|_{-}) \\
\lesssim L^w(\mathcal{O}^+ , \Omega^-) \|E_k^+ f\|_{-} + \|E_k^- f\|_{-} \lesssim L^w(\mathcal{O}^+ , \Omega^-) \|f\|_{-},
\]

up to universal constants and compact terms. The estimate \( \|f\|_{N,k,O^+} \approx \|f\|_{E,k,O} \) follows in the same way. \qed

**Lemma 5.1.8.** We have continuous Hilbert space embeddings

\[
\begin{align*}
\nu E_k^\pm \mathcal{E} & \hookrightarrow \nu N^+ \mathcal{E} \oplus \nu N^- \mathcal{E} : \nu f = \gamma_\pm F \mapsto \gamma_d F + \gamma_\delta F, \\
\nu N^+ \mathcal{E} & \hookrightarrow \nu E_k^+ \mathcal{E} \oplus \nu E_k^- \mathcal{E} : \nu f \mapsto \gamma_+(-B_k(\nu \wedge f)) + \gamma_-(B_k(\nu \vee f)), \\
\nu N^- \mathcal{E} & \hookrightarrow \nu E_k^+ \mathcal{E} \oplus \nu E_k^- \mathcal{E} : \nu f \mapsto \gamma_+(-B_k(\nu \wedge f)) + \gamma_-(B_k(\nu \vee f)),
\end{align*}
\]

where \( B_k \) denotes the Cauchy convolution operator from Proposition 1.2.6. For these embeddings we have estimates

\[
\begin{align*}
\max(\|\nu \wedge f\|_{N,k,O^\pm}, \|\nu \vee f\|_{N,k,O^\pm}) & \leq \|\nu f\|_{E,k,O^\pm}, \quad f \in E_k^\pm \mathcal{E}, \\
\|f\|_{E,k,O} & \leq C(1 + d(\Sigma, \partial O)^{-1}) \min(\|f\|_{N,k,O^+}, \|f\|_{N,k,O^-}) + \|Kf\|_{\mathcal{X}}, \quad f \in N^\pm \mathcal{E},
\end{align*}
\]

where \( C \) is a universal constant, \( d(\Sigma, \partial O) \) denotes the distance from \( \Sigma \) to \( \partial O \) and \( K : N^\pm \mathcal{E} \rightarrow \mathcal{X} \) is a compact operator. Moreover, independently of \( k \) we have

\[
\begin{align*}
\|N^\pm f\|_{N,k,O^-} & \lesssim L^w(\mathcal{O}^+ , \Omega^-) \|N^\pm f\|_{N,k,O^+}, \\
\|N^\pm f\|_{N,k,O^+} & \lesssim L^w(\mathcal{O}^- , \Omega^+) \|N^\pm f\|_{N,k,O^-},
\end{align*}
\]

where \( L^w \) denotes the constants from Definition 5.1.7.

**Proof.** (i) Let \( F \in C_k^+ \mathcal{E} \subset D(d_{\Omega^+}) \cap D(\delta_{\Omega^+}) \). Then we have

\[
\gamma_d F + \gamma_\delta F = [-d_k, \chi] F + [-\delta_k, \chi] F = -D_k(\chi F) = \gamma_+ F,
\]

and the estimate \( \max(\|\nu \wedge f\|_{N,k,O^+}, \|\nu \vee f\|_{N,k,O^+}) \leq \|\nu f\|_{E,k,O^+} \) follows directly from Definition 5.1.6. Similarly for \( E_k^- \mathcal{E} \).

(ii) Let \( \nu \wedge f \in \nu \wedge \mathcal{E} \) and pick \( F \in D(d_{R^+}) \) such that \( \|\nu \wedge f\|_{N,k,O^+} \approx \|F\|_{D(d_{\Omega^+})} \). Let \( \eta \in C_0^\infty(\mathcal{O}) \) be a cutoff function such that \( \eta = 1 \) in a neighbourhood of \( \Sigma \), \( 0 \leq \eta \leq 1 \) and \( \|\nabla \eta\|_\infty \approx d(\Sigma, \partial O)^{-1} \). Define operators \( T_0 F := \chi(\eta F) \) and \( T_1 F := \chi(d_k(\eta F)) \), where we have estimates

\[
\begin{align*}
\|T_0 F\|_{L^2(R^n)} & \leq \|F\|_{L^2(\mathcal{O}^+)} \lesssim \|\nu \wedge f\|_{N,k,O^+}, \\
\|T_1 F\|_{L^2(R^n)} & \leq \|d_k(\eta F)\|_{L^2(\mathcal{O}^+)} \leq C(1 + \|\nabla \eta\|_\infty) \|\nu \wedge f\|_{N,k,O^+},
\end{align*}
\]

86
and $T_i F$ are supported in $\Omega^+$. Using Proposition 1.2.6, we now calculate

$$G(x) = \int E_{-k}(y - x) \nu \wedge f(y) = -\int E_k(x - y) \nu \wedge f(y)$$

$$d_k G = (I - B_k d_k) [d_k, \chi](\eta F) = (B_k d_k)(T_0 F) - B_k(T_1 F),$$

which gives us

$$\gamma_+(G|_{\Omega^+}) + \gamma_-(G|_{\Omega^-}) = D_k B_k(\nu \wedge f) = \nu \wedge f,$$

since $B_k$ is convolution with the fundamental solution of $D_k$. To obtain the estimate, let $O^\prime := O \setminus \Sigma$. Then the term $B_k(T_1 F)$ is compact $N^+ E \to L^2(O^\prime)$ and we have $d_k G|_{O^\prime} = (B_k d_k)(T_1 F)$. This gives us

$$\|G\|_{D_k(O^\prime)} \leq C\|B_k d_k\|_{L^2(O^+)} \|T_0 F\|_{L^2(O^+)} + \|T_1 F\|_{L^2(O^+)} + \|K(\nu \wedge f)\|_X,$$

$$\leq C(1 + d(\Sigma, \partial O)^{-1})\|\nu \wedge f\|_{N_k, O^+} + \|K(\nu \wedge f)\|_X.$$

The proof for the norm $\|\nu \wedge f\|_{N_k, O^+}$ and for $\nu \wedge f \in \nu \wedge E$ is similar.

(iii) To estimate $\|N^+ f\|_{N_k, O^+}$ with $\|N^+ f\|_{N_k, O^-}$, extend given $F \in D(d_{O^+})$ to $\chi^{-1} F \in D(d_{O^+ \cup \Omega^-})$. Note that $\nu \wedge f = \nu \wedge (\chi^{-1} F)$ and that $d_k(\chi^{-1} F) = [d, \chi^{-1}] F + \chi^{-1}(d_k F)$. This gives

$$\|N^+ f\|_{N_k, O^+} \leq \inf(\|d_k(\chi^{-1} F)\|_{L^2(O^-)} + \|\chi^{-1} F\|_{L^2(O^-)}),$$

$$\leq L^u(O^+, \Omega^-) \inf(\|d_k F\|_{L^2(O^+)} + \|F\|_{L^2(O^+)}),$$

$$\approx L^u(O^+, \Omega^-) \|N^+ f\|_{N_k, O^+}.$$

The proof for $N^- E$ and for the opposite inequality is similar.

As in Chapter 3 we now introduce, besides the two reflection operators $E_k$ and $N$, the exterior/interior derivative operator $\Gamma_k$.

**Proposition 5.1.9.** For each wave number $k \in \mathbb{C}$, there exists a unique bounded, nilpotent operator $\Gamma_k : E \to E$, such that

$$\Gamma_k(\nu \gamma_d F) = \nu \gamma_d(d_k F), \quad F \in D(d_{\mathbb{R}^n})$$

$$\Gamma_k(\nu \delta F) = \nu \gamma_d(-\delta_k F), \quad F \in D(\tilde{d}_{\mathbb{R}^n})$$

$$\Gamma_k(\nu \gamma \pm F) = \nu \gamma \pm(d_k F) = \nu \gamma \pm(-\delta_k F), \quad F \in C^\pm_k E.$$

The operator $\Gamma_k$ commutes with $N$ and $E_k$.

In particular for the null-space, we have that $N(\Gamma_k) \cap N^+ E$ are the $k$-curl free tangential fields, $N(\Gamma_k) \cap N^- E$ are the $k$-divergence free normal fields and $N(\Gamma_k) \cap E^\pm_k E$ are the boundary traces of twosided $k$-monochromatic fields in $\Omega^\pm$ as in Proposition 3.2.5.

87
Proof. We see from Remark 5.1.2(i) that \(d_k\) acts boundedly and nilpotently in \(C^+_k\). Also, \(d_k\) induces a bounded nilpotent operator \([d_k]\) in \(N^+(\Omega^+):=D(d_\Omega)/D(d_{\Omega^-})\) and similarly for \(\delta_k\). Define \(\Gamma_k: \mathcal{E} \rightarrow \mathcal{E}\) via the first two identities in Proposition 5.1.9. Then \(\Gamma_k\) is a bounded nilpotent operator in \(\mathcal{E}\) such that \(\Gamma_k N = N \Gamma_k\). Then if \(F \in C^+_k\) we have

\[
\Gamma_k(\nu \gamma_+ F) = \Gamma_k(\nu \gamma_0 F) + \Gamma_k(\nu \gamma_0 F) = \nu \gamma_0(d_k F) + \nu \gamma_0(-\delta_k F)
\]

\[
= \nu \gamma_0(d_k F) + \nu \gamma_0(d_k F) = \nu \gamma_0(d_k F),
\]

and similarly for \(C^-_k\). Thus the last identity in Proposition 5.1.9 holds and since \(\Gamma_k\) is diagonal in the splitting \(E^+_k \mathcal{E} \oplus E^-_k \mathcal{E}\) we get \(\Gamma_k E_k = E_k \Gamma_k\).

We finish this section with some remarks on how the normal \(\nu\) acts as a multiplication operator. The basic rule is the following.

A multiplication operator involving an even number of factors \(\nu\) maps \(\mathcal{E} \rightarrow \mathcal{E}\), while a multiplication operator involving an odd number of factors \(\nu\) maps \(\mathcal{E} \rightarrow \mathcal{E}^* = \nu \mathcal{E}\).

We will not make this general statement more precise here since we can verify it in each concrete case. For a proof that indeed \(\mathcal{E}\) and \(\nu \mathcal{E}\) are dual spaces in the sense of Definition 1.4.7 with respect to the \(L_2(\Sigma; \wedge)\) pairing, see Proposition 6.1.2.

Example 5.1.10. The reflection/projection operators \(f \mapsto \nu f^\nu, f \mapsto \nu \perp (\nu \perp f)\) and \(f \mapsto \nu \perp (\nu \perp f)\) use two factors \(\nu\) and thus act in \(\mathcal{E}\).

If \(f \in \mathcal{E}\), then \(\nu \perp f = \gamma_0 F\) defines a functional on \(\mathcal{E}\) through the formula

\[
\int_\Sigma (g, \nu \perp f) := \int_{\Omega^+} (G, dF) + (\delta G, F), \quad G \in D(\delta_{\Omega^+}).
\]

For more examples, see Lemma 5.3.6.

Although we will not use it here, we remark that the Hodge complement operator \(f \mapsto f^\perp := f \perp e_{012,..n}\) acts in \(\mathcal{E}\).

5.2 Non-relativistic transmission problems

In preparation for the transmission problem for Maxwell’s equations, we generalise the formalism from Chapter 2 by introducing an operator \(T\).

Definition 5.2.1. The time reflection operator \(T\) in \(\wedge_\mathbb{C} \mathbb{R}^{n+1}\) is

\[
Tu := e_0 u^\perp e_0,
\]

where the associated projections \(T^\pm := \frac{1}{2}(I \pm T)\) are the space-like projection \(T^+ u = e_0 \perp (e_0 \wedge u)\) and the time-like projection \(T^- u = e_0 \wedge (e_0 \perp u)\). More generally let \(T_{\alpha, \beta}\), where \(\alpha, \beta \in \mathbb{C}\), denote the rescaling operator \(T_{\alpha, \beta} := \alpha T^- + \beta T^+ = \frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2} T\).
Let $\alpha^\pm, \beta^\pm \in \mathbb{C} \setminus \{0\}$ be material constants, let $\alpha := \alpha^+ / \alpha^-$ and $\beta := \beta^+ / \beta^-$ be jump parameters, let $c^\pm = \beta^\pm / \alpha^\pm$ be the propagation speed in $\Omega^\pm$, and let $k^\pm = \omega / c^\pm$ be the wave numbers, where $\omega \in \mathbb{C}$ is the frequency. Note that the jump parameters $\alpha, \beta$ and the wave numbers $k^\pm$ are not independent since they satisfy the relation $k^+ \beta = k^- \alpha$. Thus $\alpha \beta$, but not $\alpha / \beta$, is independent of $k^\pm$.

We here consider the following non-relativistic Dirac transmission problem. Given $g \in \mathcal{E}$, find $F^\pm \in C^\pm_{k^\pm} \mathcal{E}$ such that

$$
\begin{align*}
N^+ T^+ \left( \frac{1}{\beta^+} f^+ - \frac{1}{\beta^-} f^- \right) &= N^+ T^+ g, \\
N^+ T^- \left( \frac{1}{\alpha^+} f^+ - \frac{1}{\alpha^-} f^- \right) &= N^+ T^- g, \\
N^- T^+ \left( \beta^+ f^+ - \beta^- f^- \right) &= N^- T^+ g, \\
N^- T^- \left( \alpha^+ f^+ - \alpha^- f^- \right) &= N^- T^- g,
\end{align*}
$$

or in a more compact form

$$
\begin{align*}
N^+ \left( T^+ \frac{1}{\alpha^+ \beta^+} f^+ - T^- \frac{1}{\alpha^- \beta^-} f^- \right) &= N^+ g, \\
N^- \left( T^+ \alpha^+ \beta^+ f^+ - T^- \alpha^- \beta^- f^- \right) &= N^- g.
\end{align*}
$$

(5.3)

On one hand, this transmission problem generalizes that for Maxwell’s equations as explained in Example 5.2.4 below. On the other hand, it is special enough that the map $\{F^+, F^-\} \mapsto g$ commutes with an exterior/interior derivative operator.

**Proposition 5.2.2.** The time reflection operator $T$ acts boundedly in $\mathcal{E}$, and leaves the subspaces $N^\pm \mathcal{E}$ invariant since $TN = NT$. If $k = \omega \alpha / \beta$, then we have

$$
\begin{align*}
\Gamma_\omega T^+ \frac{1}{\alpha^+ \beta^+} f &= T^+ \frac{1}{\alpha^+ \beta^+} \Gamma_k f, \quad f \in N^+ \mathcal{E}, \\
\Gamma_\omega T^\pm \beta f &= T^\alpha \beta \Gamma_k f, \quad f \in N^\pm \mathcal{E}.
\end{align*}
$$

(5.4)

**Proof.** That $T$ and $N$ commutes follows since the vectors $e_0$ and $\nu$ are orthogonal. To establish the second identity for example, let $f \in N^- \mathcal{E}$ and $\nu \cdot f = \gamma \delta F, F \in \mathcal{D}(\delta_{\mathbb{R}^n})$, and calculate

$$
\begin{align*}
\Gamma_\omega T^\alpha \beta f &= -\nu \gamma \delta (\delta T^\alpha \beta F) = -\nu \gamma \delta ((\delta + \omega e_0) \alpha T^- + \beta T^+) F \\
&= -\nu \gamma \delta (T^\alpha \beta \delta F + \omega \alpha e_0 \cdot F) \\
&= -\nu \gamma \delta ((\alpha T^- + \beta T^+) (\delta F + k e_0 \cdot F)) = T^\alpha \beta \Gamma_k f.
\end{align*}
$$

In particular, applying $\Gamma_\omega$ to (5.4) gives

$$
\begin{align*}
N^+ \left( T^+ \frac{1}{\alpha^+ \beta^+} \Gamma_k f^+ - T^- \frac{1}{\alpha^- \beta^-} \Gamma_k f^- \right) &= N^+ \Gamma_\omega g, \\
N^- \left( T^+ \alpha^+ \beta^+ \Gamma_k f^+ - T^- \alpha^- \beta^- \Gamma_k f^- \right) &= N^- \Gamma_\omega g.
\end{align*}
$$

(5.5)

The main theorem in this chapter is the following.
Theorem 5.2.3. Consider the map
\[ S : N(d_{k^+} \oplus d_{k^-}) \subset C_{k^+} \mathcal{E} \oplus C_{k^-} \mathcal{E} \longrightarrow N(\Gamma_\omega) \subset \mathcal{E} : \{F^+, F^-\} \mapsto g, \]
given by (5.3). Let \( C_\Sigma \) and \( C_{k_\Sigma} \) be the constants from Theorem 5.3.1 and Theorem 5.3.5. Then \( S \) is a Fredholm operator if the jump parameters \( \alpha \) and \( \beta \) both belong to the region
\[ G(\Sigma) := C \setminus \{ix \mid x \in \mathbb{R}, 1/C_\Sigma \leq |x| \leq C_\Sigma\}, \]
and an isomorphism if \( \text{Im} k^+ > 0 \) and \( \text{Im} k^- > 0 \) and either
\[ |\arg(\alpha\beta)| + |\arg k^+ - \frac{\pi}{2}| + |\arg k^- - \frac{\pi}{2}| < \pi \quad \text{or} \quad \min(|\alpha - \beta|^2, |\frac{1}{\alpha} - \frac{1}{\beta}|^2) < 2/C_{k\Sigma}. \]
Since \( S \) preserves \( \wedge \), the same holds true for the restricted map \( S_j : N(d_{k^+} \oplus d_{k^-}, \wedge^3) \longrightarrow N(\Gamma_\omega, \wedge^3). \)

Proof. Theorem 5.3.4 and Theorem 5.3.5 below prove that the map \( S : C_{k^+} \mathcal{E} \oplus C_{k^-} \mathcal{E} \rightarrow \mathcal{E} : \{F^+, F^-\} \mapsto g \) is Fredholm/an isomorphism under the conditions above. Thus Lemma 5.3.7(iii) below shows that the same is true for \( S : N(d_{k^+} \oplus d_{k^-}) \rightarrow N(\Gamma_\omega) : \{F^+, F^-\} \mapsto g. \) \( \square \)

Example 5.2.4. The non-relativistic transmission problem (5.3) is motivated by the Maxwell transmission problem (1) in the introduction. Recall that each of the domains \( \Omega^+ \) and \( \Omega^- \) is composed of a linear, homogeneous, isotropic, possibly conducting material with permittivity \( \varepsilon^\pm > 0 \), permeability \( \mu^\pm > 0 \) and conductivity \( \sigma^\pm \geq 0 \). Define \( \varepsilon_\pm := \varepsilon^\pm + i \sigma^\pm/\omega \), \( \alpha^\pm := \sqrt{\varepsilon_\pm} \) and \( \beta^\pm := 1/\sqrt{\mu^\pm} \) so that the jump parameters are \( \alpha := \sqrt{\varepsilon^+ / \varepsilon^-} \) and \( \beta := \sqrt{\mu^- / \mu^+} \), the propagation speeds are \( c^\pm = 1/\sqrt{\varepsilon^\pm \mu^\pm} \) and the wave numbers are \( k^\pm = \omega \sqrt{\varepsilon^\pm \mu^\pm} \). Here we use the principal square roots.

We are given an incoming electromagnetic field \( F_0 : \Omega^- \rightarrow \Lambda^2 \mathbb{R}^4 \), as in (1.24), satisfying \( D_k F_0 \) in \( \Omega^- \) (but not the radiation condition (5.1)) and with boundary trace \( f_0 \in \mathcal{E} \). We want to solve for the reflected field \( F^- \in C_{k^-} \mathcal{E} \cap \Lambda^2 \mathbb{R}^4 \) and the transmitted field \( F^+ \in C_{k^+} \mathcal{E} \cap \Lambda^2 \mathbb{R}^4 \) such that the jump relations
\[ \nu \wedge (B^+ - B^-) = \nu \wedge B_0, \]
\[ \nu \wedge (E^+ - E^-) = \nu \wedge E_0, \]
\[ \nu \perp (\frac{1}{\mu^+}B^+ - \frac{1}{\mu^-}B^-) = \nu \perp (\frac{1}{\mu^+}B_0), \]
\[ \nu \perp (\epsilon^+ E^+ - \epsilon^- E^-) = \nu \perp (\epsilon^- E_0) - \rho_s, \]
holds across \( \Sigma \), where the surface charge \( \rho_s \) is determined by the continuity equation
\[ -i\omega \rho_s + \nu \perp (J_0 + J^- - J^+) \]. Therefore (1) is a special case of (5.3) with
\[ g = N^+ T_{\frac{1}{\alpha^+ \beta^+}} f_0 + N^- T_{\alpha^- \beta^-} f_0 \in \mathcal{E}(\Sigma; \Lambda^2), \]
\[ T_{\frac{1}{\alpha \beta}} F = T \frac{1}{\sqrt{\varepsilon \mu}} F = \mathcal{T}_0 \wedge E + B, \]
\[ T_{\alpha \beta} F = T \frac{1}{\sqrt{\varepsilon \mu}} F = \mathcal{T}_0 \wedge (D + \frac{i}{\omega} J) + H. \]
Since we assume that the incoming EM field $F_0$ is a pure $\Lambda^2$-field, we have $\Gamma_k f_0 = 0$. Thus, for the Maxwell transmission problem, the right hand side in (5.5) vanishes. In particular, Theorem 5.2.3 implies Theorem 0.0.1.

One can also consider the case of a prescribed tangential surface current $j_s = \vec{e}_0 \rho_s + \int J_s \in \nu \cdot \mathcal{E}(\Sigma; \Lambda^2)$, which gives a right hand side $g = \nu j_s \in \mathcal{E}(\Sigma; \Lambda^2)$ in (5.3). We then assume that the total surface charge is preserved, i.e. we have the continuity equation

$$\Gamma_\omega (\nu j_s) = \nu(-i \omega \rho_s + \nabla_{\text{tan}} \cdot J_s) = 0. \quad (5.7)$$

As before, this implies that the right hand side in (5.5) vanishes and thus we can apply Theorem 5.2.3.

Before starting with the spectral estimates in Section 5.3, we investigate in this section the ansatz

$$\mathcal{E} \rightarrow C^+_k \mathcal{E} \oplus C^-_k \mathcal{E} : f \longmapsto C^+_k f \oplus C^-_k f = F^+ \oplus F^- \quad (5.8)$$

for the non-relativistic transmission problem (5.3), which we will use to prove existence of solutions to (5.3). Since the space $\mathcal{E}$ is of “mixed regularity $\pm 1/2$”, and thus $E_{k^+} - E_{k^-}$ is not a compact operator, it is not clear if (5.8) is a Fredholm map. To overcome this difficulty, we use the following trick: instead of eliminating the leading term in the expansion of $\vec{E}_k(x)$, we eliminate the second term.

**Lemma 5.2.5.** For any finite, non-zero material constants $\alpha^\pm$ and $\beta^\pm$ in the non relativistic transmission problem (5.3), the following hold.

(i) The operators $c^+ E^{\pm}_k + c^- E^{-\pm}_k$ are Fredholm operators with index zero on $\mathcal{E}$.

(ii) The spaces $\mathcal{E}^{\pm}_k \cap \mathcal{E}^{\pm}_k$ and $\mathcal{E} / (\mathcal{E}^{\pm}_k + \mathcal{E}^{\pm}_k)$ are finite dimensional.

For a fixed speed $c^-$ (and frequency $\omega$) there exists a discrete set $S = S(c^-) \subset \mathbb{C}$ such that if $c^+ \notin S$, then $c^+ E^{\pm}_k + c^- E^{-\pm}_k$ are isomorphisms on $\mathcal{E}$ and we have exact splittings

$$\mathcal{E} = \mathcal{E}^{\pm}_k \oplus \mathcal{E}^{\pm}_k$$

**Proof.** Recall the expansion $\vec{E}_k(x) = E(x) + k e_0 \Phi(x) + O(r^{-(n-3)})$ around $x = 0$ from Proposition 1.2.5. We see that the first two terms are the non-compact part of the operator $E_k = E - k e_0 \Phi + K_c : \mathcal{E} \rightarrow \mathcal{E}$. It follows that

$$c^+ E^{\pm}_k + c^- E^{-\pm}_k = \frac{1}{2} ((c^+ + c^-) I \pm (c^+ E^+ - c^- E^-))$$

$$= \frac{1}{2} ((c^+ + c^-) I \pm ((c^+ - c^-) E + (c^+ K_{c^+} - c^- K_{c^-}))).$$

For all $c^\pm \in \mathbb{C} \setminus \{0\}$, the operator $(c^+ + c^-) I \pm (c^+ - c^-) E$ is an isomorphism, since $E$ is a reflection operator. Thus $c^+ E^{\pm}_k + c^- E^{-\pm}_k$ are Fredholm operators with index zero. Moreover, applying analytic Fredholm theory 1.4.6 to the analytic family $c^+ \mapsto \pm ((c^+ + c^-) I \pm (c^+ - c^-) E)^{-1} (c^+ K_{c^+} - c^- K_{c^-})$ of compact operators shows the existence
of a discrete set $S = S(c^-) \subset \mathbb{C}$ such that both operators $c^+ E^\pm_{k^+} + c^- E^\pm_{k^-}$ are isomorphisms when $c^+ \notin S$. Now the stated properties of the splittings of $\mathcal{E}$ follow from the inclusions

$$E^\pm_{k^+} \mathcal{E} \cap E^\pm_{k^-} \mathcal{E} \subset \mathbb{N}(c^+ E^\pm_{k^+} + c^- E^\pm_{k^-}),$$

$$R(c^+ E^\pm_{k^+} + c^- E^\pm_{k^-}) \subset E^\pm_{k^+} \mathcal{E} + E^\pm_{k^-} \mathcal{E}.$$

\[ \square \]

**Proposition 5.2.6.** The ansatz (5.8) is a Fredholm map with deficiency indices

$$\dim \mathbb{N}(C^+_{k^+} \oplus C^-_{k^-}) = \dim(C^+_{k^+} \mathcal{E} \oplus C^-_{k^-} \mathcal{E})/\mathbb{R}(C^+_{k^+} \oplus C^-_{k^-}) = \dim(E^+_{k^+} \mathcal{E} \cap E^+_{k^-} \mathcal{E}).$$

In particular, the index is zero.

**Proof.** The ansatz (5.8) is a semi-Fredholm map since composing it with the (bounded) trace map $c^+ \gamma_+ + c^- \gamma_- : C^+_{k^+} \mathcal{E} \oplus C^-_{k^-} \mathcal{E} \to \mathcal{E}$ gives the Fredholm map $c^+ E^+_{k^+} + c^- E^-_{k^-}$ from Lemma 5.2.5. If we identify $C^\pm_{k^\pm} \mathcal{E}$ with $E^\pm_{k^\pm} \mathcal{E}$ and use the dualities from Proposition 6.1.3, then we obtain dual maps

$$\gamma_- + \gamma_+ : \mathcal{E} \leftrightarrow C^-_{k^+} \mathcal{E} \oplus C^+_{k^-} \mathcal{E},$$

$$C^+_{k^+} \oplus C^-_{k^-} : \mathcal{E} \rightarrow C^+_{k^+} \mathcal{E} \oplus C^-_{k^-} \mathcal{E}.$$

Note that $\mathbb{N}(C^+_{k^+} \oplus C^-_{k^-}) = E^+_{k^+} \mathcal{E} \cap E^+_{k^-} \mathcal{E}$ and $\mathbb{N}(\gamma_- + \gamma_+) = \{(F^-, F^+) ; \gamma_- F^- = -\gamma_+ F^+ \in E^+_{k^+} \mathcal{E} \cap E^+_{k^-} \mathcal{E} \} \approx E^+_{k^+} \mathcal{E} \cap E^+_{k^-} \mathcal{E}.$

We also note the following perturbation result, proved by Mitrea–Mitrea [44], despite the mixed regularity of $\mathcal{E}$ and Example 3.2.12.

**Proposition 5.2.7.** For any $k$ and $k' \in \mathbb{C}$, the difference $N^\pm E_k N^\pm - N^\pm E_{k'} N^\pm$ is a compact operator on $\mathcal{E}$.

**Proof.** Define the operator $\Phi^\nu f(x) := \int_\Sigma \Phi(x-y) \nu(y) f(y) d\sigma(y)$. From the kernel expansions around $x = 0$ in Proposition 1.2.5, we have $N^+ E_k N^+ - N^+ E_{k'} N^+ = -(k - k')e_0 N^+ \Phi^\nu N^+$. Thus it suffices to prove that $N^+ \Phi^\nu N^+ : \mathcal{E} \to \mathcal{E}$ is a compact operator. To see this, consider the maps

$$D(d_{\Omega^+}) \xrightarrow{\gamma_d} \nu \wedge \mathcal{E} \xrightarrow{\Phi^\nu} D(d_{\Omega^+}) \xrightarrow{\gamma_d} \nu \wedge \mathcal{E},$$

where we observe that the tangential trace map $\gamma_d$ is bounded and surjective and that the convolution operator $\Phi^\nu$ is smoothing of order $-2$ on $\Omega^+$. Since $d_{\mathbb{R}^3}$ anti-commutes with $\gamma_d$ and commutes with $\Phi^\nu$, we see that $(\Phi^\nu) \gamma_d : D(d_{\Omega^+}) \to D(d_{\Omega^+})$ is a compact operator, and thus so is $N^+ \Phi^\nu N^+ \approx \gamma_d(\Phi^\nu) : \nu \wedge \mathcal{E} \to \nu \wedge \mathcal{E}$.

The proof for $N^- E_k N^- - N^- E_{k'} N^-$ is similar. \[ \square \]

Since by Proposition 2.1.2, $\lambda - E_k N$ is a Fredholm operator if and only if $c - C = c - N^+ E_k N^+ - N^- E_k N^-$ is a Fredholm operator, we get the following corollary.

**Corollary 5.2.8.** The essential spectrum $\sigma_{ess}(E_k N; \mathcal{E})$ is independent of $k$. 92
We finish this section by rewriting the non-relativistic transmission problem (5.3), with the ansatz (5.8), as a generalised type of rotation equation. These results will not be used in Section 5.3.

**Lemma 5.2.9.** The system of equations

\[
\begin{align*}
N^+T^+f_1 &= 0, & N^-T^+f_3 &= 0, \\
N^+T^-f_2 &= 0, & N^-T^-f_4 &= 0,
\end{align*}
\]

is equivalent to the single equation

\[
(f_1 + f_2 + f_3 + f_4) + N(f_1 + f_2 - f_3 - f_4) \\
+ T(f_1 - f_2 + f_3 - f_4) + NT(f_1 - f_2 - f_3 + f_4) = 0.
\]

Noting that the reflection operators \(N\) and \(T\) commutes, the proof is straightforward and we omit it. For convenience, we define operators

\[
\begin{align*}
E_k^\pm &:= \frac{1}{2}(E_k^+ + E_k^-), \\
R_k^\pm &:= \frac{1}{2}(E_k^+ - E_k^-),
\end{align*}
\]

where \(k := \{k^+, k^-\}\). Note that \(E_k^2 + R_k^2 = I\). A straightforward calculation now give us the following.

**Proposition 5.2.10.** The equation system (5.3) with the ansatz (5.8) is equivalent with

\[
\left(T_{a^+\alpha^-\beta^+\beta^-} E_k^\pm - T_{a^-\alpha^-\beta^+\beta^-} N(I + R_k^\pm)\right)f = 2\tilde{\gamma},
\]

where \(\tilde{\gamma} := (N^+T_{a^+\alpha^-\beta^+\beta^-} + N^-)g\). If \(\alpha^+ + \alpha^- \neq 0\) and \(\beta^+ + \beta^- \neq 0\), then we also have

\[
\left(I - E_k^2 T_{a^+\alpha^-\beta^+\beta^-} N(I + R_k^\pm) - R_k^2\right)f = 2E_k T_{a^+\alpha^-\beta^+\beta^-} \gamma. \tag{5.9}
\]

When the function space under consideration is such that \(E_{k_1} - E_{k_2}\) is compact for any \(k_i \in C\), as is the case for \(L_2(\Sigma; \lambda)\), then the operator \(R_k^\pm\) is compact, and \(E_k^2 = I\) modulo compact operators. Thus the principal part of the operator in (5.9) is

\[
I - E_k T_{a^+\alpha^-\beta^+\beta^-} N.
\]

This is a generalisation of the operator \(\lambda - E_k N\) used for the relativistic transmission problem.

### 5.3 Spectral estimates

We first prove spectral estimates in the relativistic special case \(c^+ = c^-\), or equivalently \(\alpha = \beta\), of the non-relativistic transmission problem (5.3) which now reduces to the system

\[
\begin{align*}
N^+(\alpha^- f^+ - \alpha^+ f^-) &= N^+g, \\
N^-(\alpha^+ f^+ - \alpha^- f^-) &= N^-g,
\end{align*} \tag{5.10}
\]


where \( f^\pm \in E_k^\pm \mathcal{E} \) and \( k := k^+ = k^- \), \( c := c^+ = c^- \) and \( \alpha := \alpha^+ / \alpha^- = \beta \). We here assume that the projective jump parameters \( \alpha^\pm \in \mathbb{C} \) and at most one of them is zero. Note that for convenience we have scaled \( g \) differently here. Recall from Chapter 2 that the system (5.10) is equivalent to the rotation operator equation \( (\lambda - E_k N) f = 2 E_k g \), where \( \lambda := (\alpha^+ + \alpha^-) / (\alpha^+ - \alpha^-) \). The spectral estimates here use the Lipschitz type constants \( L^w \) for weakly Lipschitz domains from Definition 5.1.7.

**Theorem 5.3.1.** Let \( 1 \leq C_\Sigma < \infty \) be the constant

\[
C_\Sigma := \inf_\mathcal{O} \min(L^w_0(\mathcal{O}^+, \mathcal{O}^-), L^w_0(\mathcal{O}^-, \mathcal{O}^+)),
\]

where the infimum is over all bounded, smooth neighbourhoods \( \mathcal{O} \) of \( \Sigma \). Then we have essential spectral estimates

\[
\sigma_{\text{ess}}(E_k N; \mathcal{E}) \subset \{ \lambda \in \mathbb{C} : |\lambda| = 1 \},
\]

\[
\sigma_{\text{ess}}(E_k N; \mathcal{E}) \subset (\mathcal{O}(1 + i C_\Sigma, C_\Sigma) \cap \mathcal{O}(-1 + i C_\Sigma, C_\Sigma)) \cup \left( \mathcal{O}(1 - i C_\Sigma, C_\Sigma) \cap \mathcal{O}(-1 - i C_\Sigma, C_\Sigma) \right).
\]

In particular this essential spectral estimate is independent of the wave number \( k \), and \( \sigma_{\text{ess}}(E_k N; \mathcal{E}) \cap \mathbb{R} = \emptyset \).

**Theorem 5.3.2.** Assume that \( \text{Im } k > 0 \) and let \( \eta := |\text{Re } k| / \text{Im } k \) and

\[
C_{k, \Sigma} := \min(L^w_1(\Omega^-, \Omega^+), |k| L^w_0(\Omega^-, \Omega^+), L^w_1(\Omega^+, \Omega^-), |k| L^w_0(\Omega^+, \Omega^-)) / \text{Im } k.
\]

Then we have spectral estimates

\[
\sigma(E_k N; \mathcal{E}) \subset \left( \mathcal{O}(i \eta, \sqrt{1 + \eta^2}) \cup \mathcal{O}(-i \eta, \sqrt{1 + \eta^2}) \right) \setminus \left( \mathcal{O}(i \eta, \sqrt{1 + \eta^2}) \cap \mathcal{O}(-i \eta, \sqrt{1 + \eta^2}) \right),
\]

\[
\sigma(E_k N; \mathcal{E}) \subset \left( \mathcal{O}(1 + i C_{k, \Sigma}, C_{k, \Sigma}) \cap \mathcal{O}(-1 + i C_{k, \Sigma}, C_{k, \Sigma}) \right) \cup \left( \mathcal{O}(1 - i C_{k, \Sigma}, C_{k, \Sigma}) \cap \mathcal{O}(-1 - i C_{k, \Sigma}, C_{k, \Sigma}) \right).
\]

For the geometry of the estimates (5.13) and (5.14), see Figure 1.5 and 1.3 in the Appendix. Note that Proposition 2.2.6 is a special case of the spectral estimate (5.13), and that this estimate as well as (5.11) is independent of the interface \( \Sigma \).

For the double layer type operators \( N^{\pm} E_k N^{\pm} \) we get the following essential spectral estimates, also obtained by Mitrea–Mitrea [44].

**Corollary 5.3.3.** If \( C_{\Sigma} \) denotes the constant in Theorem 5.3.1, then there exists a universal constant \( C \) such that

\[
\sigma_{\text{ess}}(N^{\pm} E_k; N^{\pm} \mathcal{E}) \subset [-1 + C / C_{\Sigma}^2, 1 - C / C_{\Sigma}^2].
\]

For the monogenic double layer type operators \( N^{\pm} E : N^{\pm} \mathcal{E} \to N^{\pm} \mathcal{E} \), we also have

\[
\sigma(N^{\pm} E; N^{\pm} \mathcal{E}) \subset [-1, 1].
\]
Proof. We have here applied the spectral map \( \frac{1}{2}(\lambda + 1/\lambda) \) from Proposition 2.1.2. From (5.13) it follows that \( \sigma(N^\pm E_k; N^\mp \mathcal{E}) \subset [-1, 1] \) when \( k \in i\mathbb{R}_+ \). Since \( k \mapsto \sigma(N^\pm E_k; N^\mp \mathcal{E}) \) is continuous we can take the limit \( k \to 0 \). The estimate of the essential spectrum follows from (5.12).

We now state the generalisations of Theorems 5.3.1 and 5.3.2 to the non-relativistic transmission problem (5.3), which give conditions on the material constants \( \alpha^\pm \) and \( \beta^\pm \) when the map
\[
C_{k^+}^+ \mathcal{E} \oplus C_{k^-}^- \mathcal{E} \longrightarrow \mathcal{E} : \{ F^+, F^- \} \longmapsto g,
\]
given by
\[
\begin{align*}
T_{\alpha^+, \beta^+} \gamma_\delta(F^+) - T_{\alpha^-, \beta^-} \gamma_\delta(F^-) &= \nu \wedge g, \\
T_{\alpha^+, \beta^+} \gamma_\delta(F^+) - T_{\alpha^-, \beta^-} \gamma_\delta(F^-) &= \nu \wedge g
\end{align*}
\]
is (Fredholm) invertible.

**Theorem 5.3.4.** Let \( C_{\Sigma} \) be the constant from Theorem 5.3.1. Then the map (5.15) is Fredholm with index zero if the jump parameters \( \alpha \) and \( \beta \) both belong to the region
\[
G(C_{\Sigma}) := C \setminus \{ ix : x \in \mathbb{R}, 1/C_{\Sigma} \leq |x| \leq C_{\Sigma} \}.
\]

**Theorem 5.3.5.** Assume that \( \text{Im} \ k^+ > 0 \) and \( \text{Im} \ k^- > 0 \), and let
\[
C_{k^+}^\Sigma := \min \left( L^w_1(\Omega^-, \Omega^+) + (\max |k^\pm|) L^w_0(\Omega^-, \Omega^+) \right), \\
L^w_1(\Omega^+, \Omega^-) + (\max |k^\pm|) L^w_0(\Omega^+, \Omega^-) / \sqrt{\text{Im} k^+ \text{Im} k^-}.
\]
Then the map (5.15) is an isomorphism if either of the following inequalities hold.
\[
\begin{align*}
|\arg(\alpha \beta)| + |\arg k^+ - \frac{\pi}{2}| + |\arg k^- - \frac{\pi}{2}| < \pi, \\
\min(|\alpha - \beta^\pm|, |\frac{1}{\alpha} - \frac{1}{\beta^\pm}|) < 2/C_{k^+}^\Sigma.
\end{align*}
\]

We now turn to the proofs of these four theorems. For the estimates of the essential spectra in Theorem 5.3.1 and Theorem 5.3.4 we will work in the range \( \mathcal{R}(\Gamma_k) \) and use a coercivity in the Krein form from Definition 6.1.1. For the estimates of the full spectra in Theorem 5.3.2 and Theorem 5.3.5 we will work in the null-space \( \mathcal{N}(\Gamma_k) \) and use a coercivity in the Dirac form from Definition 6.1.1. In both cases we obtain a priori estimates under certain assumptions on either the direction or size of the jump parameter(s).

**Lemma 5.3.6.** If \( f \in E_k^+ \mathcal{E} \), then
\[
\begin{align*}
\int_\Sigma (\overline{\epsilon_0 f}, \nu f) &= \int_{\Omega^+} 2\text{Im} k |F|^2, \\
\int_\Sigma (\overline{\epsilon_0 f}, \nu \wedge f) &= \int_{\Omega^+} \text{Im} k |F|^2 + i \text{Re} k (F, TF^c) + 2i \text{Im} (d_k F, \overline{\epsilon_0} F^c), \\
\int_\Sigma (\overline{\epsilon_0} \wedge f, \nu \wedge f) &= \int_{\Omega^+} \text{Im} k |F|^2 + i \text{Re} k (F, TF^c) \\
&\quad + (d_k F, \overline{\epsilon_0} \wedge F^c) + (\overline{\epsilon_0} \wedge F, (d_k F)^c), \\
\int_\Sigma (\overline{\epsilon_0} f, \nu \wedge f) &= \int_{\Omega^+} (d_k F, \overline{\epsilon_0} \wedge F^c) + (\overline{\epsilon_0} \wedge F, (d_k F)^c).
\end{align*}
\]
The corresponding statements for \( f \in E_\kappa^c \), replacing \( \int_{\Omega^+} \) by \( -\int_{\Omega^-} \), are also true when \( C_\kappa^c E \subset L^2(\Omega^-) \), i.e. if \( \text{Im} \ k > 0 \) or \( k = 0 \) and \( n \geq 3 \).

We omit the proof, since it is analogous to that of Lemma 3.1.7. Note that the boundary integrals here should be interpreted in the sense explained in the end of Section 5.1.

**Lemma 5.3.7.** Let \( \Gamma_i : \mathcal{H}_i \to \mathcal{H}_i \), \( i = 1, 2 \), be bounded, nilpotent operators and let \( T : \mathcal{H}_1 \to \mathcal{H}_2 \) be a bounded operator such that \( \Gamma_2 T = T \Gamma_1 \).

1. If \( \Gamma_1 \) gives a Hodge splitting of \( \mathcal{H}_1 \) and if \( T : R(\Gamma_1) \to R(\Gamma_2) \) has a priori estimates, then so does \( T : \mathcal{H}_1 \to \mathcal{H}_2 \).

2. If \( T : N(\Gamma_1) \to N(\Gamma_2) \) has exact a priori estimates, then so does \( T : \mathcal{H}_1 \to \mathcal{H}_2 \).

3. If \( T : \mathcal{H}_1 \to \mathcal{H}_2 \) is a Fredholm operator, then so is \( T : N(\Gamma_1) \to N(\Gamma_2) \). Moreover, if \( T : \mathcal{H}_1 \to \mathcal{H}_2 \) is an isomorphism, then so is \( T : N(\Gamma_1) \to N(\Gamma_2) \).

**Remark 5.3.8.** This lemma can be seen as an application of the five lemma, as in Pryde [54], by considering the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & N(\Gamma_1) \xrightarrow{i_1} \mathcal{H}_1 \xrightarrow{\Gamma_1} R(\Gamma_1) \rightarrow 0 \\
& \downarrow T & \downarrow T & \downarrow T \\
0 & \rightarrow & N(\Gamma_2) \xrightarrow{i_2} \mathcal{H}_2 \xrightarrow{\Gamma_2} R(\Gamma_2) \rightarrow 0,
\end{array}
\]

where the rows are exact.

**Proof.** (i) Write \( x = x_1 + x_0 + x_2 \in R(\Gamma_1) \oplus (N(\Gamma_1) \cap N(\Gamma_1^*)) \oplus R(\Gamma_1^*) \). By hypothesis, there exists a compact operator \( K : R(\Gamma_1) \to \mathcal{X} \), such that

\[
\|x_1\|_1 \lesssim \|Tx_1\|_2 + \|Kx_1\|_\mathcal{X}.
\]

Applying this estimate to \( \Gamma_1 x = \Gamma_1 x_2 \in R(\Gamma_1) \) and using \( \Gamma_2 T = T \Gamma_1 \) give us

\[
\|x_2\| \approx \|\Gamma_1 x_1\| \lesssim \|T \Gamma_1 x\|_2 + \|K \Gamma_1 x\|_\mathcal{X} \approx \|Tx_2\|_2 + \|Kx_1\|_\mathcal{X},
\]

since \( \Gamma_1 : N(\Gamma_1^\perp) \to R(\Gamma_1) \) is an isomorphism and \( \Gamma_2 \) is bounded. We now calculate

\[
\begin{align*}
\|x_1\|_1 & \lesssim \|Tx_2\|_2 + \|Tx_0\|_2 + \|Kx_1\|_\mathcal{X} \\
& \lesssim \|Tx_2\|_2 + \|x_0\|_1 + \|x_2\|_1 + \|Kx_1\|_\mathcal{X} \\
& \lesssim \|Tx_2\|_2 + \|x_0\|_1 + (\|Tx_2\|_2 + \|K \Gamma_1 x\|_\mathcal{X}) + \|Kx_1\|_\mathcal{X} \\
& \approx \|Tx_2\|_2 + \|x_0\|_1 + \|K \Gamma_1 x\|_\mathcal{X} + \|Kx_1\|_\mathcal{X} \\
\|x\|_1 & \approx \|x_1\|_1 + \|x_0\|_1 + \|x_2\|_1 \\
& \lesssim (\|Tx_2\|_2 + \|x_0\|_1 + \|K \Gamma_1 x\|_\mathcal{X} + \|Kx_1\|_\mathcal{X}) + \|x_0\|_1 + (\|Tx_2\|_2 + \|K \Gamma_1 x\|_\mathcal{X}) \\
& \approx \|Tx_2\|_2 + \|x_0\|_1 + (\|K \Gamma_1 x\|_\mathcal{X} + \|Kx_1\|_\mathcal{X}.
\end{align*}
\]

This proves (i), since \( K \Gamma_1 : \mathcal{H}_1 \to \mathcal{X} \) is compact because \( \Gamma_1 \) is bounded. Note that we have also proved that if the Hodge splitting induced by \( \Gamma_1 \) is exact, and if \( T : R(\Gamma_1) \to R(\Gamma_2) \) has exact a priori estimates, then \( T : \mathcal{H}_1 \to \mathcal{H}_2 \) has exact a priori estimates.
The proof of (ii) is similar.

(iii) If $T : H_1 \to H_2$ is an isomorphism, then obviously $T : N(\Gamma_1) \to N(\Gamma_2)$ is injective. Moreover, if $\Gamma_2 y = 0$, then $y = T x$ for some $x \in H_1$. But $T(\Gamma_1 x) = \Gamma_2 y = 0$, so $x \in N(\Gamma_1)$, which proves the surjectivity. A generalisation of this argument yields the Fredholm case. One can also apply the dimension theorem to the commutative diagram above, which gives

$$\alpha_N - \alpha + \alpha_R - \beta_N + \beta - \beta_R = 0,$$

where $\alpha$ denotes nullity and $\beta$ denotes deficiency. Here all dimensions except possibly $\beta_N$ are finite, and thus $\beta_N < \infty$. 

In the proofs of Theorem 5.3.1 and Theorem 5.3.2 below we establish a priori estimates for the map $\mathcal{E} \ni f \mapsto g \in \mathcal{E}$ defined by

$$\nu \wedge (\alpha^+ E_k^+ f - \alpha^+ E_k^- f) = \nu \wedge g,$$

$$\nu \vee (\alpha^+ E_k^+ f - \alpha^+ E_k^- f) = \nu \vee g.$$

Proof of Theorem 5.3.1. Fix bounded, smooth, open sets $\bar{O} \subseteq O \subseteq \Sigma$, and let $O^\pm := O \cap \Omega^\pm$ and $\bar{O}^\pm := \bar{O} \cap \Omega^\pm$. Fix extension maps $\chi^{-1}_+ : O^+ \to \bar{O}^+ \cup \Sigma \cup O^-$ for $\delta$, and $\chi^{-1}_- : O^- \to \bar{O}^- \cup \Sigma \cup O^+$ for $d$, as in Definition 5.1.7.

By Lemma 5.3.7(i) and Proposition 6.2.1 it suffices to prove a priori estimates for $f \in R(\Gamma_k)$. Consider the Cauchy extensions $F^+ \in C^+_k \mathcal{E} \subseteq D(\delta, \Sigma^+)$ and $F^- \in C^-_k \mathcal{E} \subseteq D(d, \Sigma^-)$, with Hodge potentials

$$\nabla_k \vee U^+ = F^+ \text{ in } O^+,$$

$$\nabla_k \wedge U^- = F^- \text{ in } O^-,$$

which we extend to $U^+ := \chi^{-1}_+ U^+ \in D(\delta, \Sigma^+)$ with support in $\bar{O}^+ \cup \Sigma \cup O^-$, and $U^- := \chi^{-1}_- U^- \in D(d, \Sigma^-)$ with support in $\bar{O}^- \cup \Sigma \cup O^+$. We here have bounded maps

$$E^+_k \mathcal{E} \ni f^\pm \mapsto N^+ u^\pm \in N^+ \mathcal{E},$$

$$E^+_k \mathcal{E} \ni f^+ \mapsto \nabla_k \vee U^+ \in L_2(\mathbb{R}^n),$$

$$E^-_k \mathcal{E} \ni f^- \mapsto \nabla_k \wedge U^- \in L_2(\mathbb{R}^n),$$

and, using Theorem 4.2.5, Schur’s lemma and Proposition 4.1.9, compact maps

$$E^+_k \mathcal{E} \ni f^\pm \mapsto U^\pm \in L_2(\mathbb{R}^n),$$

$$E^-_k \mathcal{E} \ni f^\pm \mapsto F^\pm \in L_2(\bar{O}^\pm \setminus O^\pm),$$

$$E^+_k \mathcal{E} \ni f^+ \mapsto \chi^{-1}_+ F^+ - \nabla_k \vee U^+ = [\chi^{-1}_+, \delta] U^+ \in L_2(\mathbb{R}^n),$$

$$E^-_k \mathcal{E} \ni f^- \mapsto \chi^{-1}_- F^- - \nabla_k \wedge U^- = [\chi^{-1}_-, d] U^- \in L_2(\mathbb{R}^n).$$

Using the jump conditions, we write

$$\int \Sigma (u^+, \nu \wedge g^c) = \alpha^{-c} \int \Sigma (u^+, \nu \wedge f^+ c) - \alpha^+ \int \Sigma (u^+, \nu \wedge f^- c), \quad (5.19)$$

$$\int \Sigma (g, \nu \wedge u^- c) = \alpha^+ \int \Sigma (f^+, \nu \wedge u^- c) - \alpha^- \int \Sigma (f^-, \nu \wedge u^- c), \quad (5.20)$$

97
and observe the coercivity
\[
\int_{\Sigma} (u^+, \nu \wedge f^+\partial_c) = \int_{\Omega^+} (U^+, \nabla \wedge F^+\partial_c) + (\nabla \cdot U^+, F^+\partial_c) = \int_{\Omega^+} \| F^+ \|^2
\]
\[+ O(\| U^+ \|_{L^2(\partial^+)} \| F^+ \|_{L^2(\partial^+)}) + \| \nabla \cdot U^+ \|_{L^2(\partial^+ \setminus \Omega^+)} \| F^+ \|_{L^2(\partial^+ \setminus \Omega^+)}),
\]
\[- \int_{\Sigma} (f^-, \nu \wedge u^-\partial_c) = \int_{\Omega^-} (F^-, \nabla \wedge U^-\partial_c) + (\nabla \cdot F^-, U^-\partial_c) = \int_{\Omega^-} \| F^- \|^2
\]
\[+ O(\| F^- \|_{L^2(\partial^-)} \| U^- \|_{L^2(\partial^-)}),
\]
and the cancellation
\[- \int_{\Sigma} (u^+, \nu \wedge f^-\partial_c) + \int_{\Sigma} (f^+, \nu \wedge u^-\partial_c)
\]
\[= \int_{\Omega^-} (U^+, \nabla \wedge F^-\partial_c) + (\nabla \cdot U^+, F^-\partial_c) + \int_{\Omega^+} (F^+, \nabla \wedge U^-\partial_c) + (\nabla \cdot F^+, U^-\partial_c)
\]
\[= \int_{\Omega^-} (\nabla \cdot U^+, \nabla \wedge U^-\partial_c) + \int_{\Omega^+} (\nabla \cdot U^+, \nabla \wedge U^-\partial_c)
\]
\[+ O(\| U^+ \|_{L^2(\partial^-)} \| F^- \|_{L^2(\partial^-)} + \| \nabla \cdot U^+ \|_{L^2(\partial^-)} \| U^- \|_{L^2(\partial^-)})
\]
\[+ O(\| U^+ \|_{L^2(\partial^-)} \| \nabla \wedge U^- \|_{L^2(\partial^-)} + \| F^+ \|_{L^2(\partial^-)} \| U^- \|_{L^2(\partial^-)} \| U^- \|_{L^2(\partial^-)}) \approx 0,
\]
modulo compact terms. We can also estimate these “off-diagonal terms” as
\[- \int_{\Sigma} (u^+, \nu \wedge f^-\partial_c) = \int_{\Omega^-} (U^+, \nabla \wedge F^-\partial_c) + (\nabla \cdot U^+, F^-\partial_c) \approx \int_{\Omega^-} (\chi_+^{-1} F^+, F^-\partial_c),
\]
\[\int_{\Sigma} (f^+, \nu \wedge u^-\partial_c) = \int_{\Omega^+} (F^+, \nabla \wedge U^-\partial_c) + (\nabla \cdot F^+, U^-\partial_c) \approx \int_{\Omega^+} (F^+, \chi_+^{-1} F^-\partial_c).
\]
Thus, choosing $\Omega$ and $\chi_+^{-1}$ optimal in the definition of $C_\Sigma$, we have modulo compact terms
\[\left| \int_{\Sigma} (u^+, \nu \wedge f^-\partial_c)x, \right| \leq C_{\Sigma} \| F^+ \|_{L^2(\partial^+)} \| F^- \|_{L^2(\partial^-)}.
\]
(i) Adding \( \frac{1}{\alpha} \) times (5.19) and \( \frac{1}{\alpha^2} \) times (5.20) give us the estimate
\[\left| \frac{1}{\alpha^2} \| F^+ \|^2_{L^2(\partial^+)} + \frac{1}{\alpha} \| F^- \|^2_{L^2(\partial^-)} \right| \lesssim \| f \| \| \xi \| \| \xi \| + \| f \| \| \xi \| K f \| \xi \|
\]
for some compact operator $K$. Thus we have a priori estimates when $\Re \alpha \neq 0$.

(ii) Adding \( \frac{1}{\alpha} \) times (5.19) and \( \frac{1}{\alpha} \) times (5.20) give us the estimate
\[\| F^+ \|^2_{L^2(\partial^+)} + \| F^- \|^2_{L^2(\partial^-)} \leq 2 \| f \| \| \xi \| \| \xi \| + \| f \| \| \xi \| K f \| \xi \|
\]
for some compact operator $K$. Thus we have a priori estimates when $\| \Im \alpha \| < 1/C_{\Sigma}$. Also, doing the same calculation using a $d_{\partial^+}$ potential $U^+$ for $F^+ \in R(d_{\partial^+})$ in (5.20) and a $\delta_{\partial^-}$ potential $U^-$ for $F^- \in R(\delta_{\partial^-})$ in (5.19), we see that we have a priori estimates when $\| \Im \alpha \| < 1/C_{\Sigma}$.

This proves the theorem since these estimates translate to (5.11) and (5.12) for the rotation operator $E_{\alpha} N$ using the conformal map $\lambda = (\alpha + 1)/(\alpha - 1)$ and the method of continuity 1.4.5.  

\[\square\]
Proof of Theorem 5.3.2. By Lemma 5.3.7(ii) it suffices to prove exact a priori estimates for \( f \in \mathcal{N}(\Gamma_k) \), i.e. we may assume that the Cauchy extensions \( F^\pm \) satisfy \( \nabla_k \land F^\pm = \nabla_k \land F^\pm = 0 \). We now consider the equations

\[
\begin{align*}
\int_{\Sigma} \langle \nabla_0 f^+, \nu \land g^+ \rangle &= \alpha^c - \alpha^c \int_{\Sigma} \langle \nabla_0 f^+, \nu \land f^+ \rangle - \alpha^c \int_{\Sigma} \langle \nabla_0 f^+, \nu \land f^- \rangle, \\
\int_{\Sigma} \langle \nabla_0 g, \nu \land f^- \rangle &= \alpha_+ \int_{\Sigma} \langle \nabla_0 f^+, \nu \land f^- \rangle - \alpha_- \int_{\Sigma} \langle \nabla_0 f^+, \nu \land f^- \rangle,
\end{align*}
\]  

(5.21)  
(5.22)

where we by Lemma 5.3.6 have coercivity

\[
\int_{\Omega} \langle \nabla_0 f^\pm, \nu \land f^\pm \rangle = \pm \int_{\Omega} \text{Im} k |F^\pm|^2 + i \text{Re} \text{Re} (F^\pm, TF^\pm).
\]

(i) To obtain the estimate (5.13) we add \( \frac{1}{\alpha^c} \) times (5.21) and \( \frac{1}{\alpha^c} \) times (5.22) which gives

\[
\frac{1}{\alpha^c} \int_{\Omega^+} \text{Im} k |F^+|^2 + i \text{Re} \text{Re} (F^+, TF^+) + \frac{1}{\alpha^c} \int_{\Omega^-} \text{Im} k |F^-|^2 + i \text{Re} \text{Re} (F^-, TF^-) = O(\|f\|_{L^2} \|g\|_{L^2}).
\]

From the coercivity properties of these integrands we deduce that if \( |\text{arg} \alpha + 2 | \text{arg} k - \frac{\pi}{2}| < \pi \), then we have exact a priori estimates.

(ii) To obtain the estimate (5.14) we add \( \frac{1}{\alpha^c} \) times (5.21) and \( \frac{1}{\alpha^c} \) times (5.22) which gives

\[
\int_{\Omega^+} \text{Im} k |F^+|^2 + i \text{Re} \text{Re} (F^+, TF^+) + \int_{\Omega^-} \text{Im} k |F^-|^2 + i \text{Re} \text{Re} (F^-, TF^-) + 2 \text{Im} \alpha \int_{\Sigma} \langle \nabla_0 f^+, \nu \land f^- \rangle = O(\|f\|_{L^2} \|g\|_{L^2}).
\]

(5.23)

To estimate the cross term, extend \( F^- \) to \( \chi^{-1} F^- \in D(d_{\Omega^+}) \) and note that \( d(\chi^{-1} F^-) = [d, \chi^{-1}] F^- + \chi^{-1}(-k \epsilon_0 \land F^-) = [d, \chi^{-1}] F^- - k \epsilon_0 \land (\chi^{-1} F^-) \). From the identity

\[
\int_{\Sigma} \langle \nabla_0 f^+, \nu \land f^- \rangle = \int_{\Omega^+} \langle \nabla_0 f^+, [d, \chi^{-1}] F^- - k \epsilon_0 \land (\chi^{-1} F^-) \rangle = \langle \nabla_0 (-k \epsilon_0 \land F^+), \chi^{-1} F^- \rangle
\]

we now get the estimate

\[
\left| \int_{\Sigma} \langle \nabla_0 f^+, \nu \land f^- \rangle \right| \leq L^w_1 \| F^+ \|_{L_2(\Omega^+)} \| F^- \|_{L_2(\Omega^-)} + L^w_0 |k| \| \epsilon_0 \land F^+ \|_{L_2(\Omega^+)} \| \epsilon_0 \land F^- \|_{L_2(\Omega^-)} + \| \epsilon_0 \land F^+ \|_{L_2(\Omega^+)} \| \epsilon_0 \land F^- \|_{L_2(\Omega^-)} \leq (L^w_1 + |k| L^w_0) \| F^+ \|_{L_2(\Omega^+)} \| F^- \|_{L_2(\Omega^-)}.
\]

Taking real parts of (5.23), we will thus have exact a priori estimates when

\[
|\text{Im} \alpha| (L^w_1 (\Omega^-, \Omega^+) + |k| L^w_0 (\Omega^+, \Omega^-)) < \text{Im} k.
\]

If we instead extend \( F^+ \) to \( \Omega^- \), we obtain an a priori estimate when \( |\text{Im} \alpha| (L^w_1 (\Omega^+, \Omega^-) + |k| L^w_0 (\Omega^+, \Omega^-)) < \text{Im} k \). Similarly, pairing with \( f^- \) in (5.21) and \( f^+ \) in (5.22) gives exact a priori estimates when \( |\text{Im} (1/\alpha)| < 1/C_{k, \Sigma} \).

Applying the spectral map \( \lambda = (\alpha + 1)/(\alpha - 1) \) and using the method of continuity 1.45 now proves (5.13) and (5.14). \( \square \)
We now give a lemma with the perturbation technique used for the non-relativistic transmission problem.

**Lemma 5.3.9.** Call the parameters \((k, \alpha, \beta)\) permissible if they can be obtained from some non-zero material constants \(\alpha^\pm\) and \(\beta^\pm\) and a frequency \(\omega \in \mathbb{C}\) as in the statement of (5.3), i.e. if \(\alpha, \beta \in \mathbb{C}\setminus \{0\}\) and \(k^+ + k^- = k^\alpha\). Assume we have a continuous path through permissible parameters \((k_t, \alpha_t, \beta_t), t \in [0, 1]\), such that the following hold.

(i) For each \(t \in [0, 1]\), the associated map (5.15) is semi-Fredholm.

(ii) For \(t = 0\), we have \(\alpha := \alpha_0 = \beta_0\) and \(k := k_0^+ = k_0^-\) and \((\alpha + 1)/(\alpha - 1) - E_k N\) is a Fredholm operator.

Then the map (5.15) corresponding to \(t = 1\) is Fredholm with the same index.

**Proof.** Let \(A_{k_t, \alpha_t, \beta_t}\) be the composition
\[
\mathcal{E} \longrightarrow C^+_k \mathcal{E} \oplus C^-_k \mathcal{E} \longrightarrow \mathcal{E},
\]
where the first map is the ansatz (5.8) and the second is the map (5.15). By Proposition 5.2.6 and the hypothesis \(A_{k_t, \alpha_t, \beta_t} : \mathcal{E} \rightarrow \mathcal{E}\) is a semi-Fredholm map for \(t \in [0, 1]\). Furthermore, we note that \(t \mapsto A_{k_t, \alpha_t, \beta_t}\) is continuous and that
\[
A_{k_0, \alpha_0, \beta_0} = (\alpha^+ N^+ + \frac{1}{\alpha^+} N^-)^{-1} \frac{1}{2} ((\alpha + 1) E_k - (\alpha - 1) N)
\]
is a Fredholm operator. By Theorem 1.4.5 it follows that \(A_{k_t, \alpha_t, \beta_t}\) is a Fredholm operator. Moreover, from Proposition 5.2.6 we see that the map (5.15) corresponding to \(t = 1\) has the same index as \((\alpha + 1)/(\alpha - 1) - E_k N\).

In the proofs of Theorem 5.3.4 and Theorem 5.3.5 below we establish a priori estimates for the map \(C^+_k \mathcal{E} \oplus C^-_k \mathcal{E} \ni \mathcal{F} := \{F^+, F^\mp\} \mapsto g \in \mathcal{E}\) defined by

\[
N^+ T^+ (\frac{1}{\beta^+} f^+ - \frac{1}{\beta^-} f^-) = N^+ T^+ g, \tag{5.24}
\]
\[
N^+ T^- (\frac{1}{\alpha^+} f^+ - \frac{1}{\alpha^-} f^-) = N^+ T^- g, \tag{5.25}
\]
\[
N^- T^+ (\beta^+ f^+ - \beta^- f^-) = N^- T^+ g, \tag{5.26}
\]
\[
N^- T^- (\alpha^+ f^+ - \alpha^- f^-) = N^- T^- g. \tag{5.27}
\]

**Proof of Theorem 5.3.4.** (i) As the proof of the a priori estimates is similar to that of Theorem 5.3.1, we will here only point out the main differences. By Lemma 5.3.7(i) and Proposition 6.2.1 it suffices to prove a priori estimates for \(\{F^+, F^\mp\} \in \text{R}(d_{k^+} \oplus d_{k^-})\), since the map (5.15) intertwines \(d_{k^+} \oplus d_{k^-}\) and \(\Gamma_\omega\) as in (5.5).

We choose the Hodge potentials so that
\[
\nabla_{k^+} U^+ = F^+, \quad \text{in} \quad \mathcal{O}^+,
\]
\[
\nabla_{k^-} U^- = F^-, \quad \text{in} \quad \mathcal{O}^-.
\]
From the jump conditions (5.25) and (5.27), we obtain
\[ (\alpha^+\alpha^-)^c \int_\Sigma (T^{-}u^+, \nu \wedge g^c) = \alpha^- \int_\Sigma (T^{-}u^+, \nu \wedge f^+ c) - \alpha^+ \int_\Sigma (T^{-}u^+, \nu \wedge f^- c), \]
\[ \int_\Sigma (T^{-}g, \nu \wedge u^- c) = \alpha^+ \int_\Sigma (T^{-}f^+, \nu \wedge u^- c) - \alpha^- \int_\Sigma (T^{-}f^-, \nu \wedge u^- c), \]
and from the jump conditions (5.24) and (5.26)
\[ (\beta^+\beta^-)^c \int_\Sigma (T^{+}u^+, \nu \wedge g^c) = \beta^- \int_\Sigma (T^{+}u^+, \nu \wedge f^+ c) - \beta^+ \int_\Sigma (T^{+}u^+, \nu \wedge f^- c), \]
\[ \int_\Sigma (T^{+}g, \nu \wedge u^+ c) = \beta^+ \int_\Sigma (T^{+}f^+, \nu \wedge u^+ c) - \beta^- \int_\Sigma (T^{+}f^-, \nu \wedge u^+ c). \]

As in the proof of Theorem 5.3.1, using these four equations and the corresponding four with \( U^\pm \) interchanged, we now estimate the time-like part
\[ \|T^{-}F^+\|_{L^2(\Omega^+)}^2 + \|T^{-}F^-\|_{L^2(\Omega^-)}^2 \lesssim \|E\|_{C^+_{k^+} \varepsilon \oplus C^-_{k^-} \varepsilon} (\|g\| \varepsilon + \|KF\| \chi) \]
when \( \alpha \in G(C_{\Sigma}) \), and the space-like part
\[ \|T^{+}F^+\|_{L^2(\Omega^+)}^2 + \|T^{+}F^-\|_{L^2(\Omega^-)}^2 \lesssim \|E\|_{C^+_{k^+} \varepsilon \oplus C^-_{k^-} \varepsilon} (\|g\| \varepsilon + \|KF\| \chi) \]
when \( \beta \in G(C_{\Sigma}) \). Adding these gives the a priori estimate \( \|E\| \lesssim \|g\| + \|KF\| \) when \( \alpha, \beta \in G(C_{\Sigma}) \).

(ii) We now use Lemma 5.3.9 to show that the index is zero. Consider the open, connected set \( G = G(C_{\Sigma}) \) from (5.16) where the permissible parameters \((k, \alpha, \beta)\) satisfy \( \alpha, \beta \in G \). Since \( G \) is connected, we can find continuous \( z : [0, 1] \to C \) such that \( z(1) = 1 \), \( z(0) = \alpha / \beta \) and \( z(t) / \beta \in G \) for all \( t \in [0, 1] \). Since the a priori estimates are independent of \( k \), there exists \( k_1 \) such that \((k_1, \alpha, z(t) \beta)\) is a continuous path through permissible parameters, where \( k_1 = k \) and the corresponding maps (5.15) are all semi-Fredholm. Since \( \alpha_0 = \alpha = z(0) / \beta = \beta_0 \), it follows from Lemma 5.3.9 and Theorem 5.3.1 that the index of the map (5.15) corresponding to \((k, \alpha, \beta)\) is zero. \( \square \)

**Proof of Theorem 5.3.5.** (i) As the proof of the a priori estimates is similar to that of Theorem 5.3.2, we will here only point out the main differences. By Lemma 5.3.7(ii) it suffices to prove exact a priori estimates for \( \{F^+, F^-\} \in N(d_{k^+} \oplus d_{k^-}) \), since the map (5.15) intertwines \( d_{k^+} \oplus d_{k^-} \) and \( \Gamma_\omega \) as in (5.5).

We now use the jump relations (5.25) and (5.26) to get identities
\[ (\alpha^+\alpha^-)^c \int_\Sigma (\vec{\varphi}_0 \wedge f^+, \nu \wedge g^c) = \alpha^- \int_\Sigma (\vec{\varphi}_0 \wedge f^+, \nu \wedge f^+ c) - \alpha^+ \int_\Sigma (\vec{\varphi}_0 \wedge f^+, \nu \wedge f^- c), \]
\[ \int_\Sigma (\vec{\varphi}_0 \wedge g, \nu \wedge f^- c) = \beta^+ \int_\Sigma (\vec{\varphi}_0 \wedge f^+, \nu \wedge f^- c) - \beta^- \int_\Sigma (\vec{\varphi}_0 \wedge f^-, \nu \wedge f^- c), \]
where we by Lemma 5.3.6 have coercivity
\[ \int_\Sigma (\vec{\varphi}_0 \wedge \vec{f}^\pm, \nu \wedge \vec{f}^\pm) = \pm \int_{\Omega^\pm} \text{Im} \, k^\pm |\vec{F}^\pm|^2 + i \text{Re} \, k^\pm (\vec{F}^\pm, T\vec{F}^\pm c). \]
The rest of the proof of the a priori estimates is now the same as for Theorem 5.3.2. By adding $\frac{1}{\alpha}$ times equation (5.28) and $\frac{1}{\beta^+}$ times (5.29) we obtain exact a priori estimates under the condition (5.17), while adding $\frac{1}{\alpha}$ times equation (5.28) and $\frac{1}{\beta^-}$ times (5.29) gives exact a priori estimates when $|\alpha - \beta^2| < 2/C_{\beta^2}$. Similarly, pairing with $f^-$ in (5.28) and $f^+$ in (5.29) gives exact a priori estimates when $|\alpha - \beta^2| < 2/C_{\beta^2}$.

(ii) We now prove that for $(\theta, \alpha, \beta)$ satisfying (5.17) the index is zero, and thus the map (5.15) is an isomorphism. Define

$$z(t) := t + (1 - t) \sqrt{k^+/k^-},$$

where $\text{Re} \sqrt{k^+/k^-} > 0$. It follows that $(k^+/z(t), z(t), \alpha/z(t), z(t), \beta)$ is a continuous path through permissible parameters. Since $\alpha t, \beta t = \alpha \beta$ and $|\arg k_t^+ - \pi/2| + |\arg k^- - \pi/2| \leq |\arg k^+ - \pi/2| + |\arg k^- - \pi/2|$, it now follows from Lemma 5.3.9, the a priori estimates in (i) and Theorem 5.3.2 that the index is zero.

(iii) We now prove that for $(\theta, \alpha, \beta)$ satisfying (5.18) the index is zero, and thus the map (5.15) is an isomorphism. Assume that $|\alpha - \beta^2| \leq |\frac{1}{\alpha} - \frac{1}{\beta^2}|$. Define a continuous path $(\theta_t, \alpha_t, \beta_t)$ through permissible parameters as follows. For $0 \leq t \leq 1/2$, let

$$(\theta_t, \alpha_t, \beta_t) := (k^+/z(2t), z(2t), \alpha/z(2t), z(2t), \beta),$$

where $z(t)$ is as in (ii) above and $\epsilon > 0$ is chosen small enough that $|\alpha_t - \beta^2| < 2/C_{\beta^2}$ for $0 \leq t \leq 1/2$. For $1/2 \leq t \leq 1$, let $(\theta_t, \alpha_t, \beta_t) := (k^+, \alpha, 1 + 2(1 - \epsilon)(t - 1), \beta, 1 + 2(1 - \epsilon)(t - 1))$. It now follows from Lemma 5.3.9, the a priori estimates in (i) and Theorem 5.3.2 that the index is zero. The case $|\alpha - \beta^2| > |\frac{1}{\alpha} - \frac{1}{\beta^2}|$ is proved similarly.
Chapter 6

Some complementary results

6.1 Duality in trace spaces

We have in previous chapters proved existence of solutions using perturbation theory in the form of Theorem 1.4.5. Another way is to use duality in the form of Theorem 1.4.8. In this section we give some duality results for the two trace spaces $L_2(\Sigma; \Lambda)$ and $\mathcal{E}(\Sigma; \Lambda)$. Besides the Hilbert space scalar products, we have the following two useful pairings.

**Definition 6.1.1.** The *Krein pairing* is the sesqui-linear pairing

$$
\langle g, f \rangle := \int_\Sigma (g(y), \nu(y) f^c(y)) \, d\sigma(y), \quad g, f : \Sigma \to \Lambda.
$$

The *Dirac pairing* is the bi-linear pairing

$$
\langle g, f \rangle_{\mathcal{E}} := \int_\Sigma (\overline{g(y)}, \nu(y) f(y)) \, d\sigma(y), \quad g, f : \Sigma \to \Lambda.
$$

These two pairings are essentially the same, but it is convenient to distinguish them. It is clear that $\langle L_2(\Sigma; \Lambda), L_2(\Sigma; \Lambda) \rangle$ and $\langle L_2(\Sigma; \Lambda), L_2(\Sigma; \Lambda) \rangle_{\mathcal{E}}$ are dualities in the sense of Definition 1.4.7. Moreover, we have the following.

**Proposition 6.1.2.** We have well defined dualities $\langle \mathcal{E}, \mathcal{E} \rangle$ and $\langle \mathcal{E}, \mathcal{E} \rangle_{\mathcal{E}}$, where

$$
\langle N^- f, N^+ g \rangle := \int_{\Omega^+} (F, dG^c) + (\delta F, G^c)
$$

$$
= - \int_{\Omega^-} (F, dG^c) + (\delta F, G^c), \quad F \in D(\delta_{\mathbb{R}^n}), G \in D(d_{\mathbb{R}^n}),
$$

and in general $\langle f, g \rangle := \langle N^- f, N^+ g \rangle + \langle N^- g, N^+ f \rangle^c$ and $\langle g, f^c \rangle_{\mathcal{E}} := \langle \overline{g}, f \rangle$ when $f, g \in \mathcal{E}$.

**Proof.** Consider the trace maps

$$
\gamma_\delta : N^+(\Omega^+) := D(d_{\Omega^+})/D(d_{\Omega^+}) \to \nu N^+ \mathcal{E} : [F] \mapsto \nu \vee f,
$$

$$
\gamma_\delta : N^- (\Omega^+) := D(\delta_{\Omega^+})/D(\delta_{\Omega^+}) \to \nu N^- \mathcal{E} : [F] \mapsto \nu \wedge f,
$$
from Theorem 5.1.3. We need to show that \( \langle [F], [G] \rangle := \int_{\Omega^+} (F, dG^c) + (\delta F, G^c) \) defines a duality \( \langle N^{-}(\Omega^+), N^{+}(\Omega^+) \rangle \). Using the (complex structure) operator \( I \) in the graph space \( L_2(\Omega^+; \Lambda)^2 \) defined by \( I(F, G)^{\dagger} := (-G, F)^{\dagger} \), we get from Definition 1.4.7 that

\[
G(d_{\Omega^+}) \oplus G(d_{\Omega^+}) = G(d_{\Omega^+}) \cap IG(-\delta_{\Omega^+}) = I(G(-\delta_{\Omega^+}) \cap IG(d_{\Omega^+})) = I(G(-\delta_{\Omega^+}) \oplus G(-\delta_{\Omega^+})).
\]

This proves that

\[
\langle [F], [G] \rangle = \int_{\Omega^+} ((F, d_{\Omega^+}F)^{\dagger}, I(G, -\delta_{\Omega^+}G)^{\dagger}C)
\]

is a duality, and since \( f \mapsto \overline{c_0}f \) is an isomorphism in \( E \) it also follows that \( \langle E, E \rangle_{\pi_0} \) is a duality. \( \square \)

One can view the Krein pairing as a natural Krein space structure (indefinite inner product) corresponding to the canonical symmetry \( J = \nu \triangle (\cdot) \) as in Azizov–Iokhvidov [5].

The reason why the Dirac pairing is useful for us is that both reflection operators \( N \) and \( E_k \) are skew-adjoint with respect to \( \langle \cdot, \cdot \rangle_{\pi_0} \). We also remark that the quantity \( \langle f, f^c \rangle_{\pi_0} \) in the Dirac electron theory is interpreted as the “flux of Dirac current through \( \Sigma^\prime \)”.)

**Proposition 6.1.3.** With respect to the Dirac pairing \( \langle E, E \rangle_{\pi_0} \) on a bounded weakly Lipschitz interface, the operators \( N, E_k \) and \( \Gamma_k \) have adjoints

\[
N' = -N, \quad E'_k = -E_k, \quad \Gamma'_k = -\epsilon_0 \Gamma_{-k} \epsilon_0,
\]

while with respect to the Krein pairing \( \langle E, E \rangle \), the operators \( N, E_k \) and \( \Gamma_k \) have adjoints

\[
N' = -N, \quad E'_k = -E_{-k}, \quad \Gamma'_k = -\Gamma_{-k}.
\]

In particular we have restricted dualities \( \langle N^+ E, N^+ E \rangle_{\pi_0} \) and \( \langle E^+_k E, E^+_k E \rangle_{\pi_0} \) with respect to the Dirac form and \( \langle N(\Gamma^{*}_{-k}, E), N(\Gamma_k; E) \rangle \) with respect to the Krein form.

**Proof.** (i) Since \( N \) is a reflection operator, the dualities \( N' = -N \) follows directly from the definition, because we have annihilators \( (N^\pm E)^a = N^\pm E \).

(ii) We consider the adjoint of \( E_k \) with respect to the Dirac form; the duality with respect to the Krein form is shown similarly. It suffices to prove that \( \langle E^+_k E, E^+_k E \rangle_{\pi_0} = \{0\} \) and \( \langle E^-_k E, E^-_k E \rangle_{\pi_0} = \{0\} \). The first identity follows directly the boundary theorem 1.2.7. To prove the second identity for all \( k \in \mathbb{C} \), assume \( F, G \in C_k^\pi E \) and let \( \Sigma \subset B_r \subset B_R \). Then the boundary theorem and the radiation condition from Definition 5.1.1 shows that

\[
\langle f, g \rangle_{\pi_0} = \int_{\partial B_R} (\overline{c_0}F(x), \nu(x)G(x)) \, d\sigma(x) = \int_{\partial B_r} d\sigma(y) \int_{\partial B_r} d\sigma(z)
\]

\[
\left( \nu(y)F(y), \left( \int_{\partial B_R} d\sigma(z) \overline{E_k}(x - y) \overline{c_0} \nu(x) \overline{E_k}(x - z) \right) \nu(z)G(z) \right).
\]

104
But from Lemma 1.2.4 we now get

\[
\int_{\partial B_R} d\sigma(x) \vec{E}_k(x - y) \vec{v}_0 \nu(x) \vec{E}_k(x - z) \\
= \int_{B_R} (\vec{E}_k(x - y) \vec{v}_0 \delta_z(x) - k e_0 \vec{E}_k(x - z)) - (\delta_y(x) - \vec{E}_k(x - y) k e_0) \vec{v}_0 \vec{E}_k^2(x - z)) \, dx \\
= \vec{E}_k(z - y) \vec{v}_0 \vec{E}_k(y - z) = 0.
\]

(iii) We consider the adjoint of \( \Gamma_k \) with respect to the Krein form; the adjoint with respect to the Dirac form follows from this. It suffices to prove that if \( F \in \mathcal{D}(\delta_{\mathbb{R}^n}) \) and \( G \in \mathcal{D}(d_{\mathbb{R}^n}) \), then \( \langle f, \Gamma_k g \rangle = \langle -\Gamma_{-k^*} f, g \rangle \). Using the boundary theorem, we see that

\[
\langle f, \Gamma_k g \rangle = \int_{\Sigma} (f, \nu \wedge (d_k G|\Sigma)^e) = \int_{\Omega^+} (F, \nabla \wedge (k e_0 \wedge G)^e) + (\nabla \perp F, (d_k G)^e) \\
= \int_{\Omega^+} (\delta F, dG^e) + k e_0 \left(-e_0 \perp F, dG^e\right) + (\delta F, e_0 \wedge G^e) \\
= -\int_{\Omega^+} (-\delta_{-k^*} F, \nabla \wedge G^e) + (\nabla \perp (k^e e_0 \perp F), G^e) \\
= -\int_{\Sigma} (-\delta_{-k^*} F|\Sigma), \nu \wedge g^e) = \langle -\Gamma_{-k^*} f, g \rangle.
\]

\[\square\]

**Proposition 6.1.4.** With respect to the Dirac pairing \( \langle L_2(\Sigma; \lambda), L_2(\Sigma; \lambda) \rangle_{\vec{v}_0} \) on a bounded Carleson–Lipschitz interface, the operators \( N, E_k \) and \( \Gamma_k \) have adjoints

\[
N' = -N, \quad E_k' = -E_k, \quad \Gamma_k' = -e_0 \Gamma_{-k^*} e_0,
\]

while with respect to the Krein pairing \( \langle L_2(\Sigma; \lambda), L_2(\Sigma; \lambda) \rangle_{\vec{v}_0} \), the operators \( N, E_k \) and \( \Gamma_k \) have adjoints

\[
N' = -N, \quad E_k' = -E_{-k^*}, \quad \Gamma_k' = -\Gamma_{-k^*}.
\]

In particular we have restricted dualities \( \langle N^\perp L_2, N^\perp L_2 \rangle_{\vec{v}_0} \) and \( \langle E_k^\perp L_2, E_k^\perp L_2 \rangle_{\vec{v}_0} \) with respect to the Dirac form and \( \langle N(\Gamma_{-k^*}; L_2), N(\Gamma_k; L_2) \rangle \) with respect to the Krein form.

The dualities for \( N \) and \( E_k \) follow exactly as in \( \mathcal{E} \). However, note also the pointwise identities

\[
(\vec{v}_0 g, \nu(v^* f)) = (\vec{v}_0 g, f^* \nu) = (\vec{v}_0 g, \nu f) = (\vec{v}_0 \nu g, \nu f) = -(\vec{v}_0 \nu g, \nu f),
\]

\[
(\vec{v}_0 g(x), \nu(x)(-\vec{E}_k(x - y) \nu(y) f(y))) = -(\vec{v}_0(-\vec{E}_k(y - x) \nu(x) g(x), \nu(y) f(y)),
\]

where we have used Lemma 1.2.4 on the last line. The dualities for \( \Gamma_k \) follow from Proposition 6.2.5 below.

We now recall the idea used in Axelsson–Grognard–Hogan–McIntosh [4] to prove that the restricted projections have index zero.

**Proposition 6.1.5.** Assume that \( A \) and \( B \) are two reflection operators in a Hilbert space \( \mathcal{H} \) equipped with a duality \( \langle \mathcal{H}, \mathcal{H} \rangle \) such that \( A' = -A \) and \( B' = -B \) with respect to this
duality. Then with respect to the restricted dualities \((B^+ \mathcal{H}, B^- \mathcal{H})\) and \((A^\perp \mathcal{H}, A^\perp \mathcal{H})\) we have adjoint operators

\[
A^-_\perp : B^+\mathcal{H} \longrightarrow A^\perp\mathcal{H}, \\
B^-_\perp : A^\perp\mathcal{H} \longrightarrow B^-\mathcal{H}.
\]

In particular, if \(A\) and \(B\) are essentially transversal, i.e. \(\sigma_{ess}(BA) \cap \{1, -1\} = \emptyset\), and \(I \pm BA\) have index zero, then both \(A^-_\perp : B^+\mathcal{H} \longrightarrow A^\perp\mathcal{H}\) and \(B^-_\perp : A^\perp\mathcal{H} \longrightarrow B^-\mathcal{H}\) are Fredholm operators with index zero and

\[
\mathcal{N}(A^-_\perp|_{B^+\mathcal{H}}) = \mathcal{N}(B^-_\perp|_{A^\perp\mathcal{H}}) = A^\perp\mathcal{H} \cap B^+\mathcal{H}.
\]

By replacing either of \(A\) and \(B\) with \(-A\) and \(-B\), the corresponding results for the remaining 6 restricted projections hold true as well.

Using this proposition with the reflection operators \(A = N\) and \(B = E_k\) and with the Dirac pairing, we obtain the following from Theorem 3.2.15 and Theorem 5.3.1.

**Corollary 6.1.6.** Let \(\mathcal{H}\) be either \(L_2(\Sigma; \wedge)\) or \(\mathcal{E}\). Then for all \(k \in \mathbf{C}\), all eight restricted projections

\[
\begin{align*}
N^+_k & : E^+_k\mathcal{H} \longrightarrow N^+\mathcal{H}, \\
N^-_k & : E^-_k\mathcal{H} \longrightarrow N^-\mathcal{H}, \\
E^+_k & : N^\pm\mathcal{H} \longrightarrow E^+_k\mathcal{H}, \\
E^-_k & : N^\pm\mathcal{H} \longrightarrow E^-_k\mathcal{H},
\end{align*}
\]

are Fredholm operators with index zero.

### 6.2 Hodge decomposition of trace spaces

In Chapter 4 we developed the abstract theory for nilpotent operators and Hodge decompositions and applied it to the nilpotent \(d\) and \(\delta\) operators in a bounded, weakly Lipschitz domain. In this section we prove Hodge decompositions for the nilpotent operator \(\Gamma_k\) in the two trace spaces \(L_2(\Sigma; \wedge)\) and \(\mathcal{E}(\Sigma; \wedge)\).

**Proposition 6.2.1.** Let \(\Sigma\) be a bounded, weakly Lipschitz interface. Then the exterior/interior derivative operator \(\Gamma_k : \mathcal{E} \rightarrow \mathcal{E}\) from Proposition 5.1.9 is a bounded Fredholm-nilpotent operator and induces a Hodge type splitting

\[
\mathcal{E} = R(\Gamma_k; \mathcal{E}) \oplus (\mathcal{N}(\Gamma_k; \mathcal{E}) \cap \mathcal{N}(\Gamma^+_k; \mathcal{E})) \oplus R(\Gamma^+_k; \mathcal{E}).
\]

**Proof.** Since \(N\Gamma_k = \Gamma_k N\), it suffices to prove that \(\Gamma_k : N^\pm\mathcal{E} \rightarrow N^\pm\mathcal{E}\) is a Fredholm-nilpotent operator. For example, in \(N^+\mathcal{E}\) it suffices to prove that \([d_{k,\Omega}]\) is a Fredholm-nilpotent operator in \(N^+([d_{k,\Omega}] := D(d_{\Omega})/D(d_{\Omega}^+)\), where \(\Omega := \Omega^+\) and \([d_{k,\Omega}]([F]) := [d_{k,\Omega}F]\), since \(\nu\gamma_{\Omega}\) is an isomorphism intertwining \([d_{k,\Omega}]\) and \(\Gamma_k|_{N^+\mathcal{E}}\). Using Proposition 1.4.4, we prove that \(R([d_{k,\Omega}])\) has finite codimension in \(\mathcal{N}([d_{k,\Omega}])\) through

\[
R([d_{k,\Omega}]) = [R(d_{k,\Omega})] \subset [\mathcal{N}(d_{k,\Omega})] \subset \mathcal{N}([d_{k,\Omega}]).
\]
The first equality follows directly from the surjectivity of the quotient map $D(d_{k}) \to N^{+}(\Omega)$, and the middle inclusion has finite codimension due to the Hodge decomposition for $d_{k,\Omega}$. We claim that the last inclusion also has finite codimension. To see this, write

$$D(d_{k,\Omega}) = N(d_{k,\Omega}) \oplus (D(d_{k,\Omega}) \cap N(d_{k,\Omega})^{\perp})$$

and define the auxiliary spaces

$$X_1 := \{ F \in N(d_{k,\Omega}) \cap D(d_{k,\Omega}) ; d_{k,\Omega}F \in N(d_{k,\Omega}) \},$$

$$X_2 := \{ F \in N(d_{k,\Omega}) \cap D(d_{k,\Omega}) ; d_{k,\Omega}F \in R(d_{k,\Omega}) \}.$$

We now verify that $[N(d_{k,\Omega}) \oplus X_1] = N([d_{k,\Omega}])$ and $[N(d_{k,\Omega}) \oplus X_2] = [N(d_{k,\Omega})]$ and since the reduced operator $\tilde{d}_{k,\Omega} : D(d_{k,\Omega}) \cap N(d_{k,\Omega})^{\perp} \to N(d_{k,\Omega})$ is injective, it follows from the Hodge decomposition for $d_{k,\Omega}$ that $X_1/X_2$ is finite dimensional and thus so is $N([d_{k,\Omega}])/[N(d_{k,\Omega})]$.

Note that by Proposition 5.1.9, it follows that $d_{k} : C_{k}^{\pm} E \to C_{k}^{\pm} E$ is a bounded Fredholm-nilpotent operator.

**Proposition 6.2.2.** Let $\Sigma$ be a bounded Carleson–Lipschitz interface. Then the exterior/interior derivative operator $\Gamma_k$ in $L_2(\Sigma; \wedge)$ is a diffuse Fredholm-nilpotent operator and induces a Hodge type splitting

$$L_2(\Sigma; \wedge) = R(\Gamma_k; L_2) \oplus (M(\Gamma_k; L_2) \cap N(\Gamma_k^*; L_2)) \oplus R(\Gamma_k^*; L_2).$$

We end this chapter with a more detailed investigation of $\Gamma_k$ in $L_2(\Sigma; \wedge)$, which in particular proves Proposition 6.2.2. First note that for the definition of $\Gamma_k$ in $L_2(\Sigma; \wedge)$, it is not necessary for $\Sigma$ to be strongly Lipschitz as in Section 3.2.

**Definition 6.2.3.** Let $\Sigma$ be a bounded Carleson–Lipschitz interface. In a chart $V_j$ we have $\Sigma \cap V_j = \Sigma_{\rho} \cap V_j$ as in Definition 1.5.1, where the Carleson–bilipschitz map $\rho = \rho_{V_j} : \mathbb{R}^n \to \mathbb{R}^n$ restricts to a bilipschitz map $\rho_{0} : \mathbb{R}^{n-1} \to \Sigma_{\rho}$.

(i) In the splitting $L_2(\Sigma; \wedge) = E_0^+ L_2 \oplus E_0^- L_2$, we define the closed operator

$$\Gamma_k(F^+|_{\Sigma} + F^-|_{\Sigma}) := (d_kF^+)|_{\Sigma} + (d_kF^-)|_{\Sigma}, \quad F^\pm \in C_{k}^{\pm} L_2,$$

with natural domain.

(ii) In $N^+ L_2$ we define closed, densely defined operators $d_k$ and $\delta_k$ with domains

$$D(d_k) := \{ f_1 + \ldots + f_N \in N^+ L_2 ; \text{supp} f_j \subset V_j, \rho_{0}^* f_j \in D(d_{R^{n-1}}) \},$$

$$D(\delta_k) := \{ f_1 + \ldots + f_N \in N^+ L_2 ; \text{supp} f_j \subset V_j, \rho_{0}^{-1} f_j \in D(\delta_{R^{n-1}}) \},$$

such that locally we have intertwining $d_{R^{n-1}} \rho_{0}^* = \rho_{0}^* d_k$ and $\delta_{R^{n-1}} \rho_{0}^{-1} = \rho_{0}^{-1} \delta_k$.

Here

$$\rho_{0}^*, \quad \rho_{0}^{-1} : N^+ L_2(\Sigma_{\rho}) \longrightarrow L_2(\mathbb{R}^{n-1}; e_{n-1} \wedge)$$

are the pullback and reduced pushforward isomorphisms from Definition 1.2.1.
Remark 6.2.4. From this definition it is straightforward to verify that $d_k$ and $\delta_k$ are diffuse Fredholm-nilpotent operators in $N^+ L_2$ and that $d_k' = -\delta_k$ in the sense of Definition 1.4.7. Thus Proposition 6.2.2 and the dualities for $\Gamma_k$ in Proposition 6.1.4 follow from Proposition 6.2.5 below.

Proposition 6.2.5. On a bounded Carleson–Lipschitz interface $\Sigma$ we have

$$D(\Gamma) = \{f_1 + \nu \wedge f_2 : f_1 \in D(d), f_2 \in D(\delta)\}$$

and $\Gamma_k(f_1 + \nu \wedge f_2) = d_k f_1 + \nu \wedge (\delta_k f_2)$.

Proof: After a straightforward reduction, we may assume that $k = 0$.

(i) To prove that $D(d) \subset D(\Gamma)$, let $f \in D(d) \subset N^+ L_2$. We may assume that $f$ is supported in a chart $V$. Consider the Cauchy extension $F := C^+ f$ to $\Omega^+$. Fix $x \in \Omega^+$ and $w \in \Lambda$ and let $\Phi(y) := E(y - x)w$. Then Lemma 1.5.18 shows that

$$\left(w, \nabla \wedge F(x)\right) = \int_{\Sigma} (\nabla \wedge \Phi(y), \nu(y) \wedge f(y)) = \int_{\mathbb{R}^{n-1}} (\tilde{\rho}^{-1}(\nu \wedge \delta \Phi), \rho^{\circ} f)$$

$$= \int_{\mathbb{R}^{n-1}} (-\delta \mathbb{R}^{n-1}, \tilde{\rho}^{-1}(\nu \wedge \phi), \rho^{\circ} f) = \int_{\Sigma} (\nu \wedge \phi, df) = \left(w, \int_{\Sigma} E(y - x)(\nu \wedge df)(y)\right).$$

The same argument applies to $C^- f$ and thus $f \in D(\Gamma)$ and $\Gamma f = df$. The proof that $\nu \wedge D(\delta) \subset D(\Gamma)$ is similar.

(ii) Conversely, assume that $F$ and $dF \in C^+ L_2$ and form the pullback $G = \rho^* F$. Since the problem is local, it suffices to prove that the tangential trace $e_n \cdot (e_n \wedge G|_{\mathbb{R}^{n-1}}) = \rho^* (N^+ f)$ locally belongs to $D(d_{\mathbb{R}^{n-1}})$. However, this follows from a version of Lemma 1.5.18 for $\Phi = F \in D(d, C^+ L_2)$. Similarly, using the reduced pushforward $\tilde{\rho}^{-1} F$, it follows that $\nu \wedge f \in D(\delta)$, and thus $D(\Gamma) \subset D(d) \oplus \nu \wedge D(\delta)$. \qed

108
Appendix A

Pictures of spectra

Below, Figure 1.1, 1.3 and 1.5 are examples of spectral estimates for the rotation operator, with corresponding spectral estimates for the double layer type operators shown in Figure 1.2, 1.4 and 1.6.

Figure 2.1 - 2.9 show the spectrum of the classical double layer potential on various logarithmic spirals as in Example 1.5.7.
Bibliography


