Transmission problems for
Dirac’s and Maxwell’s equations with
Lipschitz interfaces

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Till minnet av min morfar Gunnard Rosén
som gav mig inspiration att ta steget
Declaration

The work in this thesis is my own except where otherwise stated.

Andreas Axelsson
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Erratum

Listed below are the four papers that have been produced from this thesis.

Chapter 2:
Axelsson, A.
Transmission problems and boundary operator algebras.

Chapter 3:
Axelsson, A.
Oblique and normal transmission problems for Dirac operators with strongly Lipschitz interfaces.

Chapter 4:
Axelsson, A.; McIntosh, A.
Hodge decompositions on weakly Lipschitz domains.

Chapter 5:
Axelsson, A.
Transmission problems for Maxwell's equations with weakly Lipschitz interfaces.
(Submitted for publication)
Abstract

The aim of this thesis is to give a mathematical framework for scattering of electromagnetic waves by rough surfaces. We prove that the Maxwell transmission problem with a weakly Lipschitz interface, in finite energy norms, is well posed in Fredholm sense for real frequencies. Furthermore, we give precise conditions on the material constants $\epsilon, \mu$ and $\sigma$ and the frequency $\omega$ when this transmission problem is well posed.

To solve the Maxwell transmission problem, we embed Maxwell’s equations in an elliptic Dirac equation. We develop a new boundary integral method to solve the Dirac transmission problem. This method uses a boundary integral operator, the rotation operator, which factorises the double layer potential operator. We prove spectral estimates for this rotation operator in finite energy norms using Hodge decompositions on weakly Lipschitz domains.

To ensure that solutions to the Dirac transmission problem indeed solve Maxwell’s equations, we introduce an exterior/interior derivative operator acting in the trace space. By showing that this operator commutes with the two basic reflection operators, we are able to prove that the Maxwell transmission problem is well posed.

We also prove well-posedness for a class of oblique Dirac transmission problems with a strongly Lipschitz interface, in the $L_2$ space on the interface. This is shown by employing the Rellich technique, which gives angular spectral estimates on the rotation operator.

This thesis includes parts of the following papers.


(The main result given in Proposition 6.1.5.)


(A version included as Chapter 3.)


(A version included as Chapter 2.)
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Chapter 0

Introduction

The motivation for this thesis comes from the study of scattering of electromagnetic waves by a rough surface. We consider the following Maxwell transmission problem.

Let $\Sigma \subset \mathbb{R}^3$ be a surface/interface separating the bounded interior domain $\Omega^+$ and the exterior domain $\Omega^-$. Suppose that each of the domains $\Omega^+$ and $\Omega^-$ is composed of a linear, homogeneous, isotropic, possibly conducting material with permittivity $\varepsilon^\pm > 0$, permeability $\mu^\pm > 0$ and conductivity $\sigma^\pm \geq 0$. Define $\varepsilon^\pm := \varepsilon^\pm + i\sigma^\pm / \omega$, where $\omega$ is the frequency. In $\Omega^\pm$, monochromatic, i.e. time-harmonic, electric and magnetic fields $E(x,t) = E(x)e^{-i\omega t}$ and $B(x,t) = B(x)e^{-i\omega t}$ satisfy Maxwell’s equations

$$\nabla \cdot B = 0,$$

$$-i\omega B + \nabla \times E = 0,$$

$$-i\omega \varepsilon^+_\pm E - \nabla \times \left( \frac{1}{\mu^\pm} B \right) = 0,$$

$$\nabla \cdot (\varepsilon^+_\pm E) = 0.$$

Given an incoming field $\{E_0,B_0\}$ in $\Omega^-$, find a transmitted field $\{E^+,B^+\}$ in $\Omega^+$ and a reflected field $\{E^-,B^-\}$ in $\Omega^-$ such that $\{E^\pm,B^\pm\}$ satisfies Maxwell’s equations in $\Omega^\pm$, $\{E^-,B^-\}$ satisfies the Silver–Müller radiation condition at $\infty$ and on $\Sigma$ we have the jump relations

$$\nu \cdot (B_0 + B^- - B^+) = 0,$$

$$\nu \times (E_0 + E^- - E^+) = 0,$$

$$\nu \times \left( \frac{1}{\mu^-}(B_0 + B^-) - \frac{1}{\mu^+} B^+ \right) = 0,$$

$$\nu \cdot (\varepsilon^-_-(E_0 + E^-) - \varepsilon^+_+(E^+)) = 0,$$  \hspace{1cm} (1)

where $\nu$ denotes the outward pointing unit normal on $\Sigma$.

The main themes in this thesis are the following.

(i) Embedding of Maxwell’s equations in an elliptic Dirac equation.

(ii) A pure first order approach to Dirac’s and Maxwell’s equations. We develop methods for directly working with the first order Dirac equation, instead of reducing to a problem for the Laplace operator.
(iii) Splittings of function spaces. For example, two complementary projections $A^\pm$ in a Hilbert space induce the splitting $\mathcal{H} = A^+\mathcal{H} \oplus A^-\mathcal{H}$. Also relevant to transmission problems are Hodge type splittings $\mathcal{H} = R(\Gamma) \oplus (N(\Gamma) \cap N(\Gamma^*)) \oplus R(\Gamma^*)$ induced by a nilpotent operator $\Gamma$, i.e. $\Gamma^2 = 0$.

(iv) Fredholm operator theory.

We now give an overview of our approach to solving the Maxwell transmission problem.

- A usual approach to Maxwell’s equations is to discard the two Gauss divergence equations as redundant. However, by only using the Faraday and Maxwell-Ampère equations the partial differential system is no longer elliptic. In this thesis we show in detail how to solve the Maxwell transmission problem (1) with elliptic methods, using the Gauss equations in a natural way. The basic idea is to rewrite Maxwell’s equations as an elliptic Dirac equation

$$\mathbf{D}_kF := e_1 \triangle \partial_1 F + e_2 \triangle \partial_2 F + e_3 \triangle \partial_3 F + ke_0 \triangle F = 0.$$  

Here $F := -i\sqrt{\varepsilon}e_0 \wedge E + \frac{1}{\sqrt{\mu}}B$ is the full electromagnetic field, where we have replaced the magnetic field above with the corresponding Hodge dual field $B = B_1e_2 \wedge e_3 + B_2e_3 \wedge e_1 + B_3e_1 \wedge e_2$, and where $\{ie_0, e_1, e_2, e_3\}$ are the basis vectors for Minkowski space and $k := \omega \sqrt{\varepsilon / \mu}$ is the wave number. We write $\wedge$ for the exterior product and $\triangle$ for the Clifford product.

It is important to note that Maxwell’s equations combine to a Dirac equation, but also that the electromagnetic field $F$ is a special solution of $\mathbf{D}_kF = 0$ as it is a pure bivector field $F : \Omega \rightarrow \wedge^2 \mathbb{R}^4$. Surprisingly, it turns out that we can reformulate this constraint on the range of $F$ as a divergence/curl free condition on $F$, see Proposition 3.2.5.

- To solve the transmission problem for the Dirac operator $\mathbf{D}_k$, we develop a new boundary integral method. For simplicity, consider the “relativistic” case when $\varepsilon^+\mu^+ = \varepsilon^-\mu^-$. In this case, (1) can be written in the form

$$\begin{cases}
N^+(\alpha^+ f^+ - \alpha^+ f^-) = N^+ g & \text{on } \Sigma, \\
N^-(\alpha^+ f^+ - \alpha^- f^-) = N^- g & \text{on } \Sigma, \\
\mathbf{D}_k F = 0 & \text{in } \Omega^\pm.
\end{cases}$$

Here $g$ is determined by $\{E_0, B_0\}$, $F^\pm$ is the electromagnetic field with components $\{E^\pm, B^\pm\}$ and boundary trace $f^\pm$, the jump parameters $\alpha^\pm$ are determined by $\varepsilon^\pm$ and $\mu^\pm$ and $N^+$ and $N^-$ denote the tangential and normal projection operators respectively. In this Dirac transmission problem, $F^\pm$ and $g$ in general take values in the full exterior algebra $\wedge = \wedge \mathbb{R}^{n+1}$.

Associated with the Dirac operator $\mathbf{D}_k$ is a Cauchy type singular integral operator $E_k$. Denoting the corresponding Hardy type projection operator onto the space of
traces of monochromatic fields in $\Omega^\pm$ by $E_k^\pm$, we make the ansatz $f^\pm = E_k^\pm f$, where $f = f^+ + f^-$. We now introduce the reflection operators $E_k := E_k^+ - E_k^-$ and $N := N^+ - N^-$, where $E_k^2 = N^2 = I$, as well as the rotation operator $E_k N$. Just as one uses the double layer potential operator to solve the Dirichlet boundary value problem for the Laplace operator, so we use the rotation operator $E_k N$ to solve the Dirac transmission problem. Indeed, (2) is equivalent to the integral equation

$$(\lambda - E_k N)f = 2E_k g,$$

where $\lambda := (\alpha^+ + \alpha^-)/(\alpha^+ - \alpha^-)$. A remark here is that the rotation operator essentially factorises the double layer potential type operator, just as $D_k^2$ squares to the Helmholtz operator $D_k^2 = \Delta + k^2$, see Proposition 2.1.2.

- Note that a boundary value problem for the Dirac operator corresponds to the spectral points $\lambda = \pm 1$ in (2), and more generally a relativistic Maxwell transmission problem with non-conducting materials corresponds to real $\lambda$. Ideally, one would like to prove that the spectrum of $E_k N$ is close to the unit circle $|\lambda| = 1$, and bounded away from $\lambda = \pm 1$. In this thesis, we show how the local regularity of the interface $\Sigma$ influences the essential spectrum of the rotation operator. We consider two boundary function spaces.

(a) The space $L_2(\Sigma; \lambda)$ on a strongly Lipschitz interface $\Sigma$ together with a domain space $D(\Gamma; L_2) \subset L_2(\Sigma; \lambda)$ of mixed 0 and 1 regularity. In Chapter 3 we prove angular spectral estimates using the Rellich technique. In Section 3.1 we also prove estimates for more general oblique Dirac transmission problems in $L_2(\Sigma; \lambda)$.

(b) The energy trace space $E(\Sigma; \lambda)$ on a weakly Lipschitz interface. This is a Hilbert space of mixed $\pm 1/2$ regularity which we define and use in Chapter 5 to solve Maxwell transmission problems for fields with locally finite energy. In Section 5.3, we prove estimates on $\sigma_{\text{ess}}(E_k N; E)$ using Hodge decompositions. In Chapter 4 we give a pure first order approach, free from Laplace operators, to Hodge decompositions on bounded, weakly Lipschitz domains.

- As a last step in solving the Maxwell transmission problem, we prove that a solution to the Dirac transmission problem (2) is actually a solution to (1). We here encounter Hodge type decompositions of Hilbert spaces again. Indeed, the Maxwell transmission problem is essentially a Dirac transmission problem restricted to the divergence/curl-free Hodge component of the trace space. We prove that there exists a nilpotent exterior/interior derivative operator $\Gamma_k$ in the boundary trace spaces such that

- on normal fields, $\Gamma_k$ acts as a tangential divergence,
- on tangential fields, $\Gamma_k$ acts as a tangential curl, and
- $\Gamma_k$ commutes with both reflection operators $E_k$ and $N$. 

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We now solve (1) as follows. A function $g$ in (2) coming from a Maxwell field $\{E_0, B_0\}$ satisfies $\Gamma_k g = 0$. Since $\Gamma_k E_k = E_k \Gamma_k$ and $\Gamma_k N = N \Gamma_k$, it follows that $\Gamma_k f^? = 0$. This essentially means that $F^? \text{ are Maxwell fields solving } (1)$.

We now state our main result on the Maxwell transmission problem (1). This is a special case of Theorem 5.2.3 as explained in Example 5.2.4.

**Theorem 0.0.1.** Let $\Sigma \subset \mathbb{R}^3$ be a bounded, weakly Lipschitz surface, as in Definition 1.5.1, and consider the Maxwell transmission problem (1) in the finite energy trace space $\mathcal{E}$, defined in Section 5.1. Define jump parameters $\alpha := \sqrt{\epsilon^+_\Sigma / \epsilon^-_\Sigma}$ and $\beta := \sqrt{\mu^-_\Sigma / \mu^+_\Sigma}$, and wave numbers $k^\pm = \omega \sqrt{\epsilon^+_\Sigma \mu^\pm}$. Let $C_\Sigma$ and $C_{\Sigma, 2}$ be the constants from Theorem 5.3.1 and Theorem 5.3.5. Then (1) is well posed in Fredholm sense if

$$\alpha \notin \{ix; x \in \mathbb{R}, 1/C_\Sigma \leq |x| \leq C_\Sigma\},$$

and it is well posed if $\text{Im} \, k^+ > 0$ and $\text{Im} \, k^- > 0$ and either

$$\begin{align*}
|\arg \alpha| + |\arg k^+ - \tfrac{\pi}{2}| + |\arg k^- - \tfrac{\pi}{2}| &< \pi \quad \text{or} \\
\min(|\alpha - \beta|, |\alpha - \tfrac{1}{\beta}|) &< 2/C_{\Sigma, 2}.
\end{align*}$$

The background to this thesis and in particular Chapter 3 is the development in harmonic analysis, the theory of Calderón–Zygmund operators, wavelet theory and boundary value problems on Lipschitz domains since the late 1970’s. The central result here is $L_2(\Sigma)$-boundedness of the Cauchy singular integral operator on strongly Lipschitz surfaces. In two dimensions this was first proved by Calderón [8] when the Lipschitz constant is small and by Coifman–McIntosh–Meyer [10] in the general case. There are by now many proofs and extensions of this celebrated result. The proof which seems most relevant to us in connection with transmission problems is Li–McIntosh–Semmes’ [33] higher dimensional extension of Coifman–Jones–Semmes’ first proof in [9], which we survey in Section 1.5.

A classical method for solving the Dirichlet and Neumann boundary value problems for the Laplace operator is to solve the associated boundary integral equation, an equation of the second kind involving the double layer potential operator or its adjoint. The $L_2(\Sigma)$-boundedness of the double layer potential operator follows from that of the Cauchy singular integral operator. As for the invertibility of the double layer potential equation on strongly Lipschitz surfaces, this was proved in the important work by Verchota [63] using Rellich estimates as a substitute for Fredholm theory. There is much related work on this topic by Dahlberg, Fabes, Jerison, Kenig and others. A good reference is the book [30] by Kenig.

The Rellich estimate technique for solving boundary value problems on strongly Lipschitz domain has also been applied with success to other partial differential systems such as the Lamé system of elasticity, the Stokes system of hydrostatics and Maxwell’s equations in electromagnetic theory. Relevant to us here is the latter, where Rellich estimates were adapted to the study of Maxwell’s equations by Mitrea–Torres–Welland [51] and Mitrea [50] and to the Dirac equation by McIntosh–Mitrea [39] and McIntosh–Mitrea–Mitrea [38].
Compared with (a) above, the Maxwell transmission problem in energy norms in (b) does not require the techniques from harmonic analysis above, and a more complete spectral theory can be obtained. To show boundedness of the boundary integral operators, only Fourier theory on $\mathbb{R}^n$ is needed. For the classical double layer potential operator, early spectral estimates can be found in Kellogg’s classical book [29]. For the Dirac equation, the substitute for Rellich estimates is Hodge decompositions, and this theory goes through in the larger class of weakly Lipschitz domains. The theory presented in Chapter 4 and Chapter 5 has essentially been obtained independently here. However, the method employed to obtain Hodge decompositions on weakly Lipschitz domains has also been observed by Picard [52]. Also, Dorina and Marius Mitrea have investigated boundary value problems related to Chapter 5, on strongly Lipschitz domains in [43] and [44]. See the introduction to Chapter 5.

Finally, relevant to boundary value problems and transmission problems for Maxwell’s and Dirac’s equations, we would also like to mention the monographs Colton–Kress [11], Schwarz [57] and Jiang [27]. In the latter, it is pointed out that the elliptization procedure in (i) above is important for numerical stability and that the Gauss equations should not be neglected.