Singularity theorems
and
the abstract boundary construction

By
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Declaration

I certify that the work contained in this thesis is my own original research, produced in collaboration with my supervisor – Dr Susan M. Scott. All material taken from other references is explicitly acknowledged as such. I also certify that the work contained in this thesis has not been submitted for any other degree.

Michael Ashley
Dedication

To my Grandfather
I am sure that many doctoral candidates would agree with me that completing a Ph.D. is not only an academic victory but a personal one as well. Since the beginning of my doctoral studies many events have occurred which have allowed me to develop both intellectually and personally. Indirectly, all these events have contributed in some way to the content and form of this document and the research contained within it. I have had the great fortune of meeting some of the most amazing people. Unknowingly, these friends and colleagues have given me much more than just academic guidance. I only hope that the following thank you-s do justice to your generosity and kindness.

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Mike Ashley

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Abstract

The abstract boundary construction of Scott and Szekeres has proven a practical classification scheme for boundary points of pseudo-Riemannian manifolds. It has also proved its utility in problems associated with the re-embedding of exact solutions containing directional singularities in space-time. Moreover it provides a model for singularities in space-time — essential singularities. However the literature has been devoid of abstract boundary results which have results of direct physical applicability.

This thesis presents several theorems on the existence of essential singularities in space-time and on how the abstract boundary allows definition of optimal embeddings for depicting space-time. Firstly, a review of other boundary constructions for space-time is made with particular emphasis on the deficiencies they possess for describing singularities. The abstract boundary construction is then pedagogically defined and an overview of previous research provided.

We prove that strongly causal, maximally extended space-times possess essential singularities if and only if they possess incomplete causal geodesics. This result creates a link between the Hawking-Penrose incompleteness theorems and the existence of essential singularities. Using this result again together with the work of Beem on the stability of geodesic incompleteness it is possible to prove the stability of existence for essential singularities.

Invariant topological contact properties of abstract boundary points are presented for the first time and used to define partial cross sections, which are an generalization of the notion of embedding for boundary points. Partial cross sections are then used to define a model for an optimal embedding of space-time.

Finally we end with a presentation of the current research into the relationship between curvature singularities and the abstract boundary. This work proposes that the abstract boundary may provide the correct framework to prove curvature singularity theorems for General Relativity. This exciting development would culminate over 30 years of research into the physical conditions required for curvature singularities in space-time.
## Contents

1 Introduction  

2 A History of Boundary Constructions in General Relativity.  
   2.1 Preliminaries  
   2.2 The $g$-Boundary  
      2.2.1 Construction of the $g$-Boundary  
      2.2.2 Problems with the $g$-Boundary  
   2.3 The Bundle Boundary  
      2.3.1 Constructing the $b$-Boundary  
      2.3.2 The Bundle Metric on $L(M)$  
      2.3.3 Problems with the $b$-Boundary Approach  
      2.3.4 Alternate Versions of the $b$-Boundary Construction  
      2.3.5 Summary of the $b$-Boundary Construction  
   2.4 The Causal Boundary  
      2.4.1 Ideal Points in the $c$-Boundary  
      2.4.2 Problems with the GKP Prescription for $\mathcal{T}$ and Alternate Topologies for $M^*$  
      2.4.3 Concluding Summary of the $c$-Boundary Construction  
   2.5 Concluding Remarks on the Boundary Constructions  

3 A Review of the Abstract Boundary Construction.  
   3.1 Preliminaries  
   3.2 Introductory $a$-Boundary Concepts  
   3.3 $\mathcal{C}$-Approachability  
   3.4 Regular Boundary Points and Extendability  
      3.4.1 Non-Regular Points  
      3.4.2 Points at Infinity  
      3.4.3 Singular Boundary Points  
   3.5 Classifying Boundary Points  
      3.5.1 Open Questions in the $a$-Boundary Point Classification Scheme  
   3.6 Topological Properties of Abstract Boundary Representative Sets  
      3.6.1 Invariance of Compactness for Boundary Sets  
      3.6.2 Isolated Boundary Sets  
   3.7 The Topological Neighbourhood Property  

ix
Appendices

A  The Hierarchy of Causality Conditions for Space-time  141
B  Definitional issues for the concepts of in contact and separate  145
List of Figures

2.1 A thickening of geodesics .................................................. 6
2.2 Commutative diagram for maps between sub-bundles of a symmetric space-time ................................................. 16
2.3 TIP’s and TIF’s in a portion of Minkowski space .................. 24
2.4 Inner and pre-boundary ∂cM points may not be T2-separated .... 28
2.5 TIP’s which are T0 but not T1-separated in T#GKP ................. 29
2.6 Space-time for which TGKP separates two points which Szabados’ relation identifies .................................................. 30
2.7 Space-time for which RGKP identifies more points than the Szabados relation ......................................................... 31
3.1 Penrose diagram of the Kruskal-Szekeres extension ............... 39
3.2 The covering definition ..................................................... 40
3.3 Example explaining Theorem 3.11 ....................................... 41
3.4 Definition of a metric extension about a boundary point ......... 46
3.5 Maximally extended Penrose diagram for the Schwarzschild space-time ................................................................. 48
3.6 Flowchart for classifying boundary points of an embedding in the abstract boundary approach ................................. 52
3.7 Table of allowed boundary point coverings ........................... 53
3.8 Abstract boundary point categories ..................................... 54
3.9 Closed boundary sets may not remain closed when re-enveloped and isolated boundary sets must be bounded ................ 57
3.10 Isolated boundary sets must be closed ................................. 58
3.11 Connectedness is not an invariant property under equivalence . 61
3.12 Two boundary points of an enveloped manifold maybe coalesced into one by the choice of a suitable envelopment φ .............. 61
3.13 Connected boundary sets may not satisfy the CNP ................ 62
3.14 A boundary set which is not simply connected but obeys the SCNP ................................................................. 64
3.15 Misner 2-dimensional example ......................................... 68
4.1 Diagram exhibiting the proof of Proposition 4.6 ..................... 74
4.2 Diagram exhibiting the proof of Proposition 4.19 .................... 83
4.3 Figure illustrating the proof of Theorem 4.22 ......................... 87
4.4 Carter’s example of a space-time containing a causal curve imprisoned in a compact set ........................................... 90

xiii
<table>
<thead>
<tr>
<th>Number</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>Null geodesic approaching a manifold point without violating the</td>
<td>92</td>
</tr>
<tr>
<td></td>
<td>distinguishing condition.</td>
<td></td>
</tr>
<tr>
<td>5.1</td>
<td>Variance of approachability for b.p.p. curve families under conformal</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>transformations.</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>An example of the deviations allowed in the $C^0$-fine topology on</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td>metrics.</td>
<td></td>
</tr>
<tr>
<td>5.3</td>
<td>Imprisonment and Partial Imprisonment.</td>
<td>106</td>
</tr>
<tr>
<td>6.1</td>
<td>Example of a map which expands a boundary point to an interval in</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td>the boundary.</td>
<td></td>
</tr>
<tr>
<td>6.2</td>
<td>Definitional tree for the classification of partial cross sections.</td>
<td>121</td>
</tr>
<tr>
<td>A.1</td>
<td>Hierarchy of Causality Conditions</td>
<td>142</td>
</tr>
<tr>
<td>B.1</td>
<td>The contact relation $\uparrow$ should not be defined using limit</td>
<td>148</td>
</tr>
<tr>
<td></td>
<td>points.</td>
<td></td>
</tr>
</tbody>
</table>