Encoding Possible Final States of the Universe with Conformal Structures

By
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Declaration

I certify that the work contained in this thesis is my own original research, produced in collaboration with my supervisor – Dr. Susan M. Scott. All material taken from other references is explicitly acknowledged as such. I also certify that the work contained in this thesis has not been submitted for any other degree.

Philipp Höhn
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Philipp Höhn

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Abstract

The concept of an Isotropic (Past) Singularity (IPS) was defined by Goode and Wainwright in 1985 as a mathematical formalisation of quiescent cosmology and the Weyl Curvature Hypothesis (WCH) for the isotropic initial state of the universe.

In this thesis it is argued that the framework of an IPS is not sufficient to guarantee a future behaviour which is compatible with the future anisotropy implied by quiescent cosmology and the WCH. Therefore it is necessary to complete and combine the framework of an IPS with new definitions, in order to assure an appropriate past and future behaviour of a cosmology satisfying the respective combination of definitions. Since it is not yet clear whether our universe will expand indefinitely or recontract, it is reasonable to provide a new definition for the scenario of an ever expanding cosmos and one for a recollapsing universe.

Specific example space-times are explored for their conformal structure, future evolution and compatibility with the WCH as guidance in the quest for the new definitions. Motivated by these particular models, we present for the first time the definitions for the conformal structure of an Anisotropic Future Endless Universe (AFEU) and an Anisotropic Future Singularity (AFS). For the purpose of completeness and comparison, we furthermore define the physically less realistic Isotropic Future Singularity (IFS) and the Future Isotropic Universe (FIU).

A number of essential technical implications of the new definitions are derived. It is explicitly shown that a conformal structure, whose conformal factor is a function of cosmic time, necessarily leads to an asymptotically Ricci dominated Weyl curvature and asymptotically expansion dominated kinematics, if the conformal metric remains regular. This condition is satisfied by the IFS and FIU. Based on this, it is argued that a conformal structure for an anisotropic final state of the universe requires a degenerate conformal metric, as is the case for the AFS and AFEU.

This degeneracy complicates the derivation of physical attributes of the concepts of an AFEU and an AFS and, consequently, new approaches are unavoidable. Some physical properties are examined, such as the behaviour of the expansion scalar and the curvature. It is proven that the conformal space-times always possess a future singularity, which under reasonable assumptions corresponds to a strong curvature singularity. Finally, we reveal sufficient conditions for the AFS, as well as the IPS, to be a strong curvature singularity.

The combination of the IPS with the AFEU and the AFS could provide a possible first version of a complete mathematical formalisation of quiescent cosmology.
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Chapter 1

Introduction

It was not until 1917, with the submission of the paper “Kosmologische Betrachtungen zur Allgemeinen Relativitätstheorie” (Cosmological Considerations in the Theory of General Relativity) by Einstein [1] to the Preussische Akademie der Wissenschaften (Prussian Academy of Sciences) in Berlin, that cosmology came into being as a quantitative natural science. Its aim is to determine the evolution and large-scale structure of the physical universe.

The problem of mathematically formulating cosmological models which might appropriately describe the universe has been a rather controversial story. Due to the lack of known evidence about the evolution of the universe, there have been many different viewpoints on what features these models should possess. Most of the models are based on the Theory of General Relativity, while rather recent models arise from Quantum Gravity, Quantum Cosmology and String Theory. Elementary particle physics has also proven to be essential for the understanding of the evolution of the universe; it has contributed significant ideas to cosmology, such as inflation and other concepts for initial conditions. An understanding of cosmology requires the merging of the physics of both the microcosm and the macrocosm.

Gravity is the dominant force on cosmological scales. General Relativity has turned out to be, so far, the most reliable theory for gravitational interaction on scales of the solar system and we shall assume in this thesis that this is also the case for scales of clusters of galaxies and the universe itself. Therefore this work will be based on the Theory of General Relativity and thus here the energy content of the universe will determine our space-time’s geometry. Matter and radiation contribution to the energy content can be described in several ways; here we will adopt Einstein’s idea of treating it as a fluid.

The focus of this work will lie partially on the very beginning, but mostly on the final state of the universe. By studying initial and final singularities we will explore the limits of this theory. These limits need not be the end of the physical world, however, and we will leave it open for a more precise and appropriate theory of gravitational interaction to extend (or perhaps replace) the view gained under the assumption of the validity of General Relativity. Quantum Gravity or String Theory might be possible candidates for this task.

The vast majority of cosmological models are based on the philosophical viewpoint known as the Copernican Principle. It states that “we do not occupy a priv-
ileged position in space-time”. This implies that local physical laws are the same everywhere in the universe and, furthermore, that our view of the universe is not a preferred one, when it comes to astronomical observations. This is a reasonable and greatly simplifying assumption.

A further assumption often used in cosmology, justified by observations, is isotropy, which in the cosmological context simply means that large-scale observations and effects are direction-independent. Belief in the Copernican Principle then leads us to assume that this isotropy must be seen from every point in space-time. Homogeneity is another simplifying assumption which, in cosmology, means that physical conditions are the same everywhere in the universe and also that the metric we use to measure distances is valid everywhere in space-time. This, of course, is just an application of the Copernican Principle.

Einstein’s first model was based on an infinite, homogeneous and isotropic universe without boundary for which he needed to introduce the cosmological constant $\Lambda$ in order to enforce a constant size of the universe. In 1922, however, Friedmann published a work which showed that this constant was unnecessary if one would accept a time dependent length-scale of the universe. This work became significant after the discovery of the red-shift of galaxies by Hubble in 1929, which indicated that the universe is expanding.

There are a number of models which describe an isotropic universe. The astrophysical community most commonly uses the Friedmann-Robertson-Walker (FRW) family of models which we will briefly introduce in chapter 3. These models treat the universe as an isotropic, homogeneous, maximally symmetric and irrotational perfect fluid without shear and acceleration, but with non-vanishing expansion. They combine the metric of Robertson and Walker with the Friedmann equations which are solutions to Einstein’s field equations. In fact, it can be shown by using measurements of the Hubble factor, that if the FRW models would describe the actual universe, then it must have originated from a singular initial state - a hot singularity at which energy density, pressure and space-time curvature were divergent - the big bang.

In 1948 Gamov et al. predicted the cosmic microwave background (CMB) as a remnant radiation of the hot beginning of the universe. It was only in 1965 that Penzias and Wilson discovered this radiation by chance and the community immediately interpreted it as having originated in the big bang. From observations of the CMB via the satellites COBE (Cosmic Background Explorer) and WMAP (Wilkinson Microwave Anisotropy Probe) and of the matter distribution in our observable vicinity, it is well known that the universe is, in fact, extremely smooth and isotropic around us - at least on large scales - which appears to give credibility to the FRW models.

It has become one of the most important issues in cosmology to explain this apparent isotropy in the universe; why does it exist, where does it come from and how does it evolve? There exist several attempts to explain this fact and several names have been given to the respective schools of thought.

The idea on which this thesis will be focused is quiescent cosmology, in contrast to chaotic cosmology. The latter was suggested by Misner in 1968 [2] and is based
on the idea that the universe was extremely irregular in the very beginning. The matter distribution was conjectured to have been maximally chaotic at the big bang - similar to the situation in an explosion. This was necessary to allow an infinite variety of initial conditions which could all lead to an universe like the one we observe, instead of implementing too stringent constraints on the initial conditions of the universe. The great appeal of this idea is that one would not need to know the exact initial conditions in order to understand the large scale structure at present and in the future.

The initial irregularities were conjectured to have been smoothed out by dissipative effects, such as particle creation, hadron collisions, neutrino viscosity, inflation etc.. According to this view, we see the universe today as being so isotropic around us because we simply happen to live at a somewhat late stage of its evolution. Nevertheless, it was shown by detailed calculations [3, 4, 5] that this picture was untenable in its full generality. There would not have been enough time by now for the dissipation to occur and problems with the high entropy in the beginning due to the maximal chaos could not be ironed out.

Barrow then introduced the idea of quiescent cosmology in 1978 [6] which is also based on ideas by Penrose [7]. This picture, in fact, is the opposite of the above and states that the geometry of the universe showed initially a complete lack of chaos. The universe was initially highly regular and only evolved away from regularity due to gravitational attraction. This view explains why we see the universe today as being so isotropic around us with the fact that we still happen to live at a somewhat early stage of its evolution - in contrast to the earlier view.

From predictions of the FRW models and observations it seems reasonable that the matter and radiation of the universe were initially in thermal equilibrium, so the entropy in the initial matter must have been already rather high, like in chaotic cosmology. The apparent omnipresent increase of entropy, however, can now be explained via gravitational entropy and the increase of clumping of matter, which chaotic cosmology does not allow, but which can occur in this scenario as a consequence of gravitational attraction. Certainly, due to the initial thermal equilibrium, a form of gravitational entropy cannot be sought in the matter distribution, but in the geometry of space-time [7].

Gravity becomes the dominant force on large scales, therefore clumping will be enhanced with the growing size of the universe. In fact, gravity shows a somewhat anomalous behaviour with regard to entropy; due to its attractive nature, the natural tendency of a purely gravitating system is towards an enhanced clumping. It seems reasonable that, even in a purely gravitating system, the entropy should still always rise (or at least remain constant) and therefore, it is conjectured, that what we call gravitational entropy, increases with clumping. This seems counter-intuitive at first sight when one thinks about the usual concept of entropy, which increases with the uniformity of the matter distribution. Here we are looking for a supplementary concept, however, which takes the gravitational attraction into account to make the evolution of the universe consistent with the second law of thermodynamics, and which, as mentioned, will be sought in the geometry of the universe. It is this idea of quiescent cosmology which we will adopt for the remainder of this thesis.
as it seems to provide physically reasonable constraints to reduce the number of realistic cosmological models.

This scenario implies that the natural thermodynamic boundary condition for the universe is an absence of clumping at the very beginning. Thermal equilibrium and the absence of clumping imply something very close to spatial isotropy and homogeneity. Therefore the very beginning of the universe must have been similar to the FRW type mentioned above. Since the FRW models are completely isotropic and admit no clumping at all, the initial singularities which lead to an absence of clumping are called isotropic singularities.

Giving a mathematical definition for such an isotropic singularity proved to be a rather difficult task. Goode and Wainwright [8], however, published such a definition in 1985 which formed the basis of much research in the following years. Their definition relates the “physical” space-time in which we live conformally (via a rescaling transformation which leaves light-cones invariant) to an “unphysical” space-time in which all quantities are regular at the time of the initial singularity.

This regularity makes the definition of Goode and Wainwright a very useful mathematical tool; quantities in both space-times are conformally related, therefore the behaviour of certain quantities in the regular “unphysical” space-time allows us to analyse the behaviour of the respective quantities of the “physical” space-time at the isotropic singularity, which would otherwise be arduous due to the singular behaviour in the “physical” space-time.

Much is already known about the implications of the definition by Goode and Wainwright, such as information about physical conditions which a cosmological model needs to satisfy in order to admit such an isotropic singularity and, additionally, a number of example models have been found which possess such an initial singularity (e.g. see [9, 10, 11, 12]). Given the known results, the framework of this definition seems to provide a possible mathematical formulation of quiescent cosmology, at least for the initial state of the universe. As we will see, however, the definition by itself is not sufficient to guarantee a future evolution of a cosmological model which is compatible with the ideas of quiescent cosmology.

The cosmological singularities mentioned so far are initial singularities and up to the present day there has been no notable effort in the literature to analyse the picture of quiescent cosmology in future scenarios.

Most publications on future evolution focus on special cases, e.g. many papers (e.g. [13, 14, 15]) have been published recently on the consequences of work by Barrow in 2004 [16] which defines what he calls sudden future singularities. These are future singularities in FRW models, satisfying the strong energy condition, which can occur as a big rip with infinite pressure and curvature even in an expanding universe before reaching a maximum size, or a final big crunch. It was shown [14], however, that sudden future singularities are not strong curvature singularities, meaning that not all finite objects are necessarily squashed to zero volume as they approach the singularity. This is important, as it implies that space-time might be extendible across such singularities [14], meaning that they are not necessarily the final fate of the universe. Furthermore, since there is absolutely no physical mechanism known which could explain the behaviour of sudden future singularities,
we regard them as physically unrealistic.

Some recent work has also been done on kinematic and Weyl singularities [17] in spatially homogeneous Bianchi cosmologies and on other future singularities. The recent analyses of future singularities, however, are model-specific and, consequently, it is necessary to pursue more general considerations concerning the encoding of possible final states of the universe into mathematical definitions.

Gravitational interaction appears to be only attractive and thus time-asymmetric. Our current knowledge, and the fact that gravity becomes dominant on large scales, imply that a matter filled universe should be time-asymmetric as well. This can be explained by the above mentioned gravitational entropy which increases with clumping of matter in the universe. Quiescent cosmology therefore suggests an anisotropic future evolution of the cosmos and a final high-entropy state which corresponds to a maximum degree of clumping. In other words, if the final state is associated with a final singularity, one would look at the time-reverse of chaotic cosmology. It is, however, not yet clear whether our universe will recollapse in finite future time or expand indefinitely. In the latter case the clumps of matter would increase in size with cosmic evolution but the distances between them would become unthinkably large.

It is the goal of this thesis to complete the framework of isotropic singularities with analogous new definitions for the final state of the universe which are compatible with quiescent cosmology, in order to provide a possible full mathematical formulation of quiescent cosmology for the first time.

For this thesis it is assumed that the reader is acquainted with relativistic cosmology and basic differential geometry. Appendix A gives a brief introduction to the former with regards to definitions, relations and notations used in the main body of this work. An introduction to the general notion of space-time singularities and the specific notion of strong curvature singularities is presented in chapter 2. In chapter 3 we briefly introduce the FRW models which have provided guidance for the definition of isotropic singularities, and address some problems in using them for the description of our own universe. The definition and background of the isotropic singularity - which we will henceforth refer to as the isotropic past singularity (IPS) to avoid confusion with the new definitions - will be summarised in chapter 4 and a review of the implications of this definition will be given in chapter 5. The results presented in chapter 5 could give impetus and direction to future research on the new definitions presented in this thesis, therefore it is necessary to treat them in some detail even though they are not new. Thereupon we will leave the background knowledge and focus on new material for the remainder of this thesis. As guidance in the quest for the new definitions, we analyse the future behaviour of specific space-time models in chapters 6 and 7. In the former we find conformal structures with a vanishing conformal factor, and in the latter we analyse non-FRW models, which have previously been shown to admit an IPS, and find irregular conformal structures with diverging conformal factor. Appendix B provides some further details of the calculations concerning the example models. Motivated by the analyses of the particular cosmologies, we provide four new definitions of conformal structures in chapter 8, namely the definitions of the isotropic future singularity, the
future isotropic universe, and in the light of quiescent cosmology, more importantly, of the anisotropic future endless universe and the anisotropic future singularity. In chapter 9 we derive a number of technical properties of the new definitions which are essential for the elaboration of physical results. The implications of the new isotropic definitions on curvature and the kinematic quantities will be investigated in chapter 10, while chapter 11 provides information on the expansion scalar and the presence of strong curvature singularities for the new anisotropic definitions. Chapter 11 will, furthermore, show for the first time that the IPS is a strong curvature singularity under a reasonable assumption. The thesis will close with a summary of the presented work and an outlook on further research on the new definitions in chapter 12.

1.1 Preliminaries and conventions

Throughout this thesis we will adopt the following definition of a space-time which is based on the versions found in [18, 19].

Definition 1.1 (Space-time)
A space-time \((\mathcal{M}, g)\) is a real, four-dimensional, connected, \(C^\infty\) Hausdorff manifold with a globally defined \(C^2\) tensor field \(g\) of type \((0,2)\), which is symmetric, non-degenerate and Lorentzian. By Lorentzian is meant that, for any \(x \in \mathcal{M}\), there is a basis for the tangent space \(T_x \mathcal{M}\) to \(\mathcal{M}\) at \(x\), relative to which \(g_x\) has the matrix \(\text{diag}(-1,1,1,1)\). Furthermore, the pair \((\mathcal{M}, g)\) shall not be further extendible with the required differentiability.

This manifold \(\mathcal{M}\) is necessarily paracompact [20].

For later use it is advantageous to recall the following definition.

Definition 1.2 (Metric degeneracy)
A metric \(g\) is called degenerate at \(p \in \mathcal{M}\), if \(\exists X \in T_p \mathcal{M}: X \neq 0\) and \(g(X,Y) = 0\) \(\forall Y \in T_p \mathcal{M}\).

Furthermore, the following conventions are chosen for this thesis.

- We use geometrised units \(8\pi G = c = 1\).
- Latin letters denote 0, 1, 2, 3, Greek letters denote 1, 2, 3.
- \(^\sim\) denotes that the respective entity is defined in the unphysical space-time \((\mathcal{M}, \tilde{g})\) for the isotropic past singularity,
- \(^\sim\) denotes that the respective entity exists in the unphysical space-time \((\mathcal{M}, \tilde{g})\) (or \((\mathcal{M}, \tilde{g})\)) of the conformal structure for the final state.
- \(T\) denotes the cosmic time function used for isotropic past singularities,
- \(\tilde{T}\) is the cosmic time function used for the future evolution.
1.2. Abbreviations

- A prime denotes differentiation with respect to $T$ ($\bar{T}$ respectively).
- , denotes a partial derivative,
- ; denotes covariant differentiation with respect to the physical metric $g$,
- : denotes the covariant derivative with respect to any of the unphysical metrics $\tilde{g}, \bar{g}$.
- The effective time derivative of an entity $F$, i.e. the covariant derivative of $F$ with respect to the relevant metric in the direction of the fluid flow, will be denoted by $\dot{F} = F_{,a} u^a$.
- $A = o(B)$ means $\frac{A(x)}{B(x)} \to 0$ as $x \to x_0$.
- Two functions, $A$ and $B$, are said to be asymptotically equivalent as $x \to x_0$, written $A(x) \approx B(x)$, if $A(x) = B(x)[1 + o(1)]$ as $x \to x_0$.
- Round brackets denote symmetrisation, so $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$, and square brackets denote anti-symmetrisation, e.g. $T_{[abc]} = \frac{1}{3!}(T_{abc} + T_{cab} + T_{bca} - T_{bac} - T_{acb} - T_{cba})$.

## Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>AFEU</td>
<td>Anisotropic Future Endless Universe (see Definition 8.7)</td>
</tr>
<tr>
<td>AFS</td>
<td>Anisotropic Future Singularity (see Definition 8.10)</td>
</tr>
<tr>
<td>ASPH</td>
<td>Asymptotic Spatial Homogeneity (see Definition 5.3)</td>
</tr>
<tr>
<td>EFE</td>
<td>Einstein Field Equations</td>
</tr>
<tr>
<td>FIU</td>
<td>Future Isotropic Universe (see Definition 8.3)</td>
</tr>
<tr>
<td>FRW</td>
<td>Friedmann-Robertson-Walker (model)</td>
</tr>
<tr>
<td>FRWC</td>
<td>FRW Conjecture (see Conjecture 5.15)</td>
</tr>
<tr>
<td>GVR</td>
<td>General Vorticity Result (see Theorem 5.21)</td>
</tr>
<tr>
<td>IFS</td>
<td>Isotropic Future Singularity (see Definition 8.1)</td>
</tr>
<tr>
<td>IPS</td>
<td>Isotropic Past Singularity (see Definition 4.1)</td>
</tr>
<tr>
<td>IVC</td>
<td>Initial Value Conjecture (see Conjecture 5.1)</td>
</tr>
<tr>
<td>RM</td>
<td>Restricted Metric (see Definition 5.24)</td>
</tr>
<tr>
<td>TSCS</td>
<td>Tipler Strong Curvature Singularity (see Definition 2.3)</td>
</tr>
<tr>
<td>WCH</td>
<td>Weyl Curvature Hypothesis (see section 4.1)</td>
</tr>
<tr>
<td>ZAR</td>
<td>Zero Acceleration Result (see Theorem 5.22)</td>
</tr>
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</table>
Chapter 2

Space-time singularities

This chapter shall briefly summarise some important ideas concerning space-time singularities with the aim to provide the reader with a better general understanding of this topic in section 2.1 and to introduce the notion of strong curvature singularities, which will be needed in this thesis, in section 2.2.

2.1 General notion of a space-time singularity

Giving a clear-cut general definition for a space-time singularity has been one of the major problems in mathematical relativity for the last decades\(^1\). Common sense leads one to think of a space-time singularity as a place where the metric shows pathological behaviour, for example, infinite curvature. This idea, however, cannot easily be transformed into mathematical language. General relativity is the only theory in which the manifold and the metric are not assumed in advance. Unlike in the case of electrodynamics, we cannot say that a point where a physical quantity diverges is a singular point in space-time. Here the metric is a solution of the Einstein field equations (EFE). The idea of an event in space-time only makes physical sense when both the manifold and the metric are defined around it, i.e. when the solution to the EFE is given in that region. Otherwise the known physical laws and classical general relativity would break down at these points and measurements would become impossible. Only a more general theory of gravity could then provide information about these events. For classical general relativity, however, it is inappropriate to regard points with pathological behaviour of the metric as being part of space-time. This has led to Definition 1.1 of a space-time where the singular points are excised. Consequently the big bang singularities in FRW models and also the isotropic past singularities and future singularities treated in this thesis are not considered to be part of the physical space-time.

The fact that for certain space-time singularities all physical objects (or even the universe itself) are crushed to zero volume should furthermore not be interpreted in the sense that such a singularity is a point in a “bigger” manifold which could be obtained by extending the space-time manifold to regions where the metric violates the requirements of Definition 1.1. This would misinterpret the notion of distance

\(^1\)A more profound discussion of the topic can be found in [18], [19, ch 8], [21, ch 9].
which only makes sense in the space-time. All distance (and thus volume) measurements are performed with the metric and are therefore only meaningful where the metric is defined. In fact, as will be seen in the remainder of this thesis, the physically reasonable cosmological singularities are spacelike hypersurfaces in a bigger manifold which can be obtained as indicated above. It is then only the metric which tells us that the distance among all points of the hypersurface is zero. The notion of a point-like singularity in section 5.2 should be understood in this sense.

It should be noted here that much research has been done on defining the notion of a singular boundary of a space-time. One could, for example, add the singular points to the space-time manifold and define a manifold with boundary on the resulting set of points. In this sense one could refer to a singularity as being in a particular place. Unfortunately, such a process involves many difficulties. Nevertheless, this has led to the development of several important boundary constructions, namely the g-boundary [22], b-boundary [23], causal-boundary [24] and abstract boundary [25]. These boundary constructions, however, will not be employed in this thesis and therefore will not be further discussed.

What other possibilities of characterising a singularity do we have at our disposal? As has been pointed out in the literature [19, 21], simply a divergent tensor component may not be an appropriate characterisation of a singularity. Divergent components of the Riemann tensor \( R_{abcd} \) - the tensor field representing curvature in space-time - or its derivatives can be due to a “bad choice” of coordinates, such as in the prime example of the coordinate singularity at \( r = 2M \) in the Schwarzschild space-time. Avoiding this problem with coordinate independent curvature scalars, such as \( R, R_{ab}R^{ab} \) or other scalar polynomial expressions of the curvature tensor and its covariant derivatives is also not a sufficient characterisation. The scalar could diverge at infinite proper time of an observer - which would not be a physically reasonable singularity - or the scalar might vanish even though parallel propagated components of the curvature tensor blow up. Furthermore, there exist space-times with singularities, but with zero curvature throughout [19]. Hence, for a general definition of a singularity, curvature cannot be the characterising feature.

The most successful approach to a general notion of singularities is to exploit the fact that they are not part of space-time. Deciding whether a space-time has a singularity is then equivalent to determining whether it is incomplete in a sense, i.e. whether any singular points have been cut out [19].

In that case the incompleteness should manifest itself in incomplete geodesics, especially in incomplete causal geodesics, which are the world-lines of freely falling physical objects or photons. Such a geodesic is incomplete, if it is inextendible, but still has finite affine length (which is a generalisation of proper time). In this sense we would call a space-time singular if at least one geodesic was incomplete. Removing regular points from the manifold, thereby rendering some geodesics incomplete, is not permissible, since this would violate the inextendibility of space-time in Definition 1.1.

\(^1\)see Appendix A.1 for its definition.
\(^2\)see Appendix A.1 for the definition of \( R_{ab} \) and \( R \).
\(^3\)This is due to the Lorentzian metric and similar to the zero length of null vectors.
This approach to singularities is reasonable, nevertheless even geodesic incompleteness does not necessarily lead to “holes” for all types of geodesics in space-time as one would expect. Many models are known which are causal geodesically complete, but spacelike geodesically incomplete. In these cases there is a singularity in space-time, but no observer or light ray can ever reach it. In fact space-times can be found which are incomplete in one of the three possible ways (timelike, null, spacelike) and complete in the remaining two [19]. Geroch [26] has even found a model, which is geodesically complete, but at the same time admits future inextendible timelike curves with bounded acceleration which only exist for finite proper time. A rocket ship with a finite amount of fuel, which travels on one of these curves, would vanish from this universe in a finite proper time. Therefore one would like to generalise the incompleteness to include any continuous causal curves as well. This can be done with some more effort [19].

Even restricting our attention to causal geodesic incompleteness leads to physical pathologies and therefore singularities, as observers or light rays can end their existence in finite proper time. This incompleteness has been proven for a wide class of space-times by the famous singularity theorems of Penrose and Hawking (most of which can be found in [19]). As most of the conditions in these theorems seem physically reasonable for the universe, they strongly suggest that our universe is a singular space-time. They show that singularities are a true generic feature of cosmological and gravitational collapse solutions to the EFE.

Although singularity existence has been proven in the sense of incomplete causal geodesics, one would like to classify the singularities encountered by the incomplete geodesics. One could classify them as (a) scalar curvature singularities, if curvature scalars (discussed above) diverge, (b) parallelly propagated curvature singularities, if no scalar, but components of $R_{abcd}$ in a parallelly propagated frame diverge or (c) non-curvature singularities if they are not of type (a) or type (b). Unfortunately, the singularity theorems basically do not give any information about such behaviour as terms like “unbounded” and “near the singularity” are difficult to grasp, since singularities are not part of space-time. New approaches via the notion of the abstract boundary construction and strong curvature (see next section) are currently undertaken to prove what will be called curvature singularity theorems [27].

### 2.2 Strong curvature

In this section we will introduce the concept of strong curvature which will be used in section 11.2 to prove the presence of future strong curvature singularities in the new definitions. The following procedure is based on [27, 28, 29].

Strong curvature singularities were initially considered by Ellis and Schmidt [30] as a type of singularity at which all physical objects experience destruction due to unbounded tidal forces and thus space-time cannot be extended through it. The first formal definition was given by Tipler [31] and basically requires that every physical object will be crushed to zero volume as it approaches such a singularity. For completeness we will also mention a second definition due to Królak [32] which
is weaker than the Tipler definition. Both definitions are based on Jacobi fields which are defined below.

**Definition 2.1 (Jacobi field)**

If \( \mu : [0, t_s) \rightarrow M, t_s \in \mathbb{R}^+ \cup \{\infty\} \) is a geodesic with affine parameter \( s \), then the smooth vector field \( J : [0, t_s) \rightarrow TM \) along \( \mu \) is a Jacobi field if it satisfies the Jacobi equation (geodesic deviation equation)

\[
\nabla^2_{\mu'} J = R(\mu', J) \mu',
\]

where \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \) is the curvature operator (see Appendix A.1).

A derivation of the Jacobi equation may be found in [29]. For a causal geodesic \( \mu \) with tangent vector \( \mu' \) one can find a Jacobi field \( J \) along \( \mu \) which is orthogonal to \( \mu' \) and Lie-transported with the flow. Thus, \( J \) represents the displacement vector between \( \mu \) and another nearby geodesic. Consequently \( \nabla^2_{\mu'} J \) can be interpreted as the relative (or tidal) acceleration between these nearby causal geodesics. This leads to the interpretation of the Jacobi equation as relating the relative acceleration between neighbouring geodesics to the curvature of the space-time.

The two strong curvature definitions are expressed using Jacobi fields with specific behaviour, namely those satisfying the following definition.

**Definition 2.2**

Let \( \mu : [0, t_s) \rightarrow M, t_s \in \mathbb{R}^+ \cup \{\infty\} \) be a causal geodesic with affine parameter \( s \). Define \( J_b(\mu) \) for \( b \in [0, t_s) \) to be the set of smooth vector fields \( Z : [b, t_s) \rightarrow TM \) along \( \mu \) such that

1. \( Z(s) \in T_{\mu(s)}M \),
2. \( Z(b) = 0 \),
3. \( \nabla^2_{\mu'} Z = R(\mu', Z) \mu' \),
4. \( g(\nabla_{\mu'} Z|_b, \mu'(b)) = 0 \).

The Jacobi fields of \( J_b(\mu) \) therefore vanish at \( \mu(b) \) and are orthogonal to \( \mu' \) for each \( s \).

Along any timelike geodesic \( \gamma \) one can find three linearly independent spacelike Jacobi fields \( \{Z^1, Z^2, Z^3\} \), \( Z^\alpha \in J_b(\gamma) \) (\( \alpha = 1, 2, 3 \)). The dual 1-forms \( Z^\alpha \) to the \( Z^\alpha \) (\( \alpha = 1, 2, 3 \)) allow the definition of a spacelike 3-volume element \( V(s) = Z_1 \wedge Z_2 \wedge Z_3 \) at each \( \gamma(s) \), analogously to the definition of the volume form. Similarly, along any null geodesic \( \nu \) one can choose two linearly independent spacelike Jacobi fields \( \{\hat{Z}^1, \hat{Z}^2\} \), \( \hat{Z}^\beta \in J_b(\nu) \) (\( \beta = 1, 2 \)). Again, at every \( \nu(s) \) one can define a spacelike 2-volume element \( \hat{V}(s) = \hat{Z}_1 \wedge \hat{Z}_2 \) with the help of the dual 1-forms \( \hat{Z}_\beta \) to the \( \hat{Z}^\beta \) (\( \beta = 1, 2 \)). This specific choice of Jacobi fields is employed in the strong curvature definitions by Tipler and Królak.
Definition 2.3 (Tipler strong curvature singularity (TSCS))
Let $\gamma$ be a timelike geodesic $\gamma : [0, t_s) \rightarrow M$, $t_s \in \mathbb{R}^+$, (respectively let $\nu$ be a null geodesic) with affine parameter $s$. The Tipler strong curvature condition is said to be satisfied along $\gamma$ (respectively $\nu$) if for all $b \in [0, t_s)$ and all linearly independent Jacobi fields $Z^1, Z^2, Z^3 \in J_b(\gamma)$ (respectively, $\dot{Z}^1, \dot{Z}^2 \in J_b(\nu)$)
\[
\liminf_{s \to t_s} V(s) = 0 \quad \text{(or, respectively, } \liminf_{s \to t_s} \dot{V}(s) = 0). \tag{2.2}
\]

Definition 2.4 (Królok strong curvature singularity (KSCS))
Let $\gamma$ be a timelike geodesic $\gamma : [0, t_s) \rightarrow M$, $t_s \in \mathbb{R}^+$, (respectively let $\nu$ be a null geodesic) with affine parameter $s$. The Królok strong curvature condition is said to be satisfied along $\gamma$ (respectively $\nu$) if for all $b \in [0, t_s)$ and all linearly independent Jacobi fields $Z^1, Z^2, Z^3 \in J_b(\gamma)$ (respectively, $\dot{Z}^1, \dot{Z}^2 \in J_b(\nu)$) there exists a $c \in [b, t_s)$ such that
\[
\frac{dV}{ds}|_c < 0 \quad \text{(or, respectively, } \frac{d\dot{V}}{ds}|_c < 0). \tag{2.3}
\]

The Królok definition is clearly weaker than the Tipler definition and is sometimes referred to as the limiting focussing condition.

Clarke and Królok [33] showed that TSCS’s are parallelly propagated curvature singularities, i.e. some components of $R^a_{bcd}$, $R_{ab}$ and $C^a_{bcd}$ become unbounded in a parallelly propagated frame along $\gamma$ (respectively $\nu$). Furthermore, even the integrals over some of the parallelly propagated components will diverge, as may be seen in the following propositions (proofs may be found in [28]).

Proposition 2.5
For both the timelike and the null cases, if the TSCS condition is satisfied, then for some component $R^a_{000}$ of the Riemann tensor in a parallelly propagated frame the integral
\[
I_{\gamma}(v) = \int_0^v dv' \int_0^{v'} dv'' |R^a_{000}(v'')| \tag{2.4}
\]
does not converge as $v \to t_s$.

Proposition 2.6
If $\nu(v)$ is a null geodesic and the TSCS condition is satisfied, then with respect to a parallelly propagated frame, either the integral
\[
K(v) = \int_0^v dv' \int_0^{v'} dv'' R_{00}(v'') \tag{2.5}
\]
or the integral
\[
L_{c,d}(v) = \int_0^v dv' \int_0^{v'} dv'' \left(\int_0^{v''} dv''' |C^c_{000}(v''')| \right)^2 \tag{2.6}
\]
for some $c, d$ does not converge as $v \to t_s$. 

Similar results for the KSCS were proven with one integral less in each case and can be found in [33].
A brief introduction to FRW models

The Friedmann-Robertson-Walker (FRW) models have played a significant role in cosmology and have been widely used as a test ground for the implications of many cosmological observations, as well as a source for predictions of properties of our observable universe. The prediction of the primordial He-abundance, based on the FRW models, for example, has been in good agreement with astrophysical observations. Relative to observers who co-move with the expanding universe, these general relativistic cosmological models appear spatially homogeneous and isotropic, which is fairly consistent with large scale observations. These are not the only features of the observable universe that the FRW models encapsulate and due to their structural simplicity they are commonly reverted to in order to interpret cosmological observations, such as in the case of the evidence for an accelerating universe and a non-vanishing cosmological constant (e.g. [34, 35, 36]).

As pointed out in the Introduction, the FRW models are of a great importance in quiescent cosmology and therefore in the framework of the isotropic past singularity (IPS). It is the singularity in the FRW models that the IPS is compared to (see section 4.3.2) in order to say that it is “isotropic” and thus for a deeper understanding of the background of the IPS, it is necessary to briefly introduce the FRW models at this point. We begin by presenting the Robertson-Walker metric and the Friedmann equations in section 3.1, before we discuss some aspects related to singularities in these models in section 3.2. Section 3.3 summarises characteristic properties of the FRW models and section 3.4 explains some problems in conjunction with FRW universes and observations. A more profound discussion of the topic may be found in [19, 21, 29, 37, 38] or in any text book treating relativistic cosmology.

3.1 The Robertson-Walker metric and the Friedmann equations

Homogeneity and isotropy imply a metric with the maximum possible number of Killing vectors. This metric can be set up without the EFE and is given in
synchronous coordinates by

\[ \text{ds}^2 = -dt^2 + a^2(t) \, d\sigma^2, \quad (3.1) \]

where

\[ d\sigma^2 = \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \quad (3.2) \]

and \( k \) is independent of time and represents the curvature of the 3-manifold \( \Sigma \) on which \( d\sigma^2 \) is defined. This is known as the Robertson-Walker metric. The function \( a(t) \) is referred to as the scale factor since it is a measure for the size of \( \Sigma \) at time \( t \). \( a(t) \) may be rescaled such that \( k \) can be normalised to either \( k = -1 \), which corresponds to a constant negative curvature on \( \Sigma \), and thus an open universe, \( k = 0 \), which corresponds to no curvature, and thus a flat universe, or \( k = 1 \), which corresponds to a positive curvature, and a closed universe. Equation (3.1) is often recast as (see [37])

\[ d\sigma^2 = d\chi^2 + f(\chi) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \quad (3.3) \]

where

\[ f(\chi) = \begin{cases} 
\sin \chi & \text{if } k = +1, \\
\chi & \text{if } k = 0, \\
\sinh \chi & \text{if } k = -1.
\end{cases} \quad (3.4) \]

The scale factor \( a(t) \) is constrained by the EFE for this metric and a perfect fluid source. This leads to the Friedmann-equations\(^\dagger\) (a mathematically very clean derivation may be found in [29])

\[ \mu + \Lambda = \frac{3}{a^2}(\dot{a}^2 + k) \quad (3.5) \]
\[ p - \Lambda = -\frac{1}{a^2}(2\ddot{a}a + \dot{a}^2 + k). \quad (3.6) \]

The conservation equations for a perfect fluid source furthermore lead to another constraint equation (see [37])

\[ \dot{\mu} = -3(\mu + p) \frac{\dot{a}}{a}. \quad (3.7) \]

The Robertson-Walker metric combined with the Friedmann equations, establish the solution of the EFE which is known as the \textit{Friedmann-Robertson-Walker (FRW) models.}

\(^\dagger\)They in fact show: \( \mu = \mu(t) \) and \( p = p(t) \).
3.2 The initial singularity in FRW models

To compare cosmological observations with the FRW models, it is essential to determine the Hubble parameter \( H \), and the so-called deceleration parameter \( q \), which can be roughly measured. They are defined via the scale factor

\[
H = \frac{\dot{a}}{a}, \quad q = -\frac{\ddot{a}}{a^2}. \tag{3.8}
\]

Combining equation (3.5) and (3.6) provides a specific case of the Raychaudhuri equation [37] (see equation A.32)

\[
6\dot{a} = -(\mu + 3p)a + 2\Lambda a. \tag{3.9}
\]

\( a > 0 \), hence, if the strong energy condition holds, i.e. if \( \mu + 3p > 0 \), and \( \Lambda \leq 0 \), then irrespective of the equation of state \( \ddot{a} < 0 \), or \( q > 0 \). In fact, the condition \( q > 0 \) is sufficient to imply \( \ddot{a} < 0 \) at all earlier times even if \( \Lambda > 0 \). Thus, since \( H_0 > 0 \) (Hubble expansion) and \( a_0 > 0 \), either \( \Lambda \leq 0 \) or \( q_0 > 0 \) (where the index 0 stands for the current value) imply that \( a(t) \) must have vanished at some finite time \( t_0 \) ago. This corresponds to the big bang singularity in the FRW models. It, furthermore, follows that \( t_0 \) can be estimated with the astronomical measurements of the Hubble factor to \( t_0 \lesssim H_0^{-1} \approx 10 \ldots 20 \times 10^{10} y \) [29, 37]. Due to their simplicity, the FRW models are the standard big bang cosmologies.

Another parameter, which is often used in observational cosmology, is the density parameter \(^{\dagger} \) [12, 37, 38]

\[
\mu \Omega = \frac{\mu}{3H^2} = \frac{\mu}{\mu_{\text{crit}}}. \tag{3.10}
\]

The reason for calling \( \mu_{\text{crit}} = 3H^2 \) the critical density becomes clear when equation (3.5) is combined with equation (3.10) and \( \Lambda = 0 \)

\[
\mu \Omega - 1 = \frac{k}{a^2 H^2}. \tag{3.11}
\]

Thus,

\[
\begin{align*}
\mu &< \mu_{\text{crit}} \iff \mu \Omega < 1 \iff k = -1 \iff \text{universe open}, \\
\mu &= \mu_{\text{crit}} \iff \mu \Omega = 1 \iff k = 0 \iff \text{universe flat}, \\
\mu &> \mu_{\text{crit}} \iff \mu \Omega > 1 \iff k = +1 \iff \text{universe closed}.
\end{align*}
\]

This density parameter is frequently used in conjunction with observations and the FRW models in order to find evidence for the future development of our universe. In the case \( k = -1 \) the universe has enough energy to keep on expanding forever, for \( k = 0 \) there would be just sufficient energy to escape a recollapse and if \( k = +1 \) the cosmological model inevitably recollapses to another future singularity where \( a \to 0 \); the big crunch\(^\ddagger\). It is currently believed that \( \mu \Omega \approx 1 \). As will be seen in section 5.1 the framework of IPS offers an explanation for this. In chapter 6 we will analyse two big crunch singularities in FRW models.

\(^{\dagger} \mu \Omega \) should not be confused with the conformal factor, hence the index \( \mu \).

\(^{\ddagger} \)A detailed discussion of this may be found in [37, 38].
3.3 A characteristic property

In this section we will briefly discuss some characteristic properties of the FRW models. In the literature concerning the framework of IPS it is often stated and exploited that a vanishing Weyl tensor characterises the FRW models amongst those solutions of the EFE with barotropic perfect fluid source (e.g. [8, 9, 10]). We will justify this fact by outlining the proof of this. Moreover, the behaviour of the vorticity, shear and acceleration in FRW cosmologies will become clear. Details concerning this may be found in [11, 37].

Lemma 3.1
If a space-time \((\mathcal{M}, g)\) is a solution of the EFE with an irrotational, shear-free, geodesic, perfect fluid source, then

1. \(E_{ab} = H_{ab} = 0\), and
2. \(ds^2 = -dt^2 + a^2(t)d\sigma^2\).

Proof. 1. Use the constraint equation (A.38)

\[ H_{ad} = 2\dot{u}_a(\omega_d) - h_a^s h_d^s \left[ \omega_t^{b;\,c} + \sigma_t^{b;\,c} \right] \eta_s f_{bc} u^f = 0. \]  
(3.12)

The geometric shear propagation equation (A.33) becomes, since \(\Sigma_{rs} = 0\) for perfect fluids,

\[ E_{rs} = 0. \]  
(3.13)

2. Now \(E_{ab} = H_{ab} = 0 \Rightarrow C_{abcd} = 0\) and thus the space-time is conformally flat,

\[ ds^2 = \Omega^2(T, x, y, z)(-dT^2 + dx^2 + dy^2 + dz^2), \]  
(3.14)

and the comoving fluid can be normalized \(u^a = \frac{1}{\Omega} \delta^a_0\). This leads to the following expressions for the vorticity, shear, and acceleration,

\[ \omega = 0 \]  
(3.15)

\[ \sigma = 0 \]  
(3.16)

\[ \dot{u}^a = \frac{1}{\Omega^3} \left[ 0, \frac{\partial \Omega}{\partial x}, \frac{\partial \Omega}{\partial y}, \frac{\partial \Omega}{\partial z} \right]. \]  
(3.17)

Now \(\dot{u}^a = 0 \Rightarrow \Omega = \Omega(T)\), therefore

\[ ds^2 = -\Omega^2(T)dt^2 + \Omega^2(T)(dx^2 + dy^2 + dz^2). \]  
(3.18)

Defining \(dt = \Omega(T)dt\), \(d\chi = dx, f(\chi)d\theta = dy, f(\chi)\sin\theta d\phi = dz\) and \(a(t) = \Omega(T)\) produces

\[ ds^2 = -dt^2 + a^2(t) \left( d\chi^2 + f(\chi) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right) \]  
(3.19)

\[ = -dt^2 + a^2(t)d\sigma^2 \]  
(3.20)

where \(d\sigma^2\) is given by equation (3.3).
3.4. A remark on the applicability of FRW models

This lemma provides the proof of the if part of the following lemma.

**Lemma 3.2**
A solution of the EFE with a perfect fluid source is a FRW model if and only if the fluid congruence is irrotational, geodesic, and shear-free.

The converse can be seen by direct calculation [11]. In fact, one can furthermore easily prove [11, p 138]:

**Lemma 3.3**
If a space-time \((\mathcal{M}, g)\) is a solution of the EFE with a barotropic perfect fluid source with \(E_{ab} = H_{ab} = 0\), then the fluid flow is irrotational, shear-free, and geodesic.

### 3.4 A remark on the applicability of FRW models

As mentioned in the beginning of the chapter, the FRW models are frequently used - with considerable success - to compare observations with cosmological theory. At the end of the 1970’s, however, it became obvious that an exact FRW universe would lead to 5 principle problems [12, 40], [39, p 323].

- **The galaxy formation problem.** How can we explain the origin and evolution of galactic structures in the universe? Why does the matter appear to be distributed in filament structures and why are there voids in between?

- **The flatness problem.** Why is the density parameter observed to be so close to the critical value \(\mu \Omega = 1\)?

- **The uniqueness problem.** Is the observed universe unique or due to special initial conditions?

- **The horizon problem.** Why did causally disconnected regions of the universe evolve similarly?

- **The monopole problem.** An overproduction of magnetic monopoles has been predicted in the observable region of the universe by the application of grand unified theories of particle physics to exact FRW cosmologies. Why are these monopoles not observed?

These problems clearly suggest that more general models are necessary to describe our universe. *Inflation* is capable of answering many of these questions [12], [39, p 323], though it is based on some currently unprovable ideas. Goode, Coley and Wainwright [12] argue that *quiescent cosmology* and the related ideas of Penrose regarding the Weyl curvature can provide answers when formulated in the framework of IPS and are thus a viable alternative. This will be discussed in chapter 5.
It should be, furthermore, noted that the Friedmann-equations (3.5) and (3.6) are only valid for a perfect fluid source which is an approximation for our observable universe. But our observable vicinity is clearly not exactly isotropic, neither exactly homogeneous nor an exact perfect fluid and thus one should be careful when interpreting observations with the help of the FRW models, such as in the case of the evidence for an accelerating universe and a non-vanishing cosmological constant in [34, 35, 36].

In the light of quiescent cosmology and the WCH the importance of the FRW models is restricted to the early universe, nevertheless, the big crunch in two specific FRW models will be analysed in chapter 6 for technical interest.
Chapter 4

The definition of an isotropic past singularity

The definition of the Isotropic Singularity (IS) by Goode and Wainwright in 1985 [8] - which we will henceforth refer to as the Isotropic Past Singularities (IPS), due to new future definitions in this thesis - is based on a large amount of previous work on initial singularities in cosmological models. Motivated by quiescent cosmology and Penrose’s Weyl Curvature Hypothesis it was the aim to find a geometric, and hence coordinate independent definition for an initial singularity with similar conditions to those in the FRW universes. Furthermore, for generality, the definition was intended to be independent of the source of the gravitational field and therefore of the EFE. One possible definition fulfilling these conditions is given by the IPS. This idea generalises previous work on “quasi-isotropic” singularities [41] and “Friedmann-like” (or velocity dominated) singularities [42, 43], which where shown by Goode to be essentially equivalent [8]. The major limitations of these two approaches were the coordinate-dependence and the restriction to perfect fluid sources with an exact $\gamma$-law equation of state. Another formalisation of quiescent cosmology is the more restrictive conformal singularity [44, 45] which will not be further treated in this thesis (the relationship between velocity dominated and conformal singularities, and the framework of an IPS may be found in [11, p 9,p 12]). So far the IPS has proven to be the most promising formalisation of quiescent cosmology.

Before introducing the definition of an IPS in section 4.2 we need to recall some properties of the Weyl tensor in section 4.1 in order to understand the Weyl Curvature Hypothesis which expresses the ideas of gravitational entropy, discussed in the introduction, and which forms the fundament for the IPS. The similarity of the initial conditions in the IPS scenario with the FRW models, as well as the reason for terming these singularities “isotropic”, will become clear in section 4.3. Finally, section 4.4 will discuss how the framework of the IPS relates to the Weyl Curvature Hypothesis.
4.1 The Weyl tensor and Penrose’s Weyl Curvature Hypothesis

This section shall introduce the Weyl tensor $C^{abcd}$, its relation to the Riemann tensor $R^{abcd}$, some of its characteristics, and the significant hypothesis by Penrose regarding the initial behaviour of this tensor in a cosmological model, which forms the fundament on which this thesis is built.

The Riemann tensor possesses 20 independent components in four dimensions, 10 of which are given by the Ricci tensor $R^{ab}$. The other 10 are determined by the Weyl tensor. For dimension $n > 2$ the Weyl tensor is defined by (see [19])

$$C^{abcd} = R^{abcd} + \frac{2}{n-2} \left\{ g_{a[d}R_{c]b} + g_{b[c}R_{d]a} \right\} + \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b}, \quad (4.1)$$

where $R$ is the Ricci scalar. Equation (4.1) shows that $C^{abcd}$ possesses the same symmetries as $R^{abcd}$. Furthermore,

$$C^{\alpha a}_{\; \beta b d} = 0, \quad (4.2)$$

thus, one can think of the Weyl tensor as the traceless part of the Riemann tensor. One of its important features is its conformal invariance, i.e. for two conformally related metrics $g = \tilde{\Omega}^2 \tilde{g}$ (where $\tilde{\Omega}$ is a suitably differentiable function) one finds

$$C^{\alpha a}_{\; \beta b c d} = \tilde{C}^{\alpha a}_{\; \beta b c d}. \quad (4.3)$$

While $R^{ab}$ is determined by the EFE\(^4\), and therefore locally by the matter distribution, the interpretation of $C^{abcd}$, on the other hand, is not obvious in the first place. It can, in fact, be interpreted as that part of the curvature at a point which is not determined by the matter distribution at that very point, but by the matter distribution at other points [19]. This becomes evident when one rewrites the Bianchi identities of $R^{abcd}$\(^5\) as

$$C^{\alpha a}_{\; \beta b c d, d} = J^{abc}, \quad \text{with} \quad J^{abc} = R^{c[a;b]} + \frac{1}{6} g^{c[b} R^{a]}, \quad (4.4)$$

which is rather similar to the relativistic form of Maxwell’s equations. In fact, these equations allow one to separate the Weyl tensor into a “magnetic” and an “electric” part (see Appendices A.1 and A.7\(^x\)). Thus, in a sense, one could interpret these Bianchi identities as the field equations for the Weyl tensor.

The Weyl tensor plays a key role in the idea of quiescent cosmology, which was briefly discussed in the introduction. It was argued that, due to the initial thermal equilibrium in the universe, the initial thermodynamic low-entropy constraint at the big bang cannot be sought in the special matter distribution, but must be due to a special initial space-time geometry, which takes the absence of clumping into

\(^{1}\)see Appendix A.1 for the definition of some quantities in this section.

\(^{4}\)see Appendix A.3.

\(^{x}\)see Appendix A.7.
4.2. The definition

The above interpretation indicates that clumps of matter are surrounded by a region of non-zero Weyl curvature and as the clumping becomes enhanced, due to gravitational attraction, empty regions in space open with increased Weyl curvature. Penrose [7] argued that this curvature becomes maximal (divergent) at a future singularity and, thus, the increasing of the Weyl curvature with clumping led him to conjecture that Weyl curvature may be identified with a measure of gravitational entropy. The consequent natural thermodynamic initial condition for the universe would therefore be a Weyl curvature which either tends to zero or is at least bounded. This idea is known as the Weyl Curvature Hypothesis (WCH).

The absence of the Weyl curvature does not permit the definition of principal null directions (see [29] for definition), which is a minimum condition for spatial isotropy, and is thus compatible with the initial absence of gravitational clumping. As an example, the isotropic FRW models do not admit clumping of matter and, unsurprisingly, show a vanishing Weyl tensor throughout (see section 3.3).

As has been pointed out by Penrose [46], the Weyl tensor does not directly affect the expansion of the universe, it, however, causes distortion which in turn leads to contraction. This implies a matter (i.e. Ricci) dominated initial Weyl curvature and a Weyl dominated Ricci curvature for the final stage of cosmic evolution [7]. The WCH, furthermore, indicates that any high-entropy singularity should lead to a very large Weyl curvature.

The WCH has influenced the definition of an IPS, given in the next section, and plays a significant role in shaping the definition for the cosmological futures, presented in chapter 8.

4.2 The definition

Scott [47] has removed some inherent redundancies of the original definition by Goode and Wainwright [8]. This amended definition provides a pattern in the quest for the definition of final states in this thesis and is given in Definition 4.1.

**Definition 4.1 (Isotropic Past Singularity)**

A space-time \((\mathcal{M}, g)\) is said to admit an isotropic past singularity if there exists a space-time \((\mathcal{M}, \bar{g})\), a smooth cosmic time function \(T\) defined on \(\mathcal{M}\), and a conformal factor \(\Omega (T)\) which satisfy

1. \(\mathcal{M}\) is the open submanifold \(T > 0\),

2. \(g = \Omega^2 (T) \bar{g}\) on \(\mathcal{M}\), with \(\bar{g}\) regular (at least \(C^3\) and non-degenerate) on an open neighbourhood of \(T = 0\),

3. \(\Omega (0) = 0\) and \(\exists b > 0\) such that \(\Omega \in C^0 [0, b] \cap C^3 (0, b]\) and \(\Omega (0, b) > 0\),

4. \(\lambda \equiv \lim_{T \to 0^+} L (T)\) exists, \(\lambda \neq 1\), where \(L \equiv \frac{\Omega''}{\Omega} \left(\frac{\Omega}{T}\right)^2\) and a prime denotes differentiation with respect to \(T\).
At this point it is necessary to better explain the definition of a cosmic time function [19, 47] used in the definition of the IPS as the literature tends to become rather inconsistent and sometimes mathematically unclean regarding the exact behaviour of this function. Throughout this thesis the following definition will be used.

**Definition 4.2 (Cosmic time function)**

A cosmic time function is a function $T$ on $\mathcal{M}$, which has a past directed timelike gradient $\nabla T$ with respect to $\mathbf{g}$ everywhere on $\mathcal{M}$.

A theorem shall be given here which clarifies the characteristics of such a cosmic time function. The proof is based on [48].

**Theorem 4.3**

The cosmic time function $T$ on $\mathcal{M}$ increases along every future directed causal curve, the “slices” $\mathcal{S}_T = \{ T = \text{const} \}$ define a family of spacelike hypersurfaces which foliate $\mathcal{M}$ and to which $\nabla T$ is orthogonal. Every hypersurface $\mathcal{S}_T$ can only be intersected once by a future directed causal curve.

**Proof.** Let $\gamma : (a, b) \to \mathcal{M}$ be any future directed causal curve with tangent vector $\gamma'(t) \neq 0$. Then $\mathbf{g}(\nabla T(\gamma(t)), \gamma'(t)) = \gamma'(t)(T) > 0$ since $\nabla T$ is past directed and timelike. Thus, $T$ must strictly increase along $\gamma$. $\nabla T$ obviously must be orthogonal to the hypersurfaces $\mathcal{S}_T$ and since $\nabla T$ is timelike the $\mathcal{S}_T$ are spacelike. $\nabla T$ is non-vanishing and $dT$ is an exact 1-form. Therefore $\mathcal{M}$ can be foliated by the $\mathcal{S}_T$. As $T$ strictly increases along any of these $\gamma$, it becomes clear that each of theses curves cannot intersect the $\mathcal{S}_T$ more than once.

Hawking [19] has proven the important result that a space-time $(\mathcal{M}, \mathbf{g})$ admits a cosmic time function if and only if it is stably causal (stable causality seems to be the appropriate condition for a space-time not to admit closed timelike curves, e.g. see Wald [21] and Ashley [49] for an interesting discussion).

**Remark 4.4**

Requiring the existence of a global time function $T$ in the definition of an IPS is therefore equivalent to requiring stable causality on $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$.

It is instructive, to interpret the definition of an IPS pictorially, as is done in Fig. (4.1). We will call the cosmological solution, $(\mathcal{M}, \mathbf{g})$, of the EFE, the **physical space-time**, and, correspondingly, the conformally related space-time, $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ - which does not necessarily describe a physically reasonable universe - will be referred to as the **unphysical space-time**. It is important to note that due to this shape of the definition one should think here of the **physical space-time** to be produced from the **unphysical** one, rather than vice-versa. In this sense the initial singularity in the **physical space-time** is only due to the vanishing of $\Omega$ on the regular hypersurface $T = 0$ in $\tilde{\mathcal{M}}$. Consequently, the slice $T = 0$ will be referred to as the IPS.
4.3. Isotropy and fluid flow

As will be seen in the next section, Definition 4.1 as it stands is not sufficient to guarantee an “isotropic” behaviour of the kinematics at the IPS. A further constraint on the fluid flow will solve the problem.

4.3 Isotropy and fluid flow

Now that the definition of an IPS stands, it is necessary to analyse how “isotropic” such a singularity is. To this end we first need to introduce the fluid flow.

4.3.1 The velocity vector field and the fluid flow

From observations of the almost isotropically distributed clusters of galaxies and their movement relative to us and to each other, it is well known that the general motion of matter around us is an overall expansion on large scales. Random velocities relative to the main movement are virtually negligible. This allows the definition of an average velocity vector $u^a$, which represents the overall motion in our local vicinity [37]. Applying the Copernican Principle (see Introduction) to this vector implies its existence everywhere in the universe. Therefore we may assume the existence of a unique timelike vector field at every point in space-time which represents the average motion of matter. In order to admit comparisons of this vector field at different points of space-time we must normalise it, i.e.

$$u_a u^a = -1.$$  \hspace{1cm} (4.5)

The Copernican Principle indicates that any observer moving with this velocity field will see the isotropy of the matter and radiation distribution around him on every point of his world-line.
This velocity vector field \( u^a \) defines a congruence in space-time.

**Definition 4.5 (Congruence)**

Let \( \mathcal{O} \subset \mathcal{M} \) be open. A congruence in \( \mathcal{O} \) is a family of curves such that through each \( p \in \mathcal{O} \) there passes precisely one curve in this family.

Every continuous vector field - such as \( u^a \) - generates a congruence of curves and, conversely, the tangents to a congruence define a vector field in \( \mathcal{O} \) [21]. The congruence generated by the velocity vector field \( u^a \) is referred to as the fluid flow.

### 4.3.2 Isotropy

In their seminal paper Goode and Wainwright [8] discuss three types of exact isotropy, all of which can be found in the FRW models\(^1\), in order to justify terming the singularities of Definition 4.1 “isotropic”.

1. **Weyl isotropy:** \( C_{abcd} \equiv 0 \), i.e. there do not exist preferred directions due to the principal null directions.

2. **Ricci isotropy relative to \( u \)** (where \( u \) is the timelike congruence defined in section 4.3.1): The anisotropic parts of the Ricci tensor relative to \( u \) vanish, \( \Sigma_a = \Sigma_a^b \equiv 0 \) (see Appendix A.3 for a definition). The definition of \( \Sigma_a \) implies that its vanishing makes \( u \) an eigenvector of \( R_{ab} \), while \( \Sigma_a^b = 0 \) guarantees that the total projection of \( R_a^b \) orthogonal to \( u \) is isotropic.

3. **Kinematic isotropy relative to \( u \):** \( \sigma = \omega = \hat{\omega} \hat{u}_a = 0 \), i.e. the shear- and vorticity eigenvectors, as well as the acceleration vector cannot define a preferred direction orthogonal to \( u \).

One could require that the term “isotropic singularity” should only be given to singularities which satisfy the above types of isotropy. Restricting ourselves only to cosmologies as the FRW models, however, would not be helpful for studying the implications of quiescent cosmology and the WCH. Moreover, the universe is certainly not an exact FRW model\(^2\) and therefore we are interested in a more general class of models which do not exactly fulfil these types of isotropy.

The arguments in the introduction suggest models in which the shear, vorticity, acceleration and the anisotropic parts of the Ricci tensor are initially expansion-dominated. Unfortunately, Definition 4.1 as it stands, does not guarantee such a behaviour and thus allows singularities which are in no sense, isotropic, “quasi-isotropic” or “Friedmann-like”. The exact viscous fluid FRW cosmology by Coley and Tupper\(^3\) [50] admits an IPS according to Definition 4.1, but does not satisfy the requirement of the expansion-dominated kinematics [8].

---

\(^1\)see section 3.3.

\(^2\)see section 3.4.

\(^3\)This solution of the EFE possesses an FRW metric but a different interpretation of the energy-momentum tensor and thus is not really an FRW cosmology.
4.4. Gravitational entropy

The strongest version of the *Weyl Curvature Hypothesis* (WCH) suggests that the Weyl tensor must have been identical to zero at the beginning of the universe, to guarantee a complete absence of clumping. This is, however, a too stringent constraint on the initial state of the universe; Tod [51] has conjectured that there are no non-FRW models that admit an IPS and also satisfy the strongest version

\[ \lim_{T \to 0^+} \frac{C_{abcd}C^{abcd}}{R_{ef}R^{ef}} = 0, \quad \lim_{T \to 0^+} \frac{\Sigma^a\Sigma_a}{\theta^4} = 0, \quad \lim_{T \to 0^+} \frac{\Sigma^a_{b} \Sigma^b_{a}}{\theta^4} = 0, \quad \lim_{T \to 0^+} \frac{\Sigma^a_{b} \Sigma^b_{a}}{\theta^4} = 0, \quad \lim_{T \to 0^+} \frac{\Sigma^a_{b} \Sigma^b_{a}}{\theta^4} = 0. \]  

(*4.7*)

The first ratio is independent of \( \mathbf{u} \) and will be discussed in the next section and in chapter 10.

**4.4 Gravitational entropy**

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(*4.8*)

*Quiescent cosmology* only required that the initial state of the universe was similar to, but not exactly like an FRW model, in which these equations certainly hold. In this sense it is justified to term the more general singularities given by the conjunction of Definitions 4.1 and 4.6 “isotropic singularities” to distinguish them from singularities in models which are useless for this scenario. Henceforth it is this conjunction which is meant when we refer to the definition of an IPS. This definition implies a number of physically significant results which will be discussed in the next section and in chapter 5. The slight anisotropy allowed on the slice \( T = 0 \) turns out to be crucial for the explanation of galaxy formation (see section 5.1). As will be seen in chapter 8, Definition 4.6 will moreover become essential in the treatment of future singularities.
of the WCH. This view has been supported in an analysis by Goode [52], who showed that in the case of a geodesic fluid the initial condition of a vanishing Weyl tensor in the physical space-time admits the solution of a vanishing Weyl tensor throughout the cosmology - which corresponds to an FRW model - the uniqueness of this solution, however, has not yet been confirmed.

Nevertheless, in this light the strongest version of the WHC does not seem physically plausible, since more general cosmologies than exact FRW models will be necessary for describing the overall evolution of our universe\(^1\). As briefly discussed in section 4.1 Penrose [7] suggested that the Weyl curvature should be Ricci dominated at an initial singularity and that the converse should be true for higher entropy states at later stages of cosmic evolution. The easiest way to encode this into formulae is via the following dimensionless scalar

\[ K = \frac{C_{abcd}C^{abcd}}{R_{ef}R^{ef}}; \quad (4.9) \]

which is the simplest possible measure of the relative significance of the Weyl and Ricci curvatures. In section 4.3.2 it was already seen that in general at an IPS

\[ \lim_{T \rightarrow 0^+} K = 0. \quad (4.10) \]

An instructive proof of this behaviour will be given in section 10.1, as a by-product of a more general theorem also concerning this ratio for possible future singularities. In the light of Penrose’s suggestion, equation (4.10) seems to be an appropriate initial boundary condition for the universe - i.e. an appropriate version of the WCH - which will not only be satisfied by the FRW models, but by a much wider class of physically more reasonable cosmologies.

Goode, Coley and Wainwright [12] have started to investigate whether the scalar \( K \) could be an appropriate measure of gravitational entropy, i.e. whether

\[ \partial_u K > 0 \quad (4.11) \]

where \( \partial_u \) means differentiation along the fluid congruence. The direction of increasing \( K \) would then determine a gravitational arrow of time. In [12] it is suggested to investigate the validity of equation (4.11) in the full class of cosmologies which admit an IPS. At this point we can already state that equation (4.11) does not hold in FRW universes in which \( K \equiv 0 \). Nevertheless, equation (4.11) could be confirmed for all of cosmic time in several non-FRW models, e.g. the Szekeres dust solutions and the Kantowski-Sachs models which admit an IPS (see chapter 7) among others [12]. In some classes of the Bianchi models which admit an IPS, however, it depends on the exact shape of the \( \gamma \)-law equation of state as to whether \( K \) monotonically increases along the fluid flow for all times [12]. It remains an open task to give a general answer to this behaviour. Moreover, no other clear-cut formula has been found so far which gives a measure of gravitational entropy.

\(^1\)see section 3.4.
4.4. Gravitational entropy

As the scalar $K$ will become important in our endeavour to find definitions for cosmological futures which agree with the Weyl curvature ideas by Penrose [7] given in section 4.1, we will investigate $K$ in Theorem 10.1. Furthermore, $K$ will be analysed and plotted for some specific cosmological models, thereby showing that equation (4.11) holds for more models than discussed in [12].

Finally, we state the following for the remainder of this thesis.

Remark 4.7
We will henceforth adopt the weaker version of the WCH for the remainder of this thesis, in agreement with Penrose’s discussion in [7], namely the interpretation, that the Weyl curvature should only initially be Ricci dominated. This version allows models with isotropic initial state and anisotropic future evolution, as required by quiescent cosmology.
Chapter 5

Previous results and implications of isotropic past singularities

In this chapter we will give an overview about results which have been achieved on the grounds of the definition of an IPS by other authors. It is advantageous to treat a number of the implications of an IPS in some detail at this point, as they might provide direction and impetus for future research on the implications of the new definitions for cosmological futures, presented in chapter 8. For brevity, we will abstain from presenting proofs, except in one case in section 5.5 where a proof will be duplicated as it is an enlightening prime example for techniques which are frequently used in most proofs in the framework of the IPS. Details on other calculation can be found in the references (e.g. [8, 11, 47]).

The great analytical advantage of the definition of an IPS is the required regularity in the unphysical space-time. This allows the determination of the behaviour of kinematic and geometric quantities in the unphysical space-time and thus, by the conformal transformation, of the respective quantities in the physical cosmos, which would otherwise be arduous due to the diverging behaviour at the singularity.

Section 5.1 will show how the framework of an IPS offers explanations and answers for the first three cosmological problems discussed in section 3.4. The type of singularity and the Hubble parameter are discussed in section 5.2. Some technicalities concerning the conformal factor are shown in section 5.3 and section 5.4 treats the IPS in FRW models. The accumulated knowledge about the IPS in perfect fluid cosmologies will be summarised in section 5.5 and the endeavour for a characterising feature of the IPS is discussed in section 5.6. The chapter will be closed in section 5.7 with a brief presentation of known models which admit an IPS.

5.1 An initial value conjecture, asymptotic spatial homogeneity and galaxy formation

Investigations [53, 54] have indicated that the primordial density fluctuations, from which the current large scale structure has originated, must have been present at the big bang itself. Goode, Coley and Wainwright [12] suggest that quiescent cosmology, when formulated in the framework of the IPS, is a viable alternative
to cosmic inflation, in that it provides a solution to the difficulty of encoding such initial conditions into the theory.

Goode and Wainwright already showed in their seminal paper [8] that the intrinsic geometry of the hypersurface $T = 0$ and the value of $\lambda$ (see Definition 4.1) completely determine the behaviour of the unphysical kinematic quantities, the Weyl tensor $\tilde{C}_{abcd}$ and Einstein tensor $\tilde{G}_{ab}$ at the IPS of a perfect fluid solution of the EFE. This is significant, since the EFE as well as the Bianchi identities do not impose constraints on the intrinsic geometry, which is governed by the 3-metric $\tilde{h}$ induced on $T = 0$ by $\tilde{g}$. The curvature of this hypersurface is completely described by its 3-Ricci tensor $\tilde{R}^*_{ab}$ (e.g. see [37]). Therefore it was suggested [8, 12] that the 3-metric $\tilde{h}$ on the non-singular hypersurface $T = 0$ forms the initial data for the evolution away from an IPS in the unphysical space-time, and, by the conformal relation, also for the evolution of the physical universe.

**Conjecture 5.1 (Initial Value Conjecture (IVC))**

Specification of $\tilde{h}$, together with a barotropic equation of state $p = p(\mu)$, determines a unique solution of the EFE with perfect fluid source that admits an IPS.

In spite of the difficulties involved, there has been some progress in proving the IVC. For the case of a $\gamma$-law equation of state and a specific conformal factor, Tod [51, 55] has reduced the proof to an existence/uniqueness problem. A complete proof of the IVC restricted to polytropic perfect fluids$^1$, with $1 < \gamma \leq 2$ and a particular shape of the conformal factor was given by Tod and Anguige [56]. Again, under the assumption of a specific form of the conformal factor, they furthermore proved the IVC for spatially homogeneous, massless collisionless gas cosmologies [57], which was extended to the inhomogeneous case by Anguige [58]. A general proof of the IVC, however, still remains to be found.

Considering the necessity of slight density fluctuations in the early universe to account for galaxy formation etc., it is important to not only investigate isotropy, but also homogeneity at an IPS. Since not all cosmological models which admit an IPS are spatially homogeneous, it is necessary to derive a less stringent measure of homogeneity at the IPS, in order to better compare models. The first instructive treatment of this topic was performed by Ericksson and Scott [11]. Based on the following original definition of spatial homogeneity (e.g. [21]), they define the notion of asymptotic spatial homogeneity, given in Definition 5.3.

**Definition 5.2 (Spatially Homogeneous)**

A space-time $(\mathcal{M}, g)$ is said to be spatially homogeneous if there exists a one-parameter family of spacelike hypersurfaces $\Sigma_t$ foliating the space-time such that for each $t$ and for any points $p, q \in \Sigma_t$ there exists an isometry of the space-time metric, $g$, which takes $p$ into $q$.

**Definition 5.3 (Asymptotically Spatially Homogeneous (ASPH))**

A space-time $(\mathcal{M}, g)$ is said to be asymptotically spatially homogeneous if for the

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$^1$see Appendix A.4.
5.1. An initial value conjecture, asymptotic spatial homogeneity and galaxy formation

spacelike hypersurface $\Sigma_0 = \{T = 0\}$ and any points $p, q \in \Sigma_0$ there exists an isometry on the submanifold $\Sigma_0$ of the induced space-time metric, $\rho \tilde{h}$, which takes $p$ into $q$.

It is proven [11] for conformally related space-times with non-vanishing conformal factor that one of the space-times is spatially homogeneous if and only if its conformal relative is as well. Furthermore, it was shown [11] that if the parameter $t$, labelling the spatially homogeneous hypersurfaces, is a function of cosmic time $T$, then the two definitions above are consistent, in the sense that a spatially homogeneous space-time is also ASPH.

An observation which is essential for the analysis of ASPH, is that the Ricci scalar $R$ must be constant on a surface of homogeneity [11]. Thus, a non-constant 3-Ricci scalar, $\rho \tilde{R}^*$, on the hypersurface at $T = 0$ of a space-time with IPS would show that it is not ASPH. Example space-times show that the IPS is not in general ASPH, in fact, of the models discussed in the literature, only the spatially homogeneous models were also ASPH [11], which raises the question of whether the latter implies the former. The fact that the IPS is not in general ASPH, means that the hypersurface $T = 0$ in the unphysical space-time is not in general homogeneous, and thus not in general isotropic. This is not a surprise in the light of section 4.3.2 in which it was shown that the IPS is only asymptotically isotropic.

The allowed inhomogeneity, nevertheless, is an essential property to render non-ASPH models with IPS a viable alternative to inflation. In the standard picture, galaxies are believed to have formed as a result of the growth of local density enhancements in the cosmic fluid, which are expressed as linear perturbations in FRW models. Forming these perturbations into sufficiently large density fluctuations to account for the current large scale structure has caused major difficulties which could be circumvented by the framework of an IPS. If the IVC was valid, the intrinsic geometry of the IPS could be specified arbitrarily. This can be exploited in the expansion of the energy density $\mu$ [12],

$$\mu = \mu_F(T) \left( 1 + \frac{\gamma (3\gamma - 2)^2}{8 (9\gamma - 4)} \rho \tilde{R}^* T^2 + \ldots \right)$$

where $\mu_F(T)$ is a certain function of $T$. This expression clearly explains primordial density fluctuations, i.e. a non-constant $\mu(T = 0)$, with inhomogeneities, i.e. non-constant $\rho \tilde{R}^*$, at the IPS.

**Remark 5.4**

By the conjunction of the discussions of the IVC, ASPH and galaxy formation, we believe that the framework of an IPS admits some physically unreasonable cosmological models, namely those which are ASPH at the IPS. On the other hand, the non-ASPH cosmologies which admit an IPS seem to be applicable to the description of the initial evolution of our universe. In this sense, the discussion of ASPH could be exploited to further constrain the definition of an IPS such that it only admits models which explain primordial density fluctuations. More research needs to be done on this.
Finally, it shall be mentioned that the framework of the IPS does not only offer solutions to the uniqueness and the galaxy formation problem, addressed in section 3.4, but also an explanation for the flatness problem, since Goode and Wainwright [8, 12] proved that the density parameter $\mu \Omega$ has a limiting value of 1 at the IPS,

$$\lim_{T \to 0^+} \mu \Omega = 1. \quad (5.2)$$

### 5.2 Singularity type and Hubble parameter

As pointed out by Goode and Wainwright [8], the IPS is a scalar polynomial curvature singularity in the sense that

$$\lim_{T \to 0^+} R_{ab}R^{ab} = \infty. \quad (5.3)$$

Ericksson [11] exploited this observation to point out that vacuum cosmologies cannot admit an IPS, as they globally satisfy $R_{ab}R^{ab} = 0$.

Nolan [59] furthermore examined the behaviour of the Jacobi fields (defined in section 2.2) of past directed timelike geodesics as the IPS is approached. In fact, the behaviour at the IPS is indeed isotropic, in the sense that all Jacobi fields along a geodesic running into the IPS are crushed to zero uniformly in every direction orthogonal to the geodesic, i.e. any blob of freely moving matter would be uniformly crushed to zero volume as the IPS is approached. There is also some homogeneous behaviour in the crushing of the Jacobi fields, namely the rate at which the fields go to zero depends only on the index $\lambda$ and not on the region of space-time where the geodesic originated from. This was only shown for timelike geodesics and the general behaviour of all causal curves is still not investigated. In section 11.2.3 we will extend this strong curvature picture by showing that the IPS is a TSCS for all causal geodesics whose Jacobi fields possess a limit at $T = 0$.

Information about the general behaviour of the fluid near $T = 0$ can also be gained by analysing the singularity type of the IPS\(^1\). One can choose a fluid element which at some finite time is assumed to be spherical, and investigate its asymptotic behaviour as the IPS is approached back along the flow lines of the timelike congruence. This can be done by examining the length scales $l_\alpha$ ($\alpha = 1, 2, 3$) in the eigendirections of the expansion tensor. The $l_\alpha$ are defined, up to a multiplicative constant [60], by

$$\frac{\dot{l}_\alpha}{l_\alpha} \equiv \theta^\alpha, \quad \alpha \text{ not summed} \quad (5.4)$$

with $\theta^\alpha_\alpha$ being the components of the expansion tensor in its eigenframe. The overall scale factor of the cosmological model is then defined by (see [60])

$$l = (l_1l_2l_3)^{1/3} \quad \text{or} \quad H \equiv \frac{\dot{l}}{l} = \frac{\theta}{3} \quad \text{respectively}, \quad (5.5)$$

where $H$ is the Hubble factor. Based on these definitions one can define four types of singularities\(^2\).

\(^1\)The following discussion is based on [11, ch 5].

\(^2\)Note that not every singularity needs to be of one of these types.
5.3. The conformal factor

- a point-like singularity if all three \( l_\alpha \) approach zero,
- a barrel singularity if two of the \( l_\alpha \) approach zero and the other approaches some finite number,
- a cigar singularity if two of the \( l_\alpha \) approach zero and the other approaches infinity,
- a pancake singularity if one of the \( l_\alpha \) approaches zero and the other two approach some finite number.

From the discussion above it seems reasonable that the IPS is a point-like singularity, i.e. that our spherical fluid element travelling backwards in time would crush approximately isotropically into a point. This has indeed been proven by Ericksson [11].

**Theorem 5.5**

If the space-time \((\mathcal{M}, g)\) admits an IPS at which the fluid flow is regular, then \((\mathcal{M}, g)\) has a point-like singularity.

The proof of this theorem also showed that the expansion scalar \( \theta \), and thereby the Hubble parameter \( H \), which is important for comparing observations of the universe with theoretical cosmology, diverges at \( T = 0 \), i.e.

\[
\lim_{T \to 0^+} \theta = \infty
\]  

(5.6)

(this result is originally due to [8]).

5.3 The conformal factor

This rather technical section is devoted to a treatment of a few characteristics of the conformal factor \( \Omega \) which are not immediately obvious from Definition 4.1. Scott [47] has analysed the asymptotic behaviour of \( \Omega \). Some of the results are:

**Lemma 5.6**

If (1) the conformal factor \( \Omega(T) \) satisfies condition 3 of the definition of an IPS, and (2) \( \lim_{T \to 0^+} L(T) = \lambda \) exists, then \( \exists c \in (0, b] \), s.t.

(i) \( L \) is continuous on \((0, c]\), and

(ii) \( \Omega \) is a strictly monotonically increasing function on \([0, c]\).

**Theorem 5.7**

If (1) the conformal factor \( \Omega(T) \) satisfies condition 3 of the definition of an IPS, and (2) \( \lim_{T \to 0^+} L(T) = \lambda \) exists, then \( -\infty \leq \lambda \leq 1 \).
Corollary 5.8
If (1) the conformal factor $\Omega(T)$ satisfies condition 3 of the definition of an IPS, (2) $\lim_{T \to 0^+} L(T) = \lambda$ exists, and (3) $\lambda \neq 1$, then $\lim_{T \to 0^+} \frac{\Omega'}{\Omega} (\exists)$ exists = $\infty$.

The last two results motivated Scott [47] to amend the original definition by Goode and Wainwright [8], which required that $\lambda < 1$ and $\lim_{T \to 0^+} \frac{\Omega'}{\Omega} = \infty$, which was now shown to be automatically true (except $\lambda \neq 1$).

For the new definitions of cosmological futures we will derive similar results to Theorem 5.7 and Corollary 5.8 in section 9.1. The proofs of the results in section 9.1 serve as examples for proving Theorem 5.7 and Corollary 5.8, since sign changes in the respective proofs directly yield the above results.

Not only the asymptotic behaviour of $\Omega$ is restricted, but also its exact form. Important technical results regarding the form of $\Omega$ and its derivatives as $T \to 0^+$, which are essential for the proofs of several theorems, concerning models which admit an IPS, were proven in [11, 47]. Since these results are not further significant for this thesis they will not be presented here.

There is, however, another important result; it has already been pointed out that the conformal factor (and therefore the unphysical space-time) is not unique [8]. The following proposition due to [11] expresses this nicely.

Proposition 5.9
The conformal factor associated with a particular IPS can be multiplied by a factor $e^{k(T)}$, i.e.

$$\hat{\Omega}(T) = e^{k(T)}\Omega(T), \quad (5.7)$$

where $k(T)$ is a $C^3$ function of $T$ on $\mathbb{R}$, and still satisfy the conditions of an IPS.

Ericksson pointed out [11, p 11] that if the fluid flow $u$ is regular at an IPS associated with the space-time $(\tilde{M}, \tilde{g})$, then it is also regular at the IPS associated with the second unphysical space-time $(\tilde{M}, e^{2k(T)}\tilde{g})$. These results could be used to prove the following proposition.

Proposition 5.10
Suppose that the space-time $(M, g)$ admits an IPS at which the fluid flow $u$ is regular. Let $p$ be a point with $T \geq 0$ in the conformally related unphysical space-time $(\tilde{M}, \tilde{g})$. Then by a suitable choice of a new conformal factor $\tilde{\Omega}$ one can assure that

$$\hat{\theta}|_p = 0. \quad (5.8)$$

One may choose a conformal factor such that the expansion scalar of the unphysical space-time vanishes on the hypersurface $T = 0$. This can be exploited to constrain the behaviour of several kinematic and geometric quantities of the unphysical space-time at the IPS [11, p 29] which might be of some importance for the initial value problem.
5.4 Isotropic past singularities in FRW models

As seen in section 4.3.2, it is the singularities in the isotropic FRW models that the IPS is compared with in order to justify calling it “isotropic”. One could therefore expect that all FRW models admit an IPS \([8, 12]\). As shown by Ericksson and Scott, this is, however, not the case \([11, \text{ch } 6]\).

**Theorem 5.11 (FRW Result)**

A FRW model admits an IPS at which the fluid flow is regular if and only if \(\lim_{t \to 0^+} \frac{\ddot{a}a}{a^2}\) exists and is less than zero (where \(a(t)\) is the scale factor).

Since the deceleration parameter \(q\) for cosmological models is defined as

\[
q \equiv -\frac{\ddot{a}a}{a^2},
\]  
(5.9)

the FRW Result can be physically recast.

**Theorem 5.12 (FRW Result: Alternate statement)**

A FRW model admits an IPS, at which the fluid flow is regular, if and only if the model is initially decelerating, i.e., \(q > 0\).

The FRW result therefore actually gives the precise necessary and sufficient conditions for a FRW model to admit an IPS. An initially positive \(q\) is, consequently, the characterising feature of the IPS in FRW models. Since not all FRW models satisfy the conditions of the FRW result, this proves that in contrast to the previous expectations in the literature, not all FRW models and thereby not all isotropic cosmologies possess an IPS. This analysis indicates that the requirements of an IPS are stronger than just isotropy.

In conjunction with Theorem 5.18 (see section 5.5) the FRW result, in fact, even implies some interesting consequences on the physical properties of the FRW space-time \([11, \text{ch } 6]\).

**Corollary 5.13**

A non-dust, non-asymptotic dust, FRW model, for which \(\mu \neq o(p)\), admits an IPS, at which the fluid flow is regular, if and only if it has a limiting \(\gamma\)-law equation of state, \(p = (\gamma - 1)\mu[1 + o(1)]\) where \(\gamma > \frac{2}{3}\).

**Corollary 5.14**

A dust, or asymptotic dust, FRW model, for which \(\mu \neq o(p)\), admits an IPS at which the fluid flow is regular.

Since FRW space-times are perfect fluids, these two corollaries cover all possible FRW models.

Given these results, we can deduce that FRW models without an IPS, but with a limiting \(\gamma\)-law equation of state and positive energy density must have large negative
pressures, which seems physically unrealistic. Additionally, FRW models with exotic equations of state cannot admit an IPS. The framework of an IPS, when restricted to the FRW case, therefore precludes physically unreasonable properties of the fluid source.

For completeness we will introduce a special case of the IVC, discussed in section 5.1, namely the case where the hypersurface of the IPS is of constant curvature [12]. The FRW models are the only known models that satisfy the strongest version of the WCH, i.e. \( \lim_{T \to 0^+} C_{abcd} = 0 \) (e.g. see table 5.1 in section 5.7), which leads to the FRW conjecture (FRWC) [8, 9, 12, 51].

**Conjecture 5.15 (FRWC)**

*If a space-time is*

1. a \( C^3 \) solution of the EFE with barotropic perfect fluid source,
2. the unit timelike fluid congruence is regular at an IPS, and
3. \( \lim_{T \to 0^+} C_{abcd} = 0 \),

*then the space-time is necessarily a FRW model.*

A full proof of this conjecture still remains to be found. As seen in sections 4.4 and 5.1, this conjecture can, nonetheless, be proven for some special cases [51, 52, 55, 56, 57, 58].

### 5.5 Kinematics and the equation of state in perfect fluids

Many results and implications regarding the behaviour of the kinematic quantities and the limiting equation of state near an IPS are already known. This knowledge, however, is almost solely restricted to perfect fluid cosmologies. In this section we will give some insight into the framework of the IPS in perfect fluids.

Most of the results are based on the assumption that the fluid flow is orthogonal to the IPS. In the following theorem we will show that this is, indeed, justified for perfect fluids. The result of this theorem is due to Goode and Wainwright [8], it is instructive, however, to reproduce a proof by Scott [47] at this point, since this proof is especially enlightening in the sense that it nicely illustrates the techniques which are used in most proofs associated with an IPS.

Recall the EFE for a perfect fluid source\(^\dagger\)

\[
A = \mu, \quad B = p, \quad \Sigma_a = 0, \quad \Sigma_a^b = 0. \tag{5.10}
\]

**Theorem 5.16**

*If the space-time \((\mathcal{M}, g)\) is a \( C^3 \) solution of the EFE with perfect fluid source, and*
the unit timelike fluid congruence \( \mathbf{u} \) is regular at an IPS, then the fluid flow is orthogonal to the IPS.

**Proof.** By equation (5.10) we have \( \Sigma_a = 0 \iff \Sigma^a = 0 \), where \( \Sigma^a = -h^{ac}R_c^d u_d \) and thus \( \Sigma^a T_{,a} = 0 \).

The Ricci tensor \( R_a^b \) in \((\mathcal{M}, g)\) is related to the Ricci tensor \( \mathring{R}_a^b \) in \((\mathring{\mathcal{M}}, \mathring{g})\) via

\[
R_a^b = \Omega^{-2} \left[ \mathring{R}_a^b - (2\Omega^{-1} \Omega_{,ac}\mathring{g}^{bc} + \Omega^{-1} \Omega_{,cd} \mathring{g}^{cd}\delta_a^b) 
+ 4 \left( \Omega^{-2} \Omega_{,a}\mathring{g}^{bc} - \frac{1}{4} \Omega^{-2} \Omega_{,c}\mathring{g}^{cd}\delta_a^b \right) \right]
\]

(5.11)

where a colon denotes covariant differentiation with respect to \( \mathring{g} \). Employing this, one can find the following relation between \( \Sigma^a \) in \((\mathcal{M}, g)\) and \( \Sigma^a = -h^{ac}\mathring{R}_c^d \mathring{u}_d \) in \((\mathring{\mathcal{M}}, \mathring{g})\)

\[
\Sigma^a = \Omega^{-3} \left[ \mathring{\Sigma}^a + 2h^{ab} \left( \Omega^{-1} \Omega_{,bc} - 2\Omega^{-2} \Omega_{,b}\Omega_{,c} \right) \mathring{u}^c \right].
\]

(5.12)

Hence, \( \Sigma^a T_{,a} = 0 \) takes the form

\[
\Omega^{-3} \left[ \mathring{\Sigma}^a T_{,a} + 2h^{ab} T_{,a} \left( \Omega^{-1} \Omega_{,bc} - 2\Omega^{-2} \Omega_{,b}\Omega_{,c} \right) \mathring{u}^c \right] = 0.
\]

(5.13)

Now \( \Omega = \Omega(T) \Rightarrow \Omega_{,a} = \Omega'T_{,a} \) and \( \Omega_{,ab} = \Omega'(T_{,ab} + L\Omega'/\Omega T_{,a} T_{,b}) \) and thus

\[
\Omega^{-3} \left[ \mathring{\Sigma}^a T_{,a} + 2h^{ab} T_{,a} \frac{\Omega'}{\Omega} \left( T_{,bc} + (L-2) \frac{\Omega'}{\Omega} T_{,b} T_{,c} \right) \mathring{u}^c \right] = 0.
\]

(5.14)

\( \Omega^{-3} \) can be eliminated from equation (5.14) since by condition 3 of the definition of an IPS \( \exists b > 0 \) such that \( \Omega \) is positive on \((0, b]\). Additionally from the proof of Lemma 5.6 in [47] it is known that \( \exists c \in (0, b] \) such that \( \Omega'(T) \neq 0 \) on \((0, c] \). This allows us to multiply through by \( (\Omega/\Omega')^2 \) on \((0, c] \)

\[
\left( \frac{\Omega}{\Omega'} \right)^2 \mathring{\Sigma}^a T_{,a} + 2h^{ab} T_{,a} T_{,bc} \frac{\Omega}{\Omega'} \mathring{u}^c + 2(L-2) \mathring{h}^{ab} T_{,a} T_{,b} \mathring{u}^c = 0.
\]

(5.15)

Now \( \mathring{\Sigma}^a T_{,a} \), \( \mathring{h}^{ab} T_{,a} T_{,bc} \mathring{u}^c \) and \( \mathring{h}^{ab} T_{,a} T_{,b} \mathring{u}^c \) are at least \( C^1 \), \( C^2 \) and \( C^3 \) respectively on an open neighbourhood of \( T = 0 \) in \( \mathring{\mathcal{M}} \). By Corollary 5.8 \( \lim_{T \to 0^+} \frac{\Omega}{\Omega'} \) (exists) = 0 and therefore as \( T \to 0^+ \)

\[
\left( \frac{\Omega}{\Omega'} \right)^2 \mathring{\Sigma}^a T_{,a} + 2h^{ab} T_{,a} T_{,bc} \frac{\Omega}{\Omega'} \mathring{u}^c \to 0
\]

\[
\Rightarrow 2(L-2) \mathring{h}^{ab} T_{,a} T_{,b} \mathring{u}^c \to 0.
\]

(5.16)

(5.17)

We know \( \lambda \neq 2 \) and \( \mathring{u}^c T_{,c} \neq 0 \) on \( T = 0 \), which implies \( \mathring{h}^{ab} T_{,a} T_{,b} = 0 \) on \( T = 0 \), i.e. \( \mathring{u} \) is orthogonal to \( T = 0 \). \( \square \)
This result was extended to imperfect fluids with either vanishing heat flux or vanishing anisotropic stress by Ericksson [11]. As can be furthermore easily shown a geodesic timelike fluid congruence which is regular at an IPS is also automatically orthogonal to an IPS [8, 11].

Based on this theorem it is feasible to establish many results for perfect (and some imperfect) fluid cosmologies which admit an IPS. It was shown in [8] that for an imperfect fluid which is regular and orthogonal to an IPS there exists a limiting \( \gamma \)-law equation of state. Another more instructive proof of this result can be found in [47]. In fact the proof in [47] is just as much applicable to a more general imperfect fluid which is orthogonal to an IPS.

**Lemma 5.17**

Let the space-time \((\mathcal{M}, g)\) be a \(C^3\) solution of the EFE with imperfect fluid source. If \((\mathcal{M}, g)\) admits an IPS at which the unit timelike fluid congruence is both regular and orthogonal, and \(\lambda \neq \frac{1}{2}\), then as \(T \to 0^+\)

\[
\mu = 3 \left( \frac{\Omega'}{\Omega^2} \right)^2 (\bar{u}^a T_{a})^2 \to \infty
\]

\[
p \approx -2L \left( \frac{\Omega'}{\Omega^2} \right)^2 (\bar{u}^a T_{a})^2 \to \infty \quad \text{if} \quad \lambda = -\infty
\]

\[
p \approx (1 - 2\lambda) \left( \frac{\Omega'}{\Omega^2} \right)^2 (\bar{u}^a T_{a})^2 \to \begin{cases} 
\infty & \text{if} \quad -\infty < \lambda < \frac{1}{2}, \\
-\infty & \text{if} \quad \frac{1}{2} < \lambda < 1.
\end{cases}
\]

The expressions given by equations (5.18) and (5.20) immediately imply:

**Theorem 5.18 (Limiting \( \gamma \)-law equation of state)**

Let the space-time \((\mathcal{M}, g)\) be a \(C^3\) solution of the EFE with imperfect fluid source. If \((\mathcal{M}, g)\) admits an IPS at which the unit timelike fluid congruence is both regular and orthogonal, and \(\lambda \neq \frac{1}{2}\) or \(-\infty\), then there exists a limiting \( \gamma \)-law equation of state \(p = (\gamma - 1)\mu\) as the IPS is approached, where \(\gamma = \frac{2}{3}(2 - \lambda)\).

In the case of \(\lambda = -\infty\) one can also establish such an asymptotic relation between \(p\) and \(\mu\), however in that case it is not appropriate anymore to talk of a \( \gamma \)-law. Scott has pointed out that for the \(\lambda = \frac{1}{2}\) case it is not possible to find such a specific asymptotic relationship [47]. Nevertheless, the following can be easily proven [47].

**Lemma 5.19**

If the space-time \((\mathcal{M}, g)\) is a dust solution (\(p = 0\)) of the EFE and the unit timelike fluid congruence is regular at an IPS, then \(\lambda = \frac{1}{2}\) and \(\mu \to \infty\) as \(T \to 0^+\).

Goode [61] employed the results of Theorems 5.16 and 5.18 to prove the following.

**Theorem 5.20 (Vorticity Result)**

In any solution of the EFE with perfect fluid source and \( \gamma \)-law equation of state
which admits an IPS and which satisfies

(i) the fluid congruence is regular at \( T = 0 \), and

(ii) \( 1 < \gamma < 2 \), and

(iii) the dominant energy condition holds

the fluid is necessarily irrotational (i.e. \( \omega \equiv 0 \)).

This result holds only for such a restricted range of \( \gamma \). Based on the properties of \( \Omega \) presented in section 5.3 and the results above, Scott was able to extend this theorem to a perfect fluid with a general barotropic equation of state \( p = p(\mu) \) [47]. The theorem is referred to as the General Vorticity Result (GVR) and can be stated as:

**Theorem 5.21 (GVR)**

A barotropic perfect fluid cosmology with non-zero vorticity \( \omega \) and in which the dominant energy condition holds does not admit an IPS.

The knowledge about perfect fluid cosmologies which admit an IPS was further extended by another important theorem due to Ericksson and Scott [10, 11] which exploits the GVR and is known as the Zero Acceleration Result (ZAR).

**Theorem 5.22 (ZAR)**

If a space-time \((\mathcal{M}, g)\) is a \( C^3 \) solution of the EFE with barotropic perfect fluid source and the unit timelike fluid congruence \( \mathbf{u} \) is shear-free (\( \sigma \equiv 0 \)) and regular at an IPS with \(-1 < \lambda < 1\), then the fluid flow is necessarily geodesic, i.e. \( \dot{\mathbf{u}} \equiv 0 \).

The importance of the GVR and ZAR is that they can be combined to produce the fact that shear-free, barotropic perfect fluid cosmologies which are not FRW models, do not admit an IPS, since the FRW cosmologies are characterised by their globally vanishing vorticity, shear and acceleration\(^{\dagger}\).

The technicality \(-1 < \lambda < 1\) in the ZAR can be replaced by the requirement that the dominant energy condition holds in the space-time, since based on the results concerning the limiting \( \gamma \)-law presented above, Ericksson was able to prove the following Corollary [11] which shows that physically reasonable energy conditions hold near the IPS.

**Corollary 5.23**

If the space-time \((\mathcal{M}, g)\) is a \( C^3 \) solution of the EFE with perfect fluid source and the unit timelike fluid congruence \( \mathbf{u} \) is regular at an IPS, then there exists an open neighbourhood \( \mathcal{U} \) of the hypersurface \( T = 0 \) in \( \hat{\mathcal{M}} \) such that the weak and strong energy conditions are satisfied everywhere on \( \mathcal{U} \cap \mathcal{M} \). Furthermore, if \(-1 < \lambda < 1\), then the dominant energy condition also holds on \( \mathcal{U} \cap \mathcal{M} \).

\(^{\dagger}\)see section 3.3.
It should be noted, that the proof of the ZAR uses the dominant energy condition only to guarantee the validity of the GVR. One could therefore replace the dominant energy condition in the ZAR by the requirement that the fluid flow be irrotational, and the result would still be correct for barotropic perfect fluid models, which do not satisfy the dominant energy condition, but which are still irrotational and possess a unit timelike fluid congruence which is regular at an IPS.

5.6 Characterising feature?

There has been some endeavour in the literature to find the characterising feature of an IPS [11, 12] as this would extremely simplify the investigations whether a certain cosmological model actually admits an IPS or not. So far this can only be determined by finding an appropriate conformally related structure. Not finding such a structure, nevertheless, does not necessarily imply that it does not exist.

It was conjectured by Goode, Coley and Wainwright [12] that

$$\lim_{T \to 0^+} K = 0,$$

where $K$ is defined in section 4.4, would be the characterising feature of the IPS.

Since every FRW globally satisfies $K \equiv 0$, Ericksson and Scott [11, ch 6], however, have disproven this conjecture with the FRW result (see section 5.4), which shows that not every FRW model admits an IPS.

In [11, ch 9] it is furthermore argued that a characterising feature cannot be simply a measure of the Weyl tensor, since a bounded Weyl curvature does not necessarily imply the existence of an IPS, whereas the Weyl curvature is forced to be bounded at an IPS.

In section 5.4 it was seen that the deceleration parameter $q$ was the characterising feature of the IPS in FRW models. Based on this state of affairs Ericksson and Scott [11, ch 9] investigated whether this could be generalised to all IPS models. To this end, they defined a restricted form of metric (RM)$^\dagger$.

**Definition 5.24 (RM)**

*For a manifold $\mathbb{R} \times \mathbb{R}^3$ where $\mathbb{R}^3$ is a 3-dimensional manifold, we define the restricted metric by a metric which is given in normal coordinates by*

$$ds^2 = -\frac{dt^2}{F^2(x^\gamma)} + a^2(t)f_{\alpha\beta}(t,x^\gamma)dx^\alpha dx^\beta, \quad t > 0,$$

*where the spatial coordinates $\{x^\gamma\}$ are comoving w.r.t. the timelike congruence defined by $\frac{\partial}{\partial t}$. The following restrictions are made on this metric:*

1. $a(t)$ is at least $C^3$, $a(t) > 0$ and $\frac{da}{dt} > 0$ on $(0,t_1]$ for some $t_1 \in \mathbb{R}^+$,

2. $a(t) \to 0$ as $t \to 0^+$,

$^\dagger$Note the similarities with the FRW metric.
3. \( F \) is at least \( C^3 \) on \( ^3 \mathcal{M} \), and

4. \( f_{\alpha \beta}(T, x^\alpha) \) is at least \( C^3 \) and non-degenerate on \([0, \delta = T(t_1)]\), for some \( t_1 \in \mathbb{R}^+ \), where \( T(t) \equiv \lim_{\epsilon \to 0^+} \int_{\epsilon}^{t} \frac{1}{a(u)} \, du \).

Analogously to the proof of the FRW result, Ericksson and Scott were able to prove that the deceleration parameter \( q \) was, indeed, the characterising feature of the IPS in these RM models, which describe a wide class of possible cosmologies [11, ch 9].

**Theorem 5.25**

Consider an RM space-time \((\mathcal{M}, g)\) with comoving fluid flow. \((\mathcal{M}, g)\) admits an IPS, with \( \lambda = 1 - \beta \), at which the fluid flow is regular if and only if there exists a \( \beta \in \mathbb{R}^+ \cup \{+\infty\} \) such that along every flow-line \( \lim_{t \to 0^+} q = \beta \), where \( q = -\frac{\dot{a}}{a} \) as \( t \to 0^+ \).

As was also mentioned in [11, ch 9], however, \( q \) itself cannot be the characterising feature of the IPS for all models if it needs to satisfy the requirements of Theorem 5.25, as counter-examples are known. Due to counter-examples it is also argued that neither the weak nor the strong energy condition, nor a limiting \( \gamma \)-law equation of state, nor a point-like singularity could be the general characterising feature of the IPS [11, ch 9]. Thus, the precise characterising property of the IPS for all possible models still needs to be found.

### 5.7 Summary of example models

After the review of known results concerning the IPS, it is now natural to raise the question of what specific models actually admit such an IPS. Several models have been shown to possess an IPS at which the fluid flow is regular. The following collection of models is based on [9, 11] in which a complete list, references and a discussion of these models may be found.

The majority of the example models with an IPS are, in fact, perfect fluid cosmologies, all of which are given in table 5.1 categorised according to their physical characteristics - the fluid vorticity, shear, and acceleration; the Weyl tensor and its electric and magnetic parts\(^1\) and the equation of state of the perfect fluid. This table may be compared to the results regarding perfect fluids which were presented in section 5.5. It clearly reflects the GVR [47] which states that barotropic perfect fluids which admit an IPS must be irrotational, i.e. have zero vorticity. One can also see that the Mars models have non-geodesic fluid flow, but still admit an IPS. Thus, geodesicity of the fluid flow cannot be a necessary condition for a barotropic perfect fluid to allow the existence of an IPS.

Interestingly, all the listed non-FRW models in table 5.1 possess an exact \( \gamma \)-law equation of state, which raises the question of whether there can actually exist non-

\(^1\)see Appendix A.1.
FRW barotropic perfect fluids which admit an IPS and which do not have an exact $\gamma$-law equation of state. The table furthermore shows that the only known perfect fluid cosmologies which allow an isotropic singularity and satisfy the strongest version of the WCH, i.e. which have $C_{abcd} = 0$ at the IPS, are the FRW models. This lends weight to the FRW conjecture stated in section 5.4.

Of the example space-times in table 5.1 we will discuss the structure of the IPS in the Kantowski-Sachs, Szekeres and Mars models as examples in chapter 7 before investigating cosmological futures in these cosmologies.

It is not only instructive to look at models which admit an IPS, but also to investigate perfect fluids which do not allow an IPS. Table 5.2 shows some of these models categorised according to their physical characteristics. The Collins-Wainwright and

<table>
<thead>
<tr>
<th>Model</th>
<th>$\omega_{ab}$</th>
<th>$\sigma_{ab}$</th>
<th>$u^a$</th>
<th>$C^{a}_{bcd}$</th>
<th>$E_{ab}$</th>
<th>$H_{ab}$</th>
<th>$p = p(\mu)$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FRW</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>yes (a subclass)</td>
<td>$2 - \frac{3}{2}\gamma$</td>
</tr>
<tr>
<td>Kantowski-Sachs</td>
<td>0 (a)</td>
<td>0 (b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>yes ($p = \frac{1}{\mu}$)</td>
<td>0</td>
</tr>
<tr>
<td>Szekeres (subclass)</td>
<td>0 (a)</td>
<td>0 (b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>yes ($p = 0$)</td>
<td>$\frac{1}{\mu}$</td>
</tr>
<tr>
<td>Bondi</td>
<td>0 (a)</td>
<td>0 (b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>yes ($p = 0$)</td>
<td>$\frac{1}{\mu}$</td>
</tr>
<tr>
<td>Tabensky-Taub</td>
<td>0 (a)</td>
<td>0 (b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>yes ($p = \mu$)</td>
<td>$-1$</td>
</tr>
<tr>
<td>Collins 71</td>
<td>0 (a)</td>
<td>0 (b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>yes ($p = (\gamma - 1)\mu$)</td>
<td>$2 - \frac{3}{2}\gamma$</td>
</tr>
<tr>
<td>Mars 95</td>
<td>0 (a)</td>
<td>0 (b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>yes ($p = \mu$)</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

Table 5.2: Perfect fluid cosmological models without an IPS: (c) means that the relevant quantity is non-zero, (d) means that the behaviour of the relevant quantity is uncertain (after Ericksson [11]).

Wyman models are irrotational and shear-free, but have a non-geodesic fluid flow and, by the ZAR, are hence precluded from the class of space-times which admit an IPS.

In conclusion, much is already known about barotropic perfect fluid cosmologies and IPSs therein. As has been shown in section 5.2 the expansion scalar of the fluid must diverge at an IPS, i.e. models which do not satisfy this cannot admit an IPS. The GVR shows that barotropic perfect fluids which have non-zero vorticity
can also not admit an IPS and the ZAR implies that models which are furthermore shear-free do not have an IPS if the fluid flow is non-geodesic. For the case of a shear-free, irrotational and geodesic fluid flow there are cases with and without IPS, since the FRW result proved that not all FRW models admit an IPS. The only unknown cases are irrotational barotropic perfect fluid cosmologies with non-zero shear. There are examples with non-zero shear and an irrotational, geodesic fluid flow (e.g. Kantowski-Sachs models) and also irrotational examples with non-zero shear and non-zero acceleration (e.g. Mars models) which admit an IPS. This warrants some more research.

There are certainly also imperfect fluid cosmological models known which admit an IPS at which the fluid flow is regular. These are the Mimoso-Crawford models with an anisotropic fluid source without heat flux and the Carneiro-Marugan model whose matter source can be interpreted as a superposition of an anisotropic scalar field with radiation and dust [11]. The structure of the IPS of the latter will also be discussed in chapter 7 before its future metric singularity will be investigated.

Even though there is already much known about the implications of an IPS - especially in barotropic perfect fluid cosmologies - there are still many questions to be answered. A characterising feature of the IPS would provide us with the fundamental answer, but as mentioned in the previous section, it still needs to be found.
Chapter 6

Example space-times with vanishing conformal factor

Specific example space-times provide valuable guidance in the quest for physically reasonable definitions of future singularities. In this chapter, we investigate the existence of space-time singularities in four cosmological models which admit a vanishing conformal factor, three of which, in fact, are FRW models. Motivated by the Weyl Curvature Hypothesis (WCH), we will analyse the behaviour of the scalar $K$, as defined in section 4.4, throughout the evolution of the non-FRW cosmology. It proves to be an example in which $K$ steadily decreases. In the light of the quiescent cosmology concept and the WCH we will regard this model as physically unrealistic. Nevertheless, it is technically important to analyse these types of models to see what distinguishes them from physically reasonable models.

The use of proper time of the fluid flow will become essential in probing whether the vanishing conformal factors in this chapter, and especially the metric singularities found for particular future values of the cosmic time in the next chapter, can actually correspond to physical space-time singularities, i.e. whether they occur at finite or infinite values of the proper time. We can take advantage of the normalisation of the fluid flow; the proper time for a timelike curve with tangent vector $u^a$ and parametrisation $s$ is given by [21, p 44]

$$\tau = \int (-u_au^a)^{1/2}ds. \quad (6.1)$$

Thus, since the integrand is one in our case, the comoving coordinate time $t$ corresponds to the proper time of the fluid flow, if we choose $s = t$. The proper time between two metric pathologies at comoving coordinate times $t_{s_1}$ and $t_{s_2}$, where $t_{s_2} > t_{s_1}$, is therefore simply given by

$$\tau = t_{s_2} - t_{s_1}. \quad (6.2)$$

The focus will lie on the analysis of big crunch singularities in FRW models in sections 6.1.1, 6.1.2 and 6.2.1, before we investigate a singularity in section 6.2.2

---

\(^1K\) is trivially zero in the FRW space-times.
\(^2\)see section 4.3.1.
which can be either a past or a future singularity, depending on a constant. In the case of a past singularity, it is shown to be an IPS, according to Definition 4.1.

Several scalar quantities have been analysed with GRTensorII. Since the calculations are quite extensive, only the results will be presented here. Further details of the calculations can be found in Appendix B. In order to avoid confusion with conformal structures for past singularities, we equip the relevant quantities of the conformal structure for future evolution with a $^\dagger$.

## 6. The big crunch in two specific FRW models

For the investigations of big crunch singularities we are solely interested in closed $(k=+1)$ FRW universes$^\dagger$. In this case it will only be possible to find conformal relations in which the conformal factor vanishes at the singularity, due to the vanishing of the scale factor.

### 6.1 The big crunch in two specific FRW models

#### 6.1.1 A radiation filled, closed FRW universe

A radiation filled, closed FRW model, with equation of state $p = \frac{1}{3} \mu$, possesses the scale factor given by [38, p 227]

$$a(t) = C \sqrt{1 - \left(1 - \frac{t}{C}\right)^2}, \quad \text{with} \quad C^2 = \frac{\mu a^4}{3} = \text{const}, \quad C > 0. \quad (6.3)$$

Thus, $a = 0$ for $t_{s_1} = 0$ and $t_{s_2} = 2C$, where $t_{s_2}$ corresponds to the big crunch, which according to equations (6.2), occurs at finite proper time. Rewriting the scale factor as

$$a(t) = C \sqrt{\left(2 - \frac{t}{C}\right) \frac{t}{C}}, \quad (6.4)$$

and choosing a cosmic time function $\bar{T}$ such that $\bar{T} = 0 \Leftrightarrow t = t_{s_2}$ and which approaches zero from below by

$$\bar{T} = -\sqrt{2 - \frac{t}{C}}, \quad t \in (0, 2C), \quad (6.5)$$

gives the following conformal structure$^\dagger$

$$ds^2 = \Omega^2 (T) \left[ -4dT^2 + (2 - T^2) d\sigma^2 \right], \quad \text{where} \quad \Omega (T) = -CT. \quad (6.6)$$

The conformal metric $ds^2$ in the square brackets is $C^\infty$ and non-degenerate at $\bar{T} = 0$. The conformal factor $\bar{\Omega}$ vanishes at $\bar{T} = 0$ and is always positive and $C^\infty$ for $\bar{T} < 0$. The derivative ratios of $\bar{\Omega}$, discussed in section 5.3, behave as

$$\lim_{\bar{T} \to 0} \frac{\bar{\Omega}'}{\bar{\Omega}} = \lim_{\bar{T} \to 0} \frac{1}{\bar{T}} = -\infty; \quad \bar{L} = \frac{\bar{\Omega}''}{\bar{\Omega}^2} \equiv 0. \quad (6.7)$$

$^\dagger$see chapter 3.

$^\ddagger$Some details of the calculations concerning this model may be found in Appendix B.1.1.
6.1. The big crunch in two specific FRW models

Equation (6.6) implies that the physical fluid flow “blows up” as \( T \to 0^- \), since it is given by

\[
\mathbf{u} = -\frac{1}{2CT} \frac{\partial}{\partial T}.
\]

The unphysical fluid vector is \( C^\infty \) everywhere and takes the following form

\[
\mathbf{u} = \Omega \mathbf{u} = \frac{1}{2} \frac{\partial}{\partial T}.
\]

As one would expect for a big crunch, the expansion scalar of this congruence diverges as \( T \to 0^- \),

\[
\lim_{T \to 0^-} \theta = -\infty.
\]

Since we are dealing with an FRW model, however, we find \( C_{abcd} \equiv 0 \Rightarrow K \equiv 0 \). The asymptotic behaviour of the Ricci curvature, is

\[
\lim_{T \to 0^-} R_{ab}R^{ab} = \infty.
\]

Thus, as one would expect, the timelike fluid congruence encounters a scalar polynomial curvature singularity in finite proper time. The conformal structure and its behaviour are completely analogous to the one of the IPS.

For technical interest, considering the possible definition for a future singularity, it is instructive to analyse the behaviour at \( \bar{T} = 0 \) of some unphysical quantities of the conformally related space-time. We find

\[
\lim_{\bar{T} \to 0^+} \bar{\theta} = 0,
\]

\[
\lim_{\bar{T} \to 0^+} \bar{R}_{ab}\bar{R}^{ab} = \frac{39}{16},
\]

\[
\bar{C}_{abcd}\bar{C}^{abcd} \equiv 0 \equiv \bar{K}.
\]

6.1.2 A dust, closed FRW universe

The scale factor of a closed, dust FRW universe \((p = 0)\) is given by \([38, p \ 226]\) \((\phi(t)\) is a development angle\)

\[
a(\phi) = \frac{\bar{C}}{2} (1 - \cos \phi), \quad \text{where} \quad t = \frac{\bar{C}}{2} (\phi - \sin \phi)
\]

and \( \bar{C} = \frac{\mu a^3}{3} = \text{const}, \ \bar{C} > 0. \)

Hence, \( a = 0 \Leftrightarrow \phi = 2z\pi, \) with \( z \in \mathbb{Z} \). Furthermore,

\[
dt = \frac{\bar{C}}{2} (1 - \cos \phi) \ d\phi = a(\phi) \ d\phi.
\]
Equation (6.17) indicates that $\phi$ is a strictly monotonically increasing function of $t$ on an interval $[\frac{2\pi z}{2}, \frac{2\pi (z+1)}{2}]$, with $z \in \mathbb{Z}$, thus we can use $\phi$ as a cosmic time function. We choose $\phi \in [-2\pi, 0]$, in order to have the zero-point of the cosmic time function at the future singularity$^1$, and set

$$\bar{T} = \phi.$$ (6.18)

The *big crunch* occurs at $\bar{T} = 0$. Using this cosmic time function leads to the following conformal structure

$$ds^2 = \bar{\Omega}^2 (\bar{T}) \left[ -dT^2 + d\sigma^2 \right],$$ (6.19)

where $\bar{\Omega} (\bar{T}) = \frac{C}{2} \left( 1 - \cos \bar{T} \right) = a(\bar{T})$. (6.20)

The conformal metric $ds^2$ in the square brackets is clearly $C^\infty$ and non-degenerate. The conformal factor $\bar{\Omega}$ vanishes for both $\bar{T} = -2\pi$ and $\bar{T} = 0$, since it is the scale factor, which is always positive and $C^\infty$ on $(-2\pi, 0)$.

The conformal factor is found to behave as

$$\lim_{\bar{T} \to 0^-} \frac{\bar{\Omega}'}{\bar{\Omega}} = -\infty, \quad \bar{\lambda} = \frac{1}{2},$$ (6.21)

Equation (6.17) yields the following expressions for the physical and unphysical fluid flows

$$u = \frac{2}{C(1 - \cos \bar{T})} \frac{\partial}{\partial \bar{T}}, \quad \bar{u} = \frac{\partial}{\partial \bar{T}},$$ (6.22)

which shows that again the physical flow “blows up” as $\bar{T} \to 0^-$, while $\bar{u}$ is $C^\infty$ everywhere. The expansion scalar of the physical fluid flow has the expected asymptotic behaviour

$$\lim_{\bar{T} \to 0^-} \theta = -\infty.$$ (6.23)

Once more we find $C_{abcd} \equiv 0 \Rightarrow K \equiv 0$, since the cosmology is an FRW universe. Nonetheless, the timelike fluid congruence ends in a scalar polynomial curvature singularity in finite proper time, as the Ricci curvature diverges,

$$\lim_{\bar{T} \to 0^-} R_{ab} R^{ab} = \infty.$$ (6.24)

The conformal structure and its behaviour again are completely analogous to the one of the IPS.

The scalars $\tilde{\theta}, \bar{C}_{abcd} C^{abcd}$ and $\bar{K}$ of the unphysical, conformal space-time are found to vanish identically, while $\bar{R}_{ab} \bar{R}^{ab} \equiv 12$.

$^1$One could also choose $t = 0$ as the past singularity and reset $\bar{T} = \phi - 2\pi$, but the conformal structure remains the same.

$^1$Some details concerning the following investigations may be found in Appendix B.1.2.
6.2 The McVittie-Wiltshire models

McVittie and Wiltshire presented three classes of solutions to the EFE [62]. They describe non-static, spherically symmetric, perfect fluid solutions with non-vanishing shear, acceleration and expansion. In this section we will investigate subclasses of two of these classes of McVittie-Wiltshire models. The first is shown to be an FRW model, while the second offers a future or a past singularity at $T = 0$, depending on a constant. The latter is shown to be an IPS, according to the Definition 4.1.

6.2.1 McVittie-Wiltshire I

The metric of the first class has the following form in non-comoving coordinates

$$ds^2 = A \left[ -Adt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad \text{where } A = (ar^2 + bt)^{2/3}, \quad (6.25)$$

and $a$ and $b$ are constants. Choose $a = 0$, then $A = 0 \Leftrightarrow t = 0$ and the resulting metric is expressed in synchronous, comoving coordinates. As $t$ increases from $-\infty$ and therefore approaches zero from below, we can choose it as our cosmic time function $T = t$. Hence,

$$\bar{\Omega} (\bar{T}) = (b\bar{T})^{1/3}, \quad \text{where } b < 0. \quad (6.26)$$

The conformal metric $d\bar{s}^2$ in the square brackets is $C^0$ on $(-\infty, 0]$ and degenerate at $\bar{T} = 0$, since

$$d\bar{s}^2 \rightarrow dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad \text{as } \bar{T} \rightarrow 0^-.$$  \hspace{1cm} (6.27)

The conformal factor vanishes at $\bar{T} = 0$ and is positive for $\bar{T} < 0$ and $b < 0$. It is continuous at the singularity and $C^\infty$ for $\bar{T} < 0$. The ratios of the derivatives of $\bar{\Omega}$ are found to behave as

$$\lim_{\bar{T} \rightarrow 0^-} \frac{\bar{\Omega}'}{\bar{\Omega}} = -\infty, \quad \bar{L} = \frac{\bar{\Omega}''\bar{\Omega}}{\bar{\Omega}^2} \equiv -2. \quad (6.28)$$

The physical and unphysical fluid flow vectors are given by

$$\mathbf{u} = \frac{1}{(b\bar{T})^{2/3}} \frac{\partial}{\partial \bar{T}}, \quad \mathbf{u} = \frac{1}{(b\bar{T})^{1/3}} \frac{\partial}{\partial \bar{T}}, \quad (6.29)$$

which, interestingly, both diverge as $\bar{T} \rightarrow 0^-$. The expansion scalar of the physical fluid flow shows an expected asymptotic behaviour for a future singularity,

$$\lim_{\bar{T} \rightarrow 0^-} \theta = -\infty. \quad (6.30)$$

\footnote{McVittie and Wiltshire unfortunately do not provide any equation of state.}

\footnote{See Appendix A.9 for a definition.}

\footnote{Some details regarding the analysis in this section are found in Appendix B.2.1.}
Moreover, $K \equiv 0$ in these models, as the Weyl tensor vanishes throughout, so the specific choice of $a = 0$ renders this solution an FRW model. Indeed, the scale factor is given by $a(t') = (5bt'/3)^{1/5}$ for a flat FRW universe, where $t' = 3b^2/3^{5/3}/5$ (see Appendix B.2.1). The degeneracy in the unphysical metric is due to our desire to choose the cosmic time function such that $t = 0$ corresponds to a future singularity. One could also choose $\Omega(T) = a(t')$ and adjust $T$ appropriately to obtain a non-degenerate, conformal structure which admits an IPS at $t = 0$ (see Appendix B.2.1), but here it is our aim to focus on future singularities.

By the limiting behaviour of the Ricci curvature,

$$\lim_{T \to 0^-} R_{ab}R^{ab} = \infty,$$

it is evident that we are dealing with a big crunch type singularity. Unfortunately, equation (6.25) shows that another (past) metric singularity only occurs an infinite proper time before the big crunch, which makes the whole scenario physically unrealistic. In the quest for possible conformal definitions for future singularities, however, we are so far only interested in the conformal structures and their characteristics and thus we do not need to worry about how realistic a specific model is at this point.

The conformal space-time is flat and therefore all components of its Riemann tensor - and consequently of its Weyl and Ricci tensor - as well as its expansion scalar vanish identically.

### 6.2.2 McVittie-Wiltshire II

The metric of the second class of the McVittie Wiltshire models is given in non-comoving coordinates as well [63, p 262],

$$ds^2 = \exp [2\alpha (r) + 2\psi (t)] (dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 - dt^2),$$

where

\begin{align*}
\text{i)} & \quad \alpha_{,r} - (a + 1)\alpha_{,r}^2 + \frac{1}{2} = 0, \\
\text{ii)} & \quad \psi_{,\theta} - \frac{a + 1}{a} \psi_{,t}^2 - \frac{1}{2} = 0.
\end{align*}

A complete solution of ii) for $a \in (-1, 0)$ is given by

$$\psi = \pm \sqrt{-\frac{a}{2(a + 1)}} t + c_1 - \frac{a}{a + 1} \ln \left(-\frac{a + 1}{a} kt + c_2\right),$$

where $c_1, c_2$ and $k$ are constants. If we now choose $c_1 = c_2 = 0$, $a \in (-1, 0)$ and $k < 0$, we obtain a solution which satisfies $\psi(0) = -\infty$ and $t$ approaches zero from below (for $k > 0$ we would have the same behaviour but $t$ could only approach zero

---

*See section 3.3.

†This metric singularity, in fact, does not correspond to a big bang, since the scale factor and $\theta$ diverge to $+\infty$ (see Appendix B.2.1).

‡McVittie and Wiltshire also do not provide an equation of state for this class.

§The details of the calculations of this section, especially the solutions to the differential equations, are presented in Appendix B.2.2.
6.2. The McVittie-Wiltshire models

from above). Therefore we may choose a cosmic time function $\tilde{T}$ and a conformal factor $\tilde{\Omega}$ of the type

$$\tilde{T} = t, \quad \text{and} \quad \tilde{\Omega}(\tilde{T}) = \exp[\psi(\tilde{T})]. \quad (6.35)$$

The conformal factor is positive and $C^\infty$ for $\tilde{T} < 0 (\tilde{T} > 0)$ and vanishes at $\tilde{T} = 0$, where it is $C^0$. For $k < 0 (k > 0)$ the slice $\tilde{T} = 0$ is a future (past) singularity.

We can choose a specific solution for $\alpha(r)$ which is well-behaved for finite $r$, such as

$$\alpha_S(r) = \pm \sqrt{\frac{1}{2(a+1)}} r + c_3, \quad (6.36)$$

where $c_3 = \text{const}$. This solutions also holds for $a \in (-1,0)$. Thus, we obtain a conformal structure of the type

$$ds^2 = \tilde{\Omega}^2(\tilde{T}) \left[ \exp\{2\alpha_S(r)\} \left( dr^2 + d\theta^2 + \sin^2\theta d\phi^2 - dt^2 \right) \right], \quad (6.37)$$

where $\tilde{\Omega}$ and $\alpha_S$ are determined by (6.34), (6.35) and (6.36). The conformal metric $d\tilde{s}^2$ in the square brackets is clearly non-degenerate and $C^\infty$ for all values of $\tilde{T}$.

Furthermore, it is found that

$$\lim_{T \to 0^-} \frac{\tilde{\Omega}'}{\tilde{\Omega}} = -\infty, \quad (k < 0), \quad \lim_{T \to 0^+} \frac{\tilde{\Omega}'}{\tilde{\Omega}} = +\infty, \quad (k > 0), \quad (6.38)$$

and $\lambda < 1, \quad \text{(independent of } k). \quad (6.39)$

For the Weyl curvature, Ricci curvature and the scalar $K$ of the physical space-time, we find the following asymptotic behaviour, independent of $k$ and of the possible sign choices in (6.34) and (6.36),

$$\lim_{T \to 0^\pm} C_{abcd} C^{abcd} = \lim_{T \to 0^\pm} R_{ab} R^{ab} = +\infty, \quad (6.40)$$

$$\lim_{T \to 0^\pm} K = 0. \quad (6.41)$$

The evolution of the Ricci and Weyl curvature scalars is displayed in fig.6.1, while fig.6.2 shows the behaviour of $K$. Both figures are plotted for the special case $c_i = 0 \quad (i = 1, 2, 3), \quad a = -\frac{1}{2}$ and $k = -1$ at $r = 0^\dagger$.

Since the components of $d\tilde{s}^2$ are independent of $\tilde{T}$, the same quantities of the unphysical space-time are constant in $\tilde{T}$.

Depending on the sign choice in (6.34) one can find past (future) metric singularities for $k < 0 (k > 0)$ in (6.37). The graphs even indicate that there is a physical past singularity in the plotted space-time. Nonetheless, these singularities shall not be of further interest here.

\(^{\dagger}r = 0 \text{ corresponds to a coordinate singularity, however, we are graphing a coordinate independent expression, which shows no pathologies at } r = 0 \text{ (see equations (B.49), (B.50) and (B.51) in Appendix B.2.2).} \)
Figure 6.1: The evolution of $R_{ab}R^{ab}$ and $C_{abcd}C^{abcd}$ as $\bar{T} \to 0^-$ in a subclass of the McVittie-Wiltshire II models at $r = 0$ with $c_i = 0$ ($i = 1, 2, 3$), $a = -\frac{1}{2}$ and $k = -1$ (logarithmic plot, the y-axis labels are meaningless for our purposes and have been omitted).

Figure 6.2: The behaviour of $K$ as $\bar{T} \to 0^-$ in a subclass of the McVittie-Wiltshire II models at $r = 0$ with $c_i = 0$ ($i = 1, 2, 3$), $a = -\frac{1}{2}$ and $k = -1$ (linear plot, the y-axis labels are meaningless for our purposes and have been omitted).

The unphysical manifold is $\bar{M} = \mathbb{R}^4$, using the coordinate patch $(\bar{T}, r, \theta, \phi)$, where $\bar{T} \in \mathbb{R}$. The physical manifold $M$ is clearly the submanifold $\bar{T} < 0$ for $k < 0$ ($\bar{T} > 0$ for $k > 0$) of $\bar{M}$. For $k < 0$ this is the completely analogous future case of
the situation in Definition 4.1 and in the case of \( k > 0 \) it exactly satisfies Definition 4.1, which proves that this subclass of the McVittie-Wiltshire models admits an IPS for \( k > 0 \). This is a new result. Analysing the regularity of the fluid flow in this case, however, is somewhat difficult, since the metric is only given in non-comoving coordinates. It remains an open task to investigate the timelike fluid congruence.

6.3 Discussion

All four models presented in this chapter possess a conformal structure with a vanishing conformal factor, which is responsible for the discovered physical singularities.

The subclass of the McVittie-Wiltshire II models clearly violates the ideas of quiescent cosmology and the WCH, since the scalar \( K \) decreases with the evolution of this cosmology. In a sense, this class of models satisfies the time reverse of quiescent cosmology and is therefore compatible with chaotic cosmology. The Weyl curvature increases steadily into the past which indicates a strong initial clumping of matter and it is only with further cosmic evolution that this clumping decreases towards a stronger isotropy.

The three FRW models, however, also violate the ideas of Penrose outlined in section 4.1 regarding the Weyl dominated Ricci curvature at any high-entropy future singularity. In this sense, we will not regard the models discussed here as appropriate for the description of the evolution of our own universe. Since the first two FRW models also clearly possess an IPS but evolve isotropically for all times, it becomes evident that the framework of the IPS is not sufficient to guarantee a future evolution which is compatible with the anisotropic future behaviour required by quiescent cosmology.

It is important though, to have these conformal structures in hand to compare them to models which appear to be physically more realistic. A complete comparison and discussion of all investigated models in this thesis will be given in section 7.5 in the next chapter which examines example space-times with diverging conformal factor.
Example space-times with diverging conformal factor

The analysis of cosmological futures in specific space-time models with conformal structure needs to be continued and extended to the case in which the conformal factor diverges. All models examined in the following sections are non-FRW cosmologies and have previously been shown to admit an IPS (e.g. see [8, 9, 11, 12, 64]).

In this chapter, it will be especially helpful to investigate the proper time along the fluid flow as discussed in the previous chapter. As will be seen, degeneracy of the conformal metric will play an essential role and all models considered satisfy the WCH in the sense, that $K$ increases throughout all models. For specific cases of the cosmologies we will, furthermore, plot the evolution of $K$, in order to visualise this behaviour.

Before presenting the analysis of the future metric singularities, we will briefly discuss the conformal structure of the IPS previously found for these models by other authors, since it contributes to a better general understanding. In this case, it is particularly important to distinguish the conformal structures for the future and past analysis. As before we will equip all relevant quantities of the future behaviour with a “. In section 7.5 we will summarise and compare all analysed models of this thesis and point out some characteristics of conformal structures with physically reasonable future behaviour.

The analysis of the curvature scalars and the non-zero kinematic quantities has been done with GRTensorII. For revity, we will solely present results and give further details, concerning the calculations of the following sections, in Appendix B.

7.1 A subclass of Szekeres models

The Szekeres models [65] describe irrotational, geodesic, pressure-free dust cosmologies which do not possess any Killing vectors. These perfect fluid models are thus spatially inhomogeneous. We begin by briefly explaining the structure of the IPS in a subclass of these models, based on the discussion in [11].
In comoving coordinates, the metric of these models is given by

\[ ds^2 = -dt^2 + t^{4/3} \left( dx^2 + dy^2 + Z^2 dz^2 \right), \quad t > 0 \]  

(7.1)

where \( Z = A + k_+ t^{2/3} + k_- t^{-1} \), \( A = ax + by + c + \frac{5}{9} k_+ \left( x^2 + y^2 \right) \),  

(7.2)

and \( a, b, c, k_+ \) and \( k_- \) are arbitrary smooth functions of \( z \). To avoid divergent terms as \( t \to 0^+ \), the further analysis is only focused on the case where the decaying mode vanishes, i.e. \( k_- \equiv 0 \). One can choose

\[ T = 3t^{1/3} \]  

(7.3)

as the cosmic time function \( T \), to rewrite the metric as

\[ ds^2 = \Omega^2(T) \left[ -dT^2 + dx^2 + dy^2 + Z^2 dz^2 \right], \]  

(7.4)

where \( \Omega(T) = \frac{T^2}{9}; \ Z = A + k_+ \frac{T^2}{9} \).  

(7.5)

\( \Omega \) satisfies conditions (3) and (4) of the definition of an IPS, with \( \lambda = \frac{1}{2} \).

The conformal metric \( d\tilde{s}^2 \) in the square brackets is clearly \( C^3 \) and non-degenerate on an open neighbourhood of \( T = 0 \), in agreement with requirement (2) of the definition of an IPS. The unphysical manifold \( \mathcal{M} \) is given by \( \mathbb{R}^4 \) covered with the coordinate patch \((T, x, y, z)\) for \( T \in \mathbb{R} \). Consequently \( \mathcal{M} \) is the open submanifold \( T > 0 \) of \( \mathcal{M} \) and condition (1) of Definition 4.1 is satisfied. In conclusion, this subclass of the Szekeres models does admit an IPS. The fluid flow is easily verified to be regular and orthogonal to \( T = 0 \) and the behaviour of the Weyl tensor and the kinematic quantities may be found in table 5.1. This was first discovered by Goode and Wainwright [66].

Looking at equation (7.4) it becomes apparent, that this metric possesses a metric singularity as \( T \to \infty \). We can proceed in the same way as before to find a conformal relation. The details of the following calculations in this section may be found in Appendix B.3. We choose a cosmic time function, which rescales the future metric singularity to 0 and approaches this value from below, via

\[ \bar{T} = -\frac{1}{T}, \]  

(7.6)

and reexpress the metric in the shape

\[ ds^2 = \bar{\Omega}^2(\bar{T}) \left[ -d\bar{T}^2 + \bar{T}^4 \left( dx^2 + dy^2 + Z^2 dz^2 \right) \right], \]  

(7.7)

where

\[ \bar{\Omega}(\bar{T}) = \frac{1}{9\bar{T}^4}, \quad \text{and} \quad Z = A + k_+ \frac{1}{9\bar{T}^2}. \]  

(7.8)

At \( \bar{T} = 0 \) the conformal metric \( d\bar{s}^2 \) in the square brackets takes the form

\[ d\bar{s}^2 = -d\bar{T}^2 + \frac{k_+^2}{81}dz^2, \]  

(7.9)
i.e. it becomes degenerate as $\bar{T} \to 0^-$, with the $x$- and $y$-components vanishing. Even though we are dealing with $k_+ \neq 0$, the $\bar{g}_{zz}$ component does not diverge at $\bar{T} = 0$ and thus $d\bar{s}^2$ is $C^\infty$ for $\bar{T} \in (-\infty, 0]$.

The conformal factor $\bar{\Omega}$ diverges at $\bar{T} = 0$, but is always positive and $C^\infty$ for $\bar{T} < 0$. Its derivatives are found to behave as

$$\lim_{\bar{T} \to -0^+} \frac{\bar{\Omega}'}{\bar{\Omega}} = +\infty,$$

and

$$\bar{L} = \frac{\bar{\Omega}''}{\bar{\Omega}'} \equiv \frac{5}{4} > 1.$$  

(7.10)

Equation (7.7) implies that the physical and unphysical fluid flows are given by

$$u = 9\bar{T}^4 \frac{\partial}{\partial \bar{T}}, \quad \bar{u} = \bar{\Omega} u = \frac{\partial}{\partial \bar{T}}.$$  

(7.11)

The physical flow $u$ vanishes and is $C^\infty$ as $\bar{T} \to 0^-$. On the other hand, the unphysical flow $\bar{u}$ is again $C^\infty$ at and orthogonal to the slice $\bar{T} = 0$.

Equation (6.2) implies that the proper time $\tau$ along the fluid congruence from the IPS to the slice $\bar{T} = 0$ is infinite. Additionally, $\theta > 0$ for $\bar{T} < 0$ (see Fig. 7.1), and the asymptotic behaviour of the non-zero kinematic quantities is determined to

$$\lim_{\bar{T} \to -0^-} \theta = \lim_{\bar{T} \to -0^-} \sigma = 0.$$  

(7.12)

Their ratio follows an “anisotropic” behaviour, in the sense that the expansion is now shear dominated,

$$\lim_{\bar{T} \to -0^-} \frac{\sigma}{\theta} = +\infty.$$  

(7.13)
Keeping the behaviour of the kinematics in mind, it is not surprising that

\[
\lim_{T \to 0^-} R_{ab}R^{ab} = \lim_{T \to 0^-} C_{abcd}C^{abcd} = 0. \tag{7.14}
\]

The evolution of both these scalars is graphed in Fig. (7.2) for \( k_+ = 1 \), at \( A = 1 \), which clearly shows how the Weyl, as well as the Ricci curvature have decreased from the IPS, where they diverged, to \( T \to 0^- \) where they vanish. Hence, this subclass of the Szekeres models describes ever expanding universes without a big crunch type singularity. Nonetheless, in support of the WCH we find

\[
\lim_{T \to 0^-} \bar{K} = \frac{2}{225} \neq 0, \tag{7.15}
\]

and a steady evolution of \( \bar{K} \) towards this value (see Fig. (7.3)).

For technical interest, it is instructive to investigate the curvature of the unphysical space-time as well. Due to the degeneracy we find a curvature singularity at \( \bar{T} = 0 \) and therefore the conformal structure cannot satisfy an analogous situation to condition (1) of Defintion 4.1,

\[
\lim_{T \to 0^-} \bar{R}_{ab}R^{ab} = \lim_{T \to 0^-} \bar{C}_{abcd}C^{abcd} = +\infty, \quad \text{and} \quad \lim_{T \to 0^-} \bar{K} = \frac{2}{33}. \tag{7.16}
\]

Additionally,

\[
\lim_{T \to 0^-} \bar{\theta} = -\infty. \tag{7.17}
\]

We find that the absolute value of the determinant of the physical metric \( \bar{\Omega}^8|\bar{g}| \) diverges as \( \bar{T} \to 0^- \).
7.2 Mars models

There are three types of Mars models [64], all of which correspond to a perfect fluid solution with an Abelian, two-dimensional group of isometries acting orthogonally transitively on spacelike 2-surfaces and such that both Killing vectors are integrable. We begin by stating some of the results found in [9], concerning the IPS in the third type of these solutions.

The metric of the third type, which moreover describes an irrotational perfect fluid with non-geodesic fluid flow, is given in comoving coordinates by

$$ds^2 = -\frac{e^{at} + \epsilon e^{2a(t+x)}}{1 + \epsilon e^{-2at} + \beta e^{-6at}} dt^2 + e^{at} + \epsilon e^{2a(t+x)} dx^2$$

$$+ e^{a(t-x)+2\epsilon a_x} dy^2 + e^{a(t-x)-2\epsilon a_x} dz^2,$$

where $a, c, \beta$ and $\epsilon$ are constants with $a > 0$, $\epsilon = \pm 1$ and $\beta \geq 0$. For $\beta > 0$ one finds a limiting $\gamma$-law equation of state with $\gamma = \frac{14}{3}$, while for $\beta = 0$ the solution corresponds to a $\gamma = 2$ stiff fluid with an exact $\gamma$-law equation of state.

For the conformal structure it is important to note, that in this case one could also choose, among others, the following cosmic time functions to rescale $T = +\infty$ to 0,

$$\tilde{T} = \arctan T - \frac{\pi}{2}, \text{ or } \tilde{T} = -e^{-aT}, \quad a > 0. \quad (7.18)$$

In Appendix B.3 the conformal relations of these cosmic time functions will be presented and it is shown that - as one would expect - they are essentially the same.

Figure 7.3: The behaviour of the ratio of the Weyl to the Ricci curvature as $T \to 0^-$ in the Szekeres models with $k_+ = 1$ and $k_- = 0$, at $A = 1$ (linear plot, the y-axis labels are meaningless for our purposes and have been omitted).
All of these models show an initial singularity at \( t = -\infty \), which can be rescaled to zero via the cosmic time function \( T = e^{at} \).

The metric may now be rewritten in terms of \( T \) as

\[
\begin{align*}
    ds^2 &= \Omega^2 \left[ -\frac{e^{c_2 T^2 e^{2ax}}}{(1 + e^{T_2^{1/2} + \beta T^6}) a^2 T^2} dT^2 + e^{c_2 T^2 e^{3ax}} dx^2 \\
    &\quad + e^{-ax + 2ce^{ax}} dy^2 + e^{-ax - 2ce^{ax}} dz^2 \right],
\end{align*}
\]

where

\[
\Omega(T) = \sqrt{T}.
\]

The conformal factor \( \Omega \) satisfies conditions (3) and (4) of the definition of an IPS with \( \lambda = -1 \). The manifold \( \tilde{\mathcal{M}} \) is \( \mathbb{R}^4 \) covered with the coordinate patch \((T, x, y, z)\), where \( T \in \mathbb{R} \). The physical manifold \( \mathcal{M} \) is therefore the open submanifold \( T > 0 \) of \( \tilde{\mathcal{M}} \), as required by the definition of an IPS.

Equation (7.21) implies that only the \( \beta = 0 \) models provide a regular, non-degenerate conformal metric \( ds^2 \) in the square brackets, compatible with condition (2) of Definition 4.1. The fluid congruence is shown to be both regular and orthogonal to the IPS and the behaviour of the Weyl tensor and the kinematics is found in table 5.1. Hence, the third type of the Mars models with \( \beta = 0 \) does admit an IPS. This was originally stated in [64] but firstly presented in detail in [9].

We now proceed analogously to analyse the future metric singularity at \( T = \infty \) of theses models (the details of the following calculations in this section may be found in Appendix B.4). Clearly, we can choose our cosmic time function \( \tilde{T} \) as

\[
\tilde{T} = -\frac{1}{T},
\]

and to rescale the metric singularity to \( \tilde{T} = 0 \), while leaving the IPS at \( \tilde{T} = -\infty \). Rewriting the metric (\( \beta \geq 0 \) again) gives

\[
\begin{align*}
    ds^2 &= \tilde{\Omega}^2(\tilde{T}, x) \left[ -\frac{\tilde{T}^2}{(1 + e^{T_2^{1/2} + \beta t^6}) a^2} d\tilde{T}^2 + \tilde{T}^2 dx^2 + \tilde{T}^2 e^{-ax + 2ce^{ax}} dy^2 \\
    &\quad + \tilde{T}^2 e^{-ax - 2ce^{ax}} dz^2 \right],
\end{align*}
\]

where

\[
\tilde{\Omega}(\tilde{T}, x) = \frac{e^{c_2 e^{2ax}/2\tilde{T}^2}}{(-\tilde{T})^{3/2}}.
\]

At \( \tilde{T} = 0 \) the conformal metric \( ds^2 \) in the square bracket becomes

\[
ds^2 \to -\frac{d\tilde{T}^2}{a^2}, \quad \text{as} \quad \tilde{T} \to 0^-,
\]

(7.26)
7.2. Mars models

i.e. $d\mathbf{s}^2$ is highly degenerate at $\bar{T} = 0$. Equation (7.24) moreover indicates that $d\mathbf{s}^2$ is only $C^0$ at this value of $\bar{T}$.

The conformal factor $\Omega$ depends also on the space coordinate $x$ this time, however, as will be seen, this does not influence the analysis much. $\Omega$ diverges at $\bar{T} = 0$, independent of the $x$-value and is positive and $C^\infty$ for $\bar{T} < 0$. The time derivatives behave as

$$
\lim_{\bar{T} \to 0^-} \frac{\Omega'}{\Omega} = +\infty \quad \text{and} \quad \lim_{\bar{T} \to 0^-} \bar{L} = 1, \quad (7.27)
$$

independent of $x$.

From equation (7.24) it is readily seen that the timelike congruences in the physical and conformal space-time possess the forms

$$
\mathbf{u} = -\frac{a\bar{T}^{3/2}\sqrt{1 + \epsilon T^2 + \beta T^6}}{e^{\epsilon x^2/2T^2}} \frac{\partial}{\partial T}, \quad \mathbf{\dot{u}} = a\sqrt{1 + \epsilon T^2 + \beta T^6} \frac{\partial}{\partial T}. \quad (7.28)
$$

The physical fluid flow $\mathbf{u}$ vanishes at $\bar{T} = 0$ independent of $x$. On the other hand, the fluid flow $\mathbf{\dot{u}}$ in the unphysical space-time is $C^\infty$, regular at and orthogonal to $\bar{T} = 0$.

By equation (6.2), the proper time from the IPS to $\bar{T} = 0$ along the fluid lines is seen to be infinite. The non-zero kinematic quantities of the physical fluid flow furthermore satisfy

$$
\lim_{\bar{T} \to 0^-} \frac{\sigma}{\theta} = 0, \quad \lim_{\bar{T} \to 0^-} \frac{\dot{\mathbf{u}}^a}{\theta^2} = e^{-ax - 2\epsilon x^2}. \quad (7.30)
$$

from which $\theta$ is graphed in Fig. 7.4. Their ratios show an “anisotropic” asymptotic behaviour, i.e. the kinematics are asymptotically not expansion dominated,

$$
\lim_{\bar{T} \to 0^-} \frac{\sigma}{\theta} = \frac{1}{6}, \quad \lim_{\bar{T} \to 0^-} \frac{\dot{u}_a \dot{u}^a}{\theta^2} = e^{-ax - 2\epsilon x^2}. \quad (7.31)
$$

In the light of equations (7.29) and (7.30), it is not astonishing that\(^1\)

$$
\lim_{\bar{T} \to 0^-} R_{ab} R^{ab} = \lim_{\bar{T} \to 0^-} C_{abcd} C^{abcd} = 0. \quad (7.32)
$$

This behaviour of the curvature scalars arises, since this type of Mars models clearly describes an ever expanding universe. The metric singularity is therefore not a physical singularity.

In agreement with the WCH (the sign is not essential) we find

$$
\lim_{\bar{T} \to 0^-} K = -\infty, \quad \text{for} \quad (x \neq -\infty). \quad (7.33)
$$

Fig. 7.5 shows the evolution of $K$ and how the Weyl curvature starts to dominate over the Ricci curvature as $\bar{T} \to 0^-$ for a particular choice of constants.

\(^1\)Since the overall behaviour of these scalars is exactly as in the Szekeres models we abstain from plotting them.
Figure 7.4: The behaviour of $\theta$ as $\bar{T} \to 0^-$ in the third type of the Mars models, for $a = c = \beta = \epsilon = 1$ at $x = 0$ (linear plot, the y-axis labels are meaningless for our purposes and have been omitted).

Figure 7.5: The behaviour of the ratio of the Weyl to the Ricci curvature as $\bar{T} \to 0^-$ in the third type of the Mars models, for $a = c = \beta = \epsilon = 1$ at $x = 0$ (linear plot, the y-axis labels are meaningless for our purposes and have been omitted).

It is instructive, to examine the behaviour of the respective curvatures in the conformal space-time. $\bar{K}$ vanishes at $\bar{T} = 0$ for $x \neq -\infty$ and diverges for $x = -\infty$. The Ricci curvature scalar $\bar{R}_{ab}\bar{R}^{ab}$ diverges for any value of $x$ at $\bar{T} = 0$, while the Weyl curvature scalar $\bar{C}_{abcd}\bar{C}^{abcd}$ diverges only for $x \neq -\infty$. For $x = -\infty$ it vanishes.
Again, there is a curvature singularity at $\bar{T} = 0$ in the conformal space-time, which explains

$$\lim_{\bar{T} \to 0^-} \bar{\theta} = -\infty. \quad (7.34)$$

Interestingly, all these results are basically independent of the $x$-value, even though the conformal factor depends on it.

Furthermore, the determinant of the physical metric diverges as $\bar{T} \to 0^-$ in these models.

It should be noted, that once again we could use a different choice of cosmic time functions $\bar{T}$ to rescale the metric singularity to $\bar{T} = 0$. E.g. the choices $\bar{T} = -\exp(-at)$ and $\bar{T} = \arctan t - \frac{\pi}{2}$ provide a conformal structure which is essentially the same as the one derived above. As the procedure, however, is exactly the same as in Appendix B.3, we will not further elaborate this.

### 7.3 Carneiro-Marugan model

The Carneiro-Marugan model [67] is the irrotational subclass of the so-called RTKO metrics and describes spatially homogeneous, but not spatially isotropic cosmologies. Its matter source can be interpreted as a superposition of an anisotropic scalar field with perfect fluid radiation and dust. The fluid flow is irrotational, shear-free and geodesic. We briefly summarise some details found in [11, p 74] regarding the IPS encountered in this model.

The metric is given in comoving coordinates by

$$ds^2 = a^2(\eta)[-d\eta^2 + dx^2 + e^{2x}dy^2 + dz^2], \quad (7.35)$$

where

$$a(\eta) = \frac{D}{3} \left[ \cosh \left( \frac{\eta}{\sqrt{2}} \right) - 1 \right] + \sqrt{\frac{2}{3}} A \sinh \left( \frac{\eta}{\sqrt{2}} \right) \quad (7.36)$$

and $A$ and $D$ are non-negative constants. Defining the cosmic time function $T$ by

$$T(\eta) = \eta, \quad (7.37)$$

allows to transform equation (7.35) for $T \geq 0$ into

$$ds^2 = \Omega^2[-dT^2 + dx^2 + e^{2x}dy^2 + dz^2], \quad (7.38)$$

where $\Omega(T) = a(T). \quad (7.39)$

The conformal factor $\Omega$ is shown to satisfy conditions (3) and (4) of the definition of an IPS, with $\lambda = 0$.

The conformal metric $d\bar{s}^2$ in the square brackets is clearly $C^\infty$ and non-degenerate $\forall T$. $\bar{\mathcal{M}}$ is the $\mathbb{R}^4$ covered by the coordinate patch $(T, x, y, z)$, where $T \in \mathbb{R}$. The physical manifold $\mathcal{M}$ is certainly the open submanifold $T > 0$ of $\bar{\mathcal{M}}$. Consequently,
conditions (1) and (2) of Definition 4.1 are fulfilled as well and the Carneiro-Marugan model admits an IPS. The fluid flow is easily verified to be regular and orthogonal to the slice $T = 0$ and the behaviour of $C_{abcd}$ and the kinematics is presented in table 5.1. The existence of this conformal structure was first derived in [11, p 74f].

Equation (7.36) diverges for $\eta \to +\infty$. This metric singularity can again be rescaled to 0 by choosing the cosmic time function $\tilde{T}$ as

$$\tilde{T} = -\frac{1}{\eta}.$$ \hfill (7.40)

Rewriting yields

$$ds^2 = \tilde{\Omega}^2(\tilde{T})[-d\tilde{T}^2 + \tilde{T}^4(dx^2 + e^{2x}dy^2 + dz^2)],$$ \hfill (7.41)

where

$$\tilde{\Omega}(T) = \frac{a(\tilde{T})}{\tilde{T}^2}.$$ \hfill (7.42)

As $\tilde{T} \to 0^-$ the conformal metric $d\tilde{s}^2$ in the square brackets behaves as

$$d\tilde{s}^2 \to -d\tilde{T}^2, \quad \text{as} \quad \tilde{T} \to 0^-,$$ \hfill (7.43)

i.e., $d\tilde{s}^2$ becomes highly degenerate at $\tilde{T} = 0$, with all spatial components vanishing. Otherwise $d\tilde{s}^2$ is $C^\infty$ for $\tilde{T} < 0$. The conformal factor $\tilde{\Omega}$ diverges as $\tilde{T} \to 0^-$, but is positive and $C^\infty$ for $\tilde{T} < 0$. The behaviour of the derivatives is determined to

$$\lim_{\tilde{T} \to 0^-} \frac{\tilde{\Omega}'}{\tilde{\Omega}} = +\infty \quad \text{and} \quad \lim_{\tilde{T} \to 0^-} \tilde{L} > 1.$$ \hfill (7.44)

The fluid flows of the physical and unphysical space-time are

$$\mathbf{u} = \frac{T^2}{a(T)} \frac{\partial}{\partial T}, \quad \tilde{\mathbf{u}} = \frac{\partial}{\partial \tilde{T}}.$$ \hfill (7.45)

The physical flow $\mathbf{u}$ vanishes as $\tilde{T} \to 0^-$ and is otherwise $C^\infty$, while the unphysical flow $\tilde{\mathbf{u}}$ is both regular and orthogonal to the slice $\tilde{T} = 0$.

The proper time from the IPS to $\tilde{T} = 0$ is once more infinite along the timelike congruence, $\theta > 0$ for $\tilde{T} < 0$ and the asymptotic behaviour of the expansion scalar is

$$\lim_{\tilde{T} \to 0^-} \theta = 0.$$ \hfill (7.46)

As before$^1$,

$$\lim_{\tilde{T} \to 0^-} R_{ab}R^{ab} = \lim_{\tilde{T} \to 0^-} C_{abcd}C^{abcd} = 0,$$ \hfill (7.47)

$^1$The details of the calculations in this section are given in Appendix B.5.

$^1$For this model we will abstain from graphing $\theta$ and the curvature scalars, since they cannot be put into expressions of “acceptable” length for non-trivial choices of the constants. Therefore we will restrict ourselves to the asymptotic behaviour.
thus, the Carneiro-Marugan model describes an ever expanding cosmology without a \textit{big crunch} type singularity. Furthermore, the scalar $K$ supports the WCH in the sense that

$$\lim_{T \to 0^-} K = K_0 > 0.$$  \hfill (7.48)

The conformal curvature scalars indicate that a scalar polynomial curvature singularity occurs in the unphysical space-time at $\bar{T} = 0$,

$$\lim_{T \to 0^-} \bar{R}_{ab} \bar{R}^{ab} = \lim_{T \to 0^-} \bar{C}_{abcd} \bar{C}^{abcd} = +\infty, \quad \text{and} \quad \lim_{T \to 0^-} \bar{K} = \frac{2}{3}. \hfill (7.49)$$

Furthermore,

$$\lim_{T \to 0^-} \bar{\theta} = -\infty.$$  \hfill (7.50)

The determinant of the physical metric is moreover shown to diverge as $\bar{T} \to 0^-$. Analogously to the Szekeres and Mars models, one could choose other cosmic time functions $\bar{T}$ to work out an equivalent conformal structure.

### 7.4 The Kantowski-Sachs models

The Kantowski-Sachs models describe spatially homogeneous but not spatially isotropic cosmologies with an irrotational, geodesic, perfect fluid source, which satisfies a radiation equation of state $p = \frac{1}{3} \mu$ in the Kantowski-Sachs type [68, 69] and a dust equation of state $p = 0$ in the Kantowski case [69, 70].

To begin with, we collect some information about the IPS, found in these models, from [9, 11]. In comoving coordinates one finds the following form of the metric

$$ds^2 = -Adt^2 + t \left[ A^{-1}dx^2 + A^2b^{-2} \left( dy^2 + f^2dz^2 \right) \right] \hfill (7.51)$$

where

\begin{align*}
A &= 1 - \frac{4\epsilon b^2 t}{9}, \quad t > 0 \quad \text{and} \quad b = \text{const} \\
f(y) &= \begin{cases} 
\sin y & \text{if } \epsilon = 1 \ (\text{Kantowski-Sachs}), \\
\sinh y & \text{if } \epsilon = -1 \ (\text{Kantowski}).
\end{cases}
\end{align*}

By choosing the following cosmic time function $T$

$$T = \sqrt{2t}, \hfill (7.52)$$

and redefining $\bar{x} = \frac{1}{\sqrt{2}} x$ and $\bar{b} = \sqrt{2} b$, we obtain for $T \geq 0$

$$ds^2 = \Omega(T)^2 \left[ -AdT^2 + A^{-1}d\bar{x}^2 + A^2\bar{b}^{-2} \left( d\bar{y}^2 + f^2d\bar{z}^2 \right) \right] \hfill (7.53)$$
where

\[ \Omega(T) = T \quad \text{and} \quad A = 1 - \frac{e^{b^2T^2}}{9}. \quad (7.54) \]

\( \Omega(T) \) satisfies conditions (3) and (4) of the definition of an IPS, with \( \lambda = 0 \). The conformal metric \( ds^2 \) in the square brackets is non-degenerate and \( C^3 \) on an open neighbourhood of \( T = 0 \), thus condition (2) of the definition is also satisfied.

For \( T \in \mathbb{R} \) the manifold \( \mathcal{M} \) is the \( \mathbb{R}^4 \) covered by the coordinate patch \( (T, x, y, z) \). Thus, by equation (7.56), we can see that \( \mathcal{M} \) is the open submanifold \( T > 0 \) of \( \mathcal{M} \). Requirement (1) of the definition of an IPS is therefore satisfied as well and, consequently, the Kantowksi-Sachs models do admit an IPS. The fluid flow is examined to be regular at \( T = 0 \) and the behaviour of the Weyl tensor and the kinematics is given in table 5.1. This was first presented in [8].

In order to determine the future behaviour of these models, it is necessary to analyse them separately.

### 7.4.1 Future singularities in Kantowski models?

Choosing \( \epsilon = -1 \), we find \( A = 1 + (4b^2t)/9 \neq 0 \) \( \forall t \), but \( A \to \infty \) as \( t \to \infty \). It is helpful to pick the cosmic time function \( \bar{T} \) in the form\(^1\)

\[ \bar{T} = -A^{-1} = -\frac{1}{1 + \frac{4b^2t}{9}}, \quad (7.55) \]

since \( t = \infty \iff \bar{T} = 0 \), and the IPS occurs at \( \bar{T} = -1 \), i.e. the IPS and the future metric singularity are separated by a unit value difference of \( \bar{T} \). This yields

\[ ds^2 = \bar{\Omega}^2(\bar{T}) \left\{ -\frac{81}{16b^4} d\bar{T}^2 + \frac{9(\bar{T}^3 + \bar{T}^2)}{4b^2} \left\{ -\bar{T}^3 dx^2 + b^{-2} \left( dy^2 + \sinh^2 ydz^2 \right) \right\} \right\} \quad (7.56) \]

where

\[ \bar{\Omega}(\bar{T}) = (-\bar{T})^{-5/2}. \quad (7.57) \]

The conformal metric \( ds^2 \) in the square brackets becomes degenerate as \( \bar{T} \to 0^- \), since all spatial components vanish,

\[ ds^2 \to -\frac{81}{16b^2} d\bar{T}^2, \quad \text{as} \quad \bar{T} \to 0^- \quad (7.58) \]

Otherwise \( ds^2 \) is \( C^\infty \) for \( \bar{T} \leq 0 \).

The conformal factor \( \bar{\Omega}(\bar{T}) \) diverges as \( \bar{T} \to 0^- \), but is positive and \( C^\infty \) for \( \bar{T} < 0 \). The derivative ratios are determined to satisfy

\[ \lim_{\bar{T} \to 0^-} \frac{\bar{\Omega}'}{\bar{\Omega}} = +\infty \quad \text{and} \quad \bar{L} \equiv \frac{7}{5}. \quad (7.59) \]

\(^1\)The details of the calculations concerning this section are given in Appendix B.6.
The fluid flows of the physical and unphysical space-time are given by

\[ u = \frac{4b^2(-\bar{T})^{5/2}}{9} \frac{\partial}{\partial \bar{T}}, \quad \bar{u} = \frac{4b^2}{9} \frac{\partial}{\partial \bar{T}}. \]  

(7.60)

The physical flow \( u \) vanishes at \( \bar{T} = 0 \) and is otherwise \( C^\infty \), while the unphysical flow \( \bar{u} \) is both regular and orthogonal to the slice \( \bar{T} = 0 \).

Equation (6.2) once more indicates that the proper time of a fluid particle from the IPS to \( \bar{T} = 0 \) is infinite. As in the previous cases, \( \theta > 0 \) for \( \bar{T} < 0 \) and the non-zero kinematic quantities and the curvature scalars fulfils

\[ \lim_{T \to 0^-} \theta = \lim_{T \to 0^-} \sigma = \lim_{T \to 0^-} R_{ab}R^{ab} = \lim_{T \to 0^-} C_{abcd}C^{abcd} = 0, \]  

(7.61)

which clearly shows that the Kantowski models do not possess a physical future singularity; they correspond to ever expanding universes. The evolution of \( \theta \) and the Weyl and Ricci curvature are shown in Figs. 7.6 and 7.7, respectively, for \( b = 1 \).

![Figure 7.6: The behaviour of \( \theta \) for all of cosmic time, \( \bar{T} \in (-1, 0) \), in the Kantowski model with \( b = 1 \).](image)

The ratio of the non-zero kinematic quantities shows that the shear is asymptotically not expansion dominated, since

\[ \lim_{T \to 0^-} \frac{\sigma}{\theta} = \sqrt{\frac{1}{6}}. \]  

(7.62)

The Weyl curvature becomes slightly stronger than the Ricci curvature, in the sense that

\[ \lim_{\bar{T} \to 0^-} K = 16. \]  

(7.63)
7. Example space-times with diverging conformal factor

Figure 7.7: The behaviour of $R_{ab} R^{ab}$ and $C_{abcd} C^{abcd}$ for all of cosmic time, $T \in (-1, 0)$, in the Kantowski model with $b = 1$.

Figure 7.8: The evolution of $K$ for all of cosmic time, $T \in (-1, 0)$, in the Kantowski models ($b$ independent).

The complete evolution of $K$ is graphed in Fig. 7.8, in strong support of the WCH, independent of $b$.

A curvature singularity can, however, be encountered in the unphysical space-time as $T \to 0^-$. The conformal quantities have the following asymptotic behaviour

$$\lim_{T \to 0^-} R_{ab} R^{ab} = \lim_{T \to 0^-} C_{abcd} C^{abcd} = \infty,$$  \quad (7.64)

$$\lim_{T \to 0^-} \bar{\theta} = -\infty, \quad \text{and} \quad \lim_{T \to 0^-} \bar{K} = 0.$$  \quad (7.65)
Again, the determinant of the physical metric $\hat{g}$ diverges as $\hat{T} \to 0^-$. As before, we could have chosen another cosmic time function $\hat{T}$ to devise an equivalent conformal structure.

### 7.4.2 A future singularity in the Kantowski-Sachs models

Given equation (7.51), it is readily seen that for $\epsilon = 1$ we encounter a future metric singularity when $A = 0$, i.e. when $t \to t_s = \frac{\alpha}{4b^2}$. By the following choice of the cosmic time function $\hat{T}$

$$
\hat{T} = -A^2 = -\left(1 - \frac{4b^2t}{9}\right)^2,
$$

which approaches 0 from below and satisfies $t = t_s \leftrightarrow \hat{T} = 0$, it is evident that this case is different than the previous examples. The IPS occurs at $\hat{T} = -1$ and thus, equation (6.2) shows that this metric singularity occurs at finite proper time of the fluid particles, namely $\tau = \frac{\alpha}{4b^2}$. Rewriting the metric yields

$$
ds^2 = \hat{\Omega}^2(\hat{T}) \left[-\frac{81}{64b^4}dT^2 + \left(1 - \sqrt{-\hat{T}}\right) \frac{9}{4b^2} \left[dx^2 + (-\hat{T})^{3/2}b^{-2}(dy^2 + f^2dz^2)\right]\right],
$$

where $(\hat{T} \in (-1, 0))$

$$
\hat{\Omega}(\hat{T}) = \frac{1}{(-\hat{T})^{1/4}}.
$$

The $y$- and $z$-components of the conformal metric $ds^2$ in the square brackets vanish as $\hat{T} \to 0^-$,

$$
ds^2 \to -\frac{81}{64b^4}dT^2 + \frac{9}{4b^2} dx^2, \quad \text{as} \quad T \to 0^-,
$$

i.e. $ds^2$ becomes degenerate as $\hat{T} \to 0^-$. Equation (7.67) implies moreover that $ds^2$ is only $C^0$ at $\hat{T} = 0$.

The conformal factor $\hat{\Omega}$ diverges and becomes $C^0$ as $\hat{T} \to 0^-$, but is positive and $C^\infty$ for $\hat{T} < 0$. The time derivatives behave as

$$
\lim_{\hat{T} \to 0^-} \frac{\hat{\Omega}'}{\hat{\Omega}} = +\infty, \quad \text{and} \quad \hat{L} = \frac{\hat{\Omega}''\hat{\Omega}}{\hat{\Omega}^2} = \frac{40}{3} > 1.
$$

By equation (7.67), it is apparent that in these coordinates the timelike fluid flows in the physical and conformal space-time are given by

$$
u = \frac{8b^2}{9} (-\hat{T})^{1/4} \frac{\partial}{\partial T}, \quad \hat{\nu} = \hat{\Omega}nu = \frac{8b^2}{9} \frac{\partial}{\partial \hat{T}}.
$$

The physical flow $\nu$ vanishes and becomes $C^0$ at the metric singularity. However, the unphysical fluid flow $\hat{\nu}$ is $C^\infty$, regular at and orthogonal to the slice $\hat{T} = 0$. 

\section{Appendix: The Kantowski-Sachs models}

\begin{equation}
\end{equation}
The expansion scalar of the physical fluid flow is plotted in Fig. 7.9 for $b = 1$ and is shown to diverge to $-\infty$,

$$\lim_{T \to 0^-} \theta = -\infty, \quad (7.72)$$

while the only other non-zero kinematic quantity follows

$$\lim_{T \to 0^-} \sigma = \infty. \quad (7.73)$$

Their ratio moreover shows an “anisotropic” asymptotic behaviour

$$\lim_{T \to 0^-} \frac{\sigma}{\theta} = \sqrt{\frac{2}{15}}. \quad (7.74)$$

Analysing the Weyl and Ricci curvature scalars, it is found that

$$\lim_{T \to 0^-} R_{ab} R^{ab} = \lim_{T \to 0^-} C_{abcd} C^{abcd} = +\infty. \quad (7.75)$$

The overall behaviour of these curvature scalars between the singularities is pictured in Fig. (7.10) for $b = 1$. The Ricci curvature diverges at both singularities and it is readily seen how the Weyl curvature initially decreases and eventually increases from a finite value to infinity throughout the evolution of the model. Thus, the Kantowski-Sachs models, with $\epsilon = 1$, describe recollapsing universes in which the

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$^1$Details of the calculations for this section are presented in Appendix B.7.
timelike congruence encounters a physical, scalar polynomial curvature singularity in finite proper time.

In this class of models, the WCH seems to hold and the scalar $K$ could, indeed, be a measure of gravitational entropy, as pointed out in section 4.4, since it becomes maximal at the future singularity

$$\lim_{T \to 0^+} K = +\infty,$$

and originating from 0 at the IPS, it steadily increases, as can be seen for the case $b = 1$ in Fig. 7.11. The Weyl curvature clearly becomes dominant with growing cosmic time.

Due to the degeneracy of the conformal metric, we again find a scalar polynomial curvature singularity at $\tilde{T} = 0$ in the unphysical space-time,

$$\lim_{\tilde{T} \to 0^-} \tilde{R}_{ab}\tilde{R}^{ab} = \lim_{\tilde{T} \to 0^-} \tilde{C}_{abcd}\tilde{C}^{abcd} = +\infty.$$

The ratio of the Weyl and Ricci curvature is non-zero at this unphysical singularity,

$$\lim_{\tilde{T} \to 0^-} \tilde{K} = \frac{16}{9},$$

and, as we already expect,

$$\lim_{\tilde{T} \to 0^-} \tilde{\theta} = -\infty.$$

Unlike in the previous cases, the choice of possible cosmic time functions $\tilde{T}$, with the desired property to approach the singularity at $\tilde{T} = 0$ from below is very limited in these models.
Another essential difference to the other examples studied, is that not only the determinant of the unphysical metric vanishes as $\bar{T} \to 0^-$, but also the determinant of the physical metric,

$$\lim_{\bar{T} \to 0^-} g = \lim_{\bar{T} \to 0^-} \Omega^8 \bar{g} = 0. \quad (7.80)$$

7.5 Discussion

Comparing the models presented in this and the previous chapter, already offers some information regarding the conformal structure for a physically realistic future behaviour. In the light of *quiescent cosmology* and the WCH we would expect the scalar $K$ to increase with cosmic evolution in a physically realistic cosmological model. Apart from this - as we expect anisotropies to be formed with cosmic evolution by the enhanced gravitational clumping - we would anticipate that the asymptotic kinematic isotropy\(^1\) of the initial state does not hold anymore for the non-zero kinematics at later cosmic times. The behaviour of the curvature scalars on the other hand depends on whether a physical future singularity exists or not and is, consequently, less important, since both scenarios are compatible with *quiescent cosmology* and the WCH. From our current knowledge of the universe we have no strong indications for neither one.

To facilitate the comparison of the models it is helpful to summarise the behaviour of the characteristic quantities at $\bar{T} = 0$; Table 7.1 recapitulates the behaviour of the curvature scalars, proper time, the equation of state and the existence

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\(^1\)see section 4.3.2.
of an IPS, Table 7.2 summarises the behaviour of the fluid flow and the kinematic quantities in the physical space-time and of the fluid flow and its expansion scalar in the conformal space-time, and Table 7.3 collects the information regarding the conformal structure of the models and the behaviour of some relevant unphysical curvature scalars.

<table>
<thead>
<tr>
<th>Model</th>
<th>$K$</th>
<th>$R_{ab}R^{ab}$</th>
<th>$C_{abcd}C^{abcd}$</th>
<th>$\tau_s$</th>
<th>$p = p(\mu)$</th>
<th>IPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>rad. FRW ($k = +1$)</td>
<td>0</td>
<td>(a)</td>
<td>0</td>
<td>finite</td>
<td>$p = \frac{3}{4}\mu$</td>
<td>yes</td>
</tr>
<tr>
<td>dust FRW ($k = +1$)</td>
<td>0</td>
<td>(a)</td>
<td>0</td>
<td>finite</td>
<td>$p = 0$</td>
<td>yes</td>
</tr>
<tr>
<td>McVittie-Wiltshire I</td>
<td>0</td>
<td>(a)</td>
<td>0</td>
<td>infinite</td>
<td>(e)</td>
<td>no</td>
</tr>
<tr>
<td>McV.-Wil. II ($k &lt; 0$)</td>
<td>(d)</td>
<td>(a)</td>
<td>(a)</td>
<td>(e)</td>
<td>(e)</td>
<td>no</td>
</tr>
<tr>
<td>Szekeres (subclass)</td>
<td>(c)</td>
<td>(d)</td>
<td>(d)</td>
<td>infinite</td>
<td>$p = 0$</td>
<td>yes</td>
</tr>
<tr>
<td>Mars (3rd type)</td>
<td>(b)*</td>
<td>(d)</td>
<td>(d)</td>
<td>(e)</td>
<td>(e)</td>
<td>yes*</td>
</tr>
<tr>
<td>Carneiro-Marugan</td>
<td>(c)</td>
<td>(d)</td>
<td>(d)</td>
<td>infinite</td>
<td>(e)</td>
<td>yes</td>
</tr>
<tr>
<td>Kantowski</td>
<td>(c)</td>
<td>(d)</td>
<td>(d)</td>
<td>infinite</td>
<td>$p = 0$</td>
<td>yes</td>
</tr>
<tr>
<td>Kantowski-Sachs</td>
<td>(a)</td>
<td>(a)</td>
<td>(a)</td>
<td>finite</td>
<td>$p = \frac{3}{4}\mu$</td>
<td>yes</td>
</tr>
</tbody>
</table>

Table 7.1: Summary of the behaviour of several quantities as $\bar{T} \to 0^-$ in the physical space-time of the examples studied. $\tau_s$ denotes proper time from the initial state to the slice $\bar{T} = 0$ along the fluid flow. (a) means that the relevant quantity diverges to $+\infty$ at $\bar{T} = 0$, but is $C^\infty$ for $\bar{T} < 0$, (b) means that the relevant quantity diverges to $-\infty$ at $\bar{T} = 0$, but is $C^\infty$ for $\bar{T} < 0$, (c) means that the relevant quantity is finite and $C^\infty$ for $\bar{T} < 0$, (d) means that the relevant quantity vanishes as $\bar{T} \to 0^-$ and (e) means that I am uncertain about the relevant quantity. Furthermore, * stands for $x \neq -\infty$, and • means "only for the case $\beta = 0$".

Looking at the behaviour of $K$ in Table 7.1, the fluid flow and the asymptotic ratios of the kinematics in Table 7.2 and the conformal factor and the conformal metric in Table 7.3, it appears that a conformal structure which leads to a physically realistic future behaviour of the model must satisfy the following at $\bar{T} = 0$:

1. the conformal factor diverges,
2. the conformal metric is degenerate with some spatial components vanishing,
3. the physical fluid flow $\mathbf{u}$ vanishes, and
4. the unphysical fluid flow $\mathbf{u}$ is $C^\infty$.

In Theorem 10.1 we will confirm the second point in the sense that a regular conformal metric cannot lead do a non-vanishing $K$ at $\bar{T} = 0$. The proof of the theorem, however, does not preclude the case that a vanishing conformal factor and a degenerate conformal metric also provide a non-zero $K$ at $\bar{T} = 0$. Hence, the first point does not necessarily need to hold. The differentiabilities of the conformal metrics listed in Table 7.3 moreover display that some conformal metrics only become $C^0$ at $\bar{T} = 0$, which should be considered in the new definitions.
7. Example space-times with diverging conformal factor

<table>
<thead>
<tr>
<th>Model</th>
<th>$u$</th>
<th>$\theta$</th>
<th>$\dot{u}^a$</th>
<th>$\sigma$</th>
<th>$\omega$</th>
<th>$\ddot{u}$</th>
<th>$\theta$</th>
<th>$\dot{u}_a\dot{u}^a/\theta$</th>
<th>$\sigma/\theta$</th>
<th>$\omega/\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>rad. FRW ($k = +1$)</td>
<td>(a)</td>
<td>(b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(c)</td>
<td>(d)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>dust FRW ($k = +1$)</td>
<td>(a)</td>
<td>(b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(c)</td>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>McVittie-Wiltshire I (a)</td>
<td>(a)</td>
<td>(b)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(a)</td>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>McV.-Wil. II ($k &lt; 0$)</td>
<td>(e)</td>
<td>(e)</td>
<td>(e)</td>
<td>(e)</td>
<td>(e)</td>
<td>(e)</td>
<td>(e)</td>
<td></td>
<td>(e)</td>
<td>(e)</td>
</tr>
<tr>
<td>Szekeres (subclass)</td>
<td>(d)</td>
<td>(d)</td>
<td>(d)</td>
<td>(d)</td>
<td>(c)</td>
<td>(b)</td>
<td>0</td>
<td></td>
<td>(c)</td>
<td>0</td>
</tr>
<tr>
<td>Mars (3rd type)</td>
<td>(d)</td>
<td>(d)</td>
<td>(d)</td>
<td>(d)</td>
<td>(c)</td>
<td>(b)</td>
<td>(c)</td>
<td></td>
<td>(c)</td>
<td>0</td>
</tr>
<tr>
<td>Carneiro-Marugan</td>
<td>(d)</td>
<td>(d)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(c)</td>
<td>(b)</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Kantowski</td>
<td>(d)</td>
<td>(d)</td>
<td>0</td>
<td>(a)</td>
<td>0</td>
<td>(c)</td>
<td>(b)</td>
<td></td>
<td>0</td>
<td>(c)</td>
</tr>
<tr>
<td>Kantowski-Sachs</td>
<td>(d)</td>
<td>(b)</td>
<td>0</td>
<td>(a)</td>
<td>0</td>
<td>(c)</td>
<td>(b)</td>
<td></td>
<td>0</td>
<td>(c)</td>
</tr>
</tbody>
</table>

Table 7.2: Behaviour of the fluid flow and its kinematic quantities as $\bar{T} \to 0^-$ in the physical space-time, and of the conformal fluid flow and its expansion, in the investigated examples (see Table 7.1 for an explanation of (a), (b), etc.).

Interestingly, all the investigated non-FRW models which admit an IPS, satisfy the four requirements above, which raises the question whether all non-FRW models with an IPS do so. In contrast to this, all example models of the previous chapter show a vanishing conformal factor and a non-degenerate conformal metric\(^1\) and the physical fluid flow in these models diverges as $\bar{T} \to 0^-$. In the analysed subclass of the McVittie-Wiltshire I models even the unphysical fluid flow diverges at $\bar{T} = 0$.

The tables, furthermore, show that neither the existence of an equation of state $p = p(\mu)$, nor of an IPS, nor the behaviour of the kinematic quantities (not their ratios) can be a distinguishing characteristic between physically realistic and unrealistic future behaviours. An equation of state as well as an IPS are found for both types of models, namely those from the previous chapter which violate the idea of a Weyl dominated Ricci curvature as a high-entropy state and those models from this chapter which support it. The kinematic quantities only show behaviours which seem to occur in both types of cosmologies and only depend on whether a physical singularity is encountered or not.

The behaviour of the ratios of the non-zero kinematic quantities with the expansion scalar, however, could be a distinguishing feature. In all the non-FRW cosmologies with an IPS, we find that the asymptotic isotropy of the IPS does not hold anymore, i.e. the kinematics are not expansion and the Weyl curvature is not Ricci dominated at $\bar{T} = 0$. In this sense, these models do allow some degree of anisotropy at late cosmic times. This anisotropic behaviour makes perfectly sense in conjunction with the ideas of quiescent cosmology.

Concerning the definition for a conformal structure with realistic future behaviour, we can already see that it will not be possible to construct a similar condition to requirement (1) of the definition of an IPS; Tables 7.2 and 7.3 show that all examples with increasing $K$ possess a scalar polynomial curvature singularity and

\(^1\)or, as in the case of the McVittie-Wiltshire I models, a degenerate conformal metric, with $\bar{g}_{00}$ vanishing in diagonal form.
7.5. Discussion

Table 7.3: Some characteristic properties of the conformal structure, and the conformal space-time in the investigated examples as \( \mathcal{T} \to 0^- \). \( C^n_0 \) denotes the degree of differentiability of \( \bar{g} \) at \( \mathcal{T} = 0 \). (f) means that timelike components of the metric vanish, and (g) means that spatial components of the metric vanish. See Table 7.1 for the explanation of the other abbreviations.

<table>
<thead>
<tr>
<th>Model</th>
<th>degenerate ( \bar{g} )</th>
<th>( C^n_0 )</th>
<th>( \Omega )</th>
<th>( \Omega^5 \bar{g} )</th>
<th>( K )</th>
<th>( C_{abcd} )</th>
<th>( C^{abcd} )</th>
<th>( R_{ab} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>rad. FRW (( k = +1 ))</td>
<td>no</td>
<td>( n = \infty )</td>
<td>(d)</td>
<td>(d)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(c)</td>
</tr>
<tr>
<td>dust FRW (( k = +1 ))</td>
<td>no</td>
<td>( n = \infty )</td>
<td>(d)</td>
<td>(d)</td>
<td>/</td>
<td>0</td>
<td>0</td>
<td>(0)</td>
</tr>
<tr>
<td>McVittie-Wiltshire I</td>
<td>(f)</td>
<td>( n = 0 )</td>
<td>(d)</td>
<td>(d)</td>
<td>/</td>
<td>0</td>
<td>0</td>
<td>(0)</td>
</tr>
<tr>
<td>McV.-Wil. II (( k &lt; 0 ))</td>
<td>no</td>
<td>( n = \infty )</td>
<td>(d)</td>
<td>(d)</td>
<td>(c)</td>
<td>(c)</td>
<td>(c)</td>
<td>(0)</td>
</tr>
<tr>
<td>Szekeres (subclass)</td>
<td>(g)</td>
<td>( n = \infty )</td>
<td>(a)</td>
<td>(a)</td>
<td>(c)</td>
<td>(a)</td>
<td>(a)</td>
<td>(0)</td>
</tr>
<tr>
<td>Mars (3rd type)</td>
<td>(g)</td>
<td>( n = 0 )</td>
<td>(a)</td>
<td>(a)</td>
<td>(d)*</td>
<td>(a)*</td>
<td>(a)</td>
<td>(0)</td>
</tr>
<tr>
<td>Carneiro-Marugan</td>
<td>(g)</td>
<td>( n = \infty )</td>
<td>(a)</td>
<td>(a)</td>
<td>(c)</td>
<td>(a)</td>
<td>(a)</td>
<td>(a)</td>
</tr>
<tr>
<td>Kantowski</td>
<td>(g)</td>
<td>( n = \infty )</td>
<td>(a)</td>
<td>(a)</td>
<td>(d)</td>
<td>(a)</td>
<td>(a)</td>
<td>(0)</td>
</tr>
<tr>
<td>Kantowski-Sachs</td>
<td>(g)</td>
<td>( n = 0 )</td>
<td>(a)</td>
<td>(d)</td>
<td>(c)</td>
<td>(a)</td>
<td>(a)</td>
<td>(a)</td>
</tr>
</tbody>
</table>

a diverging expansion scalar at \( \mathcal{T} = 0 \) in the unphysical space-time, i.e. we do not necessarily find a curvature singularity for the physical space-time in such a conformal structure, but apparently always for the unphysical space-time at \( \mathcal{T} = 0 \). Thus, we cannot require that the physical manifold be a submanifold of the conformal manifold.

It is interesting to note, that there exists a wide choice of possible cosmic time functions for the ever expanding models with diverging conformal factor, examined in this chapter. This is a point which warrants further investigation since the conformal structure should explain this state of affairs.

The distinguishing characteristic between those physically reasonable models with future singularity, and those which describe ever expanding universes, seems to be the determinant of the physical metric \( \delta \hat{g} \). All non-FRW models which admit an IPS show a diverging conformal factor at \( \mathcal{T} = 0 \); of those space-times it is only the Kantowski-Sachs models in which the determinant of the physical metric vanishes, and they are the only models with a physical future singularity. This connection will be proven in section 11.2.

In addition, the behaviour of the derivatives of the conformal factor, as well as the behaviour of the expansion scalar and the question of strong curvature singularities in these conformal relations, will find some explanations and answers in the next chapters.

Finally, we recall that it should be the goal to narrow the class of cosmological models which admit an IPS, to only those cosmologies which seem physically appropriate. The discussion of ASPH in section 5.1 already suggested that only those models which are inhomogeneous at the IPS should be regarded as physically realistic. The conformal structure for the future behaviour should be another tool to downsize the class of physically realistic cosmologies and hence, the definitions given in the next chapter are supposed to be a completion to the framework of an IPS. Both discussions indicate that FRW models should be omitted from this class.
New definitions for cosmological futures

Motivated by the example models discussed in the previous two chapters, we will now proceed to give the new definitions for conformal structures with isotropic future behaviour in section 8.1 and conformal structures with anisotropic future evolution in section 8.2. The physical implications will be analysed in the following chapters.

Unlike in the case of Definition 4.1 we will require that the conformal factor and the metric be at least $C^2$ (instead of $C^3$), in agreement with Definition 1.1.

We will denote some relevant quantities with a $\sim$, in order to emphasise that we are now dealing with conformal structures for the future evolution of cosmologies.

8.1 New definitions for isotropic future behaviour

The example space-times examined in chapter 6 exhibited a structure and physical behaviour which is essentially the time-reverse of an IPS$^\dagger$. Based on this discussion, and in close analogy to definition 4.1, we find the following definition.

Definition 8.1 (Isotropic future singularity (IFS))

A space-time $(\mathcal{M}, g)$ is said to admit an isotropic future singularity if there exists a space-time $(\tilde{\mathcal{M}}, \tilde{g})$, a smooth cosmic time function $\tilde{T}$ defined on $\tilde{\mathcal{M}}$, and a conformal factor $\tilde{\Omega}(\tilde{T})$ which satisfy

1. $\mathcal{M}$ is the open submanifold $\tilde{T} < 0$,

2. $g = \tilde{\Omega}^2(\tilde{T})\tilde{g}$ on $\mathcal{M}$, with $\tilde{g}$ regular (at least $C^2$ and non-degenerate) on an open neighbourhood of $\tilde{T} = 0$,

3. $\tilde{\Omega}(0) = 0$ and $\exists c > 0$ such that $\tilde{\Omega} \in C^0[-c, 0] \cap C^2[-c, 0]$ and $\tilde{\Omega}$ is strictly monotonically decreasing and positive on $[-c, 0)$,

$^\dagger$Except the McVittie-Wiltshire I models.
4. \( \lambda \equiv \lim_{T \to 0^-} \frac{L(T)}{T} \) exists, \( \lambda \neq 1 \), where \( L \equiv \frac{\Omega'}{\Omega} \left( \frac{\Omega'}{\Omega} \right)^2 \) is continuous on \([-c, 0)\) and a prime denotes differentiation with respect to \( T \).

It should be noted that if the model admits an IPS as well, the above conformal relation will in general not be the same as for the IPS.

Additionally, in analogy to definition 4.6, we require:

**Definition 8.2 (IFS fluid congruence)**

With any timelike congruence \( \mathbf{u} \) in \( \mathcal{M} \) we can associate a timelike congruence \( \mathbf{u} \) in \( \mathcal{M} \) such that

\[
\mathbf{\bar{u}} = \tilde{\Omega} \mathbf{u} \quad \text{in} \quad \mathcal{M}.
\]  

(a) If we can choose \( \mathbf{\bar{u}} \) to be regular (at least \( C^2 \)) on an open neighbourhood of \( \Bar{T} = 0 \) in \( \mathcal{M} \), we say that \( \mathbf{u} \) is regular at the IFS.

(b) If, in addition, \( \mathbf{\bar{u}} \) is orthogonal to \( \Bar{T} = 0 \), we say that \( \mathbf{u} \) is orthogonal to the IFS.

There is, however, another possibility for a conformal structure with an isotropic future behaviour. At the moment we do not have an example model for the following definition. Nevertheless, it shall be presented here and analysed in chapter 10 for completion, as we believe that some open FRW universes might satisfy these conditions.

**Definition 8.3 (Future isotropic universe (FIU))**

A space-time \( (\mathcal{M}, g) \) is said to be a future isotropic universe if there exists a space-time \( (\mathcal{M}, \tilde{g}) \), a smooth cosmic time function \( \Bar{T} \) defined on \( \mathcal{M} \), and a conformal factor \( \tilde{\Omega}(T) \) which satisfy

1. \( \lim_{T \to 0^-} \tilde{\Omega}(\Bar{T}) = +\infty \) and \( \exists \ c > 0 \) such that \( \tilde{\Omega} \in C^2[-c, 0) \) and \( \tilde{\Omega} \) is strictly monotonically increasing and positive on \([-c, 0)\),

2. \( \bar{\lambda} \) as defined above exists, \( \bar{\lambda} \neq 1, 2 \), and \( \tilde{L} \) is continuous on \([-c, 0)\), and

3. otherwise the conditions of Definitions 8.1 and 8.2 are fulfilled.

The implications of these definitions will be studied in chapters 9 and 10 which will clarify the details and justify the names of the new definitions.

### 8.2 Anisotropic future endless universes and anisotropic future singularities

As indicated previously, the framework of an IPS as it stands is not sufficient to guarantee a future evolution which is compatible with *quiescent cosmology* and the
8.2. Anisotropic future endless universes and anisotropic future singularities

WCH. Thus, it is necessary to complete the framework of an IPS with definitions which describe a non-isotropic future evolution of the universe, in order to downsize the class of physically reasonable cosmologies. Since it is not yet clear whether our universe will expand forever or recollapse in finite proper time it is, in fact, necessary to give two definitions, one for each scenario.

Before we present the definitions for the conformal structures, we need to define the notion of a limiting causal future.

**Definition 8.4 (Limiting causal future)**

Let \((M, g)\) be a space-time, where \(M \subset \overline{M}\). We define the limiting causal future of \(M\), denoted \(F^+(M)\), as follows

\[
F^+(M) := \{ p \in \overline{M} \mid \exists \text{ a future inextendible causal curve } \gamma_p(s) : [0, a) \to M, \text{ where } a \in \mathbb{R}^+ \cup \{\infty\}, \text{ such that } p = \gamma_p(a) \equiv \lim_{s \to a^-} \gamma_p(s) \}.
\]

This enables us to furthermore define the following type of degeneracy.

**Definition 8.5 (Causal degeneracy)**

Consider \(p \in F^+(M)\). Let \(\gamma_p(s)\) be a causal curve in \(M\) as defined above with a limiting tangent vector \(\gamma_p' \neq 0\) at \(p\). The metric \(g\) is said to be causally degenerate at \(p\) if there exists such a curve \(\gamma_p\) which satisfies \(g(\gamma_p', X) = 0 \forall X \in T_p\overline{M}\) (Note that this assumes that the metric is continuous on an open neighbourhood of \(p\)).

**Remark 8.6**

We will henceforth require that \(F^+(M)\) be a non-empty set, and, for simplicity, that our space-time models do not contain any type of astrophysical singularity (e.g. black holes), i.e. that \(F^+(M)\) corresponds solely to the final cosmological state.

The following two space-time definitions are based on the discussion of the example models in chapter 7 and we believe that they deliver the demonstrated future behaviour. The conformal structures differ significantly from the isotropic case.

**Definition 8.7 (Anisotropic future endless universe (AFEU))**

A space-time \((M, g)\) is said to be an anisotropic future endless universe if there exists a larger manifold \(\overline{M} \supset M\), a space-time \((\overline{M}, \overline{g})\), a smooth function \(T\) defined on \(M \cup F^+(M)\), where \(F^+(M) \neq \emptyset\), and a conformal factor \(\overline{\Omega}(\overline{T})\) which satisfy

1. \(\overline{T} = 0\) on \(F^+(M)\), and \(\overline{T}\) is a cosmic time function on \(M\) with range \(\overline{T} < 0\),
2. \(g = \overline{\Omega}^2(\overline{T})\overline{g}\) on \(M\), and \(\overline{g}\) is \(C^0\) on \(M \cup F^+(M)\) and degenerate, but not causally degenerate, on \(F^+(M)\),
3. \(\lim_{\overline{T} \to 0} \overline{\Omega}(\overline{T}) = +\infty\), and \(\exists c > 0\) such that \(\overline{\Omega} \in C^2[-c, 0)\) and \(\overline{\Omega}\) is strictly monotonically increasing and positive on \([-c, 0)\),
4. \(\lambda\) as defined above exists, \(\lambda \neq 1\) and \(L\) is continuous on \([-c, 0)\), and
5. \( \lim_{T \to 0^-} \Omega^8 |\tilde{g}| = +\infty \) across all of \( F^+(\mathcal{M}) \), where \( \tilde{g} \) is the determinant of \( \tilde{g} \).

Note, that unlike in the case of an IPS, the physical space-time and the conformal space-time possess the same manifold \( \mathcal{M} \). Furthermore, it is redundant to require that \( \tilde{g} \) be regular (at least \( C^2 \) and non-degenerate) on \( \mathcal{M} \), since \( (\mathcal{M}, \tilde{g}) \) is a space-time by construction.

**Remark 8.8**

Given condition 1 of Definition 8.7, the smoothness of \( \tilde{T} \) implies that \( F^+(\mathcal{M}) \) is a smooth “slice” in \( \tilde{\mathcal{M}} \).

Keeping in mind that \( T = 0 \) is a “slice”, we will additionally require the following, in order to guarantee an appropriate behaviour of the fluid flow quantities:

**Definition 8.9 (AFEU fluid congruence)**

With any timelike congruence \( \mathbf{u} \) in \( (\mathcal{M}, g) \) we can associate a timelike congruence \( \tilde{\mathbf{u}} \) in \( (\mathcal{M}, \tilde{g}) \) such that

\[
\tilde{\mathbf{u}} = \tilde{\Omega} \mathbf{u} \text{ in } \mathcal{M}.
\] (8.2)

(a) If we can further choose \( \tilde{\mathbf{u}} \) to be regular (at least \( C^2 \)) on \( \mathcal{M} \cup F^+(\mathcal{M}) \), we say that \( \mathbf{u} \) is regular on the slice \( T = 0 \).

(b) If, in addition, \( \tilde{\mathbf{u}} \) is orthogonal to \( T = 0 \), we say that \( \mathbf{u} \) is orthogonal to the slice \( \tilde{T} = 0 \).

Now we will reveal the second anisotropic space-time definition.

**Definition 8.10 (Anisotropic future singularity (AFS))**

A space-time \( (\mathcal{M}, g) \) is said to admit an anisotropic future singularity if there exists a larger manifold \( \tilde{\mathcal{M}} \supset \mathcal{M} \), a space-time \( (\tilde{\mathcal{M}}, \tilde{g}) \), a smooth function \( \tilde{T} \) defined on \( \mathcal{M} \cup F^+(\mathcal{M}) \), where \( F^+(\mathcal{M}) \neq \emptyset \), and a conformal factor \( \tilde{\Omega} (\tilde{T}) \) which satisfy

1. conditions 1. - 4. of Definition 8.7, and

2. \( \lim_{T \to 0^-} \tilde{\Omega}^8 |\tilde{g}| = 0 \) across all of \( F^+(\mathcal{M}) \), where \( \tilde{g} \) is the determinant of \( \tilde{g} \).

Similarly to the previous case, and in analogy to Definition 8.2, we additionally require the following for the fluid flow:

**Definition 8.11 (AFS fluid congruence)**

With any timelike congruence \( \mathbf{u} \) in \( (\mathcal{M}, g) \) we can associate a timelike congruence \( \tilde{\mathbf{u}} \) in \( (\mathcal{M}, \tilde{g}) \) such that

\[
\tilde{\mathbf{u}} = \tilde{\Omega} \mathbf{u} \text{ in } \mathcal{M}.
\] (8.3)
8.3 Comment

(a) If we can further choose $\bar{u}$ to be regular (at least $C^2$) on $M \cup F^+(M)$, we say that $u$ is regular at the AFS.

(b) If, in addition, $\bar{u}$ is orthogonal to $\bar{T} = 0$, we say that $u$ is orthogonal to the AFS.

Remark 8.12
For simplicity, and without loss of generality, we will henceforth refer to $F^+(M)$ as $\bar{T} = 0$ in the remainder of this thesis. Equivalently, whenever we use $\lim_{\bar{T} \to 0^-}$ we will mean that the limit is taken along any future directed causal curve. Furthermore, for brevity, we will write $\Omega(0) = +\infty$ for $\lim_{\bar{T} \to 0^-} \Omega(\bar{T}) = +\infty$ from now on.

Some implications of these definitions will be investigated in chapters 9, 10 and 11 where the details will become clear and justifications for the various names are provided.

8.3 Comment

Motivated by the analysis of example space-times, we have given four new definitions for conformal structures, two for an isotropic future behaviour and two for an anisotropic future behaviour.

Except for Definition 8.3, we have example models for each new definition. As one can easily convince oneself, the radiation filled, closed FRW universe and the dust, closed FRW universe both satisfy the conditions of Definitions 8.1 and 8.2. The McVittie-Wiltshire II models fulfil the conditions of Definition 8.1, but the fluid flow in these solutions remains to be analysed in order to determine whether they also satisfy the conditions of Definition 8.2. Since the conformal structure of the McVittie-Wiltshire I models appears to be physically unreasonable, we have not further considered this case here.

Conditions 1-4 of Definition 8.7 are satisfied by a subclass of the Szekeres models, the Carneiro-Marugan model, the Kantowski models and the Kantowski-Sachs models; for a space-time $(M, g)$ one can always find a larger manifold $\bar{M}$ on which $g$ does not satisfy the conditions of Definition 1.1 everywhere. Additionally, the fact that neither of these models possesses a conformal metric which becomes \textit{causally degenerate} at $\bar{T} = 0$ can be most readily seen from the diagonal form in which all metrics are presented. The $\bar{g}_{00}$ components of these (continuous) conformal metrics never vanish, hence $\bar{g}$ cannot become \textit{causally degenerate}. The other conditions can be verified directly.

The subclass of the Szekeres models, the Carneiro-Marugan model and the Kantowski models furthermore fulfil condition 5 of Definition 8.7 and consequently are all AFEUs. This becomes clear by the expressions for $\Omega$ and $\bar{g}$ in Appendix B. The Kantowski-Sachs models, on the other hand, are the only models analysed in this thesis which admit an AFS, since only they also satisfy condition 2 of Definition 8.10.
The third type of the Mars models is a special case with $\lambda = 1$, which would match the AFEU models with its general behaviour. However, for $\lambda = 1$ we will find some difficulties in proving some results of section 9.1. Therefore we have left this case out.

As in the case of the IPS, we will refer to $(\mathcal{M}, g)$ as the physical space-time, while the conformal space-time $(\bar{\mathcal{M}}, \bar{g})$ (or $(\mathcal{M}, \bar{g})$ respectively) will be referred to as the unphysical space-time, since it is not necessarily a solution of the EFE.

It should be noted that there is most likely some redundancy in all the definitions, as was the case for the original definition of an IPS. As in the case of the IPS, the monotonicity of $L$ and $\Omega$ might already be implied by the other conditions (see Lemma 5.6). A similar analysis as in [47] will provide the answer to this question.

Quiescent cosmology and the WCH suggest that only the definitions of an AFEU and AFS are physically reasonable and, consequently, we are mainly interested in these two definitions. Nevertheless, some implications of all four definitions will be analysed in chapters 9, 10 and 11, in which the names chosen for the definitions and some technical details will be clarified.
A few technical properties of the conformal factor and the metric are necessary for the derivation of several results in the following chapters. For this purpose, we analyse the conformal factor in section 9.1 and some properties of a degenerate metric in section 9.2.

9.1 Conformal factor

We are interested in the existence and the possible values of the limit of several ratios of the conformal factor. Recall that if some quantities of the conformal factor or the cosmic time function are denoted without a , we are specifically referring to the situation in which the cosmic time function approaches 0 from above. We begin by analysing the possible values of \( \lambda = \lim_{T \to 0^-} L \), where \( L \equiv \frac{\Omega''\Omega}{(\Omega')^2} \), for the case \( \Omega(0) = 0 \).

Lemma 9.1
If \( T \) is a cosmic time function which approaches 0 from below, and

1. \( \bar{\Omega}(\bar{T}) \) is strictly monotonically increasing, positive and \( C^2 \) on \( I = [-c, 0) \) (where \( c > 0 \)),
2. \( \bar{\Omega}(0) = \infty \), and
3. \( \bar{\lambda} \) as defined above exists and \( L \) is continuous on \( I \),

then \( \bar{\lambda} \geq 1 \).

Proof. Define \( \phi = \ln \bar{\Omega} \) on \( I \). Then

\[
\phi' = \frac{\bar{\Omega}'}{\bar{\Omega}} \quad \text{and} \quad \phi'' = \bar{\Omega}^{-2} \left[ \bar{\Omega}'' \bar{\Omega} - (\bar{\Omega}')^2 \right].
\] (9.1)
From condition 1 it follows immediately that φ is a strictly monotonically increasing function on I.

Suppose \( \lambda < 1 \). Then, by condition 3, \( \exists -\eta \in I \), such that \( \bar{L}(\bar{T}) < 1 \) \( \forall \bar{T} \in J = [-\eta, 0) \).

\[
\Rightarrow \frac{\bar{\Omega}\bar{\Omega}''}{(\bar{\Omega}')^2} < 1 \quad (9.2)
\]

\[
\Leftrightarrow \bar{\Omega}''\bar{\Omega} - (\bar{\Omega}')^2 < 0 \quad (9.3)
\]

\[
\Leftrightarrow \phi'' < 0, \quad (9.4)
\]

i.e. \( \phi' \) is a strictly monotonically decreasing function on \( J \) and thus \( \phi' \) is bounded above by \( \phi'(-\eta) \) on \( J \). With the help of the mean value theorem we find that

\[
\frac{\phi(\bar{T}) - \phi(-\eta)}{\bar{T} + \eta} = \phi'(\xi) \leq \phi'(-\eta) \quad \text{where} \quad \xi \in [-\eta, \bar{T}] \quad (9.5)
\]

\[
\Rightarrow \phi(\bar{T}) < \phi(-\eta) + \eta \phi'(-\eta) \quad \forall \bar{T} \in J. \quad (9.6)
\]

The two terms on the r.h.s. of equation (9.6) are both finite by construction, therefore \( \phi \) is bounded above on \( J \). This is a contradiction to condition 2 which says that \( \phi \to \infty \) as \( \bar{T} \to 0^- \).

Hence, \( \bar{\lambda} \geq 1 \). \( \Box \)

Now it is necessary to derive a similar result for \( \bar{\Omega}(0) = 0 \); in fact, the following result is completely identical to the one for an IPS\(^\dagger\). The proof proceeds analogously.

**Lemma 9.2**

If \( \bar{T} \) is a cosmic time function which approaches 0 from below, and

1. \( \bar{\Omega}(\bar{T}) \) is strictly monotonically decreasing, positive, \( C^0 \) on \( (-c, 0] \) and at least \( C^2 \) on \( I = [-c, 0) \) (where \( c > 0 \)),

2. \( \bar{\Omega}(0) = 0 \), and

3. \( \bar{\lambda} \) as defined above exists and \( \bar{L} \) is continuous on \( I \),

then \( \bar{\lambda} \leq 1 \).

**Proof.** As before, define \( \phi = \ln \bar{\Omega} \) on \( I \). Condition 1 immediately implies that \( \phi \) is a strictly monotonically decreasing function on \( I \).

Suppose \( \bar{\lambda} > 1 \). Then, by condition 3, \( \exists -\eta \in I \), such that \( \bar{L}(\bar{T}) > 1 \) \( \forall \bar{T} \in J = [-\eta, 0) \).

\[
\Rightarrow \frac{\bar{\Omega}\bar{\Omega}''}{(\bar{\Omega}')^2} > 1 \quad (9.7)
\]

\[
\Leftrightarrow \bar{\Omega}''\bar{\Omega} - (\bar{\Omega}')^2 > 0 \quad (9.8)
\]

\[
\Leftrightarrow \phi'' > 0, \quad (9.9)
\]

\(^\dagger\)see section 5.3.
i.e. \( \phi' \) is a strictly monotonically increasing function on \( J \) and thus \( \phi' \) is bounded below by \( \phi'(-\eta) \) on \( J \). The mean value theorem implies

\[
\frac{\phi(T) - \phi(-\eta)}{T + \eta} = \phi'(\xi) \geq \phi'(-\eta) \quad \text{where} \quad \xi \in [-\eta,T] \tag{9.10}
\]

\[
\Rightarrow \phi(T) \geq \phi(-\eta) + \eta \phi'(-\eta) \quad \forall \ T \in J. \tag{9.11}
\]

The two terms on the r.h.s. of equation (9.11) are both finite by construction, therefore \( \phi \) is bounded below on \( J \). This contradicts condition 2 which indicates \( \phi \to -\infty \) as \( T \to 0^- \).

Hence, \( \lambda \leq 1 \).

\[ \square \]

**Remark 9.3**

Scott [47] has already proven that if \( T \to 0^+ \), \( \Omega(0) = 0 \) and \( \Omega \) is positive on \( (0,c) \), \( c > 0 \), then \( \lambda \leq 1 \). If one is aware of the signs, by Lemma 9.1 one can easily verify that under the same conditions, and \( \Omega(0) = \infty \), one again finds \( \lambda \geq 1 \).

The limit of \( \frac{\dot{\Omega}}{\Omega} \) of the conformal factor turns out to be an essential property in the derivation of some results. We will first prove the existence and the value of this limit for the case \( \dot{\Omega}(0) = \infty \).

**Lemma 9.4**

Let \( \bar{T} \) be a cosmic time function which approaches 0 from below and let \( \bar{\Omega}(\bar{T}) \) be positive, strictly monotonically increasing and \( C^2 \) on some interval \( (-c,0) \), \( c > 0 \). If, furthermore, \( \bar{\Omega}(0) = \infty \) and \( \bar{\lambda} \) as defined above exists, \( \bar{\lambda} \neq 1 \), then

\[
\lim_{T \to 0^-} \frac{\dot{\Omega}}{\Omega}(\exists) = +\infty.
\]

**Proof.** By Lemma 9.1 we know that, in this case, \( \bar{\lambda} > 1 \), and proceeding as in the proof of Lemma 9.1 we find an open interval \( (-\eta,0) \), \( \eta > 0 \), on which \( \phi'' > 0 \), i.e. on which \( \phi' \) is a strictly monotonically increasing function. Thus,

\[
\lim_{T \to 0^-} \phi' = \lim_{T \to 0^-} \frac{\dot{\Omega}}{\Omega} \quad \text{exists}.
\]

Now assume

\[
\lim_{T \to 0^-} \frac{\dot{\Omega}}{\Omega} = k, \quad \text{where} \quad |k| < \infty. \tag{9.12}
\]

Then

\[
\frac{\dot{\Omega}}{\Omega} = f(T), \quad \text{with} \quad f(0) = k \tag{9.13}
\]

\[
\frac{d\bar{\Omega}}{\bar{\Omega}} = f(T)\, d\bar{T} \tag{9.14}
\]

\[
\Rightarrow \bar{\Omega}(\bar{T}) = c \cdot \exp \left[ \int f(T)\, d\bar{T} \right], \quad c = \text{const.} \tag{9.15}
\]

Since \( \bar{\Omega}(0) = +\infty \),

\[
F(0) = \left. \int f(T)\, d\bar{T} \right|_{\bar{T}=0} = \infty. \tag{9.16}
\]
but then
\[ F(0) = \infty \text{ and } F'(0) = f(0) = k \] (9.17)
\[ \Rightarrow \text{ such a function } F \text{ cannot exist.} \]

Thus, we encounter a contradiction. Consequently, \(|k| = \infty\) and the sign of the assertion becomes clear by the fact that we have already shown that \(\phi'\) is a strictly monotonically increasing function on \((-\eta, 0)\) \(\Rightarrow \lim_{T \to 0^-} \frac{\Omega'}{\Omega} = +\infty. \)

Now we will turn our attention to the case \(\Omega(0) = 0\).

**Lemma 9.5**

Let \(\bar{T}\) be a cosmic time function which approaches 0 from below and let \(\bar{\Omega}(\bar{T})\) be positive, strictly monotonically decreasing and continuous on some interval \((-c, 0]\) and \(C^2\) on \((-c, 0]\), \(c > 0\). If, furthermore, \(\bar{\Omega}(0) = 0\) and \(\bar{\lambda}\) as defined above exists, \(\bar{\lambda} \neq 1\), then \(\lim_{T \to 0^-} \frac{\Omega'}{\Omega} (\text{exists}) = -\infty.\)

**Proof.** Lemma 9.2 implies that \(\bar{\lambda} < 1\) and proceeding as in the proof of Lemma 9.2 we find that \(\exists -\eta \in (-c, 0): \phi'' < 0\) on \((-\eta, 0)\), i.e. such that \(\phi'\) is a strictly monotonically decreasing function on \((-\eta, 0)\). Hence, \(\lim_{T \to 0^-} \phi' = \lim_{T \to 0^-} \frac{\Omega'}{\Omega} \) exists.

Now suppose
\[
\lim_{T \to 0^-} \frac{\Omega'}{\Omega} = k, \text{ where } |k| < \infty. \] (9.18)

Since \(\Omega(0) = 0\), by equation (9.15) we find
\[
F(0) = \int f(\bar{T}) d\bar{T} \bigg|_{\bar{T}=0} = -\infty, \] (9.19)

but then
\[
F(0) = -\infty \text{ and } F'(0) = f(0) = k \] (9.20)
\[ \Rightarrow \text{ such a function } F \text{ cannot exist.} \]

Thus, we find a contradiction. Therefore \(|k| = \infty\) and the sign of the assertion becomes evident by the fact that we have already shown that \(\phi'\) is a strictly monotonically decreasing function on \((-\eta, 0)\). Therefore \(\lim_{T \to 0^-} \frac{\Omega'}{\Omega} = -\infty.\)

**Remark 9.6**

If \(T\) approaches 0 from above, Lemmas 9.4 and 9.5 are clearly also true if the signs in the assertions are vice versa. This once more confirms that there is a redundancy in the initial definition of an IPS by Goode and Wainwright [8] as was first shown by Scott [47]. The original requirement regarding the behaviour of \(\frac{\Omega'}{\Omega}\) can thus be safely omitted.
The cases $\bar{\Omega}(0) = \infty$ and $\bar{\Omega}(0) = 0$ lead to completely different behaviours of the ratio $\bar{M} \equiv \bar{\Omega}' \bar{\Omega}^{-1}$, which will be of importance in the analysis of the expansion scalar.

**Lemma 9.7**

*Let $\bar{T}$ be a cosmic time function which approaches 0 from below and let $\bar{\Omega}(\bar{T})$, with $\bar{\Omega}(0) = \infty$, be positive, strictly monotonically increasing and $C^2$ on some interval $(-c,0)$, $c > 0$. If $\bar{\lambda}$ exists and $\bar{\lambda} \neq 1, 2$, then $\lim_{\bar{T} \to 0^-} \bar{M}(\text{exists}) = \kappa$, and $0 < \kappa \leq \infty$ if $\bar{\lambda} > 2$ and $0 \leq \kappa < \infty$ if $\bar{\lambda} < 2$.\*

*Proof.* The conditions imply $\bar{M} > 0$ on $(-c,0)$. Furthermore,

$$\bar{M}' = \frac{\bar{\Omega}'\bar{\Omega} - 2(\bar{\Omega}')^2}{\bar{\Omega}^3} = (\bar{L} - 2)\frac{(\bar{\Omega}')^2}{\bar{\Omega}^3} \quad (9.21)$$

and $\frac{(\bar{\Omega}')^2}{\bar{\Omega}^3} > 0$ on $(-c,0)$ and $\bar{\lambda} > 1$. There are two cases:

(a) $\bar{\lambda} > 2$, then $\exists \, d > 0$, such that $\bar{M}' > 0$ on $(-d,0)$. Now $\bar{M} > 0$, $\bar{M}' > 0 \Rightarrow \lim_{\bar{T} \to 0^-} \bar{M}(\text{exists}) = \alpha \in \mathbb{R}^+ \cup \{+\infty\}$.

(b) $\bar{\lambda} < 2$, then $\exists \, d > 0$, such that $\bar{M}' < 0$ on $(-d,0)$. Now $\bar{M} > 0$, $\bar{M}' < 0 \Rightarrow \lim_{\bar{T} \to 0^-} \bar{M}(\text{exists}) = \beta \in \mathbb{R}^+ \cup \{0\}$. \hfill \square

**Lemma 9.8**

*If all conditions of Lemma 9.7 are satisfied, except that here we assume that $\bar{\lambda} = 2$ and $\bar{L}$ is a continuous and strictly monotonic function on $(-c,0)$, then $\lim_{\bar{T} \to 0^-} \bar{M}(\text{exists}) \geq 0$.\*

*Proof.* Since $\bar{L}$ is continuous and strictly monotonic on $(-c,0)$, there are two possibilities:

1. $\bar{L} \to 2^+$ as $\bar{T} \to 0^-$. By equation (9.21) $\exists \, d > 0$: $\bar{M} > 0$ and $\bar{M}' > 0$ on $(-d,0)$. Thus, $\lim_{\bar{T} \to 0^-} \bar{M}(\text{exists}) = \alpha \in \mathbb{R}^+ \cup \{+\infty\}$.

2. $\bar{L} \to 2^-$ as $\bar{T} \to 0^-$. By equation (9.21) $\exists \, d > 0$: $\bar{M} > 0$ and $\bar{M}' < 0$ on $(-d,0)$. Thus, $\lim_{\bar{T} \to 0^-} \bar{M}(\text{exists}) = \beta \in \mathbb{R}^+ \cup \{0\}$. \hfill \square

**Remark 9.9**

*In the case $T \to 0^+$ Lemmas 9.7 and 9.8 are clearly also true if the signs in the limit of $\bar{M}$ are reversed.*

For the case $\Omega(0) = 0$ the analyses are much simpler.
Lemma 9.10
Let \( \bar{T} \) be a cosmic time function which approaches 0 from below and let \( \bar{\Omega}(\bar{T}) \), with \( \bar{\Omega}(0) = 0 \), be positive, strictly monotonically decreasing and continuous on some interval \((-c,0]\) and \(C^2\) on \((-c,0)\), \(c > 0\). If \( \bar{\lambda} \) exists and \( \bar{\lambda} \neq 1 \), then \( \lim_{\bar{T} \to 0^-} \bar{M} \) (exists) = \(-\infty\).

Proof. By Lemma 9.5 we know that \( \lim_{\bar{T} \to 0^-} \frac{\partial \bar{\Omega}}{\partial \bar{T}} \) (exists) = \(-\infty\). Since \( \bar{\Omega}(0) = 0 \), clearly \( \lim_{\bar{T} \to 0^-} \bar{M} \) (exists) = \(-\infty\). \[\square\]

The following two lemmas deal with the function \( \bar{N} \equiv \bar{L}^2 \bar{M}^4 \), which we will employ in the investigations concerning the physical attributes of a FIU in chapter 10.

Lemma 9.11
Let \( \bar{T} \) be a cosmic time function which approaches 0 from below and let, furthermore, \( \bar{\Omega}(\bar{T}) \) be the conformal factor which satisfies \( \bar{\Omega}(0) = \infty \) and \( \bar{\Omega} \) is positive, monotonically increasing and \(C^2\) on \((-c,0)\), \(c > 0\). Then, if \( \bar{\lambda} \) exists and \( \bar{\lambda} \neq 1, 2 \), \( \lim_{\bar{T} \to 0^-} \bar{N} \) (exists) = \( \rho \), and \( 0 < \rho \leq \infty \) if \( \bar{\lambda} > 2 \) and \( 0 \leq \rho < \infty \) if \( \bar{\lambda} < 2 \).

Proof. We know that \( 1 < \bar{\lambda} \leq \infty \) and \( \bar{\lambda} \neq 2 \). The same cases as in Lemma 9.7 apply.

(a) \( \bar{\lambda} > 2 \), \( 4 < \bar{\lambda}^2 \leq \infty \). By Lemma 9.7 \( \lim_{\bar{T} \to 0^-} \bar{M} \) (exists) = \( \alpha \in \mathbb{R}^+ \cup \{+\infty\} \). Thus, \( \lim_{\bar{T} \to 0^-} \bar{N} \) (exists) = \( \alpha' \in \mathbb{R}^+ \cup \{+\infty\} \).

(b) \( \bar{\lambda} < 2 \), \( 1 < \bar{\lambda}^2 < 4 \). From Lemma 9.7 we know that \( \lim_{\bar{T} \to 0^-} \bar{M} \) (exists) = \( \beta \in \mathbb{R}^+ \cup \{0\} \). Hence, \( \lim_{\bar{T} \to 0^-} \bar{N} \) (exists) = \( \beta' \in \mathbb{R}^+ \cup \{0\} \).

\[\square\]

Remark 9.12
The result for \( \bar{N} \) in the case \( \bar{\lambda} = 2 \) and \( \bar{L} \) continuous and strictly monotonic on \((-c,0)\) is now obvious from Lemma 9.8.

Remark 9.13
Remark 9.9 implies that the result of Lemma 9.11 is still valid for \( \bar{N} \) if \( \bar{T} \to 0^+ \) and \( \bar{\Omega}(0) = \infty \).

Lemma 9.14
Let \( \bar{T} \) be a cosmic time function which approaches 0 from below and let, furthermore, \( \bar{\Omega}(\bar{T}) \) be the conformal factor which satisfies \( \bar{\Omega}(0) = 0 \) and \( \bar{\Omega} \) is positive, monotonically decreasing and continuous on \((-c,0]\) and \(C^2\) on \((-c,0)\), \(c > 0\). Then, if \( \bar{\lambda} \) exists and \( \bar{\lambda} \neq 0, 1 \), \( \lim_{\bar{T} \to 0^-} \bar{N} \) (exists) = \(+\infty\).
Proof. We have $\tilde{\lambda} < 1$ and $\tilde{\lambda} \neq 0$, i.e. $0 < \tilde{\lambda}^2 \leq +\infty$. By Lemma 9.10, $\lim_{T \to 0^-} \tilde{M}(\text{exists}) = -\infty$. Thus, $\lim_{T \to 0^-} \tilde{N}(\text{exists}) = +\infty$.

The special cases of $\tilde{\lambda} = 0, 1, 2$ in the above lemmas require special treatment and warrant further investigation.

### 9.2 Metric degeneracy

Since we are dealing with degenerate metrics in the new definitions and the example models, it is essential to prove the following lemma concerning the determinant $g$ of the metric. The result is well known. We present the proof here, however, as it was independently derived by the author.

**Lemma 9.15**

A degenerate metric $g$ possesses a vanishing determinant $g$.

**Proof.** A metric $g$ is called degenerate at $p \in \mathcal{M}$, if $\exists X \in T_p \mathcal{M}: X \neq 0$ and $g(X, Y) = 0 \forall Y \in T_p \mathcal{M}$ to be written as $g(X, \cdot) = 0$. We choose a matrix notation for the metric, i.e. $g(X, Y)$ corresponds to $Y^T g X$, where $X, Y \in V$ represent column vectors of some vector space $V$ and $g$ represents the square matrix formed from the metric coefficients $g_{ab}$. It follows that

$$gX = 0.$$  

As $X \neq 0$ this is equivalent to an eigenvalue problem,

$$gX = \lambda X \quad \text{with} \quad \lambda = 0.$$  

Furthermore,

$$(g - \lambda I) X = 0 \iff \det(g - \lambda I) = 0.$$  

Now $\lambda = 0 \implies g = \det g = 0$. 

**Remark 9.16**

By the theorem of Lagrange and Sylvester (see [29]) there exists a coordinate basis in a neighbourhood of each $p \in \mathcal{M}$ such that the metric is in diagonal form at $p$. In such a basis, at least one component of the diagonal must clearly vanish if the metric is degenerate.

**Remark 9.17**

Since $g = 0$ at a point $p \in \mathcal{M}$ where the metric becomes degenerate, one cannot invert the metric at $p$. One could, however, construct a curve from the space-time to $p$, invert the metric in the space-time and take the limit along such a curve if the metric is continuous (some components will certainly become infinite in the limit).
In the discussion of the expansion scalar in conformal structures with degenerate metrics we depend upon the next result. The proof proceeds in a similar manner to Lemmas 9.4 and 9.5.

**Lemma 9.18**

Let the unphysical metric \( \bar{g} \) be at least \( C^2 \) on the cosmic time interval \((-c,0)\) and \( C^0 \) on \((-c,0)\), \( c > 0 \). Furthermore, let \( \bar{g} \) become degenerate as \( T \to 0^- \). If the unphysical fluid flow \( \bar{u} \) is non-vanishing and at least \( C^2 \) on \((-c,0]\), then

\[
\liminf_{T \to 0^-} \bar{g}(\bar{u}, \nabla \ln \sqrt{|\bar{g}|}) = -\infty.
\]

**Proof.** Since \( \bar{u} \) is at least \( C^2 \) on \((-c,0]\) we may choose comoving coordinates (along the flow lines of \( \bar{u} \)) for \( \bar{T} \in (-c,0] \) in which we can write \( \bar{g}(\bar{u}, \nabla \ln \sqrt{|\bar{g}|}) = \bar{u}^0 \frac{1}{2|\bar{g}|} \partial_0 |\bar{g}| \). Furthermore, if the comoving coordinate time is denoted by \( t \), then let \( t \to t_s \) as \( \bar{T} \to 0^- \). By Lemma 9.15 we know that \( \lim_{T \to 0^-} |\bar{g}| = 0 \Rightarrow \lim_{t \to t_s} |\bar{g}| = 0 \) (where the limits in this proof are taken along the flow lines of \( \bar{u} \)). Now proceeding as in Lemma 9.4 we find that

\[
\frac{1}{|\bar{g}|} \partial_0 |\bar{g}| = h(t, \bar{x}) \quad (9.22)
\]

\[
|\bar{g}| = C(\bar{x}) \exp[H(t, \bar{x})], \quad \text{where} \quad H(t, \bar{x}) = \int h(t, \bar{x}) \, dt. \quad (9.23)
\]

Since \( |\bar{g}| \) is also \( C^2 \) on \((-c,0]\) (which corresponds to some interval \((t_0, t_s), t_s > t_0, \) for \( t \)), equation (9.22) possesses a limit inferior, and by similar arguments as in the proof of Lemma 9.5, we must have that

\[
\liminf_{t \to t_s} H'(t, \bar{x}) = \liminf_{t \to t_s} h(t, \bar{x}) = -\infty \Rightarrow \liminf_{t \to t_s} \frac{1}{|\bar{g}|} \partial_0 |\bar{g}| = -\infty. \quad (9.24)
\]

Now \( \bar{u}^0 \) is non-zero and \( C^2 \) on \((-c,0]\) and can always be chosen to be positive. This leads to the assertion.

\[\square\]

**Remark 9.19**

Under the imposed conditions, both \( \bar{u}^a \) and \( \bar{u}_a \) are continuous on \((-c,0]\) and therefore \( \bar{u}^a \bar{u}_a \) must be continuous on the same interval as well. Now choosing proper time as the parametrisation gives \( \bar{u}^a \bar{u}_a = -1 \) for \( \bar{T} < 0 \), and thus by continuity of \( \bar{u} \) and \( \bar{g} \), this must also be true at \( \bar{T} = 0 \), i.e. \( \bar{u} \) does not cause the degeneracy of \( \bar{g} \).

### 9.3 Summary

In this chapter we have proven a number of lemmas which will not only be essential in the derivation of the results to come in the following chapters, but also in future investigations using the framework of the definitions given in this thesis. As in the case of the IPS, it is the technical results which will provide the key for the extension of the knowledge obtained in this thesis.
Chapter 10

Physical attributes of conformal structures with regular conformal metrics

It is time to justify the names of the new definitions, given in chapter 8, by analysing their physical characteristics. In sections 10.1, 10.2 and 10.3 we will prove that, under reasonable conditions, regular conformal metrics, which remain regular, lead to a behaviour which is similar or equivalent to the behaviour at an IPS. This provides us with a justification for terming Definitions 8.1 and 8.3 as IFS and FIU, respectively. In conclusion, this chapter, furthermore, offers a justification for the choice of a degenerate conformal metric in Definitions 8.7 and 8.10 which shall lead to an anisotropic behaviour.

In a few cases we will point out by denoting $T$, $\Omega$ and the functions of the conformal factor without a $u$ that we specifically refer to the situation $T \to 0^+$. Otherwise, since we are mainly interested in the future evolution, we will denote the relevant quantities of both the future and past case with a $u$, to keep calculations, which do not depend on whether $T \to 0^-$ or $T \to 0^+$, sufficiently brief.

10.1 Weyl versus Ricci curvature

It is essential to analyse which conditions we have to impose on the conformal metric in order to produce a non-vanishing ratio of the Weyl to the Ricci curvature as $T \to 0^-$, in accordance with Penrose’s WCH. We will now prove that under physically reasonable conditions on the conformal factor a regular conformal metric will lead to a vanishing ratio irrespective of whether $\Omega(0) = \infty$ or $\Omega(0) = 0$, or whether $T$ approaches 0 from below or above.

Recall the definition of $K$,

$$K = \frac{C_{abcd}C^{abcd}}{R_{ef}R^{ef}}. \quad (10.1)$$

\[\footnotesize{\text{\textsuperscript{\dagger}}}\text{Namely, a monotonic behaviour “near” } T = 0, \text{ since we would like to avoid an oscillatory behaviour.}\]
Theorem 10.1 (K-theorem)
Let \( \bar{T} \) be a smooth cosmic time function defined on a conformal space-time \((\bar{\mathcal{M}}, \bar{g})\) of a conformal structure \( g = \bar{\Omega}^2 (\bar{T}) \bar{g} \), where \( \bar{\Omega}(0) = 0 \) or \( \bar{\Omega}(0) = \infty \), and \( \bar{g} \) is non-degenerate and at least \( C^2 \) on an open neighbourhood of \( \bar{T} = 0 \). In the case \( \bar{T} \to 0^- \), if \( \bar{\Omega}(0) = \infty \) (\( \bar{\Omega}(0) = 0 \), respectively), let \( \bar{\Omega} \) be positive, \( C^2 \) and strictly monotonically increasing (decreasing, respectively) on some interval \([-c, 0)\), \( c > 0 \). On the other hand, if \( \bar{T} \to 0^+ \) and \( \bar{\Omega}(0) = \infty \) (\( \bar{\Omega}(0) = 0 \), respectively), let \( \bar{\Omega} \) be positive, \( C^2 \) and strictly monotonically decreasing (increasing, respectively) on some interval \((0, c]\), \( c > 0 \). If \( \lambda \) exists and \( \lambda \neq 1 \), then \( \lim_{T \to 0^\pm} K = 0 \).

**Proof.** We will analyse \( K \) directly. Using the following well known conformal relation for the Ricci tensor\(^\dag\) (a colon denotes covariant differentiation with respect to \( \bar{g} \))

\[
R_{ab} = \bar{\Omega}^{-2} \left[ \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^2 \left\{ 2 \left( 2 - \bar{L} \right) \bar{T}_a \bar{T}^b \bar{g}_{cb} - (1 + \bar{L}) \bar{g}_{ab} \bar{T}_c \bar{T}_d \bar{g}^{cd} \right\} - \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right) \left\{ 2 \bar{T}_{ab} \delta^c_b + \bar{g}_{ab} \bar{T}_{cd} \bar{g}^{cd} \right\} + \bar{R}_{ab} \right],
\]

we obtain

\[
R_{ab} R^{ab} = \bar{\Omega}^{-4} \left[ \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^4 12 \left( \bar{T}_a \bar{T}^a \right)^2 \left( \bar{L}^2 - \bar{L} + 1 \right)
- 2 \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^3 \left\{ (8 - 4 \bar{L}) \bar{T}^a \bar{T}^b \bar{T}_{ba} - (8 \bar{L} + 2) \bar{T}_a \bar{T}_b \bar{T}^{ab} \right\}
+ \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^2 \left\{ 4 \bar{T}_{ab} \bar{T}^{ba} + 8 \left( \bar{T}^a \right)^2 + 4 \left( 2 - \bar{L} \right) \bar{T}_a \bar{T}_b \bar{R}^{ba} - 2 \left( 1 + \bar{L} \right) \bar{R} \bar{T}_a \bar{T}^a \right\}
- 2 \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right) \left\{ 2 \bar{T}_{ab} \bar{R}^{ba} + \bar{R} \bar{T}^a \right\} + \bar{R}_{ab} \bar{R}^{ab} \right].
\]

Furthermore, we know that\(^\dagger\) \( g_{ab} = \bar{\Omega}^2 \bar{g}_{ab}, \ g^{ab} = \bar{\Omega}^{-2} \bar{g}^{ab} \) and \( C^a_{bcd} = \bar{C}^a_{bcd} \). Thus,

\[
C_{abcd} = \bar{\Omega}^2 \bar{C}_{abcd} \quad \text{and} \quad C^{abcd} = \bar{\Omega}^{-6} \bar{C}^{abcd}
\Rightarrow \ C_{abcd} C^{abcd} = \bar{\Omega}^{-4} \bar{C}_{abcd} \bar{C}^{abcd}.
\]

By the conjunction of equations (10.3) and (10.5) we find

\[
K = \frac{\bar{C}_{abcd} \bar{C}^{abcd}}{\left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^4 12 \left( \bar{T}_a \bar{T}^a \right)^2 \left( \bar{L}^2 - \bar{L} + 1 \right) - 2 \left( \frac{\bar{\Omega}'}{\bar{\Omega}} \right)^3 \left[ \cdots \right] + \cdots + \bar{R}_{ef} \bar{R}^{ef}}
\]

\( \bar{R}_{ef} \bar{R}^{ef} \), as well as \( \bar{C}_{abcd} \bar{C}^{abcd} \), and the derivatives of \( \bar{T} \) will be well-behaved at \( \bar{T} = 0 \), because we have required that the conformal metric be non-degenerate and

\(^\dag\)This expression can be obtained from equation (A.49) in Appendix A.8.

\(^\dagger\)see Appendix A.8.
10.1. Weyl versus Ricci curvature

Additionally, since the cosmic time function has a nowhere vanishing timelike derivative, we have $\dot{T}_a T^{a} \neq 0$ on an open neighbourhood of $\dot{T} = 0$.

In Lemmas 9.4 and 9.5 and Remark 9.6 we have seen that under the imposed conditions $\lim_{T \to 0^\pm} (\Omega')/ \Omega = \pm \infty$ (the sign depending on $\dot{T}$ and $\Omega(0)$). Asymptotically, the sign of $\Omega'/ \Omega$ will not matter, since the dominant expression in equation (10.3) is an even power. Then, as $\dot{T} \to 0^\pm$, we have two possible cases for $\lambda = \lim_{T \to 0^\pm} \ddot{L}$.

1. $\ddot{\lambda}$ finite or zero. Clearly,

$$K \approx \frac{\bar{C}_{abcd} \bar{C}^{abcd}}{\left(\frac{\alpha'}{\Omega} \right)^4 \left(12 (\bar{T}^{a} \bar{T}^{a})^2 \left(\ddot{L}^2 - \dot{L} + 1\right)\right)}.$$  \hspace{1cm} (10.7)

Since $L^2 - \dot{L} + 1 > 0 \ \forall \ L \in \mathbb{R}$, we immediately have $\lim_{T \to 0^\pm} K = 0$.

2. $\ddot{\lambda} = \pm \infty$. Then

$$K \approx \frac{\bar{C}_{abcd} \bar{C}^{abcd}}{\left(\frac{\alpha'}{\Omega} \right)^4 12 \left(\dot{T}^{a} \dot{T}^{a}\right)^2 \dot{L}^2},$$  \hspace{1cm} (10.8)

and consequently $\lim_{T \to 0^\pm} K = 0$.

The K-theorem clearly excludes regular conformal metrics for the definition of an initial or final state where $K \neq 0$.

The regular conformal structure of an IFS or a FIU could still provide an increasing Weyl curvature as $\dot{T} \to 0^-$, but one which would remain bounded by the Ricci curvature. In that case, we would have to specifically analyse a ratio, such as

$$\frac{C_{abcd} C^{abcd}|_{IFS}}{C_{abcd} C^{abcd}|_{IPS}},$$  \hspace{1cm} (10.9)

on a case by case basis, in order to determine whether the Weyl curvature scalar has evolved with cosmic evolution. It is, however, arduous to state something in general about such a ratio.

Theorem 10.1 therefore yields the important conclusion, that if we are stricter and require $K \neq 0$ as $\dot{T} \to 0^-$ from the definition of a final state, in accordance with chaotic cosmology and the WCH, we can only employ conformal metrics which become non-regular (e.g. degenerate) at $\dot{T} = 0$, if we are to maintain a conformal structure. The following observation lends weight to Definitions 8.7 and 8.10.

**Remark 10.2**

The K-theorem indicates that Goode and Wainwright did not have a wide choice in the conformal structure of the definition of an IPS in order to make it compatible with the WCH; if one attempts to follow the ideas of the WCH with a conformal structure, then clearly the conformal structure for the initial state must possess a regular conformal metric and the one for the future must contain a non-regular (e.g. degenerate) unphysical metric.
Alternatively, one could try to investigate a more general ansatz for the conformal factor $\Omega$, such as $\tilde{\Omega}(\tilde{T}, S)$ or $\tilde{\Omega}(N, \tilde{N})$, where $S$ is a cosmological scale function which has a spacelike gradient everywhere and where $N$ and $\tilde{N}$ are functions with nowhere vanishing null gradient, in order to obtain a non-zero $K$ at a cosmological future. The choice of the pairs $S, \tilde{T}$ and $N, \tilde{N}$ guarantees the cancellation of some cross terms in the calculation of the Ricci curvature scalar due to orthogonality. Such constructions by the author have indicated the possibility of satisfying the requirements of a non-zero $K$, however, proved to be too difficult for the use in a definition which admits calculations by hand. More research could be done on this, nevertheless, we will only focus on the simpler case of $\tilde{\Omega}(\tilde{T})$ in this thesis.

10.2 The Ricci and Weyl curvature invariants

The physical Ricci and Weyl curvature scalars of conformal structures with regular conformal metrics behave quite differently in the cases $\tilde{\Omega}(0) = 1$ and $\tilde{\Omega}(0) = 0$. In the latter case we (obviously) find the same results as for an IPS. The proof of Theorem 10.1 provides the key to the following results.

**Theorem 10.3**

Suppose the conditions of Theorem 10.1, for the case $\tilde{\Omega}(0) = 0$, are true. Then, $\lim_{T \to 0^\pm} R_{ab} R^{ab} = +\infty$.

**Proof.** The proof is evident by the regularity in $(\tilde{M}, \tilde{g})$, the expression given in equation (10.3) and $\lim_{T \to 0^\pm} (\tilde{\Omega})' \tilde{\Omega} = \pm \infty$. \(\square\)

**Theorem 10.4**

Suppose the conditions of Theorem 10.1 hold for the case $\tilde{\Omega}(0) = 0$. Then, $\lim_{T \to 0^\pm} C_{abcd} C^{abcd} = +\infty$, unless $\lim_{T \to 0^\pm} C_{abcd} C^{abcd} = 0$.

**Proof.** The assertion is obvious by the regularity in $(\tilde{M}, \tilde{g})$, equation (10.5) and $\lim_{T \to 0^\pm} \tilde{\Omega}(\tilde{T}) = 0$. \(\square\)

The situation for the case $\tilde{\Omega}(0) = \infty$ is somewhat more complicated and involves the functions $\tilde{M}$ and $\tilde{N}$ defined in chapter 9.

**Theorem 10.5**

Let the conditions of Theorem 10.1 hold for the case $\tilde{\Omega}(0) = \infty$. Furthermore, let $\lambda \neq 1, 2, +\infty$. Then, $\lim_{T \to 0^\pm} R_{ab} R^{ab} = \Gamma$, where $\Gamma = 0$, $0 < \Gamma < \infty$ or $\Gamma = \infty$ if $\tilde{M}_0 = 0$, $0 < |\tilde{M}_0| < \infty$ or $|\tilde{M}_0| = \infty$, respectively ($\tilde{M}_0 \equiv \lim_{T \to 0^\pm} \tilde{M}$).

**Proof.** By equation (10.3) in the proof of Theorem 10.1, we find the following asymp-
totic relation under the required circumstances

$$R_{ab} R^{ab} \approx \Omega^{-4} \left(\frac{\tilde{\Omega}}{\Omega}\right)^4 12 (\bar{T}_a \bar{T}^a)^2 (\bar{L}^2 - \bar{L} + 1)$$

(10.10)

$$= N^4 12 (\bar{T}_a \bar{T}^a)^2 (\bar{L}^2 - \bar{L} + 1).$$

(10.11)

Since $\bar{L}^2 - \bar{L} + 1 > 0 \forall \bar{L} \in \mathbb{R}$ and $\bar{T}_a \bar{T}^a \neq 0$, the relation only depends on the possible value of $\bar{N}$. In Lemma 9.7 and Remark 9.9 we have proven the existence and the possible values of $M_0$, in accordance with the assertion.

**Remark 10.6**

If $\lambda = 2$ and $\bar{L}$ is strictly monotonic and continuous on an interval $[-c, 0)$, $c > 0$, and if otherwise the conditions of Theorem 10.5 are true, then, by Lemma 9.8, we find the same result as in Theorem 10.5.

**Theorem 10.7**

Let the conditions of Theorem 10.1 hold for the case $\bar{\Omega}(0) = \infty$. Furthermore, let $\lambda = +\infty$. Then, $\lim_{\bar{T} \to 0^\pm} R_{ab} R^{ab} = \Gamma'$, where $\Gamma' = 0$, $0 < \Gamma' < \infty$ or $\Gamma' = \infty$ if $\bar{N}_0 = 0$, $0 < \bar{N}_0 < \infty$ or $\bar{N}_0 = \infty$, respectively ($\bar{N}_0 \equiv \lim_{\bar{T} \to 0^\pm} \bar{N}$).

**Proof.** Under these conditions, equation (10.3) implies the following asymptotic relation

$$R_{ab} R^{ab} \approx \bar{\Omega}^{-4} \left(\frac{\tilde{\Omega}}{\Omega}\right)^4 \bar{L}^2 12 (\bar{T}_a \bar{T}^a)^2$$

(10.12)

$$= \bar{N} 12 (\bar{T}_a \bar{T}^a)^2,$$

(10.13)

which merely depends on $\bar{N}$ since $\bar{T}_a \bar{T}^a \neq 0$. Lemma 9.11 and Remark 9.13 proved the existence of $\bar{N}_0$ and gave the range of its possible values, in accordance with the assertion.

The case of the Weyl curvature scalar is, indeed, much simpler for $\bar{\Omega}(0) = \infty$.

**Theorem 10.8**

Let the conditions of Theorem 10.1 be valid for the case $\bar{\Omega}(0) = \infty$. Then, $\lim_{\bar{T} \to 0^\pm} C_{abcd} C^{abcd} = 0$.

**Proof.** Equation (10.5) implies the assertion, since $\bar{C}_{abcd} \bar{C}^{abcd}$ is well-behaved on an open neighbourhood of $\bar{T} = 0$ and $\bar{\Omega}(0) = \infty$.

**10.3 Kinematics**

In the K-theorem we have already seen that, under physically reasonable conditions on the conformal factor, we inevitably encounter an asymptotic Weyl isotropy.
at $T = 0$ if the conformal metric is regular, irrespective of whether $T$ approaches 0 from below or above, or whether $\tilde{\Omega}(0) = \infty$ or $\tilde{\Omega}(0) = 0$. As one would expect, we also find an asymptotic kinematic isotropy under the same conditions, similar to the situation at an IPS.

**Theorem 10.9 (Asymptotic kinematic isotropy)**

Let the conditions of Theorem 10.1 be valid and let Definition 8.2 (4.6, respectively) be satisfied. Then, for a timelike congruence $\mathbf{u}$, which is orthogonal to the slice $T = 0^1$, we find that

$$\lim_{T \to 0^\pm} \frac{\sigma^2}{\theta^2} = 0, \quad \lim_{T \to 0^\pm} \frac{\omega^2}{\theta^2} = 0, \quad \lim_{T \to 0^\pm} \frac{\dot{u}^a \dot{u}_a}{\theta^2} = 0. \quad (10.14)$$

**Proof.** Employ the well known conformal relations for the kinematic quantities$^4$

$$\theta = \tilde{\Omega}^{-1} \left[ 3 \frac{\tilde{\Omega}'}{\tilde{\Omega}} T_a \dot{u}^a + \tilde{\theta} \right], \quad (10.15)$$

$$\sigma^2 = \tilde{\Omega}^{-2} \dot{\sigma}^2, \quad \omega^2 = \tilde{\Omega}^2 \omega^2, \quad (10.16)$$

$$\dot{u}^a = \dot{\tilde{u}}^a + \tilde{h}^{ab} \frac{\tilde{\Omega}'}{\tilde{\Omega}} \tilde{T}_b. \quad (10.17)$$

Realising that under the imposed conditions $\lim_{T \to 0^\pm} (\exists) \frac{\tilde{\Omega}'}{\tilde{\Omega}} = \pm \infty$ (the sign depending on $T$ and $\tilde{\Omega}(0)$) and thus that the dominant term of $\dot{u}^a \dot{u}_a$ is given by

$$\dot{u}^a \dot{u}_a \approx \left( \frac{\tilde{\Omega}'}{\tilde{\Omega}} \right)^2 \tilde{h}^{bc} \tilde{T}_b \tilde{T}_c, \quad \text{as} \quad T \to 0^\pm, \quad (10.18)$$

yields the following expressions, as $T \to 0^\pm$,

$$\frac{\sigma^2}{\theta^2} = \frac{\dot{\sigma}^2}{3 \dot{\tilde{\Omega}} \tilde{T}_a \dot{u}^a + \dot{\tilde{\theta}}^2}, \quad \frac{\omega^2}{\theta^2} = \frac{\dot{\omega}^2}{3 \dot{\tilde{\Omega}} \tilde{T}_a \dot{u}^a + \dot{\tilde{\theta}}^2}, \quad \frac{\dot{u}^a \dot{u}_a}{\theta^2} \approx \frac{\tilde{h}^{bc} \tilde{T}_b \tilde{T}_c}{9 (T_a \dot{u}^a)^2}. \quad (10.19)$$

The relative speed of observers in the congruence $\tilde{\mathbf{u}}$ and observers in the unphysical normal congruence at $T = 0$ was defined by Goode and Wainwright [8] to be

$$v^2 = \left. \frac{\tilde{h}^{bc} \tilde{T}_b \tilde{T}_c}{(T_a \dot{u}^a)^2} \right|_{T=0}. \quad (10.20)$$

Since we have required that $\mathbf{u}$ (and therefore $\tilde{\mathbf{u}}$) be orthogonal to the slice $T = 0$, i.e. $v = 0$, we find the assertion by the regularity in the unphysical space-time and the existence and the value of the limit of $\frac{\tilde{\Omega}'}{\tilde{\Omega}}$. $\square$

---

$^1$The congruence normal to $T = 0$ always exists in a space-time which admits an IPS (see section 4.3.2 and [8, p 106]). By analogy, this is also true for IFS and FIU cosmologies.

$^4$see Appendix A.8.2.
The proof provides us with other interesting results.

**Theorem 10.10**

Let the conditions of Theorem 10.1 be valid for the case $\Omega(0) = 0$. Then, $\lim_{T \to 0^\pm} \theta^2 = \lim_{T \to 0^\pm} \vartheta^a \vartheta_a = \infty$, and $\lim_{T \to 0^\pm} \sigma^2 = \lim_{T \to 0^\pm} \omega^2 = \infty$ (unless $\lim_{T \to 0^\pm} \sigma^2 = \lim_{T \to 0^\pm} \omega^2 = 0$).

*Proof.* The expressions given in equations (10.15), (10.16) and (10.18), the regularity in $(\mathcal{M}, \bar{g})$, and the existence and the value of the limit of $\frac{\Omega}{\Omega}$ immediately imply the assertion. □

**Theorem 10.11**

Let the conditions of Theorem 10.1 hold for the case $\Omega(0) = 1$. Then, $\lim_{T \to 0^\pm} \omega^2 = 0$ and $\lim_{T \to 0^\pm} \vartheta^a \vartheta_a = \infty$.

*Proof.* By equations (10.16) and (10.18), the regularity in $(\mathcal{M}, \bar{g})$, and the existence and the value of the limit of $\frac{\Omega}{\Omega}$, we find the assertion. □

**Theorem 10.12**

Let the conditions of Theorem 10.1 be valid for $\overline{T} \to 0^-$ and $\Omega(0) = \infty$. Then, if $\lambda \neq 2$, $\lim_{T \to 0^-} \theta = \vartheta$, where $\vartheta = 0$, $0 < \vartheta < \infty$ or $\vartheta = \infty$ if $\bar{M}_0 = 0$, $0 < \bar{M}_0 < \infty$ or $\bar{M}_0 = \infty$, respectively ($\bar{M}_0 \equiv \lim_{T \to 0^-} \bar{M}$).

*Proof.* Equation (10.15) implies the following asymptotic behaviour

$$\theta \approx 3\bar{M} \bar{T}_a \bar{u}^a,$$

which solely depends on $\bar{M}$ since $\bar{T}_a \bar{u}^a$ is regular on an open neighbourhood of $\overline{T} = 0$. In Lemma 9.7 we have proven the existence and the range of possible values of $\bar{M}_0$, in agreement with the assertion. □

**Remark 10.13**

Remark 9.9 implies that Theorem 10.12 is still valid in the case $T \to 0^+$ and $\Omega(0) = \infty$ if the signs of $\vartheta$ and $M$ are reversed.

### 10.4 Discussion

The above theorems clearly show that, similar to the situation at an IPS, we unavoidably obtain an asymptotic Weyl and kinematic isotropy at $\overline{T} = 0^\dagger$, if we employ conformal relations with regular conformal metrics and reasonable constraints on

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$^\dagger$Analogously one could show that an asymptotic Ricci isotropy (see section 4.3.2) also holds, but since the proof proceeds similarly and we are interested in keeping this discussion brief we abstain from presenting it.
the conformal factor, irrespective of whether the cosmic time function approaches 0 from below or above, or whether $\Omega$ diverges or vanishes at $T = 0$.

At first sight, it seems, as if the cases $\Omega(0) = \infty$ and $\Omega(0) = 0$ do not provide any major different characteristics. In Theorems 10.5, 10.7, 10.11 and 10.12 treating the case $\Omega(0) = \infty$, we have seen, however, that the curvature scalars, as well as the expansion scalar, the shear and vorticity follow an asymptotic behaviour which can be quite different from the behaviour in the case $\Omega(0) = 0$ - if one does not look at their ratios. The behaviour of some of these quantities in the case $\Omega(0) = \infty$ depends on the functions $M$ and $N$, which can vanish, and does therefore not necessarily lead to conditions which would be realistic at a cosmological singularity. In fact, for $\Omega(0) = \infty$, one always finds a vanishing Weyl invariant at $T = 0$. On the other hand, sections 10.2 and 10.3 have shown that the case $\Omega(0) = 0$ necessarily leads to a behaviour which is essentially equivalent to the behaviour at an IPS.

Since $\Omega(0) = \infty$ does not necessarily lead to a behaviour which is compatible with a singularity, while $\Omega(0) = 0$ does imply such a behaviour in any case, the above theorems furthermore support the choice of $\Omega(0) = 0$, instead of $\Omega(0) = \infty$, by Goode and Wainwright in their definition of an IPS.

By the different implications of the cases $\Omega(0) = \infty$ and $\Omega(0) = 0$, and the asymptotic isotropy, this chapter provides justification for terming the singularities given in Definition 8.1 “isotropic future singularities” and the cosmologies determined by Definition 8.3 “future isotropic universes”.

In conclusion, the conformal structures presented in this chapter do not offer any physically reasonable scenarios for the formulation of future behaviours, since they do not admit the anisotropy we are looking for, motivated by quiescent cosmology and the WCH. Thus, conformal structures with regular conformal metrics are only applicable for the initial state of the universe, but not for its future evolution\(^\dagger\) and, consequently, we will focus on the case of non-regular - and especially degenerate - conformal metrics in the remainder of this thesis.

\(\dagger\)Unless one believes in chaotic cosmology.
Chapter 11

Physical attributes of conformal structures with degenerate conformal metrics

Now that we have concluded that regular, non-degenerate conformal metrics are physically not appropriate for the use in conformal structures which describe a future behaviour compatible with quiescent cosmology and the WCH, we will investigate the case of a conformal metric which becomes degenerate as $T \to 0^-$. This degeneracy will cause many difficulties in the determination of the behaviour of many unphysical (and consequently physical) quantities, since, as we will see it corresponds to an unphysical conformal singularity. However, a few things can be said about some quantities. The derivation of the results will now be more difficult and so proceed quite differently from the proofs in the framework of regular conformal metrics.

In section 11.1.1 we will investigate the unphysical expansion scalar and prove that it necessarily diverges if the conformal structure involves a degenerate metric. Section 11.1.2 provides a number of results concerning the physical expansion scalar of conformal structures with degenerate conformal metrics in agreement with the results found in chapter 7. Thereupon we will turn our attention in section 11.2 to the difficult topic of curvature in these conformal relations. It will be proven that under certain conditions the unphysical space-times of Definitions 8.7 and 8.10 possess a future Tipler strong curvature singularity (TSCS). Furthermore, we will give sufficient conditions for the AFS to be a TSCS. As a by-product we will show in section 11.2.3 that the IPS is a TSCS.

11.1 The expansion scalar

It is possible to determine the behaviour of the expansion scalar $\theta$ by using the determinant of the metric. In an observational context it is quite useful to have some information about $\theta$, since the expansion scalar is a multiple of the Hubble parameter\footnote{see section 5.2.} $H$ which is often used for comparisons of observations with cosmological
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We begin by examining the unphysical expansion.

11.1.1 The unphysical expansion scalar

We will now prove that the unphysical expansion scalar \( \bar{\theta} \) necessarily diverges to \(-\infty\) under the conditions of degeneracy, thereby giving the theoretical background for the behaviour seen in chapter 7.

**Theorem 11.1 (Unphysical expansion)**

Let \( \bar{T} \) be a cosmic time function which approaches 0 from below and let \( \bar{g} \) be the conformal metric of some conformal relation \( g = \bar{\Omega}^2 \bar{g} \) which is \( C^2 \) on some \((c,0)\) and \( C^0 \) on \((-c,0] \), \( c > 0 \). If \( \bar{g} \) becomes degenerate at \( \bar{T} = 0 \) and the unphysical fluid flow \( \bar{u} \) is non-vanishing and at least \( C^2 \) on \((-c,0] \), then \( \liminf_{T \to 0^-} \bar{\theta} = -\infty \).

**Proof.** The expansion scalar may be written as

\[
\bar{\theta} = \bar{u}^a \cdot \bar{a} = \frac{1}{\sqrt{|\bar{g}|}} \partial_a \left( \sqrt{|\bar{g}|} \bar{u}^a \right) = \partial_a \bar{u}^a + \bar{u}^a \partial_a \ln \sqrt{|\bar{g}|}. \tag{11.1}
\]

Lemma 9.18 implies that under the imposed conditions \( \liminf_{T \to 0^-} \bar{u}^a \partial_a \ln \sqrt{|\bar{g}|} = -\infty \). Asymptotically, the first term on the R.H.S. does not matter since \( \bar{u} \) is at least \( C^2 \) on \((-c,0] \). Thence, \( \liminf_{T \to 0^-} \bar{\theta} = -\infty \). \( \square \)

11.1.2 The physical expansion scalar

The situation of the physical expansion is more complicated and we cannot give a general answer for all possible cases. Nevertheless, the following results cover a wide class of cases and thereby give an explanation for the behaviour of the physical expansion scalar found in chapter 7.

The investigations often treat constraints on the product function \( \bar{\Omega}^3 \sqrt{|\bar{g}|} \). Recall that \( \sqrt{|g|} = \bar{\Omega}^4 \sqrt{|\bar{g}|} \), where \( g \) is the determinant of the physical metric. Since we deal with \( \bar{\Omega}(0) = \infty \), \( \lim_{T \to 0^-} \bar{\Omega}^3 \sqrt{|\bar{g}|} > 0 \) implies \( \lim_{T \to 0^-} |\bar{g}| = \infty \) and \( \lim_{T \to 0^-} |\bar{g}| < \infty \) implies \( \lim_{T \to 0^-} \bar{\Omega}^3 \sqrt{|\bar{g}|} = 0 \).

We analyse five cases.

**Theorem 11.2 (AFEU expansion)**

Let \( T \) be a cosmic time function which approaches 0 from below and let \( g = \bar{\Omega}^2 \bar{g} \) be some conformal relation with \( \bar{\Omega}(0) = \infty \). Let \( \bar{g} \) and \( \bar{\Omega} \) furthermore be at least \( C^2 \) on \((-c,0) \), \( c > 0 \), and \( \bar{\Omega} \) be \( C^0 \) on \((-c,0] \). The unphysical fluid flow \( \bar{u} \) shall be non-vanishing and at least \( C^2 \) on \((-c,0] \). If \( \bar{g} \) becomes degenerate as \( \bar{T} \to 0^+ \) and \( \lim_{T \to 0^-} \bar{\Omega}^3 \sqrt{|\bar{g}|} = +\infty \), then \( \liminf_{T \to 0^-} \bar{\theta} \geq 0 \).
11.1. The expansion scalar

Proof. The expansion scalar is given by (recall \( u^a = \bar{\Omega}^{-1} \bar{u}^a \))

\[
\theta = u^a_{;a} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^a} \left( \sqrt{|g|} u^a \right) = \frac{1}{\Omega^4 \sqrt{|g|}} \frac{\partial}{\partial x^a} \left( \Omega^3 \sqrt{|g|} \bar{u}^a \right). \tag{11.2}
\]

Since \( \bar{u} \) is \( C^2 \) on \((-c, 0]\), we may choose comoving coordinates. This forms equation (11.2) into (recall \( \partial_0 = \frac{dt}{dt} \partial_T \))

\[
\theta = \frac{1}{\Omega^4 \sqrt{|g|}} \frac{d \bar{T}}{dt} \partial_T \left( \Omega^3 \sqrt{|g|} \bar{u}^0 \right). \tag{11.3}
\]

\( \bar{T} \) increases along the flow lines of \( \bar{u} \), thus \( \frac{d \bar{T}}{dt} \) will never be negative. \( \bar{u}^0 \) is non-zero and \( C^2 \), and thus the term in the parentheses will increase to \( \infty \) as \( \bar{T} \to 0^- \). Thereby it provides a non-negative and (since \( \Omega^3 \sqrt{|g|} \bar{u}^0 \) is at least \( C^1 \) on \((-c, 0]) \) continuous derivative on \((-c, 0)\). The expression in the denominator is furthermore always positive, hence the limit inferior of equation (11.3) exists and \( \liminf_{\bar{T} \to 0^-} \theta \geq 0. \)

The cosmologies of Definition 8.7 satisfy the conditions of Theorem 11.2 and therefore have \( \liminf_{\bar{T} \to 0^-} \theta \geq 0 \), which supports the name "future endless universe".

Theorem 11.3

Let \( \bar{T} \) be a cosmic time function which approaches 0 from below and let \( \bar{g} = \bar{\Omega}^2 \bar{g} \) be some conformal relation with \( \bar{\Omega}(0) = \infty \). The unphysical fluid flow \( \bar{u} \) shall be non-vanishing and at least \( C^2 \) on \((-c, 0]\), \( c > 0 \). If \( \bar{g} \) becomes degenerate as \( \bar{T} \to 0^- \), \( \Omega^3 \sqrt{|g|} \) is \( C^1 \) on \((-c, 0]\) and \( 0 < \lim_{\bar{T} \to 0^-} \Omega^3 \sqrt{|g|} < \infty \), then \( \lim_{\bar{T} \to 0^-} \theta = 0 \).

Proof. Since \( \Omega^3 \sqrt{|g|} \) is \( C^1 \) and \( \bar{u} \) is non-vanishing and at least \( C^2 \) on \((-c, 0]\) the derivative in equation (11.2) will remain finite. However, \( \lim_{\bar{T} \to 0^-} \Omega^4 \sqrt{|g|} = \infty. \)

Theorem 11.4

Let \( \bar{T} \) be a cosmic time function which approaches 0 from below and let \( \bar{g} = \bar{\Omega}^2 \bar{g} \) be some conformal relation with \( \bar{\Omega}(0) = \infty \). Let \( \bar{g} \) and \( \bar{\Omega} \) furthermore be at least \( C^2 \) on \((-c, 0] \), \( c > 0 \), and \( \bar{\Omega} \) be \( C^0 \) on \((-c, 0] \). The unphysical fluid flow \( \bar{u} \) shall be non-vanishing and at least \( C^2 \) on \((-c, 0]\). If \( \bar{g} \) becomes degenerate as \( \bar{T} \to 0^- \) and \( \lim_{\bar{T} \to 0^-} \Omega^3 \sqrt{|g|} = 0 \), then \( \liminf_{\bar{T} \to 0^-} \theta \leq 0 \).

Proof. The expression in the parentheses of equation (11.3) approaches zero from above, since \( \bar{u} \) is \( C^2 \) and \( \lim_{\bar{T} \to 0^-} \Omega^3 \sqrt{|g|} = 0 \). Thus, the derivative is negative and continuous on \((-c, 0) \) (since \( \Omega^3 \sqrt{|g|} \bar{u}^0 \) is at least \( C^1 \) on \((-c, 0)) \). By similar arguments as in the proof of Theorem 11.2 we find the assertion.

We will now investigate three special cases of \( \lim_{\bar{T} \to 0^-} \Omega^3 \sqrt{|g|} = 0 \) in more detail. The first requires that \( \Omega^3 \sqrt{|g|} \) be \( C^1 \) on the half closed interval \((-c, 0]\) and that \( \lim_{\bar{T} \to 0^-} \Omega^4 \sqrt{|g|} = \infty \).
Theorem 11.5
Let $\tilde{\Omega} \sqrt{|g|}$ be $C^1$ on $(-c,0]$, $\lim_{T \to 0^-} \tilde{\Omega} \sqrt{|g|} = \infty$ and let otherwise the conditions of Theorem 11.4 be valid, then $\lim_{T \to 0^-} \theta = 0$.

Proof. The derivative in equation (11.2) remains finite even at $T = 0$ under the imposed conditions. But $\lim_{T \to 0^-} \tilde{\Omega} \sqrt{|g|} = 1$. Unlike in some of the theorems above, the case of $\lim_{T \to 0^-} \tilde{\Omega} \sqrt{|g|} = 0$ leads to a unique result which is compatible with a big crunch type singularity, such as in the Kantowski-Sachs models.

Theorem 11.6 (AFS expansion)
Let $\lim_{T \to 0^-} \tilde{\Omega} \sqrt{|g|} = 0$. If furthermore the conditions of Theorem 11.4 are satisfied, then $\lim_{T \to 0^-} \inf \theta = -\infty$.

Proof. Use

$$\theta = u^a \cdot \frac{\partial}{\partial x^a} (\sqrt{|g|} u^a). \quad (11.4)$$

Choose comoving coordinates, i.e. $u^a = \frac{1}{\sqrt{|g|}} \delta^a_0$ and set

$$f(t,\vec{x}) = \sqrt{|g|} u^0, \quad (11.5)$$

where $t$ is the comoving coordinate time and $t \to t_s \leftrightarrow \tilde{T} \to 0^-$. Then, since $u^0 > 0$ and regular and $u^a = \Omega^{-1} \bar{u}^a \to 0$, we find $u^0 > 0$ for $\tilde{T} < 0$ and $\lim_{t \to t_s} u^0 = 0$, where $u^0 = \frac{f}{\sqrt{|g|}}$. Now $\tilde{\Omega} \sqrt{|g|} = \sqrt{|g|} \to 0^+$ and $u^0 \to 0^+$, thus the derivatives are negative and

$$\partial_t f = \sqrt{|g|} \partial_t u^0 + u^0 \partial_t \sqrt{|g|} \leq 0. \quad (11.6)$$

Therefore we find $\partial_t f \leq u^0 \partial_t \sqrt{|g|} \leq 0$ on some $(-\epsilon,0]$, $0 < \epsilon \ll 1$, and consequently in the limit (both $f$ and $\sqrt{|g|}$ are $C^1$ on $(-c,0)$, therefore both possess a limit inferior)

$$\lim_{t \to t_s} \inf \partial_t f < \lim_{t \to t_s} \inf \partial_t \sqrt{|g|} \leq 0. \quad (11.7)$$

By the proof of Lemma 9.18 the imposed conditions imply

$$\lim_{t \to t_s} \inf \partial_t \ln \sqrt{|g|} = \lim_{t \to t_s} \inf \frac{\partial_t \sqrt{|g|}}{\sqrt{|g|}} = -\infty, \quad (11.8)$$

which in conjunction with equation (11.7) implies

$$\lim_{t \to t_s} \inf \frac{\partial_t f}{\sqrt{|g|}} = \lim_{t \to t_s} \inf \theta = -\infty. \quad (11.9)$$
11.2. Degenerate metrics and strong curvature

Remark 11.7
Theorem 11.6 emphasises Definition 8.10 since it implies that the fluid flow in a space-time which admits an AFS contracts to a singularity in the fluid congruence as $\bar{T} \to 0^-$, i.e. all observers moving with the fluid flow will experience infinite crushing forces “at” the AFS. This, however, does not necessarily imply a diverging curvature.

We have covered the entire range of possible limiting values of the determinant of the physical metric $g$ in the above theorems, but have left out some special cases of the differentiability of $g$ which are very difficult to calculate.

Example 11.8
Among others, the special cases 1. $0 < \lim_{\bar{T} \to 0^-} \sqrt[3]{|g|} < \infty$ and $\Omega^3 \sqrt{|g|}$ not $C^1$ at $\bar{T} = 0$ or 2. $0 < \lim_{\bar{T} \to 0^-} \Omega^4 \sqrt{|g|} < \infty$ and $\Omega^4 \sqrt{|g|}$ not $C^1$ at $\bar{T} = 0$ are difficult to investigate, since the behaviour of the derivatives of these functions at $\bar{T} = 0$ is not obvious.

Nonetheless, the combined results - particularly Theorems 11.2 and 11.6 - determine the behaviour of $\theta$ in the AFEU and AFS cosmologies and cover a wide class of possible cases.

11.2 Degenerate metrics and strong curvature

Approaching the topic of curvature in conformal structures with degenerate metrics is significantly more difficult than in the case of regular conformal metrics seen in the previous chapter. To illustrate these difficulties we briefly discuss the following example of the curvature invariants.

11.2.1 Difficulties with the curvature invariants

From equation (4.1) we easily calculate for dimension $n = 4$

$$C_{abcd}C^{abcd} = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \frac{1}{3}R^2.$$ (11.10)

Via Riemannian normal coordinates one can readily derive expressions for the invariants on the R.H.S., e.g. of which the Riemann invariant is given by

$$R_{abcd}R^{abcd} = g_{cb,da}(g^{cb,da} + g^{da,cb} - g^{db,ac} - g^{ca,bd}).$$ (11.11)

Thus, since the conformal metric is degenerate and only $C^0$ at $\bar{T} = 0$, we can clearly see that if we apply equation (11.11) to the conformal space-time the expression will involve singular terms. However, giving a general answer as to whether the invariant diverges is not possible using the expression in equation (11.11) since the sum will in general contain both negatively and positively diverging terms. The same difficulty
applies to the remaining invariants in equation (11.10) and consequently by the
expressions above we cannot deduce more than that the invariants contain diverging
terms and that in many cases we would expect a curvature singularity to occur at
least in the unphysical space-time.

Clearly, we have to tackle the curvature problem differently by using other tech-
niques. As we would expect all physical objects to be destroyed at a cosmological
singularity, it makes sense to investigate the notion of strong curvature, defined in
section 2.2, in the framework of the definitions given in chapter 8.

11.2.2 Strong curvature singularities and degenerate met-
rics

By Definition 1.1 we already know that a region where the metric is degenerate
cannot be part of the space-time itself. We would not be able to implement a causal
structure via light cones on this region of space-time and furthermore since in any
locally oriented coordinate system the 4-volume form $dM^4$ of the space-time is given
by (see [19, 21, 29])

$$dM^4 = \sqrt{|g(y)|} dy^1 \wedge \cdots \wedge dy^m$$

we can see by Lemma 9.15 that this volume form would vanish where the metric
is degenerate$^1$. In other words, any 4-volume integration over a set of points with
degenerate metric would go to zero, i.e. the universe crushes down to zero volume
over such a set of points, compatible with a cosmological singularity. In this sense
every such point with degenerate metric must be a singularity. This observation
strongly supports the definition of an AFS.

From this it is not immediately clear, however, that this also implies the presence
of a strong curvature singularity. It only shows that all 4-volumes must vanish where
the metric becomes degenerate, but it does not mean that a 3 (or 2)-volume, as is
needed for the strong curvature singularity definition, necessarily goes to zero there
as well (e.g. one could think of a situation similar to that of a sheet of paper, where
the 3-volume is zero, but the 2-volume is finite).

Recall that the three linearly independent spacelike Jacobi fields $\{Z^1, Z^2, Z^3\}$
along a timelike geodesic $\gamma$ with affine parameter $s$ form a spacelike 3-surface $S \subset M$
orthogonal to $\gamma(s)$ for each $s$. Its 3-volume element is given by $V(s) = Z_1 \wedge Z_2 \wedge Z_3$
(where the $Z_{\alpha}$ are the dual 1-forms to the $Z^\alpha$, $\alpha = 1, 2, 3$). For arbitrary, linearly
independent vectors $v^1, v^2, v^3 \in T_{\gamma(s)}M$ which lie in $S$ (e.g. some coordinate vectors)
this takes the form

$$V(s)(v^1, v^2, v^3) = \det \begin{pmatrix}
Z_1(v^1) & Z_2(v^1) & Z_3(v^1) \\
Z_1(v^2) & Z_2(v^2) & Z_3(v^2) \\
Z_1(v^3) & Z_2(v^3) & Z_3(v^3)
\end{pmatrix},$$

$$= \det \begin{pmatrix}
g(Z^1, v^1) & g(Z^2, v^1) & g(Z^3, v^1) \\
g(Z^1, v^2) & g(Z^2, v^2) & g(Z^3, v^2) \\
g(Z^1, v^3) & g(Z^2, v^3) & g(Z^3, v^3)
\end{pmatrix}. \quad (11.14)$$

$^1$Locally oriented coordinate systems can exist metric independently.
Before we present sufficient conditions for a Tipler strong curvature singularity (TSCS) - and thereby a Królik strong curvature singularity (since it is weaker, see Definitions 2.3 and 2.4) - to occur in both the physical and conformal space-time, it is necessary to consider the following lemma.

**Lemma 11.9**

Let \( p \in \mathcal{M} \) and \( W \in T_p \mathcal{M} \), such that \( W \neq 0 \) and \( g(W, \cdot) = 0 \). Let furthermore \( x^i (i = 0, \cdots, 3) \) be a basis in \( T_p \mathcal{M} \), such that \( W = \sum_i a_i x^i \). Then for those \( a_i \) which satisfy \( a_i \neq 0 \) we find \( g(x^i, \cdot) = 0 \ (i = 0, \cdots, 3) \).

**Proof.** Not all \( a_i \) are zero since \( W \neq 0 \). Suppose that there exists at least one \( j \) so that \( a_j \neq 0 \) and \( g(x^j, \cdot) \neq 0 \). We will show that under this assumption we can construct \( \tilde{W} = \sum_i b_i x^i \in T_p \mathcal{M} \) so that \( g(W, \tilde{W}) \neq 0 \), which is a contradiction.

For all \( i \neq j \) let the \( b_i \) be determined via

\[
g(x^i, \tilde{W}) = \tilde{a}_i = \begin{cases} 0 & \text{if } g(x^i, \cdot) = 0, \\ a_i & \text{if } g(x^i, \cdot) \neq 0, \end{cases}
\]

and finally, for \( i = j \) let

\[
g(x^j, \tilde{W}) = \tilde{a}_j = \frac{-\sum_{i \neq j} a_i \tilde{a}_i + c}{a_j},
\]

where \( c \in \mathbb{R} \). Since \( g(x^i, \tilde{W}) = \sum_k b_k g(x^i, x^k) \) we need to solve the equation

\[
\begin{pmatrix}
g(x^0, x^0) & \cdots & g(x^3, x^0) \\
\vdots & \ddots & \vdots \\
g(x^0, x^3) & \cdots & g(x^3, x^3)
\end{pmatrix}
\begin{pmatrix}
b_0 \\
\vdots \\
b_3
\end{pmatrix}
= \begin{pmatrix}
\tilde{a}_0 \\
\vdots \\
\tilde{a}_3
\end{pmatrix}
\] (11.15)

in order to find our \( \tilde{W} \). A solution exists if and only if the rank of

\[
G = \begin{pmatrix}
g(x^0, x^0) & \cdots & g(x^3, x^0) \\
\vdots & \ddots & \vdots \\
g(x^0, x^3) & \cdots & g(x^3, x^3)
\end{pmatrix}
\]

is equal to the rank of

\[
[G \tilde{a}] = \begin{pmatrix}
g(x^0, x^0) & \cdots & g(x^3, x^0) & \tilde{a}_0 \\
\vdots & \ddots & \vdots \\
g(x^0, x^3) & \cdots & g(x^3, x^3) & \tilde{a}_3
\end{pmatrix}.
\]

We will now show that this is true. By construction if the \( i \)-th column of \( G \) is zero then \( \tilde{a}_i = 0 \), hence the only way in which the ranks of our two matrices may differ is if the non-zero columns of \( G \) are linearly dependent. If that was the case then there exists \( \alpha \) so that \( g(x^\alpha, \cdot) = \sum_{m \neq \alpha} d_m g(x^m, \cdot) = g(\sum_{m \neq \alpha} d_m x^m, \cdot) \). Since we have
non-zero columns this implies that \( x^\alpha = \sum_{m \neq \alpha} d_m x^m \) which contradicts that the \( x^i \) form a basis. The non-zero columns can therefore not be linearly dependent and a solution of equation 11.15 exists. This solution gives us our needed \( \bar{W} = \sum_i b_i x^i \).

For this \( \bar{W} \) we can calculate that,

\[
g(W, \bar{W}) = \sum_k a_k g(x^k, \bar{W}) = \sum_k a_k \bar{a}_k \tag{11.16}
\]

\[
= \sum_{k \neq j} a_k \bar{a}_k + a_j - \frac{a_i \bar{a}_i + c}{a_j} = c \tag{11.17}
\]

but we can choose \( c \) so that \( c \neq 0 \). This is a contradiction, and therefore for all \( i \) if \( a_i \neq 0 \) then \( g(x^i, \cdot) = 0 \).

Since we are only interested in space-times which allow a continuous orientation of the basis vectors in \( TM \) and which furthermore possess a globally defined, non-vanishing timelike vector field, i.e. which admit a global time direction, we will restrict ourselves to orientable and time orientable space-times. It is well known that in such space-times there exists an infinite number of non-vanishing, continuous spacelike vector fields. We will now make use of one such vector field in the following theorem which gives a sufficient condition for a timelike geodesic \( \gamma \) to run into a TSCS. In our case, however, it is necessary to assume the existence of the limits of the Jacobi fields at the TSCS\(^1\) which is physically not unreasonable since we would like to avoid an oscillatory behaviour.

**Theorem 11.10**

Let \( \gamma : [0, a) \to M \) be a timelike geodesic with affine parameter \( s \) which satisfies \( \lim_{s \to a} \gamma'(s) \) exists \( = \gamma'(a) \neq 0 \) and \( \lim_{s \to a} \bar{g}(\gamma'(s), \gamma'(s)) \neq 0 \), where \( \gamma(a) \in M \subseteq M \) and \( \bar{g} \) is at least \( C^0 \). Suppose there exists a continuous, non-vanishing vector field \( S \) on \( M \), which is spacelike on \( (M, \bar{g}) \) and which satisfies \( \lim_{s \to a} \bar{g}(S(\gamma(s)), \cdot) = 0 \). Furthermore assume \( \lim_{s \to a} Z^\alpha(s) \) exists \( = Z^\alpha(a) \), where \( Z^\alpha \ (\alpha = 1, 2, 3) \) are Jacobi fields to \( \gamma \). Then \( \gamma \) ends in a TSCS as \( s \to a \).

**Proof.** The \( Z^\alpha(s) \) \((\alpha = 1, 2, 3)\) are the Jacobi fields along \( \gamma \). Now assume \( Z^\alpha(a) \neq 0 \) \((\alpha = 1, 2, 3)\) (otherwise the result would be trivially obtained). There are two possibilities, either the \( \{\gamma'(a), Z^\alpha(a)\} \) \((\alpha = 1, 2, 3)\) are linearly independent or linearly dependent. We will show that either case leads to a vanishing volume.

Firstly, assume linear independence. Since \( \{\gamma'(a), Z^\alpha(a)\} \) \((\alpha = 1, 2, 3)\) are linearly independent, we may write\(^4\) \( S(\gamma(a)) = b\gamma'(a) + \sum_{\alpha} c^\alpha Z^\alpha(a) \). Now \( \lim_{s \to a} S(\gamma(s)), \cdot) = 0 \Rightarrow \lim_{s \to a} \bar{g}(b\gamma'(s) + \sum_{\alpha} c^\alpha Z^\alpha(s), \cdot) = 0 \). Since \( S \neq 0 \) and \( \lim_{s \to a} \bar{g}(\gamma'(s), \gamma'(s)) \neq 0 \), by Lemma 11.9 there is only one possibility, namely:

\(^1\)The Jacobi fields can certainly vanish in the limit.

\(^4\)In any vector space we can construct linear combinations of vectors independently of the existence of a metric.
\[ b = 0 \text{ and at least for one } \alpha = \alpha^*, \lim_{s \to a} g(Z^{\alpha^*}(s), \cdot) = 0 \] (and if there are other \( \alpha \neq \alpha^* \) which do not satisfy this, then \( c^\alpha = 0 (\alpha \neq \alpha^*) \)).

But \( \lim_{s \to a} g(Z^{\alpha^*}(s), \cdot) = 0 \) for at least one \( \alpha \) implies a vanishing column in equation (11.14) in the limit, which gives \( \lim_{s \to a} V(s) = 0 \).

Now we assume linear dependence of \( \{ \gamma'(a), Z^{\alpha}(a) \} \) (\( \alpha = 1, 2, 3 \)). We will show that this implies linear dependence of the \( \{ Z^{\alpha}(a) \} \) (\( \alpha = 1, 2, 3 \)) which is equivalent to \( \lim_{s \to a} V(s) = 0 \).

Suppose \( \gamma'(a) \in \text{span}\{Z^{\alpha}(a)\} \) and that \( \{ Z^{\alpha}(a) \} \) is a linearly independent set. The \( Z^{\alpha}(s) \) form a 3-surface \( S_a \) orthogonal to \( \gamma'(s) \) \( \forall s \in (0, a) \). By continuity (recall \( \bar{g} \) is \( C^0 \)), this is also true for \( s = a \). \( S_s \) is spacelike for \( s \in (0, a) \) and thus, by continuity, all vectors lying in \( S_a \) at \( \gamma(a) \in \mathcal{M} \) are either spacelike or degenerate.

But since \( \bar{g}(\gamma'(a), \gamma'(a)) \neq 0 \), \( \gamma'(a) \) is timelike by continuity and does not lie in \( S_a \), which contradicts \( \gamma'(a) \in \text{span}\{Z^{\alpha}(a)\} \). We conclude that this assumption and the linear independence of the \( \{ Z^{\alpha}(a) \} \) are false.

In either case the volume of the Jacobi fields vanishes at \( \gamma(a) \in \mathcal{M} \), i.e. \( \gamma \) encounters a TSCS as \( s \to a \).

Now we can apply the theorem to the conformal space-time. Again, we have to assume the existence of the limits of the Jacobi fields of the timelike geodesics on the slice \( \bar{T} = 0 \) to render the above theorem valid for the unphysical space-time. This, as seen above, is physically not unreasonable. For convenience we define the following set of timelike geodesics:

**Definition 11.11**

*For a space-time \((\mathcal{M}, \bar{g})\) we define the following set of future inextendible timelike geodesics, denoted \( \Gamma^+_\bar{g}(\mathcal{M}) \), with existent Jacobi fields and non-vanishing tangent vector in the limit, by*

\[
\Gamma^+_\bar{g}(\mathcal{M}) := \{ \gamma : [0, a) \to \mathcal{M} \mid \gamma \text{ is a future inextendible timelike geodesic with } \gamma(a) = \lim_{s \to a} \gamma(s) \in F^+(\mathcal{M}) \text{ exists}, \gamma'(a) \neq 0, \text{ and } \lim_{s \to a} Z^\alpha(s) \in T_{\gamma(a)}\mathcal{M} \}
\]

\( (\alpha = 1, 2, 3) \) exist, where the \( Z^\alpha(s) \) are the Jacobi fields along \( \gamma \}.

The following theorem now proves that the slice \( \bar{T} = 0 \) corresponds to a future TSCS for the conformal space-time of Definitions 8.7 and 8.10 if \( \Gamma^+_\bar{g}(\mathcal{M}) \) is non-empty and if the vector field \( S \) of Theorem 11.10 causes degeneracy on all of \( \bar{T} = 0 \).\footnote{In fact, the (non-causal) degeneracy of \( \bar{g} \) is a sufficient condition for the following theorem and we do not need the same \( S \) for all of \( \bar{T} = 0 \). For simplicity, however, we will require that there exists such a vector field \( S \) which causes degeneracy on all of \( \bar{T} = 0 \) since each of the example models in chapter 7 admits such a vector field, as can be seen by the respective expressions for the conformal metrics.}

On the other hand, if \( \Gamma^+_\bar{g}(\mathcal{M}) \) was empty, the pathology would be even worse and we would still be justified to talk of a space-time singularity in such a case. In this sense, the following theorem therefore proves that the slice \( \bar{T} = 0 \) corresponds to a singularity for the conformal space-time in any case.
Recall the notion of \emph{causal degeneracy} given in Definition 8.5.

**Theorem 11.12 (Conformal TSCS)**

Let \((\mathcal{M}, \bar{g})\) be the conformal space-time of Definition 8.7 (8.10 respectively) with \(\bar{g}\) at least \(C^0\) for \(T \leq 0\). Suppose there exists a continuous, non-vanishing vector field \(S\) on \(\mathcal{M} \ni M\), which is spacelike on \((\mathcal{M}, \bar{g})\) and which satisfies \(\lim_{T \to 0^-} \bar{g}(S, \cdot)|_T = 0\) (where \(|_T\) denotes evaluation anywhere on the hypersurface \(\{T = \text{const}\}\)). If \(\Gamma^+_\bar{g}(\mathcal{M})\) is a non-empty set, then every \(\gamma \in \Gamma^+_\bar{g}(\mathcal{M})\) ends in a TSCS as \(T \to 0^-\).

**Proof.** Consider any \(\gamma \in \Gamma^+_\bar{g}(\mathcal{M})\). Since \(\gamma(a) \in F^+(\mathcal{M})\), condition 1 of Definition 8.7 implies that the parameter value \(a\) corresponds to \(T = 0\). Since \(\bar{g}\) does not become causally degenerate as \(T \to 0^-\) (see Definition 8.7), every \(\gamma \in \Gamma^+_\bar{g}(\mathcal{M})\) will satisfy \(\lim_{s \to a} \bar{g}(\gamma'(s), \gamma'(s)) \neq 0\). Hence, the imposed requirements render the conditions of Theorem 11.10 applicable to the entire slice \(T = 0\) and every \(\gamma \in \Gamma^+_\bar{g}(\mathcal{M})\), which implies the assertion. □

We will now provide sufficient conditions for the conformal TSCS to translate into a TSCS for the physical space-time. We need to be careful since in general geodesics and Jacobi fields in the physical space-time do not correspond to geodesics and Jacobi fields, respectively, in the conformal space-time. However, since conformal transformations leave angles invariant\(^1\), a timelike geodesic of the physical space-time is still a timelike curve in the conformal space-time and its Jacobi fields are vectors in the conformal space-time which are orthogonal to the respective timelike curve. This will now be exploited.

If we are to follow the idea of Theorem 11.10 we have to furthermore ensure that in the case of degeneracy of the Jacobi fields the determinant in equation (11.14) still vanishes even though \(\Omega(0) = \infty\). The following theorem provides us with sufficient conditions for the AFS in Definition 8.10 to be a TSCS. This time, we assume \(\Gamma^+_{\bar{g}}(\mathcal{M})\) to be a non-empty set\(^2\).

**Theorem 11.13 (Physical TSCS)**

Let \(g = \bar{g}^2\) be some conformal relation satisfying Definition 8.10. Suppose there exist two linearly independent, continuous, non-vanishing vectorfields \(S_i\) \((i = 1, 2)\) on \(\mathcal{M} \ni M\), which are spacelike on \((\mathcal{M}, \bar{g})\) and which satisfy \(\lim_{T \to 0^+} \bar{g}(S_i, \cdot)|_T = 0\) on all of \(\bar{T} = 0\). If \(\Gamma^+_{\bar{g}}(\mathcal{M})\) is a non-empty set, then every \(\gamma \in \Gamma^+_{\bar{g}}(\mathcal{M})\) ends in a TSCS as \(T \to 0^-\).

**Proof.** We will apply the proof of Theorem 11.10 to the whole slice \(\bar{T} = 0\).

The conformal metric is not causally degenerate at \(\bar{T} = 0\), i.e. every tangent vector \(\mu'\) of a timelike curve in \((\mathcal{M}, \bar{g})\) which is non-zero at \(\bar{T} = 0\) satisfies \(\lim_{s \to a} \bar{g}(\mu'(s), \mu'(s)) \neq 0\) (where \(a\) corresponds to \(\bar{T} = 0\)). But all \(\gamma \in \Gamma^+_{\bar{g}}(\mathcal{M})\) are

\(^1\)see Appendix A.8

\(^2\)Even if \(\Gamma^+_{\bar{g}}(\mathcal{M})\) was empty, we could still speak of a singularity (see discussion above).
timelike curves in \((\mathcal{M}, \bar{g})\) with non-vanishing tangent vector. Thus, \(\lim_{s \to a} \bar{g}(\gamma'(s), \gamma'(s)) \neq 0\). Choose any timelike geodesic \(\gamma \in \Gamma_{\bar{g}}(\mathcal{M})\) and assume that its Jacobi fields satisfy \(\lim_{s \to a} Z^\alpha(s) = Z^\alpha(a) \neq 0\) (\(\alpha = 1, 2, 3\)) (otherwise the result is trivial). There are two possibilities, either the \(\{\gamma'(a), Z^\alpha(a)\}\) \((\alpha = 1, 2, 3\) are linearly independent or linearly dependent.

We firstly treat the case of linear dependence. The timelike geodesic \(\gamma\) in \((\mathcal{M}, \bar{g})\) is a timelike curve in \((\mathcal{M}, \bar{g})\) and the Jacobi fields orthogonal to \(\gamma\) in \((\mathcal{M}, \bar{g})\) correspond to vectors orthogonal to \(\gamma\) in \((\mathcal{M}, \bar{g})\) which define a volume element in \((\mathcal{M}, \bar{g})\). In the proof of Theorem 11.10 we have only made use of the fact that the \(Z^\alpha(s)\) are orthogonal to \(\gamma(s) \forall s \in (0, a)\), but not that \(\gamma\) and the \(Z^\alpha(s)\) actually satisfy the geodesic or Jacobi equation, respectively. Thus, we can simply take \(\gamma\) to be some timelike curve in \((\mathcal{M}, \bar{g})\) which transports the \(Z^\alpha(s)\) orthogonally along and apply Theorem 11.10 to this case. By the last part of the proof of Theorem 11.10 and the continuity of \(\bar{g}\), the linear dependence of \(\{\gamma'(a), Z^\alpha(a)\}\) \((\alpha = 1, 2, 3\) implies the linear dependence of the \(\{Z^\alpha(a)\}\) \((\alpha = 1, 2, 3\) But this is equivalent to \(\lim_{s \to a} V(s) = 0\) for \(\gamma\) in both \((\mathcal{M}, \bar{g})\) and \((\mathcal{M}, g)\).

Now we only need to show that in the case of linear independence the expression given by equation (11.14) still vanishes in the limit. Due to the linear independence we may write \(S_i(\gamma(a)) = b_i\gamma'(a) + \sum_k c^k_i Z^k(a) (i = 1, 2)\). Now \(\lim_{s \to a} \bar{g}(S_i(\gamma(s)), \cdot) = 0 \Rightarrow \lim_{s \to a} \bar{g}(b_i\gamma'(s) + \sum_k c^k_i Z^k(s), \cdot) = 0\). Since \(S_i \neq 0\) and \(\lim_{s \to a} \bar{g}(\gamma'(s), \gamma'(s)) \neq 0\), by Lemma 11.9 there is only the possibility that \(b_i = 0 (i = 1, 2)\) and that for each \(S_i\) there exists at least one \(\alpha = \alpha^i\) such that \(\lim_{s \to a} \bar{g}(Z^{\alpha^i}(s), \cdot) = 0\) (and if there are other \(\alpha \neq \alpha^i\) which do not satisfy this, then \(c^\alpha_i = 0 (\alpha \neq \alpha^i)\)).

We will now show that there exist at least two linearly independent \(Z^{\alpha^i}\) which satisfy \(\lim_{s \to a} \bar{g}(Z^{\alpha^i}(s), \cdot) = 0\). We only need to consider the case in which for each \(S_i\) there exists only one such \(\alpha^i\). But in that case, since the \(S_i\) are linearly independent the \(Z^{\alpha^i}(a)\) must be linearly independent as well. That is, in any case we have two linearly independent Jacobi fields which satisfy \(\lim_{s \to a} \bar{g}(Z^{\alpha^j}(s), \cdot) = 0 (j = 1, 2)\) and thus, in the limit, there can maximally be one non-zero column \(k\) in equation (11.14). Denote the columns which vanish in the limit, by \(l\) and \(m\),

\[
\begin{align*}
V(s)(v^1, v^2, v^3) &= \text{det} \begin{pmatrix}
\bar{g}(Z^k, v^1) & \bar{g}(Z^l, v^1) & \bar{g}(Z^m, v^1) \\
\bar{g}(Z^k, v^2) & \bar{g}(Z^l, v^2) & \bar{g}(Z^m, v^2) \\
\bar{g}(Z^k, v^3) & \bar{g}(Z^l, v^3) & \bar{g}(Z^m, v^3)
\end{pmatrix}.
\end{align*}
\]

But this is equivalent to

\[
\begin{align*}
V(s)(v^1, v^2, v^3) &= \text{det} \begin{pmatrix}
\bar{g}(Z^k, v^1) & \bar{g}(Z^l, v^1) & \bar{g}(Z^m, v^1) \\
\bar{g}(Z^k, v^2) & \bar{g}(Z^l, v^2) & \bar{g}(Z^m, v^2) \\
\bar{g}(Z^k, v^3) & \bar{g}(Z^l, v^3) & \bar{g}(Z^m, v^3)
\end{pmatrix}.
\end{align*}
\]

However, \(\lim_{s \to a} |\bar{g}(Z^k(s), v^n)| < \infty\) and \(\lim_{s \to a} \bar{g}(Z^{\alpha^j}(s), \cdot) = 0 (\alpha^j = l, m)\), i.e. we have no diverging but at least two vanishing columns in equation (11.19) in the
limit. And thus, \( \lim_{s \to a} V(s) = 0 \) for our timelike geodesic (where \( a \) corresponds to \( \tilde{T} = 0 \)).

By construction the \( S_i \) are degenerate on the whole slice \( \tilde{T} = 0 \) and consequently the above considerations apply to any timelike geodesic \( \gamma \in \Gamma^+_{\Omega^2\bar{g}}(\mathcal{M}) \), i.e. any \( \gamma \in \Gamma^+_{\Omega^2\bar{g}}(\mathcal{M}) \) ends in a TSCS as \( \tilde{T} \to 0^- \).

We have now proven that the AFS of Definition 8.10 corresponds to a physical TSCS - at least for the timelike geodesics - if \( \Gamma^+_{\Omega^2\bar{g}}(\mathcal{M}) \) is non-empty and the two \( S_i \) of the requirement are existent.

**Example 11.14**

*By the behaviour of the \( y \)- and \( z \)-components of \( \bar{g} \) at \( \tilde{T} = 0 \) in the Kantowski-Sachs models and the power of \( \tilde{T} \) in the respective conformal factor one can see that the Kantowski-Sachs models do admit such vector fields \( S_i \) \((i = 1, 2)\) which satisfy \( \lim_{\tilde{T} \to 0^-} \Omega^2 \bar{g}(S_i, \cdot)\bigg|_{\tilde{T}} = 0 \) on all of \( \tilde{T} = 0 \). If one could furthermore show that \( \Gamma^+_{\Omega^2\bar{g}}(\mathcal{M}) \) is non-empty in these models, one would prove that \( \tilde{T} = 0 \) is a future TSCS for the Kantowski-Sachs models.*

Analogously one could extend the theorems above to the null case. However, if we construct a basis system along a null geodesic \( \nu \), which involves its two Jacobi fields, it will be trickier to show that the linear dependence of this basis system at \( \nu(a) \) implies the linear dependence of the two Jacobi fields. Nevertheless, one should be able to extend the result without too many complications.

Lastly, it was our goal to conclude something about curvature in the cases of degenerate conformal metrics. Now that we have shown that the unphysical space-times in Definitions 8.7 and 8.10 and the physical space-time in Definition 8.10 admit a future TSCS under some conditions we are able to make this step for these cases. Propositions 2.5 and 2.6 indicate that TSCSs imply the divergence of not only various components of the Riemann tensor (and Ricci and Weyl tensor if we are dealing with a null geodesic) in a parallelly propagated frame, but also the integrals over them. Therefore the TSCSs are clearly *parallelly propagated curvature singularities* (see section 2.1).

### 11.2.3 Strong curvature singularities and regular conformal metrics

We will now briefly discuss the case of TSCS's in conformal structures with \( \Omega(0) = 0 \).

Recall that the two linearly independent spacelike Jacobi fields \( \{\hat{Z}^1, \hat{Z}^2\} \) along a null geodesic \( \nu \) with affine parameter \( s \) form a spacelike 2-surface \( \hat{S} \) orthogonal to \( \nu(s) \) for each \( s \). Its 2-volume element is given by \( \hat{V}(s) = \hat{Z}_1 \wedge \hat{Z}_2 \). Analogously to equation (11.14), for arbitrary linearly independent vectors \( v^1, v^2 \in T_{\nu(s)}\mathcal{M} \) which
lie in $\hat{S}$ this takes the form
\[
\tilde{V}(s)(v^1, v^2) = \det \begin{pmatrix} \tilde{\Omega}^2 g(\tilde{Z}^1, v^1) & \tilde{\Omega}^2 g(\tilde{Z}^2, v^1) \\ \tilde{\Omega}^2 g(\tilde{Z}^1, v^2) & \tilde{\Omega}^2 g(\tilde{Z}^2, v^2) \end{pmatrix}. \tag{11.20}
\]

By the expressions given in equations (11.19) and (11.20) we readily see that the volume along a causal geodesic whose Jacobi fields possess a limit at $\bar{T} = 0$ automatically vanishes identically at $\bar{T} = 0$ in the physical space-time if $\Omega(0) = 0$. Thus, we conclude the following:

**Remark 11.15**

The equations (11.19) and (11.20) immediately prove that the IPS, defined by Goode and Wainwright, is a TSCS for past directed causal geodesics whose Jacobi fields possess a limit at $T = 0$. Furthermore, by the same arguments, the IFS given in Definition 8.1 is clearly also a TSCS for future directed causal geodesics whose Jacobi fields have a limit at $\bar{T} = 0$, hence emphasising the analysis of the curvature invariants in chapter 10.

It was first shown by Goode and Wainwright [8] that the IPS are curvature singularities. The analysis, however, did not treat causal geodesics. Nolan [59] showed that all Jacobi fields of a past directed timelike geodesic $\gamma$ vanish identically as $\gamma$ approaches the IPS. Nonetheless, so far the case of null geodesics has not been treated yet in the literature, and thus this is the first time, that it is shown that the IPS are strong curvature singularities for past directed causal geodesics which approach $T = 0$, according to Definition 2.3 of a TSCS.

### 11.2.4 Characteristic differences between the definitions of an anisotropic future endless universe and an anisotropic future singularity

We have now proven that the cosmologies satisfying the conditions of Definition 8.10 possess future TSCSs in the physical space-time if certain conditions are fulfilled. On the other hand, it is obvious, that the models given by Definition 8.7 do not satisfy the conditions of the theorems in section 11.2.2. This by itself, nevertheless, does not allow us to conclude that Definition 8.7 does not admit future TSCSs.

But in section 11.1.2 we have seen that (in the limit) $\theta \geq 0$ in models which satisfy Definition 8.7. This, in fact, is enough to deduce that these models do not admit a future TSCS at $\bar{T} = 0$; the expansion scalar of a congruence of geodesics can be derived from the volume element $V(s)$ via [27]
\[
\theta = \frac{1}{V(s)} \frac{dV(s)}{ds}, \tag{11.21}
\]
thus, a TSCS necessarily implies $\lim_{s \to a} \inf \theta = -\infty$. It is therefore not unreasonable

---

\[\dagger\] If these conditions are not satisfied we still have a pathology (see discussion above).

\[\dagger\] This can be proven similarly to Lemma 9.5.
Remark 11.16

There is another possibility in probing without too much effort whether a metric pathology at a certain set of points in $\mathcal{M} \supset \mathcal{M}$ (where $\mathcal{M}$ is the space-time manifold) corresponds to a cosmological singularity, in the sense that the 4-volume vanishes at these points, or to a future (or past) infinity, in the sense that the 4-volume diverges at these points. Recall equation (11.12) for the volume form which is given for any (locally) oriented coordinate basis which we can always find independently of the metric. We have seen that the volume form vanishes at a set of points where the metric is degenerate and thus that the 4-volume integration over this set of points will deliver a zero 4-volume, compatible with a cosmological singularity. In the same way we can see that the 4-volume integration over a set of points would yield an infinite 4-volume - consistent with a future infinity - if the determinant of the metric (expressed in an oriented basis which is valid at these points) diverges at this set of points. Since the determinant of the transformation matrix into another oriented basis is finite, this can be verified in any oriented basis at this set of points. This lends weight to condition 5 of the definition of an AFEU which guarantees both a non-negative limiting $\theta$ value and an infinite 4-volume at $T = 0$.

11.3 Discussion

The degeneracy of the conformal metric at $T = 0$ has forced us to pursue different ways in deriving information about the behaviour of several quantities. Even though the obstacles are much greater in the framework of conformal structures with degenerate conformal metrics we were able to calculate a number of interesting results concerning the expansion scalar and the curvature. We were able to prove that due to the degeneracy the unphysical expansion scalar of the definition of an AFEU and an AFS diverges to $-\infty$, we derived the sign of the physical expansion scalar for a wide class of cases and furthermore showed that the physical expansion scalar necessarily diverges to $-\infty$ if the determinant of the physical metric becomes zero as well - as in the case of the AFS.

The first and the last result are very well compatible with the theorems in section 11.2.2 in which we showed that under certain conditions the unphysical space-time of the definitions of an AFEU and an AFS possesses a future TSCS while the definition of an AFS admits such a singularity in the physical space-time as well if further conditions are fulfilled. Moreover, by the behaviour of the expansion scalar we were able to conclude that Definition 8.7 does not admit future TSCSs. This chapter therefore partially justified the names given to Definitions 8.7 and 8.10.

As a by-product of the discussion in this chapter we furthermore could present for the first time that the IPS is a TSCS for past directed causal geodesics whose Jacobi fields possess a limit at $T = 0$. The time-reverse is true for the IFS.

It should be noted that the conditions of the theorems in section 11.2.2 can probably be weakened for proving the existence of a TSCS. The assumption that
the limits of the Jacobi fields exist at the TSCS can possibly be dropped without changing the result of the theorems. The requirement of Theorem 11.13 that two linearly independent vector fields $S_i$ satisfy $\lim_{T \to 0} \Omega^3 g(S_i, \cdot) \big|_T = 0$ seems somewhat artificial and can probably be amended.

Even though we now know under certain conditions of the presence of a curvature singularity in the conformal space-time of Definitions 8.7 and 8.10 and in the physical space-time of Definition 8.10, we are, unfortunately, still not able to deduce more about the curvature invariants of section 11.2.1. More research needs to be done on this. Also, we cannot say much about the Weyl curvature in our case since by the known propositions only a TSCS for a null geodesic implies the divergence of several components of $C_{abcd}$. The behaviour of the scalar $K$ seems to be out of reach for the moment and therefore it is difficult to judge whether Definitions 8.7 and 8.10 always force $K$ to increase in cosmological models satisfying the respective conditions, i.e. whether Definitions 8.7 and 8.10 generally satisfy the ideas of the WCH.

Lastly, it shall be mentioned that the same difficulties as discussed in the treatment of the curvature invariants in section 11.2.1 apply to the case of the other kinematic quantities as well. In fact, the situation is somewhat worse, since the kinematic quantities furthermore contain covariant derivatives which, due to the degeneracy, certainly create diverging terms. A completely different approach needs to be pursued in order to be able to discuss the behaviour of the vorticity, shear and acceleration of the fluid flow in these models which will be necessary for finally treating the topic of anisotropy in detail in the framework of Definitions 8.7 and 8.10.

There are still many open questions concerning the implications of Definitions 8.7 and 8.10 which need to be answered by further research.
Chapter 12

Conclusion

The first chapters of this thesis have indicated that the framework of the IPS is not sufficient to guarantee cosmological models which are capable of entirely describing the characteristics of our universe, e.g. the problem of galaxy formation seems to be explainable only by models which are, furthermore, ASPH. The behaviour of the example models in chapters 6 and 7 has moreover shown that the framework of an IPS does not necessarily lead to a future evolution of the respective cosmology which is compatible with quiescent cosmology and the WCH, since the FRW models with an IPS lead to an isotropic future behaviour. By the time-asymmetry of gravitational interaction, quiescent cosmology suggests an anisotropic future evolution of the universe. Therefore it is necessary to complete the picture of the IPS with new definitions to guarantee an appropriate past and future behaviour of the respective cosmology satisfying these definitions.

This led us to analyse the future behaviour of specific example space-times in chapters 6 and 7, as they provide us with valuable guidance in the quest for physically reasonable definitions. Motivated by these particular example models, we defined new conformal structures with isotropic future evolution and regular conformal metric, namely the isotropic future singularity (IFS) in Definition 8.1, and the future isotropic universe (FIU) in Definition 8.3 and, most importantly, we defined new conformal structures with anisotropic future evolution and degenerate conformal metric, namely the anisotropic future endless universe (AFEU) in Definition 8.7 and the anisotropic future singularity (AFS) in Definition 8.10. Only the latter two structures seem physically reasonable in the light of quiescent cosmology, nevertheless, the isotropic definitions are important for completeness and comparison.

Example space-times with an IFS are the radiation and the dust filled, closed FRW universes and possibly a subclass of the McVittie-Wiltshire II models. For the definition of an AFEU we have found a subclass of the Szekeres models, the Carneiro-Marugan models and the Kantowski models as example cosmologies and the only analysed models which admit an AFS are the Kantowski-Sachs models. The scalar $K$ was moreover shown to increase with cosmic time in the AFEU and AFS example space-times of chapter 7, and plotted for particular cases, in strong

\footnote{e.g. chapters 3, 4 and 5}

\footnote{The fluid flow of these models remains to be analysed.}
support of the WCH. Thereby we could show the increase of $K$ for more specific models than mentioned in [12].

In chapter 9 we derived a number of useful technical properties, such as the asymptotic behaviour of several ratio functions of the conformal factor and its derivatives, and some characteristics of the conformal metric in the case of degeneracy, which were essential in the subsequent elaboration of the physical properties of the new definitions.

As one could expect, in analogy to the IPS, the definitions of the IFS and the FIU lead to asymptotic Weyl and kinematic isotropy, i.e. the scalar $K$ vanishes asymptotically and the shear, vorticity and the acceleration of the fluid flow are asymptotically expansion dominated. In chapter 10 we even proved that this is the case for conformal structures with regular conformal metric, independently of whether $T$ approaches 0 from below or above, and whether the conformal factor vanishes or diverges asymptotically. Sections 10.2 and 10.3 furthermore justified the names “IFS” and “FIU” by showing that, apart from the asymptotic isotropy, the definition of an IFS implies a curvature and kinematic behaviour which is essentially the time-reverse of an IPS, and that the definition of a FIU does not necessarily lead to a curvature or kinematic behaviour which is compatible with a singularity.

The most important result of chapter 10, however, is the conclusion that if we are to follow the ideas of quiescent cosmology and the WCH with a conformal structure involving a conformal factor as a function of cosmic time, then we have to require a regular conformal metric for the initial state of the universe and a conformal structure with degenerate conformal metric for the future infinity or future singularity. This conclusion justifies the choice of the conformal metrics in the definitions of an AFS and an AFEU which become degenerate and possibly only $C^0$ at $T = 0$.

This irregularity of the conformal metric renders calculations difficult. In contrast to the regular conformal structure for the IPS, which facilitates investigations greatly, the new conformal structures do not provide us with an analytical advantage anymore. In fact, the analytical situation is even more difficult for the new definitions, since we have proven that the slice $T = 0$ necessarily corresponds to a singularity for the unphysical space-time, while it does not necessarily correspond to a future singularity for the physical space-time.

The degeneracy in Definitions 8.7 and 8.10 has forced us to pursue new and different ways to derive their physical implications. Even though the difficulties are now much greater, we were able to present some information about the expansion scalar and the curvature. We proved that the unphysical expansion scalar necessarily diverges to $-\infty$ while the physical expansion scalar is non-negative in the limit for the AFEU models and $-\infty$ in the limit for cosmologies which admit an AFS.

Thereupon we proved that the slice $T = 0$ corresponds to a future TSCS for the unphysical space-time of Definitions 8.7 and 8.10, if reasonable conditions are met. We furthermore provided sufficient conditions for the AFS to be a future TSCS for the physical space-time. By the behaviour of the expansion scalar and the volume form we were able to argue that the AFEU cannot admit a future TSCS, and as a by-product of the discussion of strong curvature, we could show for the first time
that the IPS, as well as the IFS, are TSCSs for causal geodesics which approach the slice $T = 0$, if the limits of their Jacobi fields exist.

The analysed AFEU and AFS example space-times have already indicated that these two definitions admit an “asymptotically anisotropic” behaviour when compared to the asymptotic isotropy of section 4.3.2. Indeed, in contrast to the IPS, these new definitions seem to allow a large variety of different asymptotic behaviours of $K$ and the kinematic ratios as their limits can vanish, remain finite or diverge\footnote{depending on whether the cosmology is shear-free, irrotational etc.}.

This variety of possibilities is well compatible with a high-entropy final state of the universe and therefore emphasises quiescent cosmology. It is, however, this variety of possible behaviours which makes the derivation of the behaviour of $K$ and the kinematics much more complicated in this scenario.

Since it is not yet clear whether our universe will expand indefinitely or recollapse in finite future time, we suggest that Definitions 8.7 and 8.10 should both be regarded as a completion to the definition of an IPS, and conjecture that the conjunction of these definitions provides a possible version of a complete mathematical formulation of quiescent cosmology and the WCH. In this sense, the combined definitions may be used as a tool to downsize the class of physically reasonable cosmologies.

As has been discussed previously, there are some problems in using FRW models for the description of our own universe. The analysis in this thesis has furthermore indicated that FRW models are most likely, in general, to be incompatible with Definitions 8.7 and 8.10. This state of affairs motivates us to suggest that the FRW universes should henceforth be regarded as physically unrealistic.

### 12.1 Further research

This thesis has opened a new field of research and consequently there are many open and unanswered questions which warrant further research in the future. We will now address some of these open questions regarding the definitions of an AFS and an AFEU.

First of all, the matter of curvature in the framework of Definitions 8.7 and 8.10 requires more detailed investigations. It seems probable that one does not necessarily need to assume that the Jacobi fields possess limits at $T = 0$ in order to prove the presence of a TSCS in the unphysical or physical space-time. Furthermore, the theorems of section 11.2.2 need to be extended to the null case in order to obtain some information about the Weyl curvature via the TSCS. The requirement of the two linearly independent vector fields in Theorem 11.13 seems somewhat artificial and can possibly be amended to a weaker condition, without changing the result of the theorem.

The behaviour of the curvature invariants and, in the light of the WCH, more importantly, of the scalar $K$, remains to be analysed in general for the definitions of an AFS and an AFEU. The considerations in section 11.2 do not allow any conclusion regarding $K$ and, consequently, it needs to be investigated whether Definitions
8.7 and 8.10 always imply its growth with cosmic evolution, as was observed for the example space-times of chapter 7.

It is necessary to finalise the discussion of anisotropy with a detailed treatment of the kinematics to complete the justification of the names AFS and AFEU. In the example models of chapter 7 it was observed that a non-vanishing shear or acceleration does not become expansion dominated at $T = 0$. It would be interesting to determine whether this is the case, in general.

Some technical properties require further research, such as the monotonicity of $\bar{\Omega}$ and $\bar{L}$, which is most likely a redundancy in all the new definitions, as was the case for the original definition of an IPS (see Lemma 5.6). Also, is $\bar{\Omega}$ (and therefore the conformal space-time) unique or does there exist some freedom of choice as in the case of the IPS? The example models for an AFEU all showed a number of possible cosmic time functions which all lead to essentially the same conformal structure. Can this fact be explained by Definition 8.7? Furthermore, are specific $\bar{\lambda}$ values related to certain (perhaps asymptotic) $\gamma$-law equations of state, as in the framework of an IPS? Under which conditions can we deduce the existence of an equation of state?

Most of the above problems can only be tackled with a new approach, since the derivations using the framework of an IPS are not applicable for these investigations, due to the degeneracy in the unphysical space-time.

The notion of “future endless” only makes sense when the implications of the AFS and the AFEU on the proper time of the fluid flow are clear. The analysis of the expansion scalar might provide some information on the proper time; e.g. it has been proven for hypersurface orthogonal congruences of timelike geodesics in space-times satisfying the strong energy condition, that if the expansion of the congruence takes a negative value at a point, it diverges to $-\infty$ at a finite proper time later (assuming the congruence extends that far)[21, p 220]. Furthermore, equation (11.21) might provide assistance in determining which conditions on the expansion scalar necessarily imply a TSCS. The derivation could proceed similarly to the proof of Lemma 9.5.

The definitions of an IFS and a FIU, on the other hand, are physically less appealing. Nevertheless, some open questions concerning these isotropic definitions should be investigated. Among these, do all results of an IPS hold for the IFS as well (except some sign changes)? Which models satisfy the conditions of a FIU? Possibly some open FRW universes?

There are certainly many more questions to be answered regarding the framework of the new definitions and, consequently, we regard this thesis as the basis for further research on the provided conformal structures, in order to finally round off the picture of isotropic past singularities and finalise a possible version of a complete mathematical formulation of quiescent cosmology.
Appendix A

Relativistic Cosmology

The work presented in this thesis is based on relativistic cosmology. This appendix will briefly introduce the framework of relativistic cosmology with regards to definitions, relations and notations. It is summarised from [8, 11, 19, 29, 37, 71].

A.1 Riemann tensor, Ricci tensor, Ricci scalar and Weyl tensor

The curvature operator of a semi-Riemannian manifold represents the non-commutativity of the covariant derivatives. It is defined via

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [X,Y]Z,$$

(A.1)

where $X, Y, Z$ are vector fields. The Riemann curvature tensor represents the curvature of space-time in general relativity; using the same symbol $\mathcal{R}$ [29] it is defined by

$$\mathcal{R}(X,Y,Z,V) = g(\mathcal{R}(X,Y)Z,V), \quad \text{where } V \text{ is any vector field},$$

and is usually expressed via its components $R_{abcd}$. The Riemann tensor has the symmetry properties

$$R_{[ab][cd]} = R_{abcd} = R_{cdab}, \quad R_{a[bc]} = 0.$$  

(A.3)

The Ricci tensor $R_{ab} = R^c_{\phantom{c}acb}$ and the Ricci scalar $R = R^a_a$ are the contractions of the Riemann tensor.

The Weyl tensor $C_{abcd}$ is the trace-free part of the Riemann tensor and defined via equation (4.1). It is common to analyse the Weyl tensor by splitting it into its “electric”, $E_{ab}$, and “magnetic”, $H_{ab}$, components, namely

$$E_{ab} = C_{acbd}u^c u^d, \quad H_{ab} = \frac{1}{2} \eta_{ac} \eta^{rs} C_{rsbd} u^c u^d,$$

(A.4)

where $u^a$ is the velocity vector field defined in equation (4.5) and $\eta^{abcd}$ is the volume element defined by $\eta^{abcd} = \eta^{[abcd]}$ and $\eta_{0123} = |g|^{-1/2} (g = \det g_{ab})$. The Weyl tensor
is trace-free, hence \( H^a_a = E^a_a = 0 \), and by the symmetry properties one finds \( E_{ab} = E_{(ab)} \), \( E_{ab}u^a = 0 \), \( H_{ab} = H_{(ab)} \) and \( H_{ab}u^a = 0 \). Inverting the equations [11] gives the Weyl tensor as a whole

\[
C_{abcd} = (-\eta_{abpq}\eta_{cdrs} + g_{abpq}g_{cdrs})u^p u^r E^{qs} - (\eta_{abpq}\eta_{cdrs} + g_{abpq}\eta_{cdrs})u^p u^r H^{qs},
\]

(A.5)

where \( g_{abcd} := g_{ac}g_{bd} - g_{ad}g_{bc} \).

### A.2 Kinematic quantities

For the timelike velocity vector field presented in equation (4.5) one can define

\[
h_{ab} := g_{ab} + u_a u_b
\]

(A.6)
as the 3-dimensional projection tensor into the rest space of an observer moving with \( u^a \), thus, providing a splitting of space-time into space and time for this observer.

Using \( h_{ab} \) it is possible to define the expansion tensor [19] as

\[
\theta_{ab} := h_a^c h_b^d u_{(c;d)}
\]

(A.7)

and the expansion scalar as

\[
\theta = h^{ab} \theta_{ab} = u^a_{;a}.
\]

(A.8)

\( \theta_{ab} \) and \( \theta \) determine the rate-of-change of distance of neighbouring particles in the fluid (e.g. clusters of galaxies). Moreover the shear tensor and shear scalar are defined [37] as

\[
\sigma_{ab} := \theta_{ab} - \frac{1}{3} h_{ab} \theta, \quad \sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab},
\]

(A.9)

and determine volume invariant distortions arising in the fluid flow. The eigenvectors of \( \sigma_{ab} \) represent the principal axes of shear and are left invariant by the distortion.

The fluid can furthermore rotate. The quantities which determine a rigid rotation of clusters of galaxies with respect to a local inertial rest frame are the vorticity tensor and the vorticity scalar [19], defined by

\[
\omega_{ab} := h_a^c h_b^d u_{[c;d]}, \quad \omega^2 = \frac{1}{2} \omega_{ab} \omega^{ab}.
\]

(A.10)
The vorticity vector, given by \( \omega^a = \frac{1}{2} \eta^{abcd} u_b u_{c;d} \), determines the axis of rotation of the fluid.

Finally, one can also define the acceleration of the fluid, which represents the combined effects of gravitational and inertial forces [37], by

\[
u^a := u^a_{;b} u^b.
\]

(A.11)
A.3. The Einstein field equations and the energy-momentum tensor

The above defined quantities define the first covariant derivative of the velocity vector completely:

\[ u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3} \theta h_{ab} - \dot{u}_a u_b. \quad (A.12) \]

Given equations (A.7), (A.9) and (A.10) one can derive

\[ \theta_{ab} u^b = \sigma_{ab} u^b = \omega_{ab} u^b = 0 \quad (A.13) \]

and \( \theta_{ab} = \theta_{(ab)} \), \( \sigma_{ab} = \sigma_{(ab)} \), \( \omega_{ab} = \omega_{[ab]} \) and \( \sigma^2 = 0 \). Additionally \( \sigma^2 = 0 \leftrightarrow \sigma_{ab} = 0 \) and \( \omega^2 = 0 \leftrightarrow \omega_{ab} = 0 \).

Using the normalisation of \( u^a \) one can furthermore easily show that the acceleration vector is spacelike, namely that

\[ \dot{u}_a u^a = 0. \quad (A.14) \]

A.3 The Einstein field equations and the energy-momentum tensor

The Einstein field equations (EFE) are given by

\[ G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = T_{ab} \Leftrightarrow R_{ab} = T_{ab} - \frac{1}{2} T g_{ab} + \Lambda g_{ab}, \quad (A.15) \]

where \( T_{ab} \) is the symmetric stress-energy-momentum tensor and \( \Lambda \) is the cosmological constant.

The Einstein tensor \( G_{ab} \) is commonly decomposed [8] relative to the velocity vector field \( u^a \) into

\[ G_{ab} = A u_a u_b + B h_{ab} + \Sigma_a u_b + \Sigma_b u_a + \Sigma_{ab} \quad (A.16) \]

where \( A \) and \( B \) are given by

\[ A = G_{cd} u^c u^d, \quad B = \frac{1}{3} h^{cd} G_{cd} \quad (A.17) \]

and \( \Sigma_a \) and \( \Sigma_{ab} \) are the anisotropic parts of the Ricci tensor relative to \( u^a \), defined by

\[ \Sigma_a = -h_{ac} R^{cd} u_d, \quad \Sigma_{ab} = h_{ac} h_{db} R^{cd} - \frac{1}{3} h_{ab} h_{cd} R^{cd}. \quad (A.18) \]

Analogously one uniquely decomposes \( T_{ab} \) [37] relative to \( u^a \) into

\[ T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab} \quad (A.19) \]

where \( \mu \) is the total energy density of matter measured by \( u^a \), \( q_a \) is the energy flux (such as heat conduction and diffusion) relative to \( u^a \), and \( p \) is the isotropic pressure,
and $\pi_{ab}$ is the anisotropic matter pressure which can be caused by processes such as viscosity. Furthermore, $q_a u^a = 0$, $\pi^a_a = 0$, $\pi_{ab} u^b = 0$.

Combining equation (A.16) and (A.19), the EFE can now be expressed as

$$A = \mu, \quad B = p, \quad \Sigma_a = q_a, \quad \Sigma_{ab} = \pi_{ab}. \quad (A.20)$$

The energy-momentum tensor has to obey the conservation equations, $T^{ab} ; b = 0$, which, when using equation (A.12) [37], deliver

$$\mu + (\mu + p) \theta + \pi_{ab} \sigma^{ab} + q^{a} ; a + \dot{u}_a q^a = 0, \quad (A.21)$$

$$(\mu + p) \dot{u}_a + h_a^c (p_c + \pi^{b} ; b + \dot{q}_c) + \left( \omega_a^b + \sigma_a^b + \frac{4}{3} \theta h_a^b \right) q_b = 0. \quad (A.22)$$

These equations provide no other information than what is already contained in the EFE, however, they express this information in a useful way for the kinematic quantities.

Note that the EFE, the energy-momentum tensor and the conservation equations simplify greatly in the case of a perfect fluid source, since it is viscosity free and admits no heat conduction, i.e. $q_a = 0$ and $\pi_{ab} = 0$. A perfect fluid is therefore isotropic in its rest frame. This isotropy and the symmetry of $T_{ab}$ furthermore imply that all off-diagonal components of $T_{ab}$ are zero and that the three space components on the diagonal must be equal, if $T_{ab}$ is represented in the rest frame of the perfect fluid.

In order for the matter to show a reasonable behaviour one normally puts energy conditions on the fluid source which are not provided by the EFE. A very general restriction would be that the energy density be positive. But it is also physically realistic to require that the pressure does not take large negative values, hence

$$\mu + p > 0 \quad \text{and} \quad \mu + 3p > 0 \quad (A.23)$$

(the there are a number of other conditions, the interested reader is urged to consult [19, p 88]).

### A.4 Equations of state

In order to put physics into the picture we need to specify the energy-momentum tensor and the matter source further. This is usually done by using equations of state.

For perfect fluid sources one most commonly uses barotropic equations of state, i.e. $p = p(\mu)$, and of these most frequently the so-called $\gamma$-law equations of state, $p = (\gamma - 1)\mu$, where $\gamma$ is a constant. Perfect fluids with $\gamma$-law equation of state are usually called polytropic which is a bit misleading as this terminology is used for a different type of equation of state in astrophysics\(^1\).

In approximating our universe as a perfect fluid, one has identified three types of $\gamma$-laws to be appropriate, namely: (1) $\gamma = 1$, pressure-free dust, (2) $\gamma = \frac{4}{3}$, a highly relativistic gas or isotropic radiation, and (3) $\gamma = 2$, a stiff fluid in which the speed of sound equals the speed of light.

\(^1\)Namely for $p = \kappa \varrho^\gamma$, where $\varrho$ is the classical matter density and $\kappa, \gamma$ are constants.
A.5 Propagation equations for the kinematic quantities

The last term in equation (A.1) vanishes in a torsion-free space-time. Substituting the velocity vector into equation (A.1) gives the Ricci identity for $u^a$ which in its components is given by

$$u_{a;dc} - u_{a;cd} = R_{abcd} u^b.$$  

(A.24)

The velocity field provides the direction that an observer moving with this velocity ascribes to time. Thus, projecting this equation onto the velocity field, i.e. contracting it with $u^a$, yields the propagation equations for the kinematic quantities. However, recalling equations (A.7), (A.9) and (A.10), we only obtain the kinematic quantities if we further contract equation (A.24) with $h^r_a$ and $h^s_c$. By doing so and noting that (see [11])

$$h^r_a h^s_c R_{abcd} u^b u^d = R_{rb sd} u^b u^d, \quad \text{and}$$  

(A.25)

$$h^r_a h^s_c u^a_{;d} u^d_{;c} = (\theta_{rd} + \omega_{rd})(\theta^d_s + \omega^d_s), \quad \text{and}$$  

(A.26)

$$(u_{ac})^* = \dot{\theta}_{ac} + \dot{\omega}_{ac} - \ddot{u}_a u_c - \ddot{u}_c u_a,$$  

(A.27)

one obtains

$$h^r_a h^s_c (\dot{\theta}_{ac} + \dot{\omega}_{ac} - \ddot{u}_a u_c) - \ddot{u}_r \ddot{u}_s + (\theta_{rd} + \omega_{rd})(\theta^d_s + \omega^d_s) = -R_{rb sd} u^b u^d. \quad \text{(A.28)}$$

Multiplying by $g^{rs}$ [11] and using

$$h^{ac}(\dot{\theta}_{ac} + \dot{\omega}_{ac}) = \dot{\theta}, \quad \text{and}$$  

(A.29)

$$(\theta_{rd} + \omega_{rd})(\theta^d_s + \omega^d_s) = 2\sigma^2 + \frac{1}{3} \theta^2 - 2\omega^2, \quad \text{and}$$  

(A.30)

$$-h^{ac} \ddot{u}_{a;c} = \dddot{u}_a u_c - \dddot{u}_c u_a,$$  

(A.31)

yields the important propagation equation for $\theta$, known as the Raychaudhuri equation,

$$\dot{\theta} + \frac{1}{3} \theta^2 - \ddot{u}_a^{;a} + 2(\sigma^2 - \omega^2) = -R_{bd} u^b u^d. \quad \text{(A.32)}$$

We can also obtain propagation equations for the shear and vorticity from equation (A.28). By using equation (4.1) to replace $R_{rb sd}$, the Raychaudhuri equation to eliminate $\theta$ and equation (A.18), we find the symmetric, trace-free parts of equation (A.28) given by

$$h^r_a h^s_c (\dot{\sigma}_{ac} - \ddot{u}_{(a;c)}) + \frac{1}{3} h^{rs} (\dddot{u}_a^{;a} - 2\sigma^2 + 2\omega^2)$$

$$- \frac{1}{2} \Sigma_{rs} - \ddot{u}_r \ddot{u}_s + \sigma_{rd} \sigma^d_s + \frac{2}{3} \theta \sigma_{rs} + \omega_{sd} \omega^d_r + E_{rs} = 0. \quad \text{(A.33)}$$

This is the shear propagation equation.

Mere anti-symmetrisation of equation (A.28) yields the vorticity propagation equation

$$h^b a \omega^b = \left( \sigma^a b - \frac{2}{3} \delta^a b \theta \right) \omega^b + \frac{1}{2} \eta^{abcd} u^b u^c d. \quad \text{(A.34)}$$
A.6 Constraint equations for the kinematic quantities

We can obtain three further sets of equations from equation (A.24). Projecting it with $h^{de}$ into the space orthogonal to $u^a$, i.e. into a 3-surface which an observer moving with $u^a$ identifies as space, we obtain the constraint equations for the kinematics, which determine the kinematic measurements of the observer.

Multiplying equation (A.24) with $g^{ae}h^{de}$ provides us with the first constraint equation [37],

$$h^e_b \left( \omega^{bc} - \sigma^{bc} + \frac{2}{3} \theta^{eb} \right) + (\omega^e_b + \sigma^e_b) \hat{u}^b = \Sigma^e.$$  \hspace{1cm} (A.35)

The vorticity vector $\omega_a$ satisfies a constraint equation by itself. We calculate it by contracting the equation on the right side in equation (A.3) with $u^a$. This yields

$$u_{[a;bc]} = 0.$$  \hspace{1cm} (A.36)

Further contracting it with $\eta^{abcd}u_d$ [37] gives

$$\omega^a :a = 2\omega^b \hat{u}_b.$$ \hspace{1cm} (A.37)

The third constraint equation can be obtained by contracting equation (A.24) with $sfdc u_f$, elimination of $R_{abcd}$ with equation (4.1) and symmetrisation [37],

$$H_{ad} = 2\hat{u}_{(a} \omega_{d)} - h_a^t h_d^s (\omega^t_{(b} \sigma^c_{c)} + \sigma^t_{(b} \sigma^c_{c)}) \eta_{s} f_{bc} u_f.$$ \hspace{1cm} (A.38)

A.7 Bianchi identities - the “Maxwell equations for the Weyl tensor”

So far we have only discussed constraint and propagation equations for the kinematic quantities. There exist, however also constraint equations on the curvature tensor. The second Bianchi identities of the Riemann tensor

$$R_{abcde} = 0$$  \hspace{1cm} (A.39)

lead to the Bianchi identities for the Weyl tensor which are given by equation (4.4) and resemble much the relativistic form of the Maxwell equations. The terminology of the “electric” and “magnetic” parts of the Weyl tensor becomes clearer when equation (A.5) is substituted into equation (4.4) and further contractions are performed, namely [11, 37]:

“$\text{div } E$”: further contraction with $u_b u_c$ gives

$$h^t_a E^{as} :d h_s^d - \eta^{tbpq} u_b \sigma^d_p H_{qd} + 3H_{s}^t \omega^s =$$

$$\frac{1}{3} h^t_a A_{a} - \frac{1}{2} h^t_c \Sigma^c_{cb} - \frac{3}{2} \omega^t_b \Sigma^b + \frac{1}{2} \sigma^t_b \Sigma^b + \frac{1}{2} \Sigma^t_b \hat{u}^b - \frac{1}{3} \theta \Sigma^t.$$ \hspace{1cm} (A.40)
“\( \text{div } H \)”: further contraction with \( h^{rs} u^s u_b \eta_{acrs} \) yields
\[
h_a^{rs} \dot{h}_{as}^d + \eta^{tbpq} u_b \sigma_p^d E_{qd} - 3 E^t_s \omega_s = (A + B) \omega^t + \frac{1}{2} \eta^{tbe} u_b \Sigma_{[e;f]} + \frac{1}{2} \eta^{tbf} u_b \Sigma_{ec} (\omega^e_f + \sigma^e_f). \tag{A.41}
\]

“\( \dot{E} \)”: further contraction with \( h_a^m h_c^t u_b \) leads to
\[
h_a^m h_c^t \dot{E}^{ac} + h_a^{(m} \eta^{t)rsd} u_r H_s^a_{s;dt} - 2 H_q^a \eta^{t} h^{bpq} u_b \dot{u}_p + h^{mt} \sigma^{ab} E_{ab} + \theta E^{mt} - 3 E^s_s (m \sigma^t) = - \frac{1}{2} (A + B) \sigma^{tm} - \dot{u}^{(t} \Sigma^{m)} - \frac{1}{2} h^{t a} h^{m c} \Sigma_{(a;c)} - \frac{1}{2} h^a h^m \tilde{\Sigma}^{ac}
\]\[
- \frac{1}{2} \Sigma^{b(m} \omega_{b)} - \frac{1}{2} \Sigma^{b(m} \omega_{b)} - \frac{1}{2} \Sigma^{tm} \theta + \frac{1}{2} (\Sigma^{a}_{ta} + \dot{u}_a \Sigma^a + \Sigma^{ab} \sigma_{ab}) h^{mt}. \tag{A.42}
\]

“\( \dot{H} \)”: further contraction with \( h^{rs} u^s u_b \eta_{rsab} \) provides us with
\[
h_a^m h_c^t \dot{H}^{ac} - h_a^{(m} \eta^{t)rsd} u_r E_s^a_{s;dt} + 2 E_q^a \eta^{t} h^{bpq} u_b \dot{u}_p + h^{mt} \sigma^{ab} H_{ab} + \theta H^{mt} - 3 H^s_s (m \sigma^t) = - \frac{1}{2} \sigma^{(t} \eta^{m)be} u_b \Sigma_f - \frac{1}{2} h_c^{(t} \eta^{m)be} u_b \Sigma_{e,f} + \frac{1}{2} (h^{mt} \omega_c \Sigma^c - 3 \omega^m \Sigma^t). \tag{A.43}
\]

### A.8 Conformal transformations

The work presented in this thesis utilises conformal transformations, such as
\[
g = \Omega^2 \tilde{g}. \tag{A.44}
\]

Since for any vectors \( W, X, Y, Z \) at a point \( p \)
\[
\frac{g(W, X)}{g(Y, Z)} = \frac{\tilde{g}(W, X)}{\tilde{g}(Y, Z)}, \tag{A.45}
\]

we find that angles and ratios of magnitudes are preserved under conformal transformations, i.e. conformal transformations preserve the null cone structure in \( T_pM \).

In this section we will recall a few well known conformal relationships for quantities of relativistic cosmology which are needed in the main body of this thesis. A full list of the conformal relations for the flow quantities and the geometrical quantities may be found in [11, ch 3].

A colon denotes covariant differentiation with respect to \( \tilde{g} \).
A.8.1 Conformal relationships for geometrical quantities

\[ g_{ab} = \Omega^{2}g_{ab} \] (A.46)
\[ \tilde{g}^{ab} = \Omega^{-2}\tilde{g}^{ab} \] (A.47)
\[ \delta_{a}^{b} = \tilde{\delta}_{a}^{b} \] (A.48)
\[ R^{b}_{d} = \Omega^{-2}\tilde{R}^{b}_{d} + 4\Omega^{-4}\Omega_{d}^{\epsilon}\Omega_{\epsilon r}^{b}g^{br} - 2\Omega^{-3}\Omega_{r\epsilon d}^{b}g^{br} \] (A.49)
\[ C^{a}_{bcd} = \tilde{C}^{a}_{bcd} \] (A.50)

If the conformal factor is merely a function of cosmic time \( \Omega = \Omega(T) \), as in our case, we furthermore find

\[ \Omega_{,a} = \Omega T_{a} \] (A.51)
\[ \Omega_{,ab} = \Omega'(T_{ab} + \Omega''/\Omega T_{a} T_{b}) \] (A.52)

A.8.2 Conformal relationships for flow quantities

\[ \tilde{u}^{a} = \Omega u^{a} \] (A.53)
\[ \tilde{u}_{a} = \Omega^{-1}u_{a} \] (A.54)
\[ \tilde{h}_{ab} = \Omega^{-2}h_{ab} \] (A.55)
\[ \tilde{h}_{a}^{b} = h_{a}^{b} \] (A.56)
\[ \tilde{\omega}_{ab} = \Omega^{-1}\omega_{ab} \] (A.57)
\[ \tilde{\omega}^{2} = \Omega^{2}\omega^{2} \] (A.58)
\[ \tilde{\sigma}_{ab} = \Omega^{-1}\sigma_{ab} \] (A.59)
\[ \tilde{\sigma}^{2} = \Omega^{2}\sigma^{2} \] (A.60)
\[ \tilde{\theta}_{ab} = \Omega^{-1}\theta_{ab} - \Omega^{-1}h_{ab}(\ln \Omega)_{,d}u^{d} \] (A.61)
\[ \tilde{\theta} = \Omega\theta - 3\Omega(\ln \Omega)_{,d}u^{d} \] (A.62)
\[ \tilde{u}_{a} = u_{a} - h_{a}^{b}(\ln \Omega)_{,b} \] (A.63)
\[ \tilde{E}_{ab} = E_{ab} \] (A.64)
\[ \tilde{H}_{ab} = H_{ab} \] (A.65)

A.9 Special coordinates

In general relativity it is often useful to derive specific results with the help of special coordinates which can greatly simplify calculations. In this section we briefly introduce the concepts of normal coordinates, comoving coordinates, comoving normal coordinates and synchronous comoving coordinates which are frequently used in relativistic cosmology. More details may be found in [11, 37, 71] and a list of
many flow and geometrical quantities expressed in these coordinates is given in [11, ch 3].

Definition A.1 (Normal coordinates)
If the normals $\frac{\partial}{\partial t}$ to the hypersurfaces of constant coordinate time $t$ are orthogonal to the spatial coordinate derivatives $\frac{\partial}{\partial x^\alpha}$ of a coordinate system, $(t, x^\alpha)$, then the coordinates are called normal coordinates.

Due to the orthogonality of $\frac{\partial}{\partial t}$ to the $\frac{\partial}{\partial x^\alpha}$ we find $g_{0\alpha} = 0$ and

$$ds^2 = -\frac{1}{F^2(t, x^\gamma)}dt^2 + g_{\alpha\beta}(t, x^\gamma)dx^\alpha dx^\beta$$  \hspace{1cm} (A.66)

if we express the metric in these coordinates.

Most commonly used are comoving coordinates.

Definition A.2 (Comoving coordinates)
We will now explain how to construct comoving coordinates. Let $u^\alpha$ be the velocity vector of a fluid filled space-time. Choose any spacelike hypersurface $S$. To every event on $S$ assign the coordinate time $t_0$. In any manner desired equip $S$ with a grid of space coordinates $(x^1, x^2, x^3)$. These coordinates may be propagated into the rest of space-time via the flow lines of the fluid. Label the fluid particles by $(x^1, x^2, x^3)$ on $S$ and at all later times label the same fluid particles by the same space coordinates $(x^1, x^2, x^3)$. This assignment has the useful consequence that the fluid is always at rest relative to the space coordinates, i.e. the space coordinates are comoving, they are merely labels for the world lines of the fluid.

Definition A.3 (Comoving normal coordinates)
If a coordinate system satisfies definition A.1 and A.2, it is said to be a comoving normal coordinate system.

A special type of comoving normal coordinates is given by the synchronous coordinates (also referred to as gaussian normal coordinates).

Definition A.4 (Synchronous coordinates)
If the coordinate time $t$ of a comoving normal coordinate system measures proper time along the lines of constant $(x^1, x^2, x^3)$, then it is called a synchronous coordinate system, since the surfaces of constant $t$ are (locally) surfaces of simultaneity for the observers who move with constant $(x^1, x^2, x^3)$.

Synchronous coordinates have very useful implications. Since the proper time along the flow lines of the fluid does not depend on the space coordinates $(x^1, x^2, x^3)$ we may write the line element for space-time in synchronous coordinates as

$$ds^2 = -\frac{1}{F^2(t)}dt^2 + g_{\alpha\beta}(t, x^\gamma)dx^\alpha dx^\beta.$$  \hspace{1cm} (A.67)
Rescaling the time coordinate via $d\hat{t} = F^{-1}(t)dt$ yields

$$ds^2 = -d\hat{t}^2 + g_{\alpha\beta}(\hat{t}, x^\gamma)dx^\alpha dx^\beta.$$  \hspace{1cm} (A.68)

This form, for instance, is often used in FRW models\(^\dagger\).

The normalised fluid flow in synchronous coordinates with coordinate time $\hat{t}$ is simply given by

$$u^\alpha = \delta^\alpha_0$$  \hspace{1cm} (A.69)

and is necessarily irrotational and geodesic since all derivatives of $u^\alpha$ vanish.

Consequently, if the fluid flow has non-zero vorticity it is not possible to find a global synchronous (or even normal) coordinate system. Nevertheless, it is always possible to locally choose normal coordinates on a sufficiently small neighbourhood of a flow line.

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\(^\dagger\)see chapter 3.
Appendix B

Conformal structures for the example space-times

In the discussions of the example models in chapters 6 and 7 we had omitted the details of the calculations. They will be found in this appendix.

B.1 Two closed FRW models

B.1.1 A radiation filled, closed FRW universe

The choice of the cosmic time function $\tilde{T}$ in (6.5) leads to

$$d\tilde{T} = \frac{dt}{2C\sqrt{2 - \frac{t}{\tilde{T}}}} \Rightarrow dt^2 = 4C^2\tilde{T}^2 d\tilde{T}^2$$ (B.1)

$$\tilde{T}^2 = 2 - \frac{t}{C} \Rightarrow a(t) = -C \tilde{T} \sqrt{2 - \tilde{T}^2}$$ (B.2)

$$ds^2 = -4C^2\tilde{T}^2 d\tilde{T}^2 + C^2\tilde{T}^2 (2 - \tilde{T}^2) d\sigma^2$$ (B.3)

where $\tilde{\Omega}(\tilde{T}) = -C \tilde{T}.$ (B.5)

GRTensorII calculated the expansion scalars as

$$\theta = -3 \frac{T^2 - 1}{C(T^4 - 2T^2)}, \quad \tilde{\theta} = \frac{3}{2} \frac{T}{T^2 - 2},$$ (B.6)

and the Ricci invariants as

$$R_{ab}R^{ab} = \frac{12}{C^4T^8(T^2 - 2)^4}, \quad \bar{R}_{ab}\bar{R}^{ab} = \frac{352 - 42T^2 + 9T^4}{4(T^2 - 2)^4}.$$ (B.7)

The Weyl curvature of the physical and unphysical space-times is identically zero throughout. The above expressions clearly imply the asymptotic behaviour stated in section 6.1.1.
B. Conformal structures for the example space-times

B.1.2 A dust, closed FRW universe

Putting $T = \phi$ and the scale factor into (3.1) gives

\[ dt = a(\phi) d\phi = a(\bar{T}) d\bar{T} \]

\[ ds^2 = \left[ \frac{\dot{C}}{2} (1 - \cos \bar{T}) \right]^2 d\bar{T}^2 + \left[ \frac{\dot{C}}{2} (1 - \cos \bar{T}) \right]^2 d\sigma^2 \]

with $\bar{\Omega}(\bar{T}) = \frac{\dot{C}}{2} (1 - \cos \bar{T})$.

Now, analysing the conformal factor, we find that

\[ \bar{\Omega}' = \frac{\dot{C}}{2} \sin \bar{T} \quad \text{and} \quad \bar{\Omega}'' = \frac{\dot{C}}{2} \cos \bar{T}, \]

\[ \frac{\bar{\Omega}'}{\bar{\Omega}} = \frac{1 - \cos \bar{T}}{\sin \bar{T}}, \]

\[ \bar{L} = \frac{\cos \bar{T}(1 - \cos \bar{T})}{\sin^2 \bar{T}}. \]

Using l'Hôpital's rule we obtain

\[ \lim_{\bar{T} \to 0^-} \frac{\bar{\Omega}'}{\bar{\Omega}} = \lim_{\bar{T} \to 0^-} \frac{\cos \bar{T}}{\sin \bar{T}} = -\infty \]

\[ \lim_{\bar{T} \to 0^-} \bar{L} = \lim_{\bar{T} \to 0^-} \frac{2 \cos \bar{T} - 1}{2 \cos \bar{T}} = \frac{1}{2}. \]

The following expressions were calculated using GRTensorII. The Ricci scalar is

\[ R_{ab} R^{ab} = \frac{192(4 \cos^2 \bar{T} - 2 \cos \bar{T} + 1)}{\bar{C}^4(\cos^6 \bar{T} - 6 \cos^5 \bar{T} + 15 \cos^4 \bar{T} - 20 \cos^3 \bar{T} + 15 \cos^2 \bar{T} - 6 \cos \bar{T} + 1)}, \]

which diverges to $+\infty$ at $\bar{T} = 0$, as can be easily verified numerically. The expansion scalar of the physical fluid flow is given by

\[ \theta = \frac{6 \sin \bar{T}}{\bar{C}(1 - \cos \bar{T})}. \]

L'Hôpital’s rule yields

\[ \lim_{\bar{T} \to 0^-} \theta = \lim_{\bar{T} \to 0^-} \frac{6 \cos \bar{T}}{\bar{C} \sin \bar{T}} = -\infty. \]

GRTensorII calculates $\bar{R}_{ab} \bar{R}^{ab} \equiv 12$, and shows that the scalars $\bar{\theta}, \bar{C}_{abcd} \bar{C}^{abcd}$ and $\bar{K}$ vanish identically.
B.2 The McVittie-Wiltshire models

B.2.1 McVittie-Wiltshire I

The conformal factor given in (6.26) satisfies
\[ \Omega' = \frac{b^{1/3}}{3T^{2/3}} \quad \text{and} \quad \Omega'' = -\frac{2b^{1/3}}{9T^{5/3}} \] (B.20)
\[ \lim_{T \to 0^-} \frac{\Omega'}{\Omega} = \frac{1}{3T} = -\infty \] (B.21)
\[ \bar{L} = \frac{\Omega''\Omega}{\Omega^2} = -2. \] (B.22)

GRTensorII determined the expressions for the expansion scalar and the Ricci invariant to be \((b > 0)\)
\[ \frac{\theta}{b^{2/3}T^{5/3}}, \quad R^b_a R^a_b = \frac{52}{27b^{8/3}T^{20/3}}. \] (B.23)

The Weyl curvature, as well as the respective quantities for the unphysical space-time, are found to be zero throughout.

We will now prove that the choice of \(a = 0\) makes this type of the McVittie-Wiltshire models a closed FRW model. Choose the scale factor \(a(t) = (bt)^{1/3} = A^{1/2}\) and solve
\[ dt^2 = (bt)^{4/3}dt^2 \Rightarrow t' = \frac{3b^{2/3}t^{5/3}}{5}. \] (B.24)

Substituting \(t'\) into (6.25) yields \(a(t') = (5bt'/3)^{1/5}\) (so this universe had an endless past!) and using \(t'\) as the comoving time coordinate, we find that the metric is given by (3.1) for the case \(k = 0\).

We can now also briefly show that this subclass (with \(a = 0\)) of the McVittie-Wiltshire models actually admits an IPS at which the fluid flow is regular if we choose \(t > 0\) and \(b > 0\). To this end we calculate
\[ (bt)^{2/3}dt^2 = dT^2 \Rightarrow T = \frac{3}{4}b^{1/3}t^{4/3}. \] (B.25)

Thus, \(T \geq 0\), and \(T = 0 \iff t = 0\), and substituting \(T\) into (6.25) gives \(A(T) = \sqrt{\frac{4}{3}bT}\). Substituting this into (6.25) provides a \(C^\infty\), non-degenerate conformal metric,
\[ ds^2 = \sqrt{\frac{4}{3}bT}[-dT^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \] (B.26)

\(^{\dagger}\) \(a\) being the constant in the metric, not to be confused with the scale factor.
with \( \Omega(T) = \left( \frac{4bT}{3} \right)^{1/4} \). Furthermore,

\[
\Omega' = \frac{\left( \frac{4b}{3} \right)^{1/4}}{4} T^{-3/4}, \quad \Omega'' = -\frac{3\left( \frac{4b}{3} \right)^{1/4}}{16} T^{-7/4},
\]

\[
\Rightarrow \frac{\Omega'}{\Omega} = \frac{1}{4T} \rightarrow +\infty \quad \text{as} \quad T \to 0^+, \quad \text{(B.27)}
\]

and \( L \equiv -3 \). \text{(B.28)}

From (B.26) it is evident that the fluid flow is regular at \( T = 0 \) and that (1) of definition 4.1 is also satisfied. Hence, this FRW subclass of the McVittie-Wiltshire models does admit an IPS at which the fluid flow is regular.

**B.2.2 McVittie-Wiltshire II**

We will try to solve ii) of (6.33). First calculate the homogeneous equation and set

\[
\psi_t = u(t)v(\psi) \Rightarrow \psi_t = vu' + u^2v \frac{dv}{d\psi}. \quad \text{(B.30)}
\]

Substituting into (6.33) gives

\[
vu' + u^2v \frac{dv}{d\psi} = \frac{a + 1}{a}u^2v^2 = 0 \quad \text{(B.31)}
\]

set \( u' = 0 \) \( \Rightarrow u = \text{const} \) \text{(B.32)}

\[
\frac{dv}{d\psi} = \frac{a + 1}{a}v \Rightarrow v = \text{const} \cdot e^{\frac{a+1}{a}\psi}. \quad \text{(B.33)}
\]

By (B.30)

\[
\psi_t = \frac{d\psi}{dt} = ke^{\frac{a+1}{a}\psi}, \quad k = \text{const}, \quad \text{(B.34)}
\]

therefore the homogeneous solution is given by

\[
\psi_H = -\frac{a}{a+1} \ln \left( -\frac{a+1}{a}kt + c_2 \right), \quad c_2 = \text{const}. \quad \text{(B.35)}
\]

Now we need to find a specific solution, so we can look at the solution of the following equation

\[
-\frac{a + 1}{a} \psi_{,t}^2 - \frac{1}{2} = 0, \quad \text{(B.36)}
\]

which will give a constant \( \psi_{,t} \) as a special solution to ii).

\[
\psi_{,t} = \pm \sqrt{-\frac{a}{2(a+1)}}, \quad \text{(B.37)}
\]

\[
\Rightarrow \psi_S(t) = \pm \sqrt{-\frac{a}{2(a+1)}}t + c_1, \quad c_1 = \text{const}. \quad \text{(B.38)}
\]
B.2. The McVittie-Wiltshire models

This solution is given for $a \in (-1, 0)$ (since we do not want any complex solutions). So the total solution of ii) is given by

$$\psi = \psi_H + \psi_S = \pm \sqrt{\frac{-a}{2(a+1)}} t + c_1 - \frac{a}{a+1} \ln \left( -\frac{a+1}{a} kt + c_2 \right). \quad (B.39)$$

To ensure that the factor

$$\exp \left[ 2\alpha(r) \right] \quad (B.40)$$

in the metric behaves well for finite $r$, we can again choose a special solution for $\alpha(r)$ from i). Take

$$-(a+1)\alpha_r^2 + \frac{1}{2} = 0. \quad (B.41)$$

Analogously to (B.38) we find a solution to this via

$$\alpha_S(r) = \pm \sqrt{\frac{1}{2(a+1)}} r + c_3, \quad (B.42)$$

(where $c_3 = \text{const}$) which is also given for $a \in (-1, 0)$ (we want to avoid complex solutions).

Now analyse the behaviour of the conformal factor given in (6.37) (recall the constraint on $a$).

\[
\begin{align*}
\tilde{\Omega}' &= e^\psi \psi' \quad \text{and} \quad \tilde{\Omega}'' = \left( \psi'^2 + \psi'' \right) e^\psi \\
\Rightarrow \quad \tilde{\Omega}' &= \psi' \quad \text{and} \quad \tilde{L} = 1 + \frac{\psi''}{\psi'^2}
\end{align*}
\]

$$\psi' = \pm \sqrt{-\frac{a}{2(a+1)}} - \frac{a}{a+1} \frac{1}{T}, \quad \Rightarrow \quad \psi'' = \frac{a}{a+1} \frac{1}{T^2} \quad (B.45)$$

$$\Rightarrow \quad \lim_{T \to 0} \frac{\tilde{\Omega}'}{\tilde{\Omega}} = \begin{cases} 
\infty & \text{if } k > 0, \\
-\infty & \text{if } k < 0. 
\end{cases} \quad (B.46)$$

$$\tilde{L} = 1 + \left( a+1 \right) \left( \pm \sqrt{\frac{a}{2(a+1)}} T - \frac{a}{a+1} \right)^2 \quad (B.47)$$

$$\Rightarrow \quad \lim_{T \to 0^\pm} \tilde{L} = 1 + \frac{a+1}{a} < 1. \quad (B.48)$$

The full expressions for $K$, $R_{ab} R^{ab}$ and $C_{abcd} C^{abcd}$ calculated by GRTensorII are only presented for the special case $a = -\frac{1}{2}$, $k = -1$ and $c_3 = 0$ ($c_1, c_2$ were already chosen to be 0), since they take very lengthy forms otherwise, without providing more information than given here. The overall behaviour of the general case of (6.37) can be checked with GRTensorII to be the same. The following expressions
hold for the choice of positive signs in both (6.34) and (6.36).

\[
R_{ab}R^{ab} = \frac{\bar{T}^4 - 4\sqrt{2}\bar{T}^3 + 30\bar{T}^2 + 12\sqrt{2}\bar{T} + 12}{T^8 \exp[4r + 2\sqrt{2}T]} \tag{B.49}
\]

\[
C_{abcd}C^{abcd} = \frac{4}{3}\bar{T}^4 \exp[4r + 2\sqrt{2}T] \tag{B.50}
\]

\[
K = \frac{4}{3} \frac{T^4}{T^4 - 4\sqrt{2}\bar{T}^3 + 30\bar{T}^2 + 12\sqrt{2}\bar{T} + 12} \tag{B.51}
\]

For different sign choices in (6.34) and (6.36) the signs of the odd powers of \(\bar{T}\) in the above expressions change. The equations are, in fact, exactly the same for \(k = +1\). For the unphysical space-time one furthermore finds that

\[
\bar{R}_{ab}\bar{R}^{ab} = 6 \exp[-4r], \quad \bar{C}_{abcd}\bar{C}^{abcd} = \frac{4}{3} \exp[-4r] \quad \Rightarrow \quad \bar{K} = \frac{2}{9}, \tag{B.52}
\]

using the above choice of constants. All these expressions clearly prove the asymptotic behaviour presented in section 6.2.2.

### B.3 A subclass of Szekeres models

We choose the cosmic time function as

\[
\bar{T} = -\frac{1}{T} \Rightarrow d\bar{T} = \frac{1}{T^2}dT. \tag{B.53}
\]

Clearly, by equation (7.4), the metric transforms into

\[
ds^2 = \frac{1}{81T^8} \left[ -d\bar{T}^2 + T^4 \left( dx^2 + dy^2 + Z^2 dz^2 \right) \right]. \tag{B.54}
\]

where \(Z\) is now

\[
Z = A + k + \frac{1}{9T^2}. \tag{B.55}
\]

The conformal factor given in equation (7.8) behaves as

\[
\Omega' = -\frac{4}{9T^5} \quad \text{and} \quad \Omega'' = \frac{20}{9T^6} \tag{B.56}
\]

\[
\frac{\bar{\Omega}'}{\bar{\Omega}} = -\frac{4}{T} \Rightarrow \lim_{T \to 0} \frac{\bar{\Omega}'}{\bar{\Omega}} = \infty \tag{B.57}
\]

\[
\bar{L} = \frac{\bar{\Omega}'\bar{\Omega}}{\bar{\Omega}^2} = \frac{5}{4} > 1. \tag{B.58}
\]
GRTensorII calculated the curvature and kinematic scalars of the physical space-time to

\[
R_a^b R_b^a = 157464 \tilde{T}^1 12 \frac{486A^2 \tilde{T}^4 + 180Ak_+ \tilde{T}^2 + 25k_+^2}{(9A\tilde{T}^2 + k_+)^2}, \quad (B.59)
\]

\[
C^{ab}_{\,cd} C_{ab}^{\,cd} = 34992 \tilde{T}^1 12 \frac{k_+^2}{(9A\tilde{T}^2 + k_+)^2}, \quad (B.60)
\]

\[
K = \frac{2}{9} \frac{486A^2T^4 + 180Ak_+T^2 + 25k_+^2}{4k_+^2}, \quad (B.61)
\]

\[
\theta = -\frac{18(27AT^2 + 4k_+)T^3}{9AT^2 + k_+}, \quad (B.62)
\]

\[
\sigma = 6\sqrt{6} \sqrt{\frac{k_+^2 \tilde{T}^2}{9A\tilde{T}^2 + k_+}}. \quad (B.63)
\]

These expressions clearly imply the asymptotic behaviour, presented in section 7.1.

From equation (B.62) and (B.63) one can obtain the following ratio of the non-zero kinematic quantities

\[
\frac{\sigma}{\theta} = \sqrt{\frac{2}{3}k_+} \sqrt{\frac{9A\tilde{T}^2 + k_+}{(27AT^2 + 4k_+)^2T^4}}, \quad (B.64)
\]

through which it becomes apparent that the ratio diverges as \( \tilde{T} \to 0^- \).

The expressions for the unphysical space-time are given by

\[
\tilde{K} = \frac{2}{3} \frac{k_+^2}{3402A^2T^4 + 324Ak_+T^2 + 11k_+^2}, \quad (B.65)
\]

\[
\tilde{R}_a^b \tilde{R}_b^a = \frac{8}{\tilde{T}^4} \frac{3402A^2T^4 + 324Ak_+T^2 + 11k_+^2}{(9A\tilde{T}^2 + k_+)^2}, \quad (B.66)
\]

\[
\tilde{C}^{ab}_{\,cd} \tilde{C}_{ab}^{\,cd} = \frac{16}{3} \frac{k_+^2}{\tilde{T}^4(9A\tilde{T}^2 + k_+)^2}, \quad (B.67)
\]

\[
\tilde{\theta} = \frac{2}{\tilde{T}(9A\tilde{T}^2 + k_+)}. \quad (B.68)
\]

One can readily deduce the asymptotic behaviour, presented in section 7.1, from these expressions.

By equation (7.7) we find the following expression for the determinant of the conformal metric

\[
\tilde{g} = -\left( A^2 \tilde{T}^1 12 + \frac{2k_+AT^{10}}{9} + \frac{k_+^2 \tilde{T}^8}{81} \right), \quad (B.69)
\]

and since \( \tilde{\Omega}^8 = 9^{-8} \tilde{T}^{-32} \) we realise that the absolute value of the determinant of the physical metric \( \Omega^8 |\tilde{g}| \) must diverge to \( +\infty \) at \( \tilde{T} = 0 \).
As mentioned in section 7.1 we could also choose a cosmic time function of the type

\[
\bar{T} = \arctan T - \frac{\pi}{2} \Rightarrow T = -\cot \bar{T} \tag{B.70}
\]

\[
dT^2 = (1 + \cot^2 T)^2 d\bar{T}^2 \tag{B.71}
\]

This again approaches \( T \to \infty \) as \( \bar{T} \to 0^- \). Now the metric looks like

\[
ds^2 = \frac{\cot^4 \bar{T} (1 + \cot^2 \bar{T})^2}{81} \left[ -d\bar{T}^2 + \frac{dx^2 + dy^2 + Z^2 dz^2}{(1 + \cot^2 \bar{T})^2} \right], \tag{B.72}
\]

where \( Z \) and the conformal factor are now given by

\[
Z = A + k_+ \frac{\cot^2 \bar{T}}{9}, \quad \bar{\Omega} (\bar{T}) = \frac{\cot^2 \bar{T} (1 + \cot^2 \bar{T})}{9}. \tag{B.73}
\]

At \( \bar{T} = 0 \) the metric has the shape

\[
ds^2 = \frac{\cot^4 \bar{T} (1 + \cot^2 \bar{T})^2}{81} \left[ -d\bar{T}^2 + \frac{k^2 dz^2}{81} \right], \tag{B.74}
\]

i.e. the conformal relation is basically the same as the one presented in section 7.1. The behaviour at \( \bar{T} = 0 \) is essentially equivalent, as only our cosmic time coordinate looks a bit different, but the way the conformal relation is arranged is exactly the same as before. Therefore we do not need to look at the behaviour of \( K \), the Ricci and Weyl curvature and the kinematic quantities at \( \bar{T} = 0 \), which will behave as shown in the first case. Another choice for a cosmic time function could be

\[
\bar{T} = -e^{-aT}, \quad a > 0 \Rightarrow T = -\frac{1}{a} \ln (-\bar{T}) \tag{B.75}
\]

\[
dT^2 = \frac{d\bar{T}^2}{a^2 T^2}. \tag{B.76}
\]

Proceeding exactly as before, we obtain

\[
ds^2 = \frac{\ln^4 (-\bar{T})}{a^6 T^2} \left[ -d\bar{T}^2 + \bar{T}^2 a^2 (dx^2 + dy^2 + Z^2 dz^2) \right], \tag{B.77}
\]

where

\[
Z = A + k_+ \frac{\ln^2 (-\bar{T})}{9a^2}, \quad \bar{\Omega} (\bar{T}) = \frac{\ln^2 (-\bar{T})}{a^3 \bar{T}}. \tag{B.78}
\]

Again, this conformal relation is in principal equivalent to the previous case and it is not instructive to perform any further investigations on its behaviour. It just emphasises that in this model we have quite a number of possible transformations at hand, to rescale the future metric singularity to \( \bar{T} = 0 \). There are of course many more.
B.4 Mars models

Choosing the cosmic time function as before

\[ \tilde{T} = -\frac{1}{T} \Rightarrow dT = \frac{1}{T^2} d\tilde{T}, \]  

and substituting this into equation (7.21) immediately yields the form of equation (7.24)

\[ ds^2 = -\frac{e^{2e^2 e^2aT^2}}{T^3} \left[\frac{dT^2}{(1+\epsilon T^2 + \beta T^6)} a^2 + T^2 dx^2 \right. \]
\[ \left. + T^2 e^{-ax+2ce^{ax} - e^2 e^2 aT^2} dy^2 + T^2 e^{-ax-2ce^{ax} - e^2 e^2 aT^2} dz^2 \right]. \]  

The time derivatives of the conformal factor behave as

\[ \dot{\Omega} = e^{\epsilon e^2 e^2 aT^2} \left[ \frac{e^{2e^2 e^2 aT^2}}{(-\tilde{T})^{9/2}} + \frac{3}{(-\tilde{T})^{5/2}} \right] \text{ and} \]
\[ \ddot{\Omega} = e^{\epsilon e^2 e^2 aT^2} \left[ \frac{e^{2e^2 e^2 aT^2}}{(-\tilde{T})^{15/2}} + \frac{15e^{2e^2 e^2 aT^2}}{2(-\tilde{T})^{11/2}} + \frac{15}{2(-\tilde{T})^{7/2}} \right]. \]  

Independent of the value of \( x \) this leads to

\[ \frac{\dot{\Omega}}{\Omega} = - \left[ \frac{e^{2e^2 e^2 aT^2}}{T^3} + \frac{3}{T} \right] \Rightarrow \lim_{T \rightarrow 0^-} \frac{\dot{\Omega}}{\Omega} = \infty \text{ and} \]
\[ \lim_{T \rightarrow 0^-} \dot{L} = 1. \]  

GRTensorII calculated the kinematic scalars of these models to

\[ \theta = \frac{a}{2(-\tilde{T})^{3/2}}(2e^{2e^2 e^2 aT^2} + 3T^2)\sqrt{1 + \epsilon T^2 + \beta T^6 e^{-\epsilon e^2 e^2 aT^2}}, \]  

\[ \sigma = \sqrt{\frac{2}{3}} \frac{ae^{2e^2 e^2 aT^2}}{2(-\tilde{T})^{3/2}} \sqrt{1 + \epsilon T^2 + \beta T^6 e^{2e^{2a} e^{-\epsilon e^2 e^2 aT^2}}}, \]  

\[ \dot{u}^a = -\frac{ae^{2e^2 e^2 aT^2}}{T} e^{2e^2 e^2 aT^2 - 2ce^{ax} e^{-\epsilon e^2 e^2 aT^2}} \delta^a_3, \]  

which clearly imply the asymptotic behaviour presented in section 7.2. Using these equations one can easily calculate

\[ \frac{\sigma}{\theta} = \sqrt{\frac{2}{3}} \frac{e^{2e^2 e^2 aT^2}}{2e^{2e^2 e^2 aT^2} + 3T^3}, \]  

\[ \frac{\dot{u}_a \dot{u}_a}{\theta^2} = \frac{4e^{4e^2 e^2 aT^2 - 2ce^{ax}}}{(2e^{2e^2 e^2 aT^2} + 3T^2)^2(1 + \epsilon T^2 + \beta T^6)}, \]  

and thus

\[ \lim_{T \rightarrow 0^-} \frac{\sigma}{\theta} = \sqrt{\frac{1}{6}}, \quad \lim_{T \rightarrow 0^-} \frac{\dot{u}_a \dot{u}_a}{\theta^2} = e^{-ax - 2ce^{ax}}. \]
The expressions for $K$, $R^a_b R^b_a$ and $C^{ab}_{cd} C^{cd}_{ab}$ are rather lengthy and will not be given here in detail. The approximate behaviour is

$$R^a_b R^b_a \propto \bar{T}^2 e^{-2a \epsilon^2 e^{2ax}/T^2} \rightarrow 0 \text{ as } \bar{T} \rightarrow 0^-.$$  \hfill (B.91)

$$C^{ab}_{cd} C^{cd}_{ab} \propto -e^{4ax-2ae^{2ax}/T^2} \rightarrow 0 \text{ as } \bar{T} \rightarrow 0^-.$$  \hfill (B.92)

The same is true for the expressions in the unphysical space-time. We will only present the approximate behaviour of $C^{abcd} C^{abcd}$,

$$C^{abcd} C^{abcd} \propto -e^{6ax} a^4 e^6 e^3 T^6 \rightarrow \begin{cases} -\infty & \text{as } \bar{T} \rightarrow 0^- \text{ and } x \neq -\infty, \\ 0 & \text{as } \bar{T} \rightarrow 0^- \text{ and } x = -\infty. \end{cases}$$  \hfill (B.93)

Equation (7.24) implies that the determinant of the unphysical metric is given by

$$\bar{g} = -\frac{\bar{T}^6}{a^2(1 + \epsilon^2 T^2 + \beta T^6)} e^{-2ax -2ae^{2ax}/T^2},$$  \hfill (B.94)

and since $\bar{\Omega}^8 = \bar{T}^{-12} \exp[4ae^{2ax}/T^2]$ we see that the absolute value of the determinant of the physical metric, $\bar{\Omega}^8 |\bar{g}|$, diverges to $+\infty$ as $\bar{T} \rightarrow 0^-.$

### B.5 Carneiro-Marugan model

Substituting the cosmic time function

$$\bar{T} = -\frac{1}{\eta}, \Rightarrow \ dy = dT \frac{T^2}{T^2},$$  \hfill (B.95)

into equation (7.35) and factoring out all diverging terms gives the desired conformal relation

$$ds^2 = a^2(\bar{T}) \left[ -\frac{dT^2}{T^4} + dx^2 + e^{2x} dy^2 + dz^2 \right]$$  \hfill (B.96)

$$= \frac{a^2(\bar{T})}{T^4} [-dT^2 + T^4(dx^2 + e^{2x} dy^2 + dz^2)],$$  \hfill (B.97)

where

$$a(T) = \frac{D}{3} \left[ \cosh \left( -\frac{1}{\sqrt{2}T} \right) - 1 \right] + \sqrt{\frac{2}{3}} A \sinh \left( -\frac{1}{\sqrt{2}T} \right),$$  \hfill (B.98)

which itself diverges as $T \rightarrow 0^-.$
B.6. The Kantowski models

The conformal factor given by equation (7.42) follows

\[ \bar{\Omega}' = -2 \frac{a(T)}{T^3} + \frac{1}{T^2} \left[ \frac{D}{3\sqrt{2T^2}} \sinh \left( -\frac{1}{\sqrt{2T}} \right) + \frac{A}{\sqrt{3T^2}} \cosh \left( -\frac{1}{\sqrt{2T}} \right) \right] \]

(B.99)

\[ \Rightarrow \frac{\bar{\Omega}'}{\bar{\Omega}} = -2 \frac{T}{T^2} \left[ \frac{D}{3\sqrt{2T^2}} \sinh \left( -\frac{1}{\sqrt{2T}} \right) + \frac{A}{\sqrt{3T^2}} \cosh \left( -\frac{1}{\sqrt{2T}} \right) \right] \]

(B.100)

\[ \Rightarrow \lim_{T \to 0^-} \frac{\bar{\Omega}'}{\bar{\Omega}} = \infty \quad \text{(B.101)} \]

and \( \lim_{T \to 0^-} \bar{L} > 1 \).

In the following, we will only give the detailed expressions of the unphysical quantities, since the expressions for the physical quantities exceeded the display in GRTensorII. We only note, that by approaching \( \bar{T} \to 0^- \) with particular \( \bar{T} \) values, GRTensorII calculated \( R_{ab}R^{ab}, C_{abcd}C^{abcd} \) and \( \bar{\theta} \) to vanish and \( K \) to take some finite value \( K_0 \), depending on the constants \( A, D \). We find the following expressions

\[ C_{abcd}C^{abcd} = \frac{4}{3T^8}, \quad \text{(B.103)} \]

\[ R_{ab}R^{ab} = \frac{2}{T^8} \left( 168 \bar{T}^4 - 20 \bar{T}^2 + 1 \right), \quad \text{(B.104)} \]

\[ K = \frac{2}{3} \frac{1}{168 \bar{T}^4 - 20 \bar{T}^2 + 1}, \quad \text{(B.105)} \]

\[ \bar{\theta} = \frac{6}{T}, \quad \text{(B.106)} \]

which readily imply the asymptotic behaviours given in section 7.3.

By equation (7.41), the determinant of the conformal metric takes the form

\[ \bar{\Omega}^8 = a^8(\bar{T})/\bar{T}^{16} \]

but since \( \bar{\Omega}^8 = a^8(\bar{T})/\bar{T}^{16} \) one readily sees that the absolute value of the determinant of the physical metric \( \bar{\Omega}^8|\bar{g}| \) diverges to \( +\infty \) as \( \bar{T} \to 0^- \).

B.6 The Kantowski models

The cosmic time function given in equation (7.55) provides

\[ \bar{T} = -A^{-1} = -\frac{1}{1 + \frac{4b^2}{9t}} \Rightarrow t = -\frac{9}{4b^2} \left( 1 + \frac{1}{\bar{T}} \right) \Rightarrow dt = \frac{9}{4b^2T^2}d\bar{T}. \quad \text{(B.108)} \]

\[ ^{\dagger} \text{The second derivative and the full expression for } \bar{L} \text{ will not be given here, since both are very long and do not provide any interesting information.} \]
Thus, substituting this into equation (7.51) and factoring out the diverging terms, gives equation (7.56)

\[ ds^2 = -\frac{81}{16b^4(-T)^5}dT^2 - \frac{9}{4b^2}\left(1 + \frac{1}{T}\right)\left[-Tdx^2 + \frac{1}{(-T)^2b^2}(dy^2 + f^2dz^2)\right] \]

which, after some simplification, results in equation (B.109)

\[ ds^2 = \frac{1}{(-T)^5}\left[-\frac{81}{16b^4}(\bar{T}^2 + 9)\{(-T)^3dx^2 + b^{-2}(dy^2 + f^2dz^2)\}\right]. \]

Now we analyse the conformal factor in equation (7.57), in order to prove the asymptotic behaviour presented in equation (7.59).

\[ \Omega' = \frac{5}{2}(-T)^{-7/2} \quad \Rightarrow \quad \Omega'' = \frac{35}{4}(-T)^{-9/2} \]

\[ \frac{\bar{\Omega}'}{\bar{\Omega}} = -\frac{5}{2T} \quad \text{and} \quad \bar{L} = \frac{\Omega''\bar{\Omega}}{\bar{\Omega}''} = \frac{7}{5}. \]

GRTensorII, furthermore, calculated the following expressions for the curvature and non-zero kinematic scalars of the physical space-time

\[ \theta = -\frac{2b^2(\bar{T} + 2)(-\bar{T})^{5/2}}{3T^2 + T}, \]

\[ \sigma = \frac{2}{9}\sqrt{6b^2(-\bar{T})^{3/2}}, \]

\[ R_{ab}R^{ab} = \frac{64b^8\bar{T}^8}{2187(T + 1)^4}, \]

\[ C_{abcd}C^{abcd} = \frac{1024b^8\bar{T}^8}{2187(T + 1)^2}, \]

\[ K = 16(T + 1)^2. \]

By equation (B.113) and (B.114), one obtains

\[ \frac{\sigma}{\bar{\theta}} = \sqrt{\frac{2\bar{T} + 1}{3\bar{T} + 2}} \quad \Rightarrow \quad \lim_{T \to 0^-} \frac{\sigma}{\bar{\theta}} = \sqrt{\frac{1}{6}}. \]

For the unphysical space-time, GRTensorII calculated

\[ \bar{\theta} = \frac{2b^2(4\bar{T} + 3)}{3T^2 + T}, \]

\[ \bar{R}_{ab}\bar{R}^{ab} = \frac{64b^8 \frac{525T^4 + 1520\bar{T}^3 + 1596\bar{T}^2 + 725\bar{T} + 125}{2187}}{T^4(T + 1)^4}, \]

\[ \bar{C}_{abcd}\bar{C}^{abcd} = \frac{1024b^8}{2187T^2(T + 1)^2}, \]

\[ \bar{K} = 16 \frac{T^2(T + 1)^2}{525T^4 + 1520T^3 + 1596T^2 + 725T + 125}. \]
The above expressions imply the asymptotic behaviour, presented in section 7.4.1. Equation (7.56) implies the following form of the determinant of the conformal metric

$$g = \frac{9(81)^2 \sinh^2 y}{4(16)^2b^4}(\tilde{T}^{12} + 3\tilde{T}^{11} + 3\tilde{T}^{10} + \tilde{T}^9),$$  \hspace{1cm} (B.123)

and since $\Omega^8 = T^{-20}$ we immediately see that the absolute value of the determinant of the physical metric, $|\tilde{\Omega}|$, diverges to $+\infty$ as $\tilde{T} \to 0$.

### B.7 Kantowski-Sachs models

The cosmic time function given in equation (7.66), leads to

$$\tilde{T} = A^2 = \left(1 - \frac{4b^2t}{9}\right)^2 \Rightarrow t = \frac{9}{4b^2} \left(1 - \sqrt{-\tilde{T}}\right)$$  \hspace{1cm} (B.124)

and $d\tilde{T} = 2 \left(1 - \frac{4b^2t}{9}\right) \left(\frac{4b^2}{9}\right) dt = \frac{8b^2}{9} \sqrt{-\tilde{T}} dt$. \hspace{1cm} (B.125)

Substituting this into equation (7.51) immediately yields equation (7.67)

$$ds^2 = \frac{1}{\sqrt{-T}} \left[-\frac{81}{64b^4}d\tilde{T}^2 + \left(1 - \sqrt{-T}\right) \frac{9}{4b^2} \left[dx^2 + \left(-T\right)^{3/2} b^{-2} (dy^2 + f^2dz^2)\right]\right].$$  \hspace{1cm} (B.126)

The conformal factor in equation (7.68) follows

$$\Omega' = \frac{1}{4 \left(-T\right)^{5/4}} \quad \text{and} \quad \Omega'' = \frac{5}{6 \left(-T\right)^{9/4}},$$  \hspace{1cm} (B.127)

$$\frac{\Omega'}{\Omega} = -\frac{1}{4\tilde{T}} \Rightarrow \lim_{T \to 0^-} \frac{\Omega'}{\Omega} = \infty,$$  \hspace{1cm} (B.128)

and $\bar{L} = \frac{\Omega''\Omega}{\Omega^2} \equiv \frac{40}{3}$. \hspace{1cm} (B.129)

The kinematic and curvature scalars of the physical space-time have been calculated by GRTensorII to be

$$\theta = \frac{2b^2}{3} \frac{-2\tilde{T} - 3\sqrt{-\tilde{T}} + 1}{(-\tilde{T})^{7/4} + 2(-\tilde{T})^{5/4} - (-\tilde{T})^{3/4}},$$  \hspace{1cm} (B.130)

$$\sigma = \frac{2\sqrt{6b^2}}{9} \sqrt{1 - 4\sqrt{-\tilde{T}} - 6\tilde{T} - 4\tilde{T}^{3/2} + \tilde{T}^2 - \tilde{T}(\tilde{T})^{3/2} + 2\tilde{T} + \sqrt{-T})^2},$$  \hspace{1cm} (B.131)

$$R_{\alpha\beta}R^{\alpha\beta} = \frac{64b^8 \tilde{T}^2 - 4 \left(-\tilde{T}\right)^{3/2} - 6\tilde{T} - 4\sqrt{-T} + 1}{2187 \left(\sqrt{-T} - 1\right)^4 \left(\sqrt{-T} + \tilde{T}\right)^4},$$  \hspace{1cm} (B.132)

$$C_{abcd}C^{abcd} = \frac{1024b^8}{2187} \times$$
\[
\frac{\bar{T}^3 + 6 \left( -\bar{T} \right)^{5/2} - 15\bar{T}^2 + 20 \left( -\bar{T} \right)^{3/2} + 15\bar{T} + 6\sqrt{-\bar{T}} - 1}{\bar{T}^3 \left( \bar{T}^4 - 8 \left( -\bar{T} \right)^{7/2} - 28\bar{T}^3 - 56 \left( -\bar{T} \right)^{5/2} + 70\bar{T}^2 - 56 \left( -\bar{T} \right)^{3/2} - 28\bar{T} - 8\sqrt{-\bar{T}} + 1 \right)}
\]

(B.133)

\[
K = \frac{16 \bar{T}^3 + 6 \left( -\bar{T} \right)^{5/2} - 15\bar{T}^2 + 20 \left( -\bar{T} \right)^{3/2} + 15\bar{T} + 6\sqrt{-\bar{T}} - 1}{\bar{T} \left( \bar{T}^2 - 4 \left( -\bar{T} \right)^{3/2} - 6\bar{T} - 4\sqrt{-\bar{T}} + 1 \right)}.
\]

(B.134)

Combining equations (B.130) and (B.131) gives

\[
\sigma \theta = \sqrt{\frac{2}{3}} \sqrt{\frac{\bar{T}^2 - 4 \left( -\bar{T} \right)^{3/2} - 6\bar{T} - 4\sqrt{-\bar{T}} + 1}{-12 \left( -\bar{T} \right)^{3/2} - 13\bar{T} - 6\sqrt{-\bar{T}} + 5}}
\]

(B.135)

\[
\implies \lim_{\bar{T} \to 0^-} \frac{\sigma}{\theta} = \sqrt{\frac{2}{15}},
\]

(B.136)

for the only non-zero ratio of the kinematic quantities.

GRTensorII furthermore gave the following expressions for the unphysical curvature scalars and the unphysical expansion scalar

\[
\bar{\theta} = \frac{2b^2(3\bar{T} + 5\sqrt{-\bar{T}} - 2)}{3(\bar{T}^2 - 2\left( -\bar{T} \right)^{3/2} - \bar{T})},
\]

(B.137)

\[
\bar{C}_{abcd}\bar{C}^{abcd} = -\frac{1024b^8 \bar{T}^3 + 6 \left( -\bar{T} \right)^{5/2} - 15\bar{T}^2 + 20 \left( -\bar{T} \right)^{3/2} + 15\bar{T} + 6\sqrt{-\bar{T}} - 1}{2187 \bar{T}^2 \left( \sqrt{-\bar{T}} - 1 \right)^4 \left( \bar{T} + \sqrt{-\bar{T}} \right)^4},
\]

(B.138)

\[
\bar{R}_{ab}\bar{R}^{ab} = \frac{64b^8}{2187} \times \frac{23\bar{T}^4 - 157 \left( -\bar{T} \right)^{7/2} - 470\bar{T}^3 - 808 \left( -\bar{T} \right)^{5/2} + 876\bar{T}^2 - 617 \left( -\bar{T} \right)^{3/2} - 278\bar{T} - 74\sqrt{-\bar{T}} + 9}{\bar{T}^2 \left( \sqrt{-\bar{T}} - 1 \right)^4 \left( \bar{T} + \sqrt{-\bar{T}} \right)^4},
\]

(B.139)

The above expressions imply the asymptotic behaviour, shown in section 7.4.2, as can be checked numerically.

By equation (7.67), we find the following for the determinant of the unphysical metric

\[
\bar{g} = \frac{9(81)^2 f^2}{\left(64b^8\right)^{1/4}} \left[ \left( -\bar{T} \right)^{9/2} - 3\bar{T}^4 + 3\left( -\bar{T} \right)^{7/2} + \bar{T}^3 \right],
\]

(B.141)

and since \( \Omega^8(\bar{T}) = \bar{T}^{-2} \) it is obvious that

\[
\lim_{\bar{T} \to 0^-} \bar{g} = \lim_{\bar{T} \to 0^-} \Omega^8 \bar{g} = 0.
\]

(B.142)
References


