Analogue Hawking Radiation, Superradiance and Sonic Horizons in Bose-Einstein Condensates

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A thesis submitted for the degree of
Bachelor of Philosophy with Honours in Theoretical Physics at
The Australian National University

May, 2005

Declaration

This thesis is an account of research undertaken between July 2004 and May 2005 at the Department of Physics, Faculty of Science, The Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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May, 2005

Acknowledgements

The past year has been a very rewarding experience, if somewhat hectic at times. I would like to thank my supervisor, Dr. Craig Savage, for his support and encouragement, and his willingness to spend hours discussing my project. I would also like to acknowledge the support provided by my family and friends. Particular thanks go to my mother Robyn for agreeing to proof-read multiple drafts of my Honours thesis. Finally, I would like to thank the other members of the Physics Department and the Australian Centre for Quantum Atom Optics for their encouragement and advice, in particular Sebastian Wuester, Susan Scott, Hans Bachor, Joseph Hope and John Close.

Abstract

The propagation of fluctuations in the velocity potential of an inviscid, irrotational fluid is governed by a wave equation very similar to that governing the propagation of massless scalar fields in curved spacetime. This similarity allows certain fluid systems to be used as experimentally realisable models for objects that cannot be studied under controlled conditions, such as black holes and ultrarelativistic rotating stars.

The analogy applies only to the propagation of fields in a fixed background, not to the behaviour of the background itself. However, even this limited similarity has excited a great deal of interest from researchers, since it suggests that phenomena such as superradiance and Hawking radiation may have analogues in superfluids.

In recent years, experimental advances in the creation and manipulation of Bose-Einstein condensates have provided researchers with very low-temperature systems possessing a uniquely simple microscopic theory. Furthermore, under certain long-length-scale approximations, Bose-Einstein condensates can be described as classical irrotational inviscid fluids, and therefore can be treated as analogues of appropriate gravitational systems.

We present a comprehensive and detailed critical review of the mean-field theory of Bose-Einstein condensates, the conditions under which Bose-Einstein condensates can be treated as an analogue gravity system, the construction of analogue black holes, and several derivations of Hawking radiation and analogue Hawking radiation. The interdisciplinary nature of analogue gravity means that the relevant results are widely scattered in the literature, and we consider this critical review and discussion of the relevant literature to be a central and noteworthy element of the project.

We then analyse in detail a recent derivation of analogue Hawking radiation in Bose-Einstein condensates as a consequence of quantum depletion. We elucidate the methodology and the physical meaning of the results by comparison with the derivations of Hawking radiation considered previously, and discuss possible extensions and improvements.

We also present an original investigation of analogue superradiance in simple vortices, with the vortex density profile chosen to mimic a realistic vortex in a Bose-Einstein condensate. We find that superradiant amplification occurs and is likely to be significant, but our long-length-scale approximation is of dubious validity near the scattering point.

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Chapter 1

Introduction

1.1 Overview

Over the past century Einstein's General Theory of Relativity has been enormously successful, passing every experimental test set for it. Unfortunately, some of the more surprising and interesting predictions of the theory are difficult or impossible to test under controlled conditions, because they occur only in the presence of very massive or rapidly accelerating bodies.

In particular, the General Theory of Relativity allows the formation of *black holes*, regions in spacetime from which not even light can escape [1, 2]. In 1974, Hawking [3] investigated the behaviour of quantum fields surrounding a star collapsing to a black hole. He predicted that the black hole would radiate particles in a thermal spectrum, behaving as a perfect black body of a temperature dependent on the black hole mass.

While fascinating, for many years this prediction seemed to be beyond experimental investigation. In 1981, however, Unruh [4] developed an analogy between fields propagating in curved space-time on the one hand, and fluctuations of the velocity potential in inviscid irrotational fluids on the other. Unruh demonstrated that the derivation of Hawking radiation could be carried across to the fluid system analogous to a spherically symmetric black hole, but the predicted temperature of the resulting radiation was far too low to be measurable.

The past ten years have seen the development of experimental techniques for creating and manipulating *Bose-Einstein condensates*, gases of bosons in which a single quantum state is macroscopically occupied [5, 6, 7, 8, 9]. Such systems have a uniquely simple microscopic theoretical description, and are created at extremely low temperatures. Furthermore, under certain conditions the condensate can be described as an inviscid irrotational fluid like those considered by Unruh. This combination of properties suggests that Bose-Einstein condensates may be suitable as an analogue gravity system, and in particular might exhibit a measurable analogue of Hawking radiation. The prospect of experimental investigation of the Hawking effect, and related effects such as black hole superradiance, has sparked renewed interest in analogue gravity over the past few years.

Strictly speaking, Unruh's analogy between fluid systems and curved spacetime can only be applied to Bose-Einstein condensates when the length scale of density variation in the condensate, and also the wavelengths of any density perturbations, are large compared to the characteristic *coherence length* of the condensate. In the presence of steep density gradients or short wavelength fluctuations, the approximations required for analogue gravity break down [5, 10].

Consequently, a rigorous investigation into analogues of Hawking radiation (and related effects) in Bose-Einstein condensates should treat the analogy with astrophysical systems as a motivation only. The analogy suggests that an effect analogous to Hawking radiation might occur, but to confirm the existence of such an effect and study its properties, the detailed microscopic theory of Bose-Einstein condensates must be invoked.

We initially believed that no one had yet employed the microscopic theory of Bose-Einstein condensates to treat analogue Hawking radiation, although there had been some proposals for creating analogue black holes in Bose-Einstein condensates which employed the microscopic theory to analyse questions of stability [11, 12]. The goal of this project was to fill that gap. A literature search eventually revealed one paper [13] which interpreted analogue Hawking radiation as a consequence of depletion of the condensate, a well-known effect in Bose-Einstein condensates.

In this thesis we draw together a number of disparate strands from the literature, outlining the derivation of Unruh's analogy between inviscid irrotational fluids and curved spacetime and then developing the analogy from the fundamental theory for Bose-Einstein condensates. We review a number of proposals for the creation of black hole analogues, first in general systems and then specialising to Bose-Einstein condensates. We then outline several derivations of the Hawking effect in astrophysics and analogues of Hawking radiation in condensed-matter systems, and discuss the relationships between the various derivations. Finally we employ these methods to elucidate Leonhardt's interpretation of Hawking radiation as depletion of a Bose-Einstein condensate [13], and discuss modifications to this approach.

This work is a continuation of an Advanced Studies Course I completed in 2004, in which I investigated analogue superradiance in simple vortices. The numerical work for that project was completed as the first part of my Honours project, and will be reviewed in this document. Our results have been submitted for publication, and may be viewed online [14]. Since this work was made public on the arXiv site in January 2005, it has been cited on eight occasions, according to Citebase.

1.2 Thesis Plan

In Chapter 2 we present a derivation of the analogy between phase fluctuations in irrotational fluids and scalar fields in curved space-time, including a discussion of the approximations and limitations of the analogy. The definitions of a sonic horizon and an analogue ergoregion are stated and discussed. Hawking radiation and superradiance are described, and the possibility of observing analogous effects in fluid systems is briefly discussed.

Several published proposals for creating sonic horizons in general fluid systems are reviewed in Chapter 3. A criterion for regularity at the ergoregion boundary is given. Specific examples of sonic horizon configurations in systems other than Bose-Einstein condensates are outlined and discussed.

The advantages of a Bose-Einstein condensate as an analogue gravity system are outlined as motivation for Chapter 4, which deals with the microscopic theory of Bose-Einstein condensates. We first derive the semiclassical theory that generates analogue gravity, and discuss the additional approximations necessary in the special case of a Bose-Einstein condensate. We then analyse the resulting approximate wave equation in several special cases: in particular, we extend the solution used by Leonhardt et al [13] to the case of a time-dependent background and varying speed of sound, and examine the case in which a sonic horizon is generated by a varying interaction strength.

Having developed the theory of analogue gravity in Bose-Einstein condensates, in Chapter 5 we present an original investigation of analogue superradiance in a system resembling a realistic vortex in a Bose-Einstein condensate. An exact criterion for the onset of superradiance is derived, and the magnitude of the superradiant amplification is investigated numerically. We employ a long-length-scale approximation and give estimates for where the approximation breaks down, based on the results of Chapter 4.

In Chapter 6 we review the quantum theory of Bose-Einstein condensates, expressing the system as a macroscopically occupied condensed state plus a small quantum field of excitations. We diagonalise the part of the Hamiltonian corresponding to this excitation field and derive the equations of motion for the excitations. Approximate solutions are derived and discussed, particularly in the context of sonic horizons. The proper treatment of solutions with negative norm is discussed.

A number of proposals for the creation of sonic horizons in Bose-Einstein condensates are outlined in Chapter 7. Garay et al [11, 12] employ the quantum theory of Bose-Einstein condensates to analyse the stability of their proposed configurations. In this chapter we also outline the forms of instability associated with sonic horizons in Bose-Einstein condensates, and describe the onset of anomalous modes with positive norm and negative energy.

Having discussed the general theory of analogue gravity and the specific properties of our candidate system, in Chapter 8 we outline several derivations of Hawking radiation. Beginning with an outline of Hawking's original calculation, we review the methods used by Unruh, Corley and Jacobson [15, 16, 17, 18] to study Hawking radiation in the case where the high-wavenumber behaviour of the relevant modes is modified.

One of the most popular heuristic descriptions of Hawking radiation involves the creation of particle-antiparticle pairs and tunnelling of one partner through the event horizon. In Chapter 9 we review a recent semiclassical derivation of Hawking radiation as a tunnelling effect [19], and discuss the relevance of this work to fluid analogue systems. Two derivations of Hawking radiation as a tunnelling effect in fermionic superfluid systems are reviewed and compared to the methods presented in Chapter 8.

Chapter 10 presents a formalism derived by Leonhardt et al [13] for analogue Hawking radiation in Bose-Einstein condensates. Detailed calculations are given, and extensions of the method to other sonic horizon configurations are considered. The methodology is compared to the other derivations of Hawking radiation given earlier and an explanation of the physical origin of analogue Hawking radiation in Bose-Einstein condensates is given.

We conclude with a summary of the work presented in this thesis, and a discussion of possible future extensions.

Chapter 2

Superfluid Flow and Curved Spacetime

2.1 Principles of the Hydrodynamic Analogy

There is a remarkable analogy between massless minimally coupled scalar fields propagating in (3+1)-dimensional curved spacetime and acoustic fluctuations of the velocity potential in barotropic, inviscid, irrotational fluids, first noted by Unruh [4] and reviewed and extended by Visser [20, 21]. Assuming that in the second case the fluid is described by the continuity equation, the zero-viscosity Euler equation and some barotropic equation of state, both fields are described by the Klein-Gordon equation,

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right) = 0. \tag{2.1}$$

In the hydrodynamic case, the spacetime metric $g_{\mu\nu}$ is replaced by an effective metric which can be calculated from the density and velocity profiles of the background fluid.

This suggests the possibility of manipulating fluid flows to create effective metrics that mimic spacetime metrics derived from general relativity. The behaviour of quantum fields in various metrics could then be studied by observing sound waves in the analogue system. While the analogy is not perfect, such an approach might yield a better understanding of astrophysical phenomena that at present cannot be observed, much less created under controlled conditions.

2.1.1 The Klein-Gordon Equation

We consider a non-relativistic, irrotational and barotropic fluid. Here “irrotational” means that the velocity field is curl-free, while “barotropic” means that the pressure at any point in the fluid is a function of the density only. The requirement of barotropicity guarantees that an initially irrotational fluid will remain irrotational [20]. As the velocity field is curl-free, there is a scalar potential θ such that $\vec{v} = \nabla\theta$.

Assuming zero viscosity, the equations of motion for the system are the continuity equation (Equation 2.2), the zero-viscosity Euler equation (Equation 2.3), and some barotropic equation of state (Equation 2.4).

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0, \tag{2.2}$$

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p, \tag{2.3}$$

$$p = p(\rho). \tag{2.4}$$

We now select some exact background solution to this system of equations, representing the bulk flow of the liquid, and consider small fluctuations around this background field, denoted $(\hat{\rho}, \hat{p}, \hat{\theta})$. Then letting c denote the speed of sound, defined by $c^2 = \partial p / \partial \rho$, the equations of motion for these fluctuations become

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot \left(\hat{\rho} \nabla \theta + \rho \nabla \hat{\theta} \right) = 0, \tag{2.5}$$

$$\rho \left(\frac{\partial \hat{\theta}}{\partial t} + \nabla \theta \cdot \nabla \hat{\theta} \right) = \hat{p}, \tag{2.6}$$

$$\hat{p} = c^2 \hat{\rho}. \tag{2.7}$$

These differential equations can be combined into a single second-order differential equation,

$$\frac{\partial}{\partial t} \left(\frac{\rho}{c^2} \left(\frac{\partial \hat{\theta}}{\partial t} + \vec{v} \cdot \nabla \hat{\theta} \right) \right) = \nabla \cdot \left(\rho \nabla \hat{\theta} - \frac{\rho \vec{v}}{c^2} \left(\frac{\partial \hat{\theta}}{\partial t} + \vec{v} \cdot \nabla \hat{\theta} \right) \right). \tag{2.8}$$

Now let us introduce four-dimensional coordinates,

$$(t, x, y, z) \to (x_0, x_1, x_2, x_3).$$

Then the previous differential equation can be written in the form

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-g} \partial_\nu \hat{\theta} \right) = 0, \tag{2.9}$$

where $g^{\mu\nu}$ is the effective inverse metric given by

$$g^{\mu\nu} = \frac{1}{\rho c} \left(\begin{array}{ccc} -1 & \vdots & -v_j \\ \cdots & \cdots & \cdots \\ -v_i & \vdots & (c^2 \delta_{ij} - v_i v_j) \end{array} \right) \tag{2.10}$$

and $g = [det\,(g^{\mu\nu})]^{-1}$.

This is exactly the Klein-Gordon equation, describing a minimally coupled massless scalar field propagating in a spacetime with inverse metric $g^{\mu\nu}$. Note that the Einstein summation convention (the presence of one raised and one lowered index indicating implicit summation over that index) is used in this equation.

2.1.2 The Effective Metric

In Cartesian coordinates, the metric, inverse metric and metric determinant are given by

$$g_{\mu\nu} = \frac{\rho}{c} \left(\begin{array}{ccc} -(c^2 - v^2) & \vdots & -v_j \\ \cdots & \cdots & \cdots \\ -v_i & \vdots & \delta_{ij} \end{array} \right), \tag{2.11}$$

$$g^{\mu\nu} = \frac{1}{\rho c}\begin{pmatrix} -1 & \vdots & -v_j \\ \cdots & \cdots & \cdots \\ -v_i & \vdots & (c^2\delta_{ij} - v_i v_j) \end{pmatrix}, \tag{2.12}$$

$$g = -\frac{\rho^4}{c^2}. \tag{2.13}$$

In some circumstances it is easier to work in cylindrical polar coordinates. In this case the four-dimensional coordinates are related to the polar coordinates by

$$(t, r, \theta, z) \rightarrow (x_0, x_1, x_2, x_3)\,.$$

The metric, inverse metric and metric determinant then become

$$g_{\mu\nu} = \frac{\rho}{c}\begin{pmatrix} -(c^2 - v^2) & -v_r & -rv_\theta & -v_z \\ -v_r & 1 & 0 & 0 \\ -rv_\theta & 0 & r^2 & 0 \\ -v_z & 0 & 0 & 1 \end{pmatrix}, \tag{2.14}$$

$$g^{\mu\nu} = \frac{1}{\rho c}\begin{pmatrix} -1 & -v_r & -\frac{v_\theta}{r} & -v_z \\ -v_r & c^2 - v_r^2 & -\frac{v_r v_\theta}{r} & -v_r v_z \\ -\frac{v_\theta}{r} & -\frac{v_r v_\theta}{r} & \frac{c^2 - v_\theta^2}{r^2} & -\frac{v_z v_\theta}{r} \\ -v_z & -v_r v_z & -\frac{v_z v_\theta}{r} & c^2 - v_z^2 \end{pmatrix}, \tag{2.15}$$

$$g = -\frac{\rho^4}{c^2} r^2. \tag{2.16}$$

2.1.3 Sonic Horizons and Ergoregions

Investigations into the hydrodynamic analogy have been motivated by the hope of using fluid systems to model astrophysical phenomena such as black holes and ultrarelativistic rotating stars [4, 10, 21, 22, 23, 24]. Unruh's original proposal [4] dealt specifically with so-called "dumb holes" (as they do not permit sound to escape), the hydrodynamic analogue of black holes, and others have extended and generalised his results [20, 21, 25]. Several authors [20, 22, 26, 27] have investigated fluid analogues of rotating black holes and "white holes". In this context, it is essential to have some way to identify fluid flows that correspond to black holes.

The defining feature of a black hole is the presence of an *event horizon*. An event horizon is the boundary of a region in space-time where all trajectories of particles moving forward in time ("timelike geodesics") point inwards, away from the region boundary. Thus no particle ever inside the event horizon can escape at some future time.

The hydrodynamic analogues of event horizons are termed "sonic horizons", and are surfaces where the velocity of the flow perpendicular to the surface is equal in magnitude to the speed of sound [20, 21]. At "black hole" sonic horizons, the perpendicular flow velocity increases in the direction of the flow (Figure 2.1), while at "white hole" sonic horizons the opposite is true (Figure 2.2). Thus inside a sonic black hole horizon the speed of sound is less than the flow velocity directly away from the horizon, and even directly outgoing sound waves cannot propagate back towards the horizon, as that would require their propagation speed relative to the flow to exceed the local speed of sound.

Figure 2.1: Schematic diagram of flow velocity near a black hole sonic horizon. The length of the arrows indicates the magnitude of the velocity.

Figure 2.2: Schematic diagram of flow velocity near a white hole sonic horizon. The length of the arrows indicates the magnitude of the velocity.

Rotating black holes and dense, rapidly rotating stars may possess *ergoregions*, within which no particle may remain stationary relative to an observer far from the rotating body, because such a trajectory would involve exceeding the speed of light. This effect is called *frame dragging*, as the rotation of the body is considered to "drag" the reference frame of a nearby observer around with it.

The hydrodynamic analogues of ergoregions (which will also be simply termed ergoregions) are those areas where the magnitude of the fluid velocity exceeds the local speed of sound, since it is then impossible for a particle to remain stationary in the laboratory frame without exceeding the local speed of sound [20, 21]. The boundary of an ergoregion, in both the astrophysical and hydrodynamic cases, is called an *ergosphere*.

In general, any event horizon must be contained in an ergoregion: if the magnitude of some component of the fluid velocity exceeds the speed of sound then certainly the magnitude of the velocity itself must also exceed c. In a one-dimensional or effectively one-dimensional system, the ergosphere collapses to the sonic horizon, as the sonic horizon must collapse to a point and the flow velocity perpendicular to the horizon is simply the flow velocity. This argument also applies to spherically symmetric systems, where the flow velocity is necessarily radial everywhere, and the sonic horizon must be spherical.

For an example of a system where the two notions do not coincide, consider a cylindrically symmetric draining vortex [20, 22]. The density and speed of sound are constants, and the velocity profile is [20]

$$\vec{v} = \frac{A\vec{r} + B\vec{\theta}}{r}. \tag{2.17}$$

The sonic horizon is the cylindrical surface where the radial component of the velocity $v_r = A/r$ equals the speed of sound c in magnitude, that is at

$$r_h = \frac{|A|}{c}. \tag{2.18}$$

The ergosphere, however, is the cylindrical surface where the magnitude of the velocity $|v| = \sqrt{A^2 + B^2}/r$ equals the speed of sound, defined by

$$r_e = \frac{\sqrt{A^2 + B^2}}{c}. \tag{2.19}$$

Clearly the two radii coincide if and only if $B = 0$, in which case the fluid flow is purely radial.

2.1.4 Analogue Hawking Radiation and Superradiance

Hawking radiation is the spontaneous emission of thermal radiation from black holes, at a temperature determined by the *surface gravity* of the black hole. The Hawking radiation mechanism was first proposed by Hawking in 1974 [3]. In Hawking's original derivation, the evolving structure of spacetime is treated as predetermined, and any backreaction of the emitted radiation on the black hole is not taken into consideration. The emitted radiation is a consequence of the behaviour of quantum fields in a spacetime metric containing an event horizon, and therefore can be expected to possess an analogue in hydrodynamic analogue gravity systems.

Unruh [4] demonstrates that the effective metric can be cast in a form identical to the Schwarzschild metric (describing a spherically symmetric, uncharged black hole) near the horizon, under the usual assumptions of an irrotational, non-viscous, barotropic fluid obeying the hydrodynamic equations of motion, and the additional constraint that the speed of sound is constant. Quantising the sound field, expanding the field in normal modes, and assuming that an observer travelling with the fluid as it passes the event horizon observes the field to be in a vacuum state, Unruh shows that the normal modes have a time dependence analogous to the behaviour of normal modes in Schwarzschild coordinates as seen by a freely falling observer, which in turn implies the existence of Hawking radiation. This derivation relies on the behaviour of normal modes near (but outside) the sonic horizon, and also on the usual assumptions required for the hydrodynamic analogy.

Visser [21] examines analogue Hawking radiation in the case where the speed of sound may vary in space, under the usual assumptions required for the hydrodynamic analogy. He derives an expression for the analogue surface gravity, and hence for the Hawking radiation temperature.

In the case of a rotating black hole, while Hawking radiation is still present, a new effect occurs due to the presence of an ergoregion outside the event horizon. Wavepackets incident on the ergoregion may be reflected with increased amplitude, for frequencies below a certain cutoff [1, 28, 29]. This reflection with amplification is called *superradiance.* It differs from Hawking radiation in that it is a classical effect that follows directly from the equation of motion for a classical scalar field in curved spacetime: there is no need to quantise the field. Analogue superradiance in hydrodynamic analogue gravity systems has been studied by Basak and Majumdar [22] and Slatyer and Savage [14]. In particular, analogue superradiance in the ergoregions of vortices without event horizons will be discussed in Chapter 5.

2.2 Potential Flaws in the Analogy

2.2.1 Validity of Inherent Assumptions

The calculations presented above rely on several assumptions about the nature of the fluid and the fluctuation field. Firstly, we have assumed that the density and velocity can be treated as continuous classical fields. This requires the relevant length scales to be long enough that the microscopic structure of the fluid can be ignored. Thus behaviour occurring at very small length scales lies beyond the scope of the analogy, but this is itself of interest as the behaviour of astrophysical systems at very small length scales is also unknown. As a result, the dependence of results at larger length scales on the small-scale structure of the system is an issue in both the astrophysical and hydrodynamic regimes, and it is possible that studying the problem in fluid systems where the microscopic theory is well understood may shed some light on possible approaches in the astrophysical case [15, 16, 17, 30].

If the fluid is to some degree rotational, viscous or non-barotropic, then clearly the analogy will not hold exactly, but the degree to which these deviations cause problems is not obvious. Failure of the irrotationality assumption is problematic as it makes it impossible to define the velocity potential, which is the analogue for the scalar field, and failure of barotropicity leads to failure of irrotationality as the system evolves [20]. The presence of non-zero viscosity seems to be less of a difficulty, and has been investigated by Visser [21] and Schutzhold and Unruh [31].

The analogy also relies crucially on the assumption that the fluctuations are small. Any process which leads to a runaway growth in the fluctuation field will rapidly pass beyond the regime of validity of this perturbative approach.

Some fluid systems, in particular Bose-Einstein condensates, only approximately obey the zero-viscosity Euler equation. In such a case the analogy fails in parameter regimes where the zero-viscosity Euler equation no longer adequately describes the system. This issue will be discussed in detail in the case of Bose-Einstein condensates (Section 4.2).

2.2.2 Failure to Reproduce Dynamics of the Background

Even if all the conditions given above for validity of the analogy are satisfied, it is important to note that the hydrodynamic analogy is a mapping *only* between the behaviour of sound waves in a fixed background fluid flow and the behaviour of scalar fields in a fixed background spacetime. There is no such mapping between the behaviour of the backgrounds themselves. The effective hydrodynamic metric is calculated from the background density and phase, and the evolution of these quantities is determined by the continuity equation and zero-viscosity Euler equation, while the astrophysical metric is governed by the Einstein field equations. Another way to put this is that only kinematic and not dynamical properties are preserved by the hydrodynamic analogy [20]. In particular, if backreaction of the fluctuation field on the background is taken into account then the resulting changes in the background cannot be mapped from the hydrodynamic regime to the astrophysical, or vice versa. As another example, the notion of black hole entropy is an intrinsically dynamical effect [20] and is not carried through to the hydrodynamic analogue.

Furthermore, for the same reasons it is generally impossible to produce effective metrics identical to solutions of the Einstein equations: for example, no choice of velocity and density profiles will give rise to an analogue of the Schwarzschild metric, which describes an uncharged spherically symmetric black hole [20]. Thus when studying the properties of relativistic astrophysical systems it is necessary to identify the important features of the system (such as the presence of an event horizon for the case of the Schwarzschild black hole) and aim to create analogue metrics that mimic this feature, rather than attempting to replicate the metric exactly.

2.2.3 Quantisation and Commutation Relations

Unruh and Schutzhold [32] demonstrate that in order to reproduce intrinsically quantum effects such as Hawking radiation, it is not sufficient to reproduce the Klein-Gordon equation with a suitably chosen effective metric. They examine a "slow light" system as an optical black hole analogue in considerable detail, and conclude that while this system reproduces classical effects such as the presence of an event horizon and mode mixing at the horizon, it does not give rise to an analogue of Hawking radiation. This is due to the behaviour of the commutation relations for the relevant field in this particular system.

The key distinction is that while the classical phenomenon of mode mixing is preserved, this may not equate to particle production in outgoing modes. The quantum commutation relations link the "Bogoliubov" modes to the notion of "particles", and the particle production associated with Hawking radiation occurs only if the commutation relations allow a physically reasonable interpretation where excitations of the various modes can be viewed as particles.

However, Unruh and Schutzhold also demonstrate that for Bose-Einstein condensates and nondispersive linear dielectric media, the commutation relations (within the validity of the approximations required for the analogy) are equivalent to those for a quantum scalar field in the corresponding curved space-time. This property is a substantial advantage to working in Bose-Einstein condensates, as opposed to more complicated sonic analogue systems.

Chapter 3

Constructing Sonic Horizons

3.1 General Horizon Configurations

3.1.1 The Laval Nozzle

The simplest and most studied scheme for the creation of sonic horizons is probably the Laval nozzle geometry [20, 33, 34]. The Laval nozzle is based on an axisymmetric tube with varying cross-section. It can be shown that subsonic flow accelerates with decreasing cross-section, while supersonic flow is accelerated as the cross-section is increased. Thus if the tube has a narrow throat, where the fluid reaches supersonic speed, then downstream from the throat the flow will remain supersonic (while upstream the flow is subsonic), giving rise to a sonic horizon (Figure 3.1).

Sakagami and Ohashi [33] investigate the classical analogue of Hawking radiation in such a system (under the assumptions that all processes are adiabatic and the fluid is well described by the hydrodynamic equations of motion), where a wave prepared as a pure-frequency plane wave in the vicinity of the horizon has a Planckian-distributed power

Figure 3.1: Schematic diagram of the Laval nozzle.

spectrum when observed in the asymptotic region at future infinity.

Barcelo, Liberati and Visser [34] consider a pair of Laval nozzles, with the second nozzle being used to return the flow to subsonic speeds. Again, they assume that the working fluid is well described by the continuity equation and the zero-viscosity Euler equation. They calculate the analogue surface gravity in the specific case of a double Laval nozzle, using the general equation for analogue surface gravity [21]. Their calculation can be applied to a general nozzle with a viscous fluid and arbitrary external potential, so long as the transverse components of the velocity are small compared to the velocity in the nozzle direction and the system has a barotropic equation of state.

They note that there is a fine-tuning condition on the system required to keep the physical acceleration at the sonic horizon finite, relating the shape of the nozzle to the external body force and specific friction at the horizon:

$$c^2\left(\frac{A'}{A}\right) - \Phi' + \frac{f_v}{\rho} \to 0. \tag{3.1}$$

Here Φ is the external potential, c is the speed of sound, f_v is the internal viscous friction, ρ is the fluid density and A is the cross-sectional area of the nozzle (all quantities being measured at the horizon).

The authors claim that experience with wind tunnels has shown that the flow will attempt to self-adjust, shifting the location of the sonic horizon, to satisfy this fine-tuning condition. If this condition is satisfied, then the surface gravity also remains finite at the horizon.

For a Laval nozzle there is no external potential, and in the case of a superfluid flow there is no viscous friction. The fine-tuning condition then reduces to the requirement that the nozzle cross-section is at a minimum at the sonic horizon, which is automatically satisfied in the double Laval nozzle. The authors obtain expressions for the surface gravity g_H and Hawking radiation temperature:

$$g_H = \pm\frac{c_H^2}{\sqrt{A_H}}\sqrt{3A''_H/4}, \tag{3.2}$$

$$k_B T_H = \hbar\frac{c_H}{2\pi\sqrt{A_H}}\sqrt{\frac{3A''_H}{4}}. \tag{3.3}$$

3.1.2 Vortices

As mentioned in the discussion of ergoregions and sonic horizons (Section 2.1.3), a sonic horizon can form in a vortex containing a central sink [14, 20, 22]. In this case, as the system is cylindrically symmetric, it is simplest to work in cylindrical polar coordinates using the analogue metric given by Equation 2.14.

Visser [20] considers an effectively two-dimensional vortex configuration with constant density and speed of sound, and velocity profile given (in cylindrical polar coordinates) by

$$\vec{v} = \frac{A\vec{r} + B\vec{\theta}}{r}. \tag{3.4}$$

As discussed in Section 2.1.3, this configuration possesses an event horizon at $r = |A|/c$, and an ergosphere at $r = \sqrt{A^2 + B^2}/c$.

The necessity for a central sink makes this configuration somewhat difficult to achieve experimentally in very cold superfluids. Furthermore, the presence of an ergoregion separate to the horizon (in the case where $B \neq 0$) is expected to give rise to an analogue of superradiance, in addition to the analogue Hawking radiation from the event horizon. The possibility of superradiance in such vortices has been investigated by Basak and Majumdar [22], using techniques from the study of superradiance in rotating black holes [29].

In the case where $A = 0$, on the other hand, no central sink is required, but there is no sonic horizon present. For B non-zero, however, an ergoregion still exists. While Hawking radiation cannot occur in such a system, superradiance may still be present: the simple vortex without a drain is in some ways analogous to ultrarelativistic rotating stars that may possess ergoregions but not event horizons [29, 35, 36]. The first part of my Honours work involved an investigation of superradiance in such vortices [14]: our methods and results will be outlined in Chapter 5.

3.1.3 Regularity of the Flow at the Sonic Horizon

The regularity requirement noted by Barcelo, Liberati and Visser [34] in the case of the Laval nozzle has been further explored by Liberati, Sonego and Visser [37] in the case of a general sonic horizon. They assume a continuous classical fluid obeying the hydrodynamic equations of motion, but do not invoke the usual assumptions of a barotropic irrotational viscosity-free flow. The authors prove that unless a particular regularity condition is met, the fluid acceleration in the direction of the flow and the density gradient diverge as the fluid speed v approaches the speed of sound c, i.e. at the ergoregion boundary. If $\vec{n}$ is the unit vector in the direction of the flow, ρ is the fluid density, Φ is the external potential and $\vec{f_v}$ gives the friction force, then the condition is

$$\frac{d\Phi}{dn} - c^2\vec{\nabla}\cdot\vec{n} - \frac{1}{\rho}\vec{n}\cdot\vec{f_v} + \frac{\partial v}{\partial t} - \frac{c}{\rho}\frac{\partial\rho}{\partial t} = 0. \tag{3.5}$$

All quantities are evaluated at the boundary of the ergoregion. In the case of a stationary flow, this condition is also required if the surface gravity is not to diverge at the ergosphere. In a stationary non-viscous flow, we obtain the simpler condition

$$\frac{d\Phi}{dn} = c^2\vec{\nabla}\cdot\vec{n}. \tag{3.6}$$

Liberati, Sonego and Visser consider a number of examples, including spherically symmetric stationary flow, the case where the speed of sound is constant, the case of a linear external potential, an analogue to the Schwarzschild geometry, and an analogue to the Reissner-Nordstrom geometry. They demonstrate that for a broad class of potentials there is one particular flow that is everywhere regular, but for some potentials this is impossible. They also show that adding viscosity terms has the effect of removing the singularities at the ergosphere.

While this effect may make it more difficult to construct sonic horizons, Liberati, Sonego and Visser suggest that it also presents an opportunity, by effectively making the surface gravity a free parameter. It is therefore possible that the surface gravity in some suggested systems may be considerably larger than previously thought.

3.2 Sonic Horizons in Specific Analogue Systems

3.2.1 Superfluid Helium

The hydrodynamic analogy applies to superfluid helium, with the relevant perturbations being phonons in superfluid ^{4}He and low-energy Bogoliubov fermions in superfluid ^{3}He-A. From a theoretical point of view, the main deficiency of helium systems is that the interparticle interaction is relatively strong and hence a mean-field description is problematic [5]. Volovik has studied analogue gravity in superfluid ^{4}He and ^{3}He-A in some detail [23, 27, 38]. He proposes an analogue black hole [27] consisting of a two-dimensional "draining bathtub" flow in a thin film of superfluid ^{3}He-A, placed on top of a superfluid ^{4}-He film to eliminate friction interactions of the flowing ^{3}He-A with a solid substrate. Volovik also suggests using a toroidal geometry [27], with a film of ^{3}He-A resting on an inner film of ^{4}He, to close the streamlines of the flow. If the inner radius of the torus is small, the flow around the meridians (minor circles) is accelerated close to the inner circle, and can there exceed the speed of sound. This arrangement gives rise to both a black hole and white hole sonic horizon. The black hole horizon can be viewed as equivalent to the "draining bathtub" geometry on the curved upper surface of the torus, as the meridional flow drains down into the torus centre.

Volovik analyses the 2D "draining bathtub" case in some detail, examining the vacuum in the comoving and rest frames, the classical trajectories of particles approaching the event horizon, and the presence of tunnelling at the horizon giving rise to thermal radiation of quasiparticles. This last effect is the analogue of Hawking radiation. Volovik also explores the idea of "negative temperature" behind the event horizon, and briefly discusses the dissipation of the event horizon due to Hawking radiation.

3.2.2 Slow-Light Systems

One practical problem with the sonic analogues of black holes is that methods for detection of sound are considerably less advanced than methods for detection of light. Furthermore, the hydrodynamic analogy incorporates scalar fields only, and is not readily generalisable to spin-one fields such as the electromagnetic field. As a result, a number of groups have considered optical analogues of black holes, usually involving dielectrics where the speed of light is slowed to a small fraction of its normal value. As discussed in Section 2.2.3, however, the analogy may fail at a quantum level in these systems.

Schutzhold, Plunien and Soff [39] explore in detail the possibility of creating analogue black holes in flowing linear nondispersive dielectrics. As might be expected, the analogue horizon occurs where the speed of the medium exceeds the local speed of light. The authors develop an expression for the effective metric, and thus derive a formula for the surface gravity at the horizon, allowing an estimate of the Hawking radiation temperature. They note that as in this case the microscopic theory is well understood, it may be feasible to investigate the analogue of the trans-Planckian problem, but do not pursue this line of thought any further.

Novello et al [40] investigate a possible analogue black hole in a nonlinear flowing isotopic dielectric (in an applied electric field). Their treatment is classical and relativistic. They examine the propagation of photons in a non-linear medium (with birefringence), which gives rise to *two* effective metrics. Each photon "sees" one metric or the other, depending on its polarisation. However, the authors prove that in a special configuration (pure radial flow with an applied electric field) the existence of a sonic horizon in one metric implies its existence in the other, and the location of the horizon is the same in both metrics. They also demonstrate that the qualitative behaviour of the photons is very similar in the two metrics, and in the case of a constant flux velocity, the shape of the effective potential for both metrics agrees qualitatively with that for photons in the geometry of a Schwarzschild black hole.

In this case, the horizon radius and black hole temperature depend on the zero order permittivity of the fluid, the charge generating the external field and the linear susceptibility of the fluid. The expression for the surface gravity (from which the Hawking temperature follows) contains a term mixing the acceleration of the fluid with its dielectric properties, which acts as a generalisation of the usual acceleration term. However, there is also a new term depending only on the dielectric properties of the fluid, suggesting a mechanism for increasing the Hawking radiation temperature.

3.2.3 Water Waves as an Analogue Gravity System

A very simple but novel analogue system was proposed by Schutzhold and Unruh [31], consisting of a container of viscosity-free, irrotational, incompressible fluid, with the relevant fluctuations being surface waves. The shape of the bottom surface of the container and the depth of the fluid are adjustable, however it is assumed that the depth of the fluid is much smaller than the horizontal length scales over which the features of the flow profile change significantly (this is a long-wavelength approximation). Furthermore, it is assumed that the curvature of the bottom of the tank is on a long length scale compared to the water depth. Given a stationary background flow, the equation for the surface waves (that is perturbations in the fluid depth) can again be cast in the form of the Klein-Gordon equation, with an effective metric determined by the acceleration due to gravity g, the velocity and depth of the background flow, and the shape of the container base.

Schutzhold and Unruh [31] discuss the effects of including surface tension and viscosity in the model. They show that the surface tension effects only become significant for small wavelengths, but can give rise to a "superluminal" dispersion relation, where the group velocity of the perturbations exceeds the background speed of sound. This only holds over a certain range of wavelengths, but this range can be adjusted by varying the surface tension coefficient of the fluid (for example by changing the temperature, changing the working fluid or adding surfactants).

Adding viscosity to the model causes damping of the perturbations: however high frequencies are damped faster, so viscosity may be employed to reduce potential high-frequency noise. The presence of viscosity makes the calculation considerably more complex and means that an effective metric cannot readily be identified. Schutzhold and Unruh discuss experimental scenarios where the problems associated with viscosity (in particular, failure of the assumption that the flow is irrotational) might be resolved.

The authors demonstrate systems with effective metrics mimicking those of a (non-rotating) black hole and white hole, as well as a system possessing both an ergoregion and an event horizon, which is analogous to a rotating black hole. They discuss analogue superradiance in the latter system, and the possible resulting instability. They also note a related instability which can occur even in non-rotating analogue black holes, and indeed in the case of purely one-dimensional flow, between a white hole horizon and black hole horizon. This instability arises from mixing of modes with positive and negative norm at the horizons. It is observed that this system is unlikely to replicate quantum effects such as Hawking radiation, but may well exhibit analogues of classical effects such as superradiance.

Chapter 4

Analogue Gravity in Bose-Einstein Condensates

4.1 Introduction and Motivation

4.1.1 Bose-Einstein Condensation

A Bose-Einstein condensate (BEC) consists of a large number of indistinguishable bosons, cooled to a sufficiently low temperature that the single-particle ground state is macroscopically occupied. Although the theory of an ideal gas of identical bosons was developed by Bose [41] and Einstein [42] in 1924-25, Bose-Einstein condensation was not observed experimentally until relatively recently, in 1995 [7, 8, 9].

Bose-Einstein condensation has, historically, been most easily achieved in very cold and dilute gases of alkali atoms such as rubidium [7], lithium [8] and sodium [9]. In such systems, the interaction between the atoms can be approximated by an effective contact potential. Furthermore, the macroscopic occupation of the condensate state and the low density of the gas lends itself to a mean-field approach. These approximations give rise to a powerful and accurate description of dilute gas BECs [5].

4.1.2 Analogue Gravity in Bose-Einstein Condensates

BECs are a particularly attractive system for studying low-temperature effects that rely heavily on quantum theory, such as Hawking radiation. Classical phenomena such as superradiance may be easier to reproduce in less experimentally challenging systems (such as the water-based model investigated by Schutzhold and Unruh [31]), but must contend with viscosity, turbulence and environmental effects that can be largely neglected in BEC systems. The relatively simple microscopic theory of BECs also make it plausible that the behaviour of the system can be described in parameter regimes where the hydrodynamic analogy may break down, particularly close to sonic horizons where very short length scales become important.

As we will show, the condensed part of a BEC behaves as a zero-viscosity irrotational classical fluid in the limit where the variation of number density with position is small. However additional terms are present in the equation of motion and become significant in the case of large density gradients, or when dealing with short-wavelength fluctuations in the condensate field. The usual approach when applying the hydrodynamic analogy to Bose-Einstein condensates is simply to neglect these additional terms: this is called the *hydrodynamic approximation* and will be discussed in more detail later. Most authors also seem to make this approximation when writing down the equations of motion for the field, and then linearise the approximate equations of motion: this method can fail when the additional terms are negligible for the background field but non-negligible for the fluctuations. Such a case occurs when the background density is constant but the wavelengths of the fluctuations are very short.

A more rigorous approach is to retain the additional terms throughout the calculation of the equation of motion for the linearised phase fluctuations. Visser [10] shows that in this case, the "effective metric" becomes a matrix of differential operators, rather than a matrix of functions. In the long-wavelength limit, the terms arising from the linearisation of the quantum potential (which contain the differential operators) can be neglected, and this approximation recovers the usual effective metric. Our approach here will yield equivalent results, but we will bypass the metric description, since for most purposes only the equation of motion is needed and the metric formulation is unnecessary.

4.2 Dynamics of the Condensate: the Gross-Pitaevskii Equation

4.2.1 Derivation of the Gross-Pitaevskii Equation

Let the Bose field operator for the condensate be denoted $\hat{\Psi}$. In the mean-field approximation, we define the classical field $\psi(\vec{r},t) = \langle\hat{\Psi}(\vec{r},t)\rangle$, and the fluctuation field operator $\hat{\Psi}'(\vec{r},t)$ by

$$\hat{\Psi}(\vec{r},t) = \psi(\vec{r},t) + \hat{\Psi}'(\vec{r},t). \tag{4.1}$$

When the fluctuations are small (i.e. the condensate fraction is large) equations of motion for both $\psi(\vec{r},t)$ and $\hat{\Psi}'(\vec{r},t)$ can be derived by expanding the equation of motion for the Bose field operator to lowest orders in $\hat{\Psi}'(\vec{r},t)$.

In terms of the Bose field operator, the single-particle Hamiltonian H^{sp} and the interparticle interaction V, the many-body Hamiltonian operator $\hat{H}$ can be written as

$$\hat{H} = \int d\vec{r}\,\hat{\Psi}^\dagger(\vec{r})\,H^{sp}\,\hat{\Psi}(\vec{r}) + \frac{1}{2}\int d\vec{r}\,d\vec{r}'\,\hat{\Psi}^\dagger(\vec{r})\,\hat{\Psi}^\dagger(\vec{r}')\,V(\vec{r}-\vec{r}')\,\hat{\Psi}(\vec{r}',t)\,\hat{\Psi}(\vec{r},t). \tag{4.2}$$

Writing $H^{sp} = -(\hbar^2\nabla^2)/2m + V_{trap}$ and employing the commutation relations for the Bose field operator, the equation of motion for the field operator becomes,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\vec{r},t) &= \left[\hat{\Psi},\hat{H}\right] \\ &= \left[-\frac{\hbar^2\nabla^2}{2m} + V_{trap} + \int d\vec{r}'\,\hat{\Psi}^\dagger(\vec{r}')\,V(\vec{r}-\vec{r}')\,\hat{\Psi}(\vec{r}',t)\right]\hat{\Psi}(\vec{r},t). \end{aligned} \tag{4.3}$$

Approximating the interparticle potential V by an effective contact interaction parametrised by the s-wave scattering length a,

$$V(\vec{r}-\vec{r}') = U_0\delta(\vec{r}-\vec{r}'), \tag{4.4}$$

$$U_0 = \frac{4\pi\hbar^2 a}{m}, \tag{4.5}$$

and replacing the Bose field operator with its expectation value, we obtain the equation of motion,

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t)=\left(-\frac{\hbar^2\nabla^2}{2m}+V_{trap}+U_0|\psi(\vec{r},t)|^2\right)\psi(\vec{r},t). \tag{4.6}$$

This equation is termed the *Gross-Pitaevskii equation.* It describes the evolution of the classical field corresponding to the condensate, provided backreaction of the fluctuation field on the bulk fluid is negligible.

4.2.2 Hydrodynamic Form of the Gross-Pitaevskii Equation

The Gross-Pitaevskii equation describes the evolution of the condensate under the assumption that fluctuations are negligible. It is instructive to express the condensate "wavefunction" ψ in the form

$$\psi=\sqrt{\rho}\,e^{i\theta}, \tag{4.7}$$

where ρ is a real positive function and θ is some real function.

In terms of the new variables ρ and θ, the derivatives of ψ take the form

$$\nabla\psi=\nabla(\sqrt{\rho}\,e^{i\theta})=i\psi\nabla\theta+\psi\frac{\nabla\rho}{2\rho}, \tag{4.8}$$

$$\nabla^2\psi=\nabla\psi\cdot\left(i\nabla\theta+\frac{\nabla\rho}{2\rho}\right)+\psi\left(i\nabla^2\theta+\frac{\nabla^2\rho}{2\rho}-\frac{|\nabla\rho|^2}{2\rho^2}\right), \tag{4.9}$$

$$\frac{\partial\psi}{\partial t}=i\frac{\partial\theta}{\partial t}\psi+\frac{\psi}{2\rho}\frac{\partial\rho}{\partial t}. \tag{4.10}$$

Substituting these three expressions into the Gross-Pitaevskii equation (Equation 4.6) yields,

$$\begin{aligned}i\hbar\left(i\frac{\partial\theta}{\partial t}+\frac{1}{2\rho}\frac{\partial\rho}{\partial t}\right) &= -\frac{\hbar^2}{2m}\left[\left(i\nabla\theta+\frac{\nabla\rho}{2\rho}\right)\cdot\left(i\nabla\theta+\frac{\nabla\rho}{2\rho}\right)+\left(i\nabla^2\theta+\frac{\nabla^2\rho}{2\rho}-\frac{|\nabla\rho|^2}{2\rho^2}\right)\right]\\ &\quad +V+U_0\rho.\end{aligned} \tag{4.11}$$

Comparing real and imaginary parts of this equation leads to the pair of equations,

$$-\hbar\frac{\partial\theta}{\partial t}=-\frac{\hbar^2}{2m}\left(-|\nabla\theta|^2-\frac{|\nabla\rho|^2}{4\rho^2}+\frac{\nabla^2\rho}{2\rho}\right)+V+U_0\rho, \tag{4.12}$$

$$\frac{\hbar}{2\rho}\frac{\partial\rho}{\partial t}=-\frac{\hbar^2}{2m}\left(\frac{1}{\rho}\nabla\theta\cdot\nabla\rho+\nabla^2\theta\right). \tag{4.13}$$

Now define $\vec{v}(\hbar/m)\nabla\theta$ [5]. Noting that any gradient is curl-free,

$$\nabla\times\vec{v}=\frac{\hbar}{m}\nabla\times\nabla\theta=0. \tag{4.14}$$

Taking the gradient of Equation 4.12 and rewriting Equations 4.12 and 4.13 in terms of $\vec{v}$ yields

$$\frac{\partial\vec{v}}{\partial t}=\nabla\left(-\frac{1}{2}\vec{v}\cdot\vec{v}+\frac{\hbar^2}{2m^2\sqrt{\rho}}\nabla^2\sqrt{\rho}-\frac{V}{m}-\frac{U_0\rho}{m}\right), \tag{4.15}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \tag{4.16}$$

Equation 4.16 is identical to the continuity equation (Equation 2.2). Now as $\nabla \times \vec{v} = 0$,

$$\nabla (\vec{v} \cdot \vec{v}) = 2 (\vec{v} \cdot \nabla) \vec{v}, \tag{4.17}$$

and Equation 4.15 can be rewritten in the form

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = \frac{\hbar^2}{2m^2} \nabla \left(\frac{1}{\sqrt{\rho}} \nabla^2 \sqrt{\rho} \right) - \nabla \left(\frac{V + U_0 \rho}{m} \right). \tag{4.18}$$

This equation closely resembles the zero-viscosity Euler equation (Equation 2.3), except that it possesses an additional spatial derivative term. This extra term is called the *quantum pressure term* [5].

Thus when the quantum pressure term is small, the classical condensate field ψ behaves like a continuous fluid with zero viscosity. The assumption that the quantum pressure term is negligible and may be neglected is the *hydrodynamic approximation.*

Denoting the spatial scale of variations of the condensate number density ρ by l, the pressure term $\nabla(U_0\rho/m)$ in Equation 4.18 is of order $U_0\rho/ml$, while the quantum pressure term is of order $\hbar^2/m^2l^3$ [5]. Very roughly, therefore, the quantum pressure term can be neglected where the spatial length scale of variations in ρ satisfies

$$l \gg \frac{\hbar}{\sqrt{mU_0\rho}}. \tag{4.19}$$

Thus the hydrodynamic analogy can be expected to apply to BECs, but only when a further long-length-scale approximation holds true.

4.2.3 Fluctuations in the Gross-Pitaevskii Wavefunction

Under the hydrodynamic approximation, the equation of motion for fluctuations of the phase θ of the Gross-Pitaevskii field ψ can be derived from the hydrodynamic equations (Equations 4.16, 4.18) just as in Section 2.1.1. The restriction on the length scale of variations in the density then applies to variations in the perturbed density also, meaning that the wavelength of the density fluctuations must be long compared to the coherence length, $\xi = \hbar/\sqrt{mU_0\rho}$.

In the absence of the hydrodynamic approximation, an additional term originating from the linearisation of the quantum pressure term appears in the coupled equations for the phase and number density. We now give the derivation of the equations of motion for the phase and number density directly from the Gross-Pitaevskii equation, without the hydrodynamic approximation.

Let the background field be denoted ψ_0, and consider a perturbation to the field denoted $\epsilon\hat{\psi}$, with ϵ real and positive and $\epsilon \ll 1$. We impose the constraint that both the background wavefunction ψ_0 and the total wavefunction $\psi = \psi_0 + \epsilon\hat{\psi}$ must satisfy the Gross-Pitaevskii equation.

To obtain the "hydrodynamic" equations of motion, the wavefunctions must be written in terms of the phase and number density of the condensate. Denoting the background phase and density by ρ_0 and θ_0 respectively, the perturbed phase and density can be written in the form $\rho = \rho_0 + \epsilon\hat{\rho}$ and $\theta = \theta_0 + \epsilon\hat{\theta}$. Thus,

$$\psi_0 = \sqrt{\rho_0}\, e^{i\theta_0}, \tag{4.20}$$
$$\psi = \sqrt{\rho}\, e^{i\theta}. \tag{4.21}$$

The perturbation to the wavefunction can be written in terms of the perturbations in phase and density, by comparing terms of first order in ϵ in the two expressions for ψ.

$$\begin{aligned}
\psi_0 + \epsilon\hat{\psi} &= \sqrt{\rho_0 + \epsilon\hat{\rho}}\, e^{i(\theta_0 + \epsilon\hat{\theta})} && (4.22)\\
&= \left(\sqrt{\rho_0} + \epsilon\frac{\hat{\rho}}{2\sqrt{\rho_0}}\right) e^{i\theta_0}\left(1 + \epsilon i\hat{\theta}\right) + O(\epsilon^2) && (4.23)\\
&= \sqrt{\rho_0}\, e^{i\theta_0} + \epsilon\, e^{i\theta_0}\left(\frac{\hat{\rho}}{2\sqrt{\rho_0}} + i\hat{\theta}\sqrt{\rho_0}\right) + O(\epsilon^2) && (4.24)
\end{aligned}$$

$$\hat{\psi} = e^{i\theta_0}\left(\frac{\hat{\rho}}{2\sqrt{\rho_0}} + i\hat{\theta}\sqrt{\rho_0}\right) = \psi_0\left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right). \tag{4.25}$$

We can now substitute $\psi = \psi_0 + \epsilon\hat{\psi}$ into the Gross-Pitaevskii equation (Equation 4.6), i.e.

$$i\hbar\frac{\partial\psi}{\partial t}(\vec{r},t)\left(-\frac{\hbar^2\nabla^2}{2m} + V(\vec{r},t) + U_0(\vec{r},t)|\psi(\vec{r},t)|^2\right)\psi(\vec{r},t), \tag{4.26}$$

$$\begin{aligned}
i\hbar\frac{\partial\psi_0}{\partial t} + \epsilon i\hbar\frac{\partial\hat{\psi}}{\partial t} &= \left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\psi_0 + \epsilon\left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\hat{\psi} + U_0\\
&\quad \left(|\psi_0|^2 + \epsilon\left(\psi_0\hat{\psi}^* + \psi_0^*\hat{\psi}\right) + O(\epsilon^2)\right)\left(\psi_0 + \epsilon\hat{\psi}\right). && (4.27)
\end{aligned}$$

We now compare terms of zero and first order in ϵ, yielding

$$\begin{aligned}
i\hbar\frac{\partial\psi_0}{\partial t} &= \left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\psi_0 + U_0|\psi_0|^2\psi_0, && (4.28)\\
i\hbar\frac{\partial\hat{\psi}}{\partial t} &= \left(-\frac{\hbar^2\nabla^2}{2m} + V\right)\hat{\psi} + U_0\left(\psi_0\hat{\psi}^* + \psi_0^*\hat{\psi}\right)\psi_0 + U_0|\psi_0|^2\hat{\psi}. && (4.29)
\end{aligned}$$

Employing Equations 4.20 and 4.25 for ψ_0 and $\hat{\psi}$ in terms of the density and phase, Equation 4.29 gives rise to equations of motion for the phase and density perturbations:

$$\begin{aligned}
i\hbar\left[\left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right)\frac{\partial\psi_0}{\partial t}\right. &= -\frac{\hbar^2}{2m}\left[\psi_0\nabla^2\left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right) + \left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right)\nabla^2\psi_0\right.\\
\left.+\psi_0\frac{\partial}{\partial t}\left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right)\right] &\quad \left.+2\nabla\left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right)\cdot\nabla\psi_0\right] + V\psi_0\left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right)\\
&\quad +U_0|\psi_0|^2\psi_0\left[\left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right) + \left(\frac{\hat{\rho}}{2\rho_0} + i\hat{\theta}\right)\right.\\
&\quad \left.+\left(\frac{\hat{\rho}}{2\rho_0} - i\hat{\theta}\right)\right]. && (4.30)
\end{aligned}$$

Some of these terms can be collected to yield Equation 4.28 multiplied by $\hat{\psi}/\psi_0$. Eliminating these terms, dividing the remaining terms by ψ_0 and using the fact that $|\psi_0|^2 = \rho_0$, we obtain a much simpler equation:

$$i\hbar\frac{\partial}{\partial t}\left(\frac{\hat{\rho}}{2\rho_0}+i\hat{\theta}\right) = -\frac{\hbar^2}{2m}\left[\nabla^2\left(\frac{\hat{\rho}}{2\rho_0}+i\hat{\theta}\right)+2\nabla\left(\frac{\hat{\rho}}{2\rho_0}+i\hat{\theta}\right)\cdot\left(\frac{\nabla\psi_0}{\psi_0}\right)\right]+U_0\hat{\rho}. \quad (4.31)$$

Now inserting Equation 4.8 for $\nabla\psi_0$,

$$\begin{aligned}\Rightarrow i\hbar\frac{\partial}{\partial t}\left(\frac{\hat{\rho}}{2\rho_0}+i\hat{\theta}\right) &= -\frac{\hbar^2}{2m}\left[\nabla^2\left(\frac{\hat{\rho}}{2\rho_0}+i\hat{\theta}\right)+2\nabla\left(\frac{\hat{\rho}}{2\rho_0}+i\hat{\theta}\right)\right.\\ &\quad\left.\cdot\left(i\nabla\theta_0+\frac{\nabla\rho_0}{2\rho_0}\right)\right]+U_0\hat{\rho}. \end{aligned} \quad (4.32)$$

Taking the real and imaginary parts of this equation, we obtain two coupled equations of motion for the phase and density fluctuations.

$$-\hbar\frac{\partial\hat{\theta}}{\partial t} = -\frac{\hbar^2}{m}\left[\nabla^2\left(\frac{\hat{\rho}}{4\rho_0}\right)+\nabla\left(\frac{\hat{\rho}}{2\rho_0}\right)\cdot\left(\frac{\nabla\rho_0}{2\rho_0}\right)-\nabla\hat{\theta}\cdot\nabla\theta_0\right]+U_0\hat{\rho}, \quad (4.33)$$

$$\hbar\frac{\partial}{\partial t}\left(\frac{\hat{\rho}}{\rho_0}\right) = -\frac{\hbar^2}{m}\left[\nabla^2\hat{\theta}+\nabla\left(\frac{\hat{\rho}}{\rho_0}\right)\cdot\nabla\theta_0+\frac{1}{\rho_0}\nabla\rho_0\cdot\nabla\hat{\theta}\right]. \quad (4.34)$$

Expanding the derivatives involving ρ_0, and applying the continuity equation (Equation 4.16) allows us to simplify the second equation further:

$$\begin{aligned}\frac{1}{\rho_0}\frac{\partial\hat{\rho}}{\partial t}+\frac{\hbar}{m}\left[\nabla^2\hat{\theta}+\frac{1}{\rho_0}\nabla\hat{\rho}\cdot\nabla\theta_0+\frac{1}{\rho_0}\nabla\rho_0\cdot\nabla\hat{\theta}\right] &= \frac{\hat{\rho}}{\rho_0^2}\frac{\partial\rho_0}{\partial t}+\frac{\hbar}{m}\frac{\hat{\rho}}{\rho_0^2}\nabla\rho_0\cdot\nabla\theta_0\\ &\quad-\frac{\hbar}{m}\frac{\hat{\rho}}{\rho_0}\nabla^2\theta_0. \end{aligned} \quad (4.35)$$

Again writing $\vec{v} = (\hbar/m)\nabla\theta_0$, the two coupled equations for the phase and velocity fluctuations become,

$$\frac{\partial\hat{\theta}}{\partial t}\frac{\hbar}{4m\rho_0}\nabla\cdot\left[\rho_0\nabla\left(\frac{\hat{\rho}}{\rho_0}\right)\right]-\vec{v}\cdot\nabla\hat{\theta}-\frac{U_0}{\hbar}\hat{\rho}, \quad (4.36)$$

$$\frac{\partial\hat{\rho}}{\partial t}+\frac{\hbar}{m}\nabla\cdot\left(\rho_0\nabla\hat{\theta}\right)+\nabla\cdot(\hat{\rho}\vec{v}) = 0. \quad (4.37)$$

In general, it is difficult to obtain an equation for the phase or density fluctuations alone from this system. As previously, if the second-derivative term can be neglected, then Equation 4.36 gives an expression for $\hat{\rho}$ in terms of $\hat{\theta}$ and the background variables. This means that a single equation of motion describing the phase fluctuations (and consequently the density fluctuations) can be written down, in the case where the second-derivative term is negligible.

$$\hat{\rho} = -\frac{\hbar}{U_0}\left(\frac{\partial\hat{\theta}}{\partial t}+\vec{v}\cdot\nabla\hat{\theta}\right), \quad (4.38)$$

$$\frac{\partial}{\partial t}\left[\frac{1}{U_0}\left(\frac{\partial\hat{\theta}}{\partial t}+\vec{v}\cdot\nabla\hat{\theta}\right)\right]-\frac{1}{m}\nabla\cdot\left(\rho_0\nabla\hat{\theta}\right)+\nabla\cdot\left[\frac{1}{U_0}\left(\frac{\partial\hat{\theta}}{\partial t}+\vec{v}\cdot\nabla\hat{\theta}\right)\vec{v}\right] = 0. \quad (4.39)$$

Now let us *define* the (real and positive) quantity c by

$$c = \sqrt{\frac{\rho_0 U_0}{m}}. \tag{4.40}$$

Substituting into Equation 4.39 yields,

$$\frac{\partial}{\partial t}\left[\frac{\rho_0}{c^2}\left(\frac{\partial\hat{\theta}}{\partial t} + \vec{v}\cdot\nabla\hat{\theta}\right)\right] + \nabla\cdot\left[\frac{\rho_0}{c^2}\left(\frac{\partial\hat{\theta}}{\partial t} + \vec{v}\cdot\nabla\hat{\theta}\right)\vec{v}\right] - \nabla\cdot\left(\rho_0\nabla\hat{\theta}\right) = 0. \tag{4.41}$$

4.3 The Fluctuation Field in the Hydrodynamic Approximation

4.3.1 One-Dimensional Systems with Constant Number Density

There are several important special cases of Equation 4.41. Leonhardt et al [13, 43] consider the case where ρ_0 is constant with respect to both space and time, and the problem is one-dimensional. In this case the equation of motion (Equation 4.41) reduces to,

$$\left[\partial_t\left(\frac{1}{c^2}\left(\partial_t + v\partial_z\right)\right) + \partial_z\left(\frac{v}{c^2}\left(\partial_t + v\partial_z\right) - \partial_z\right)\right]\hat{\theta} = 0. \tag{4.42}$$

This equation can be rewritten in the form,

$$\left[\partial_t + (v\pm c)\partial_z + \partial_z(v\pm c)\right]\left[\frac{1}{c^2}\left(\partial_t + (v\mp c)\partial_z\right)\right]\hat{\theta} \pm \frac{1}{c^2}\left(\partial_z c\,\partial_t\hat{\theta} - \partial_t c\,\partial_z\hat{\theta}\right) = 0. \tag{4.43}$$

In the case where c is constant, the last term vanishes and Equation 4.43 constitutes a factorisation of the equation of motion. Any function $\hat{\theta}$ satisfying

$$\left(\partial_t + (v\mp c)\partial_z\right)\hat{\theta} = 0 \tag{4.44}$$

is therefore a solution to Equation 4.41. This first-order partial differential equation may be solved by the method of characteristics. Writing $d\hat{\theta} = (\partial_t\hat{\theta})dt + (\partial_z\hat{\theta})dz$, the characteristic curves along which the solution $\hat{\theta}$ is constant satisfy $d\hat{\theta} = 0$, and consequently, along these curves,

$$\frac{dz}{dt} = -\frac{\partial_t\hat{\theta}}{\partial_z\hat{\theta}}. \tag{4.45}$$

But Equation 4.44 implies that $\partial_t\hat{\theta} = -(v\mp c)\partial_z\hat{\theta}$, so the characteristic curves are given by

$$\frac{dz}{dt} = v \mp c. \tag{4.46}$$

This result indicates that the fluctuations propagate with a position-dependent velocity given by $v \mp c$, as might be expected from physical arguments. The choice of $+$ or $-$ sign corresponds to waves travelling in the positive or negative z directions respectively, relative to the flow.

To obtain the solution $\hat{\theta}$, the ordinary differential equation for the characteristic curves (Equation 4.46) must first be solved for z as a function of t. As usual for first-order ordinary differential equations, the solution set will be a one-parameter family of curves. As $\hat{\theta}$ is constant along each characteristic curve, $\hat{\theta}$ can be written as a function of the parameter only.

This method constitutes a generalisation of the solution obtained by Leonhardt et al [13], as they consider only the case where v is time-independent. Our method allows the formation of the horizon to be studied analytically, provided the time-dependent velocity profile is chosen so that Equation 4.46 can be solved exactly.

To demonstrate the method, let us apply it to the case studied by Leonhardt et al [13], where v is independent of time (and c is constant). Then Equation 4.46 can be integrated to give

$$t = \int^z \frac{1}{v \pm c} dz + t_0. \tag{4.47}$$

The characteristic curves are parametrised by the constant t_0, so as discussed previously, we can write down an explicit general solution for $\hat{\theta}$ as an arbitrary function f of t_0.

$$\hat{\theta} = f(t_0) = f\left(t - \int^z \frac{1}{v \pm c} dz\right). \tag{4.48}$$

In the case where c is not constant, the general solution obtained from Equation 4.46 may still be approximately valid if the last term in Equation 4.46 (containing the derivatives of c) is suppressed compared to the other terms.

4.3.2 One-Dimensional Systems with Constant Density and Varying Interaction Strength

Returning to Equation 4.42, consider now the case where c has arbitrary dependence on the position and time coordinates. As the density has been assumed to be constant, this can occur only if the interaction strength U_0 can be varied throughout the condensate. Barcelo, Liberati and Visser [44, 45] have studied BECs with variable interaction strength as an analogue cosmological system.

In the fluid rest frame, that is where $v = 0$, Equation 4.42 simplifies dramatically:

$$\partial_{zz}\hat{\theta} = \partial_t \left(\frac{1}{c^2}\partial_t\hat{\theta}\right). \tag{4.49}$$

This equation can also be obtained from the general wave equation in the case where v is constant, by a simple Galilean transformation. In that case, if c was initially a function of the position coordinates only, the effect of the transformation is to introduce a time dependence into the speed of sound, as the function describing c is "moving through" the fluid at velocity $-v$. The presence or absence of a sonic horizon cannot depend on our choice of inertial reference frame. Thus we see that in the presence of a time-dependent c, a sonic horizon can occur even when c is strictly positive and the fluid velocity is uniformly zero.

To understand this, consider a speed of sound profile moving through the fluid from left to right at a steady velocity v (that is $c = c(z - vt)$) such that at some point on the profile, c passes through the constant value v, with $c > v$ on the right-hand side of the crossover and $c < v$ on the left-hand side. Then right-travelling wavepackets originating on the right-hand side of this point will have velocity greater than the speed at which the profile travels, and so will escape away from the crossover, but right-travelling wavepackets originating on the left-hand side will have velocity less than the speed at which the crossover point is moving, and thus will never overtake the crossover.

It has not been possible to solve Equation 4.49 in the general case. To simplify matters, assume that c can be expressed as a (time- and space-dependent) small perturbation on a constant background, that is,

$$c = c_0 + \epsilon\hat{c}(z,t). \tag{4.50}$$

Then writing the solution to the wave equation as an expansion in powers of ϵ, that is $\hat{\theta} = \hat{\theta}_0 + \epsilon\hat{\theta}_1 + ...$, and comparing the terms of zero and first order in ϵ, we obtain a pair of readily solvable partial differential equations:

$$\partial_{zz}\hat{\theta}_0 = \frac{1}{c_0^2}\partial_{tt}\hat{\theta}_0, \tag{4.51}$$

$$\partial_{zz}\hat{\theta}_1 - \frac{1}{c_0^2}\partial_{tt}\hat{\theta}_1 = -\frac{2}{c_0^3}\partial_t\left(\hat{c}\partial_t\hat{\theta}_0\right). \tag{4.52}$$

The first equation is simply the usual wave equation and is trivially solvable. Once a solution has been obtained, substituting it into the second equation yields an inhomogeneous wave equation with constant sound velocity, for which an exact solution exists [46].

4.3.3 One-Dimensional Systems with Constant Interaction Strength

In contrast to the previous section, perhaps the most natural simplifying assumption is that the interaction strength is constant through the condensate. In the case where the condensate is homogeneous in number density, this condition is simply equivalent to the speed of sound being constant throughout the condensate, and the analysis in the last section for the case of constant density and constant speed of sound is applicable.

However, in a truly one-dimensional and stationary system (that is where the background flow is time-independent), variation in the condensate flow velocity necessarily implies variation in the density, by the continuity equation (Equation 4.16). This is an advantage of systems with varying interaction strength, as in these cases a sonic horizon may occur even when the flow velocity and density are both constant, due to variations in the speed of sound.

Leonhardt et al [13] assume a constant interaction strength and position-dependent flow velocity but circumvent the restriction imposed on the density by the continuity equation by examining a particular streamline in a three-dimensional flow (which is perpendicular to the horizon close to the horizon) rather than a truly one-dimensional flow. Garay et al [11, 12] present several one-dimensional systems which contain horizons due to variation in the flow velocity, and satisfy the continuity equation by variations in density.

In the case of constant interaction strength and possibly varying density, ρ_0/c^2 is a constant and Equation 4.41 reduces to

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial\hat{\theta}}{\partial t} + \vec{v}\cdot\nabla\hat{\theta}\right)\right] + \nabla\cdot\left[\left(\frac{\partial\hat{\theta}}{\partial t} + \vec{v}\cdot\nabla\hat{\theta}\right)\vec{v}\right] - \nabla\cdot\left(c^2\nabla\hat{\theta}\right) = 0. \tag{4.53}$$

In a stationary background flow, the continuity equation (Equation 4.16) requires that

$$\nabla\cdot(\rho_0\vec{v}) = 0. \tag{4.54}$$

In the case of constant interaction strength, this condition can be written in terms of the speed of sound,

$$\nabla \cdot \left(c^2 \vec{v}\right) = 0. \tag{4.55}$$

In a one-dimensional system, this equation can only be satisfied if $c^2 v$ is constant with respect to position, so we can write

$$v(z) = A/c(z)^2. \tag{4.56}$$

The wave equation then becomes

$$\partial_t \left[\left(\partial_t + \frac{A}{c^2} \partial_z \right) \hat{\theta} \right] + \partial_z \left[\frac{A}{c^2} \left(\partial_t + \frac{A}{c^2} \partial_z \right) \hat{\theta} \right] - \partial_z \left(c^2 \partial_z \hat{\theta} \right) = 0. \tag{4.57}$$

Chapter 5

Analogue Superradiance in Simple Vortices

5.1 Introduction

Having developed the theory of the condensate as an analogue gravity system, we now present an analysis of analogue superradiance in a simple vortex with density profile chosen to resemble that of realistic vortices in a BEC. This work was initiated as part of an Advanced Studies course carried out in 2004, and completed in the last six months of 2004 as part of my Honours research.

The work presented here is an edited excerpt of a paper which has been submitted for publication [14]. The paper was co-written by myself and Dr. Craig Savage. I carried out and wrote up the analytic work and wrote the basic code for the numerical simulations; Dr. Savage refined and tested the simulations and used them to investigate analogue superradiance in a range of parameter regimes.

5.2 Analytical Description of Wave Propagation in a Simple Vortex

We analyse a cylindrically symmetric vortex with purely circumferential flow velocity $\vec{v} = v_\theta \hat{\theta}$, hence removing the requirement for a central drain. In the case of irrotational flow, $v_\theta = \alpha/r$, for some constant α. The density and speed of sound depend only on the radial coordinate r, and approach asymptotic values ρ_∞ and c_∞ for $r \to \infty$.

We consider cylindrical wave solutions to the wave equation (2.1) of the form $\phi(t, r, \theta, z) = \psi(t, r)e^{-im\theta}$, with angular wavenumber m. We insert the effective metric (Equation 2.14) into the wave equation, assuming that our system is well described by the continuity equation and zero-viscosity Euler equation. Further assuming that the square of the speed of sound is proportional to the density, as is the case for a BEC with constant interaction strength [5], the density may be eliminated from the wave equation, which then becomes

$$\frac{\partial^2 \psi}{\partial t^2} - 2i \frac{m v_\theta}{r} \frac{\partial \psi}{\partial t} - \frac{1}{r} \frac{\partial}{\partial r} \left(r c^2 \frac{\partial \psi}{\partial r} \right) + \frac{m^2}{r^2} \left(c^2 - v_\theta^2 \right) \psi = 0 \,. \tag{5.1}$$

For single frequency waves of the form $\psi(t, r) = r^{-1/2} G(r^*) e^{i\omega t}$, and defining a "tortoise coordinate" r^* by $dr/dr^* = \tilde{c}^2$, with $r^* \to r$ for $r \to \infty$, where $\tilde{c} = c/c_\infty$, we find

$$\frac{d^2 G(r^*)}{dr^{*2}} + \frac{\tilde{c}^2}{c_\infty^2} (\omega^2 - V_{eff}) G(r^*) = 0 \,, \tag{5.2}$$

$$V_{eff} = \frac{2m\omega v_\theta}{r} + \frac{m^2}{r^2}\left(c^2 - v_\theta^2\right) - \frac{1}{2r}\left(\frac{c^2}{2r} - \frac{dc^2}{dr}\right) \ , \tag{5.3}$$

where we have introduced an effective potential V_{eff} to emphasise the similarity to the time-independent Schrödinger equation.

In order to analyse superradiance we consider the two limiting cases where the tortoise coordinate r^* approaches $\pm\infty$. For r large, $r^* \approx r$, and eliminating all terms except those of highest order in r yields the asymptotic form

$$\frac{d^2G(r^*)}{dr^{*2}} + \frac{\omega^2}{c_\infty^2}G(r^*) \approx 0 \ , \tag{5.4}$$

with general solution

$$G(r^*) = Ae^{i(\omega/c_\infty)r^*} + Be^{-i(\omega/c_\infty)r^*} \ , \tag{5.5}$$

where A and B are constant amplitudes of incoming and outgoing waves respectively.

The analysis of the other limit, where $r^* \to -\infty$, requires that the density profile of the vortex be specified. In order to present analytical calculations we use the density profile

$$\rho(r) = \begin{cases} \rho_\infty \frac{[(r-r_0)/\sigma]^2}{2+[(r-r_0)/\sigma]^2} & , r > r_0 \\ 0 & , r \le r_0 \end{cases} \ . \tag{5.6}$$

This is similar to the charge $l = 1$ vortex density profile for a BEC [5], but with the scale length given by the free parameter σ, rather than by the healing length χ. In the Thomas-Fermi limit of dominant atom interactions, the length scale for a charge l vortex, with angular momentum $l\hbar$ per particle, is $l\chi$ [47]. Although a vortex with $l > 1$ is unstable to decay into l single charge vortices, it may be stabilised by a pinning optical potential [48]. Since the density profile (Equation 5.6) was chosen primarily for analytical convenience, we do not expect it to be exactly achievable experimentally, although something close to it should be. For example, the dipole potential generated by far detuned light has been used to engineer density profiles in BECs [49]. Although our profile does not reproduce any specific astrophysical metric, it does have an analogue ergoregion, and we shall show that it displays the associated wave scattering physics. The other free parameter is the radius of the zero density core r_0, which might also be engineered by applied potentials. Since we have assumed that the square of the speed of sound is proportional to the density, $\tilde{c}^2 = \rho/\rho_\infty$, and the tortoise coordinate is

$$r^* = \int \frac{1}{\tilde{c}^2}\, dr = r - \frac{2\sigma^2}{r - r_0} \ . \tag{5.7}$$

For $r \approx r_0$, $r^* \approx -2\sigma^2/(r - r_0)$ and $c^2 \approx c_\infty^2(r - r_0)^2/(2\sigma^2)$. Using these approximations and retaining only terms of lowest order in $r - r_0$, we find that asymptotically

$$r^{*2}\frac{d^2G(r^*)}{dr^{*2}} + \frac{2\sigma^2\Omega^2}{c_\infty^2}G(r^*) = 0 \ , \tag{5.8}$$

$$\Omega \equiv \omega - m\alpha/r_0^2 \ . \tag{5.9}$$

This is an Euler-type equation and has solutions of the form $G(r^*) = |r^*|^\delta$, where

$$\delta = \frac{1}{2} \pm \gamma, \quad \gamma \equiv \frac{1}{2}\mathrm{sign}(\Omega)(1 - 8\sigma^2\Omega^2/c_\infty^2)^{1/2} \ . \tag{5.10}$$

The general solution to the asymptotic form of the wave equation is then

$$G(r^*) = |r^*|^{1/2}\left(C|r^*|^{\gamma} + D|r^*|^{-\gamma}\right) \ , \tag{5.11}$$

where C and D are constants. According to Equation 5.10, γ is either real or purely imaginary. For γ imaginary this asymptotic solution is oscillatory and may be written in the form

$$G(r^*) = |r^*|^{1/2}\left(Ce^{\gamma \ln |r^*|} + De^{-\gamma \ln |r^*|}\right) \ . \tag{5.12}$$

By making $|r^*|$ the argument of a logarithm we have anticipated that r^* will be made dimensionless. This last form allows us to identify the two linearly independent solutions as outgoing and ingoing waves, with amplitudes C and D, respectively. Note that in this asymptotic limit $r^* < 0$ and hence $\ln |r^*|$ increases with decreasing r^*. Note also, that it is the sign of the group velocity, not of the phase velocity, which determines the identification of outgoing and ingoing waves [29], and the sign of the group velocity is independent of the sign of Ω.

Let the Wronskians (of the solution and its complex conjugate) corresponding to the asymptotic solutions given by Equations 5.5, 5.12 be denoted $W(\infty)$ and $W(-\infty)$. Then

$$W(\infty) = \frac{2i\omega}{c_\infty}\left(|B|^2 - |A|^2\right) \ . \tag{5.13}$$

If γ is imaginary,

$$W(-\infty) = 2\gamma(|C|^2 - |D|^2) \ . \tag{5.14}$$

Following Vilenkin [29], consider a solution to the wave equation for $G(r^*)$ representing a wave originating at $r^* = +\infty$ and having the asymptotic forms, for imaginary γ,

$$G(r^*) = \begin{cases} e^{i(\omega/c_\infty)r^*} + Re^{-i(\omega/c_\infty)r^*} & , r^* \to \infty \\ T|r^*|^{1/2}e^{-\gamma \ln |r^*|} & , r^* \to -\infty \end{cases} \ . \tag{5.15}$$

Then by Abel's Theorem, the Wronskian of this solution is constant, since there is no first derivative term in the differential equation for $G(r^*)$, Equation 5.2. Thus, setting $A = 1$ and $B = R$ in Equation 5.13, and $C = 0$ and $D = T$ in Equation 5.14, and equating $W(\infty) = W(-\infty)$,

$$\frac{2i\omega}{c_\infty}(|R|^2 - 1) = -2\gamma|T|^2 \ , \tag{5.16}$$

and therefore,

$$|R|^2 = 1 - \text{sign}(\Omega)|\gamma||T|^2 c_\infty/\omega \ . \tag{5.17}$$

In particular, if Ω is negative, and $T \neq 0$, then $|R| > 1$, and superradiance occurs, since then, in the asymptotic region far from the vortex, the amplitude of the reflected wave exceeds that of the incident wave. This superradiance inequality $\Omega < 0$ expands to

$$\omega < m\alpha/r_0^2 \ , \tag{5.18}$$

which requires $m\alpha > 0$, so that the cylindrical waves propagate in the same direction as the vortex flow.

In contrast to the draining vortex examined by Basak and Majumdar [22] and the rotating black hole [29], here there are two independent necessary conditions for superradiance: γ must be purely imaginary and Ω must be negative. Combining these conditions gives a

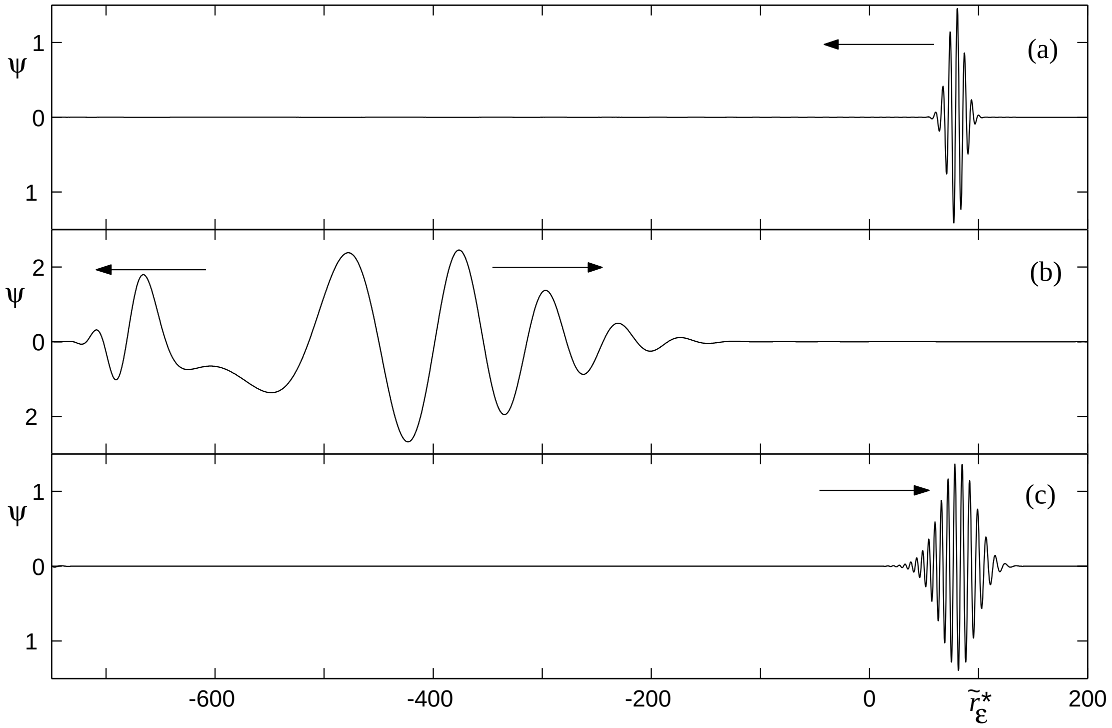

Figure 5.1: Wavepacket propagation from numerical solution of Equation 5.1. The real part of the wave packet is plotted versus the dimensionless modified radial tortoise coordinate $\tilde{r}^*_\epsilon$, defined in the text, with $\epsilon = 800/801$ so that $\tilde{r}^*_\epsilon = -800$ at $r = r_0$. The initial wavepacket, at $\tilde{t} = 0$, was given by Equation 5.19 with $\tilde{r}_{init} = 200$ and $\tilde{w} = 10$. (a) $\tilde{t} = 114$: the wavepacket is propagating towards the vortex. (b) $\tilde{t} = 330$: the wavepacket has just split into reflected and transmitted parts. (c) $\tilde{t} = 504$: the reflected wavepacket. Dimensionless parameters: $m = 1$, $\tilde{\alpha} = 2$, $\tilde{c}_\infty = 1$, $\tilde{\sigma} = 20$. The spatial grid had 8192 points over $\tilde{r}^*_\epsilon \in [-800, 400]$, and the time steps were 0.015 dimensionless time units.

frequency-independent condition for superradiance. For since $\omega \geq 0$, then $\Omega \geq -m\alpha/r_0^2$, and superradiance requires, $0 \geq \Omega \geq -m\alpha/r_0^2$, and therefore $\Omega^2 \leq m^2\alpha^2/r_0^4$. If γ is to be imaginary, we must have $1 < 8\sigma^2\Omega^2/c_\infty^2$, and since $m\alpha > 0$, the frequency-independent necessary condition for superradiance is $\sigma m\alpha > c_\infty r_0^2/(2\sqrt{2})$.

For a charge l vortex in a BEC we can substitute expressions [5] for the vortex velocity constant $\alpha = l\hbar/m_{atom}$, where m_{atom} is the mass of an atom, and for c_∞ in terms of the healing length χ, $c_\infty = \hbar/(\sqrt{2}\chi m_{atom})$, to obtain $\chi > r_0^2/(4\sigma l m)$. This is always possible to fulfill, at least in principle, since the healing length is given by $\chi = (8\pi n a)^{-1/2}$, where n is the atomic number density and a is the s-wave scattering length, and both n and a are under experimental control.

5.3 Numerical Analysis of Superradiant Scattering

To determine the magnitude of the superradiant amplification, we numerically solved the wave equation (5.1) using the XMDS package [50]. We used an initial Gaussian wavepacket of the form $\psi(0,r) = A(r)$, with frequency ω_0. Specifically,

$$\begin{aligned} A(r) &= e^{-(r-r_{init})^2/w^2} e^{i\omega_0 r/c_\infty} \ , \\ \frac{\partial \psi}{\partial t}(0,r) &= \left(i\omega_0 - 2c_\infty (r - r_{init})/w^2\right) A(r) \ . \end{aligned} \tag{5.19}$$

We found that superradiance occured for a wide range of parameters, as long as the relevant inequalities were fulfilled. Figure 5.1 shows a particularly strong and clean example of the scattering of an $m = 1$ wavepacket. Comparing the Fourier components of the real parts of the incident and reflected wavepackets in the asymptotic region, we found that the dominant Fourier power in the reflected wavepacket is approximately doubled.

The parameters are given in the caption, and are representative of conditions for a trapped dilute gas BEC. They are made dimensionless, indicated by a tilde, by measuring distance and time in units of r_0 and ω_0^{-1} respectively. For example, choosing $r_0 = 1$ μm and $\omega_0^{-1} = 10^{-3}$ s, the parameters of Figure 5.1 correspond to $c_\infty = 10^{-3}$ ms^{-1} and $\sigma = 20$ μm. For the case of a Rb^{87} BEC, $m_{atom} = 1.4 \times 10^{-25}$ kg, and the asymptotic healing length $\tilde{\chi} \approx 0.5$, and $\tilde{\alpha} = 2$ corresponds to a vortex charge of about $l = 3$. Note that the real part of the wave is plotted against the modified tortoise coordinate $\tilde{r}^*_\epsilon = \tilde{r} - 2\tilde{\sigma}^2/(\tilde{r} - 1 + \epsilon)$. This magnifies the scale as $r \to r_0$ $(\tilde{r} \to 1)$, while limiting the lower bound of $\tilde{r}^*_\epsilon$ to $1 - 2\tilde{\sigma}^2/\epsilon$. The ergoregion boundary, or static limit, is at $\tilde{r} \approx 8$ or $\tilde{r}^*_\epsilon \approx -92$.

The local healing length determines the validity of the hydrodynamic approximation (Equation 4.19), and is larger than the asymptotic value by the factor $\sqrt{\rho_\infty/\rho(r)}$. Because of this the BEC interpretation fails for Figure 5.1(b). However, we have found a tradeoff between the strength of the superradiance and the validity of the BEC hydrodynamic approximation. With increasing m, $\tilde{\alpha}$, and sound wavelength, reflection occurs at larger values of r, and hence for smaller local healing lengths, while the superradiance decreases in strength. We define the reflection point to be at the maximum of the effective potential Equation 5.3. Compared to the case of Figure 5.1, a transmitted wavelength comparable to the local healing length at the reflection point, accompanied by a 70% reduction in superradiance power, occurs for an initial wavelength an order of magnitude longer, and $\tilde{\alpha}$ a factor of five larger.

Numerical simulations of BECs under the proposed experimental conditions, and without the hydrodynamic approximation, could explore the effect of its breakdown as the wavelength of the transmitted wave decreases due to its inward propagation. This is related to the "trans-Planckian" problem of Hawking radiation [16].

In summary, our analytical and numerical work has shown superradiant sound scattering from hydrodynamic vortices. We have also argued for its experimental realization in dilute gas Bose-Einstein condensates. This might provide a useful step towards the ultimate goal of observing the sonic analogue of Hawking radiation.

Chapter 6

Quantum Theory of Excitations in Bose-Einstein Condensates

6.1 Derivation of the Bogoliubov-deGennes Equations

Up until this point we have worked entirely within a semiclassical approximation where the Bose field operator is replaced by its expectation value. The fluctuation field has been represented by linearised perturbations to the solution of the Gross-Pitaevskii equation. Having obtained the classical phase fluctuation field analogous to a massless scalar field in curved space-time, calculations of quantum mechanical effects such as Hawking radiation may proceed by quantising this field (within the regime of validity of the hydrodynamic approximation, since that was a requirement of the hydrodynamic analogy).

In the microscopic theory of BECs, however, there is already a field operator representing small fluctuations of the Bose field operator around the condensate, and hence describing creation of particles in non-condensate modes. We will now consider the evolution of that fluctuation field, $\hat{\Psi}'$, assuming that the classical mean field ψ obeys the Gross-Pitaevskii equation. Again approximating the interparticle potential by a delta function, as in Equations 4.4 and 4.5, and comparing terms of first order in the fluctuation field in the equation of motion for the field operator (Equation 4.3), we obtain an approximate equation of motion for the (necessarily small) fluctuation field,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\hat{\Psi}'(\vec{r},t) &= \left[-\frac{\hbar^2\nabla^2}{2m} + V_{trap} + U_0|\psi(\vec{r},t)|^2\right]\hat{\Psi}'(\vec{r},t) + \\ & \quad U_0\hat{\Psi}'^{\dagger}(\vec{r},t)\psi(\vec{r},t)^2 + U_0\hat{\Psi}'(\vec{r},t)|\psi(\vec{r},t)|^2 \\ &= \left[-\frac{\hbar^2\nabla^2}{2m} + V_{trap} + 2U_0|\psi(\vec{r},t)|^2\right]\hat{\Psi}'(\vec{r},t) + \\ & \quad U_0\psi(\vec{r},t)^2\hat{\Psi}'^{\dagger}(\vec{r},t). \end{aligned} \tag{6.1}$$

Now the fluctuation field operator $\hat{\Psi}'$ can be decomposed as a sum of quasiparticle annihilation and creation operators. We write

$$\hat{\Psi}'(\vec{r},t) = \sum_{\omega}\left(u_\omega(\vec{r})a_\omega e^{-i\omega t} + v^*_\omega(\vec{r})a^\dagger_\omega e^{i\omega^* t}\right). \tag{6.2}$$

Here a_ω is the operator which annihilates one quasiparticle from the eigenstate with energy $\hbar\omega$. The inclusion of the creation operator terms is necessary because of the dependence on $\hat{\psi}^\dagger$ in the equation of motion for $\hat{\psi}$.

Inserting this expansion into Equation 6.1 and comparing the coefficients of the exponential terms yields a pair of coupled equations for u_ω and v_ω,

$$
\begin{aligned}
\hbar\omega u_\omega(\vec{r}) &= \left[-\frac{\hbar^2\nabla^2}{2m} + V_{trap} + 2U_0|\psi(\vec{r},t)|^2\right] u_\omega(\vec{r}) + \\
&\quad U_0 v_\omega(\vec{r})\psi(\vec{r},t)^2, \qquad (6.3)\\
-\hbar\omega v_\omega(\vec{r}) &= \left[-\frac{\hbar^2\nabla^2}{2m} + V_{trap} + 2U_0|\psi(\vec{r},t)|^2\right] v_\omega(\vec{r}) + \\
&\quad U_0 u_\omega(\vec{r})\psi^*(\vec{r},t)^2. \qquad (6.4)
\end{aligned}
$$

These equations are called the *Bogoliubov-deGennes* (BdG) equations. Writing $\mathcal{L} = -(\hbar^2/2m)\nabla^2 + V_{trap} + 2U_0|\psi|^2$, we can write these equations more simply in matrix form,

$$
\begin{pmatrix} \mathcal{L} & U_0\psi^2 \\ -U_0\psi^{*2} & -\mathcal{L} \end{pmatrix} \begin{pmatrix} u_\omega \\ v_\omega \end{pmatrix} = \hbar\omega \begin{pmatrix} u_\omega \\ v_\omega \end{pmatrix}. \qquad (6.5)
$$

The operators a_ω and $a_\omega^\dagger$ satisfy the usual commutation relations for bosonic annihilation and creation operators provided that the mode functions u_ω and v_ω satisfy certain orthonormality conditions [5, 6, 13]:

$$
\begin{aligned}
\int d^3\vec{r}\,(u_\omega v_{\omega'} - v_\omega u_{\omega'}) &= 0, \qquad (6.6)\\
\int d^3\vec{r}\,(u_\omega^* u_{\omega'} - v_\omega^* v_{\omega'}) &= \delta_{\omega\omega'}. \qquad (6.7)
\end{aligned}
$$

Under these conditions, and retaining terms of up to second order in the fluctuations in the Hamiltonian, the system Hamiltonian can be rewritten in the form [5, 6],

$$
\begin{aligned}
\hat{H} &= E_0 + \sum_\omega \omega a_\omega^\dagger a_\omega, \qquad (6.8)\\
E_0 &= N_0 \int d^3\vec{r}\,\psi^*\mathcal{L}\psi. \qquad (6.9)
\end{aligned}
$$

6.2 The BdG Equations and the Hydrodynamic Variables

In the case where the background flow is time-independent, it is straightforward to eliminate the external potential from the BdG equations, replacing it with hydrodynamic variables (that is the density and phase). This was done by Garay et al [12] and is preferable when the background flow is already specified. This formulation also allows any sonic horizons to be easily identified.

The modified Euler equation (Equation 4.18) relates the velocity and density of the bulk fluid to the external potential. In the case where $\partial\vec{v}/\partial t = 0$, the relationship reduces to

$$
V_{trap} = V_0 - \frac{m}{2}v^2 + \frac{\hbar^2}{2m\sqrt{\rho_0}}\nabla^2\sqrt{\rho_0} - U_0\rho_0, \qquad (6.10)
$$

where V_0 is some arbitrary (and physically irrelevant) constant that we will set to zero for ease of notation. In the case where U_0 is constant it is convenient to rewrite this equation in terms of the speed of sound c,

$$V_{trap} = -m\left(\frac{v^2}{2} + c^2\right) + \frac{\hbar^2}{2mc}\nabla^2 c. \tag{6.11}$$

In terms of the velocity and sound speed in the condensate, the BdG equations then become

$$\begin{aligned} \hbar\omega u_\omega(\vec{r}) &= \left[-\frac{\hbar^2\nabla^2}{2m} - m\left(\frac{v^2}{2} - c^2\right) + \frac{\hbar^2}{2mc}\nabla^2 c\right] u_\omega(\vec{r}) + \\ & mc^2 v_\omega(\vec{r}) e^{2i\theta_0}, \end{aligned} \tag{6.12}$$

$$\begin{aligned} -\hbar\omega v_\omega(\vec{r}) &= \left[-\frac{\hbar^2\nabla^2}{2m} - m\left(\frac{v^2}{2} - c^2\right) + \frac{\hbar^2}{2mc}\nabla^2 c\right] v_\omega(\vec{r}) + \\ & mc^2 u_\omega(\vec{r}) e^{-2i\theta_0}, \end{aligned} \tag{6.13}$$

where θ_0 is the phase of the condensate satisfying $\vec{v}(\hbar/m)\nabla\theta_0$.

6.3 The Bogoliubov Dispersion Relation for Bose-Einstein Condensates

The behaviour of solutions to the BdG equations can be expressed in terms of a modified dispersion relation that is phononic at low wavenumber but asymptotes to the free particle dispersion relation at high wavenumber. This dispersion relation was first derived from the microscopic theory of BEC by Bogoliubov [51], for waves propagating in a homogeneous condensate.

In the case where the interaction strength is constant and the speed of sound has zero second derivative, the BdG equations reduce to

$$\hbar\omega u_\omega(\vec{r}) = \left[-\frac{\hbar^2\nabla^2}{2m} - m\left(\frac{v^2}{2} - c^2\right)\right] u_\omega(\vec{r}) + mc^2 v_\omega(\vec{r}) e^{2i\theta_0}, \tag{6.14}$$

$$-\hbar\omega v_\omega(\vec{r}) = \left[-\frac{\hbar^2\nabla^2}{2m} - m\left(\frac{v^2}{2} - c^2\right)\right] v_\omega(\vec{r}) + mc^2 u_\omega(\vec{r}) e^{-2i\theta_0}. \tag{6.15}$$

In the case where variation of $\vec{v}$ is negligible, let the direction of $\vec{v}$ define the x-axis, and write $\theta_0 = mvx/\hbar$. The equation system then admits simple plane-wave solutions propagating in the x-direction. Setting $u_\omega(x) = u_\omega e^{i(k_\omega + mv/\hbar)x}$, we obtain from Equation 6.14,

$$\hbar\omega u_\omega = \left[\frac{\hbar^2(k_\omega + mv/\hbar)^2}{2m} - m\left(\frac{v^2}{2} - c^2\right)\right] u_\omega + mc^2 v_\omega(x) e^{i(mv/\hbar - k_\omega)x}. \tag{6.16}$$

Consequently

$$v_\omega(x) = e^{i(k_\omega - mv/\hbar)x}\frac{u_\omega}{mc^2}\left(\hbar\omega - \frac{\hbar^2(k_\omega + mv/\hbar)^2}{2m} + m\left(\frac{v^2}{2} - c^2\right)\right). \tag{6.17}$$

Writing $v_\omega(x) = v_\omega e^{i(k_\omega - mv/\hbar)x}$ and substituting into Equation 6.15 yields

$$-\hbar\omega v_\omega = \left[\frac{\hbar^2(k_\omega - mv/\hbar)^2}{2m} - m\left(\frac{v^2}{2} - c^2\right)\right] v_\omega + mc^2 u_\omega. \tag{6.18}$$

Having derived two expressions relating u_ω and v_ω, consistency requires that

$$\begin{aligned} \left(mc^2\right)^2 &= \left(-\hbar\omega - \frac{\hbar^2(k_\omega - mv/\hbar)^2}{2m} + m\left(\frac{v^2}{2} - c^2\right)\right) \\ &\quad \times \left(\hbar\omega - \frac{\hbar^2(k_\omega + mv/\hbar)^2}{2m} + m\left(\frac{v^2}{2} - c^2\right)\right). \end{aligned} \tag{6.19}$$

Expanding gives the dispersion relation

$$\left(\frac{\hbar}{2m}\right)^2 k_\omega^4 + \left(c^2 - v^2\right) k_\omega^2 + 2\omega v k_\omega - \omega^2 = 0. \tag{6.20}$$

This equation is quadratic in ω, so solving we obtain the energy expression

$$\omega = v k_\omega \pm \sqrt{c^2 k_\omega^2 + \left(\frac{\hbar}{2m}\right)^2 k_\omega^4}. \tag{6.21}$$

Now if the magnitude of v is allowed to vary along the x-axis, and we replace the constant coefficients u_ω, v_ω with slowly varying coefficient functions $U_\omega(x), V_\omega(x)$, then $\theta_0 = \int (mv/\hbar) dx$ and the appropriate plane-wave trial solution becomes

$$u_\omega(x) = U_\omega(x) e^{i\int k_\omega(x) + (mv(x)/\hbar) dx}. \tag{6.22}$$

The calculation then proceeds as previously except that additional terms are introduced proportional to the derivatives of $v(x)$, $U_\omega(x)$ and $k_\omega(x)$, as

$$\begin{aligned} \nabla^2 u_\omega(x) &= \left(\frac{1}{U_\omega(x)}\frac{d^2 U_\omega(x)}{dx^2} + 2i\frac{1}{U_\omega(x)}\frac{dU_\omega(x)}{dx} + i\frac{dk_\omega(x)}{dx} + i\frac{m}{\hbar}\frac{dv(x)}{dx} \right. \\ &\quad \left. - \left(k_\omega(x) + \frac{mv(x)}{\hbar}\right)^2\right) u_\omega(x). \end{aligned} \tag{6.23}$$

If we assume that these additional terms are small (the WKB approximation) then the dispersion relation has the same form as previously (Equation 6.20).

6.4 Analysis of the Bogoliubov Dispersion Relation

As a quartic equation, it is possible to solve Equation 6.20 exactly for k_ω, but the general solutions are complicated and unwieldy to study analytically. For sufficiently small frequencies ω, however, the dispersion relation simplifies markedly and admits simple solutions.

In the case where $\omega = 0$, the dispersion relation becomes simply

$$\left(\frac{\hbar}{2m}\right)^2 k_0^4 + \left(c^2 - v^2\right) k_0^2 = 0, \tag{6.24}$$

with trivial solutions

$$k_0 = 0, \pm \frac{2m}{\hbar}\sqrt{v^2 - c^2}. \tag{6.25}$$

Now write $\omega = \epsilon\Omega$, $\epsilon \ll 1$, and to first order in ϵ, $k_\omega = k_0 + \epsilon\hat{k}$. Then expanding the dispersion relation and retaining terms of up to second order in ϵ, we obtain

$$\begin{aligned} 0 \;=\; & \left(\frac{\hbar}{2m}\right)^2 k_0^4 + \left(c^2 - v^2\right) k_0^2 + \epsilon\left[4\left(\frac{\hbar}{2m}\right)^2 k_0^3\hat{k} + 2\left(c^2 - v^2\right) k_0\hat{k} + 2\Omega v k_0\right] + \\ & \epsilon^2\left[6\left(\frac{\hbar}{2m}\right)^2 k_0^2\hat{k}^2 + \left(c^2 - v^2\right)\hat{k}^2 + 2\Omega v\hat{k} - \Omega^2\right]. \end{aligned} \tag{6.26}$$

In the case where $k_0 \neq 0$, setting the first-order term equal to zero yields an expression for $\hat{k}$,

$$\begin{aligned} \hat{k} &= \frac{-2\Omega v}{2\left(c^2 - v^2\right) + 4\left(\frac{\hbar}{2m}\right)^2 k_0^2} \\ &= \frac{-\Omega v}{v^2 - c^2}. \end{aligned} \tag{6.27}$$

In the case where $k_0 = 0$, on the other hand, the first-order term is trivially zero independent of $\hat{k}$, so we must pass to the second-order term, which in the case $k_0 = 0$ yields

$$\begin{aligned} \left(c^2 - v^2\right)\hat{k}^2 + 2\Omega v\hat{k} - \Omega^2 &= 0, \\ \Rightarrow \hat{k} &= \frac{\Omega}{v \pm c}. \end{aligned} \tag{6.28}$$

Thus in the limit of small ω, the solutions to the dispersion relation assume the form

$$k_\omega = \frac{\omega}{v \pm c}, \frac{-\omega v}{v^2 - c^2} \pm \frac{2m}{\hbar}\sqrt{v^2 - c^2}. \tag{6.29}$$

Given our interest in sonic horizons, it is important to note that the behaviour of these approximate solutions undergoes a qualitative change where $|v|$ becomes equal to c. Outside the sonic horizon (that is where $|v| < c$) two of the approximate solutions are real and two are imaginary, while inside the sonic horizon all four approximate solutions are real. Note however that our small-frequency approximation breaks down close to the horizon for any non-zero frequency (this is most easily seen in the case of the solution that approaches infinity close to the horizon), and in fact the onset of the additional roots occurs just inside the horizon, at a position depending on ω (Figure 6.1).

The first two solutions, with $k_\omega = \omega/(v \pm c)$, are termed the "acoustic" branches of the dispersion relation. The other two solutions are called the "trans-acoustic" branches. These analytic expressions for the solutions make it straightforward to compute the group velocity of each branch in terms of ω:

$$v_g = \frac{d\omega}{dk_\omega} = v \pm c, \frac{c^2 - v^2}{v}. \tag{6.30}$$

The acoustic solutions clearly correspond to phonons propagating either in the direction of the fluid flow or against it. For v negative, copropagating phonons correspond to the $-$ sign in Equation 6.30 and have negative group velocity everywhere, while counter-

Figure 6.1: The Bogoliubov dispersion relation, with $m = \hbar = c = 1$, $\omega = 0.1$.

propagating waves correspond to the $+$ sign and have positive group velocity outside the horizon (where $c > |v|$). Inside the sonic horizon (where $|v| > c$) the counterpropagating wave has group velocity of the same sign as v, and hence is swept away by the supersonic flow.

The trans-acoustic solutions, on the other hand, have group velocity of opposite sign to v inside the sonic horizon, as $c^2 - v^2$ is negative there. This indicates that they represent supersonic modes which can travel out through the event horizon. However the approximate solutions given here are not sufficient to investigate the behaviour of these solutions near the horizon, and even the exact solutions may not be an accurate description in regions where the WKB approximation breaks down.

6.5 Bogoliubov Modes with Negative Norm

The BdG equations have a symmetry which naturally gives rise to solutions which cannot be normalised according to Equation 6.7, instead their calculated norm is negative. In this section we will discuss the inclusion of these modes in the expansion of the field operator. This discussion will pave the way for the proper treatment of modes of negative frequency and positive norm, which occur in the presence of supersonic flow.

Given a positive frequency solution (u_ω, v_ω) of the BdG equations, the pair (v^*_ω, u^*_ω) is a single frequency solution with frequency $-\omega$. This is most easily seen by examining the matrix form of the BdG equations (Equation 6.5). Conjugating both sides and transposing both rows and columns, we obtain

$$\begin{pmatrix} \mathcal{L} & U_0\psi^2 \\ -U_0\psi^{*2} & -\mathcal{L} \end{pmatrix} \begin{pmatrix} v_\omega^* \\ u_\omega^* \end{pmatrix} = -\hbar\omega \begin{pmatrix} v_\omega^* \\ u_\omega^* \end{pmatrix}. \tag{6.31}$$

Now under the transformation $(u, v) \to (v^*, u^*)$, the normalisation integrand (Equation 6.7) transforms according to

$$u_\omega^* u_{\omega'} - v_\omega^* v_{\omega'} \to v_{\omega'}^* v_\omega - u_{\omega'}^* u_\omega. \tag{6.32}$$

In particular, in the case where $\omega = \omega'$, the sign of the normalisation integral is reversed under this transformation. Thus exactly one of the solutions (u, v) and (v^*, u^*) can have positive norm: the other must have negative norm. The operator a_ω associated with a negative norm solution (u_ω, v_ω) does not have the commutation relations of a boson annihilation operator: instead it behaves as a creation operator [12, 52].

We will call the solution (v^*, u^*) the *conjugate solution* of (u, v). Now consider the mode expansion of the fluctuation field operator (Equation 6.2),

$$\hat{\Psi}'(\vec{r}, t) = \sum_\omega \left(u_\omega(\vec{r}) a_\omega e^{-i\omega t} + v_\omega^*(\vec{r}) a_\omega^\dagger e^{i\omega^* t} \right). \tag{6.33}$$

If this expansion includes a term of the form $u_\omega(\vec{r})a_\omega e^{-i\omega t} + v_\omega^*(\vec{r})a_\omega^\dagger e^{i\omega t}$ and also a term arising from the conjugate solution, $v_\omega^*(\vec{r})a_{-\omega}e^{i\omega t} + u_\omega(\vec{r})a_{-\omega}^\dagger e^{-i\omega t}$, then adding these terms together we obtain

$$u_\omega(\vec{r}) \left(a_\omega + a_{-\omega}^\dagger \right) e^{-i\omega t} + v_\omega^*(\vec{r}) \left(a_\omega + a_{-\omega}^\dagger \right) e^{i\omega t}. \tag{6.34}$$

Thus we can eliminate one of every pair of conjugate modes from the sum by redefining our annihilation and creation operators. The operator $\left(a_\omega + a_{-\omega}^\dagger \right)$ may be defined as either an annihilation or a creation operator: we determine which is appropriate by examining the norm of the associated mode function (u_ω, v_ω). If this norm is positive then both a_ω and $a_{-\omega}^\dagger$ have the commutation relations of annihilation operators, by the discussion above, and therefore their sum also behaves as an annihilation operator. If the norm is negative then the opposite is true. In this way any mode with negative norm is subsumed into its conjugate mode (with necessarily positive norm) in the expansion of the field operator. Consequently we need only consider modes with positive norm in the expansion of the field operator.

Chapter 7

Sonic Horizons in BECs

7.1 Proposed Sonic Horizon Configurations

7.1.1 Black and White Hole Horizons in a Ring

Garay et al propose creating a sonic horizon in a Bose-Einstein condensate confined in a narrow ring [11], which would effectively behave as a one-dimensional periodic system. They select a specific density profile for the condensate,

$$\rho(\theta) = \frac{N}{2\pi}(1 + b\cos\theta). \tag{7.1}$$

Applying the continuity equation $\partial_\theta(\rho v) = 0$ gives the dimensionless flow-velocity field as a function of the effective interaction between particles, the winding number and the parameters b and θ. The Gross-Pitaevskii equation can then be employed to calculate the required trapping potential. Adjusting the interaction strength and the value of b allows the existence, location and stability of the sonic horizon(s) to be varied. Garay et al study the stability of analogue black / white holes in this system using the Bogoliubov equations and eliminating high-frequency components of the solutions, and thus develop a strategy for creating a sonic black / white hole state adiabatically from an initial stable state. They also discuss the consequences of moving from a stable black hole state into the unstable region, suggesting that this would give rise to disappearance of the sonic horizon in an "explosion of phonons".

While this paper employs both the Gross-Pitaevskii equation and the BdG equations, it does not study analogue Hawking radiation beyond the hydrodynamic approach. Garay et al demonstrate only that it should be possible to create this profile in a Bose-Einstein condensate, that it is expected to be stable, and in the hydrodynamic approximation it possesses an analogue event horizon.

7.1.2 One-Dimensional Sink-Generated Horizon

A second proposal by Garay et al [12] deals with a sonic horizon in a one-dimensional system with a sink at $z = 0$. It is experimentally challenging to produce controlled outcoupling from a BEC, but recent "atom laser" experiments suggest that this may be feasible in future [53, 54, 55, 56].

Garay et al propose a sound-speed profile of the form

$$c(z) = \begin{cases} c_0, & |z| < L, \\ c_0\left[1 + (\sigma - 1)(|z| - L)/\epsilon\right], & L < |z| < L + \epsilon, \\ \sigma c_0, & L + \epsilon < |z|. \end{cases} \tag{7.2}$$

Assuming that the interaction strength is constant and the background flow is stationary, the continuity equation (Equation 4.16) gives $\rho_0 v =$ constant and $c^2 \propto \rho_0$. Choosing the flow velocity to be directed inwards, setting $\sigma > 1$, and defining v_0 to be the absolute value of the flow velocity for $|z| < L$, we obtain the velocity profile

$$v(z) = -\frac{v_0 c_0^2}{c(z)^2} \frac{z}{|z|}. \tag{7.3}$$

The sonic horizon occurs where $|v| = c$, that is where

$$v_0 c_0^2 = c(z)^3. \tag{7.4}$$

If this condition is satisfied somewhere in the region $L < |z| < L + \epsilon$, where the velocity and speed of sound are both position-dependent, then the sonic horizon occurs at the points z satisfying

$$|z| = L + \frac{\epsilon}{\sigma - 1} \left(\sqrt[3]{\frac{v_0}{c_0}} - 1 \right). \tag{7.5}$$

Thus if a sonic horizon is to exist at all in this region, the parameters ϵ, v_0, c_0 and σ must obey the relation

$$\sigma > \sqrt[3]{\frac{v_0}{c_0}} > 1. \tag{7.6}$$

Garay et al then write down the BdG equations for this system (Equations 6.12, 6.13). An advantage of this simple system is that the terms of the form $\nabla^2 c$ vanish except at the points $z = \pm L, \pm(L + \epsilon)$. In the regions of constant velocity, the WKB approximation is exact and the solutions of the BdG equations are simply plane waves governed by the Bogoliubov dispersion relation (Equation 6.20).

The authors give an approximate solution in the intermediate regions, $L < |z| < L + \epsilon$, by neglecting terms of second order or higher in ϵ, and then use this approximate solution to obtain connection formulae between the modes in the inner and outer regions of constant flow velocity. Unfortunately the calculations stemming from this approximation cannot be used to investigate analogue Hawking radiation in this system, because the approximation immediately removes the characteristic thermal spectrum.

This effect occurs because the Hawking temperature is dependent on the surface gravity, which in this system is given by

$$\begin{aligned} \kappa &= \frac{\partial}{\partial z} (v + c) (z_{horizon}), \\ &= \frac{3c_0(\sigma - 1)}{\epsilon}. \end{aligned} \tag{7.7}$$

Thus neglecting terms of greater than second order in ϵ is equivalent to neglecting terms of greater than second order in $1/\kappa$. In this approximation, the thermal spectrum at temperature $T \propto \kappa$ has the form

$$\frac{1}{e^{A\omega/\kappa} - 1} \approx \frac{\kappa}{A\omega}. \tag{7.8}$$

Rather than investigating Hawking radiation, Garay et al [12] study the complex-frequency solutions of the Bogoliubov dispersion relation (Equation 6.20), in order to investigate dynamical instabilities of the system. They find that unstable modes correspond to bound states in the region between the two sonic horizons. Thus true stability occurs only for very small values of L, so that no bound states exist: however perturbations with wavelengths of this size are not well described by the hydrodynamic approximation. For large values of L, however, the system may become quasistable, as the rate at which the instability grows is lower for larger L.

7.1.3 Acceleration of a BEC through a Potential Barrier

Giovanazzi et al [57] discuss a system in which a Bose-Einstein condensate is pushed along a waveguide of variable area by an applied potential ("optical piston") and through a potential barrier, where the flow is accelerated to supersonic speed. Both potentials (piston and barrier) can be supplied by the foci of blue-detuned light beams. Giovanazzi et al work entirely within the hydrodynamic approximation, and study the general case of one-dimensional gas flow with variable area under the influence of external potentials.

They demonstrate that the critical potential, above which a sonic horizon forms, is given by

$$U_c = m\left(c_0^2 + \frac{v_0^2}{2} - \frac{3}{2}(v_0 c_0^2)^{2/3}\right), \tag{7.9}$$

where m is the atomic mass, c_0 is the initial speed of sound in the condensate and v_0 is its initial flow velocity. For potentials above U_c the condensate turns from subsonic to supersonic speed at the potential maximum U_m. A more tightly confined potential barrier gives rise to a larger velocity gradient and a higher Hawking radiation temperature: the Hawking temperature is given by,

$$T = \frac{\hbar\omega_0}{2\pi k_B}\frac{\sqrt{3}}{2}, \tag{7.10}$$

$$m\omega_0^2 = -\frac{\partial^2 U}{\partial x^2}\bigg|_{horizon}. \tag{7.11}$$

These relations suggest that it may be possible to substantially increase the predicted Hawking radiation temperature in an experiment by appropriate tuning of the external potential, potentially making it easier to measure Hawking radiation.

Giovanazzi et al consider the general case where both the area of the waveguide and the applied potential vary in space, under the approximation that the area varies slowly and is large enough that the Thomas-Fermi (hydrodynamic) approximation is valid over most of the volume of the condensate. In the usual Laval nozzle case there is no external potential and only the area of the waveguide varies. It is shown that if a Gaussian light beam is used to confine the flowing condensate, either no sonic horizon forms or two horizons form around the beam waist.

Finally, they present numerical simulations of the quasi-stationary flow density using the Gross-Pitaevskii equation, and obtain good agreement with the hydrodynamic theory.

7.2 Stability of Sonic Horizons in BECs

7.2.1 Dynamical Instability

The instabilities considered by Garay et al [11, 12] are dynamical instabilities, originating from complex-frequency solutions to the BdG equations. A general theory of such instabilities is outlined by Leonhardt et al [43] and Garay et al [12].

If the system possesses eigenmodes of complex frequency, then the terms associated with these modes in the expansion of the field operator (Equation 6.2) do not immediately admit an interpretation in terms of particle creation, because the norm (Equation 6.7) of such modes is zero. Consequently, the associated operators do not obey the commutation relations of either annihilation or creation operators.

However, new operators can be defined as linear combinations of the operators associated with the complex-frequency modes, in such a way that these new operators possess the commutation relations of annihilation operators. Rewriting the part of the Hamiltonian describing the complex eigenmodes in terms of these new operators gives rise to terms representing self-amplifying creation of positive and negative frequency pairs.

In general, Leonhardt et al [43] find that black hole sonic horizons are stable, but there are dynamical instabilities associated with white hole sonic horizons. In most proposals both black hole horizons and white hole horizons are created (because the flow must be decelerated back down to subsonic speeds) so these instabilities may be significant.

7.2.2 Anomalous Modes and Energetic Instability

Garay et al [12] also briefly discuss the existence of energetic instabilities. In cases where the fluctuations are linearised about an excited state of the Hamiltonian, the mode expansion of the fluctuation field operator (Equation 6.2) may admit modes with negative frequency and positive norm, termed *anomalous modes.* Such modes do not exist when the mean field ψ describes macroscopic occupation of the ground state of the full Hamiltonian (describing both the condensate and the fluctuation field), as such an occurrence would violate the definition of the ground state. Thus the presence of anomalous modes indicates that the system is energetically unstable.

Anomalous modes have been studied in the context of vortices [58] and dark solitons [59, 60]. They are also associated with any configuration in which the BEC flow velocity exceeds the local speed of sound. In this context they are associated with the Landau criterion, which states that an object moving through a superfluid at supersonic velocity will spontaneously create excitations in the fluid [5].

The energy of a mode measured in a frame comoving with the fluid is not equal to the energy of the same mode measured in the laboratory reference frame. For an excitation with momentum $\vec{p}$ and energy E_p in the comoving frame, and a fluid moving with constant velocity $vecv$, the difference in the two energies is given by $\vec{v} \cdot \vec{p}$. Thus in the laboratory frame, the energy necessary to create an excitation is $E_p - \vec{v} \cdot \vec{p}$, and the energy of the excitations in the laboratory frame may become negative for $|v| > E_p/|p|$. According to the Bogoliubov dispersion relation (Equation 6.21), in one-dimensional systems the energy of an excitation of momentum $\hbar k$ in the rest frame is given by

$$E_k = \hbar\omega = c\hbar k_\omega \sqrt{1 + \left(\frac{\hbar}{2mc}\right)^2 k_\omega^2}, \tag{7.12}$$

so the onset of negative energy modes occurs at $|v| \approx c$ for modes of small momentum.

In a flow confined by walls, these negative energy modes are spontaneously populated by the simultaneous excitation of the wall material and the negative energy modes [61]. If the walls are sufficiently smooth, and the gas is sufficiently cold and dilute, then the rate at which the system decays via this energetic instability may be made very small [12]. In a liquid helium system, where the Landau criterion also applies, Volovik [27] suggests employing a second superfluid to confine the flow in order to reduce dissipation.

Chapter 8

Hawking Radiation in Black Holes

8.1 Principles of Hawking Radiation in General Relativity

The Hawking effect can be characterised as the spontaneous emission of thermal radiation from the event horizon of a black hole, at a temperature (termed the *Hawking temperature* or *Hawking radiation temperature*) depending on the black hole mass. In Hawking's original treatment of the effect [3] this radiation is a consequence of mixing of positive and negative-frequency modes (due to the time dependence of the metric) during the formation of the black hole from a collapsing star. Employing Hawking's method, therefore, the black hole cannot be considered even approximately as a system in equilibrium for the purposes of studying Hawking radiation.

The original calculation dealt explicitly only with massless Hermitian scalar fields in a spherically symmetric collapse geometry, but can be extended to electromagnetic fields, charged black holes and non-spherically-symmetric cases [3, 28].

More significantly, in Hawking's analysis backreaction of the Hawking radiation on the black hole is neglected: the evolution of the background geometry is predetermined and drives the evolution of the field modes. Parikh and Wilczek [19] demonstrate that by considering a dynamical Schwarzschild geometry in which the hole mass is allowed to vary, Hawking radiation may be described as a tunnelling process. This calculation imposes energy conservation, meaning that the black hole mass decreases as Hawking radiation is emitted, and results in corrections to the usual emission rate.

A potential problem with Hawking's original derivation is its reliance on modes of very high wavenumber (and hence short wavelength), which arise due to blueshifting as modes pass through the collapsing star close to the formation of the event horizon. At very short length scales, normal quantum mechanics is expected to break down, and hence the behaviour of these modes may not be accurately described by low-energy theories. It has been suggested that observation of Hawking radiation might therefore provide a means to probe physics at the Planck scale [15, 16, 17, 18, 30, 62].

The dependence of Hawking radiation on high-frequency modes has been investigated by computing the particle production rate and quantum state arising from various high-frequency dispersion relations in 1+1-dimensional black holes [15, 16, 18, 62]. In most cases, the spectrum of emitted radiation appears to be largely independent of the exact high-frequency behaviour: however Unruh and Schutzhold [62] give several examples in which modifying the dispersion relation at high frequencies can lead to significant deviations.

8.2 Derivation of Hawking Radiation

In this section we will outline the original derivation of the Hawking effect [3]. The calculation deals with a minimally coupled Hermitian massless scalar field obeying the Klein-Gordon equation, propagating in an asymptotically flat space-time containing a star which collapses to form a black hole.

Quantising the field, we write the field operator as $\hat{\phi} = \sum_i \left(f_i \hat{a}_i + f_i^* \hat{a}_i^\dagger\right)$, where the f_i are a complete orthonormal family of (complex-valued) solutions to the wave equation, and the operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ are annihilation and creation operators respectively for incoming scalar particles. We impose the condition that the f_i must be ingoing solutions of positive frequency on past null infinity (that is, if we propagate the solutions far enough backwards, they will be asymptotically ingoing and of positive frequency). The initial vacuum can then be defined as the state annihilated by the operators $\hat{a}_i$.

In this case the mode expansion of the field operator was determined by the asymptotic behaviour of the solutions to the wave equation far from the black hole and far in the past. Alternate mode expansions can be chosen. In particular, as we are interested in the behaviour of the field far from the horizon and at late times, it is appropriate to re-express the field operator in terms of modes that behave in a particular way when propagated far forward in time. We write

$$\hat{\phi} = \sum_i \left(p_i \hat{b}_i + p_i^* \hat{b}_i^\dagger + q_i \hat{c}_i + q_i^* \hat{c}_i^\dagger\right). \tag{8.1}$$

Here the p_i are solutions of the wave equation which are asymptotically outgoing positive frequency waves (when propagated far forward in time), while the q_i are modes which contain no outgoing component in the limit of late time.

The functions p_i and q_i can be written as linear combinations of the functions f_i and f_i^*. For example, we can define coefficients α_{ij} and β_{ij} by $p_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right)$. Comparing the two expressions for the field operator, the creation and annihilation operators for the outgoing modes $\hat{b}_i$ and $\hat{b}_i^\dagger$ can then be written as linear combinations of the annihilation and creation operators for the ingoing modes $\hat{a}_i$ and $\hat{a}_i^\dagger$. In particular,

$$\hat{b}_i = \sum_j \left(\alpha_{ij}^* \hat{a}_j + \beta_{ij}^* \hat{a}_j^\dagger\right). \tag{8.2}$$

The particle creation in the outgoing modes can be computed by taking the expectation value of the number operator $\hat{b}_i^\dagger \hat{b}_i$ in the vacuum state defined by the ingoing modes, using this expansion.

$$\langle 0|\hat{b}_i^\dagger \hat{b}_i|0\rangle = \sum_j |\beta_{ij}|^2. \tag{8.3}$$

Hawking calculates the coefficients β_{ij} for the case where the collapse is spherically symmetric. The calculation proceeds by considering the propagation of a positive-frequency wave at future infinity backwards through spacetime. Two early-time components can be identified which contribute to the final wave, and the final wave is written as an integral over frequency of the early-time fields with the same angular momentum quantum numbers l and m (rather than the discrete sum employed earlier). It is assumed that the time dependence of infalling modes is given by $\omega^{-1/2}e^{i\omega v}$, where v is the advanced time, and the time dependence of outgoing modes is similarly $\omega^{-1/2}e^{i\omega u}$, where u is the retarded time (see the glossary for definitions of advanced and retarded time).

Firstly, as the wave is propagated backwards, part of it will be scattered off the spacetime curvature near the black hole, giving rise to an early-time wave with the same frequency as the original wave. This scattering will give rise to a term of the form $\delta(\omega-\omega')$ in the coefficient $\alpha_{\omega\omega'}$, and will not contribute to the $\beta_{\omega\omega'}$ coefficient.

The non-scattered part, on the other hand, will propagate back through the collapsing star, with strong blueshifting as it passes through the collapsing region. However, for modes entering the star at a sufficiently late time, passing through the star will be impossible due to the formation of the event horizon, so there will be a cutoff beyond which this blueshifted component is zero.

Let v_0 be the last advanced time at which a particle can begin falling towards the star, pass through the origin and escape. Then the blueshifted waves propagated back through the star will have the asymptotic form

$$p_\omega = \begin{cases} C\omega^{-1/2}e^{-i(\omega/\kappa)\ln(v_0-v)+i\omega v} & v < v_0, \\ 0 & v \geq v_0. \end{cases} \tag{8.4}$$

Here κ is the surface gravity of the black hole. We write this solution as a superposition of the infalling single-frequency modes,

$$p_\omega = \int \alpha_{\omega\omega'}(\omega')^{-1/2}e^{i\omega' v} + \beta_{\omega\omega'}(\omega')^{-1/2}e^{-i\omega' v}d\omega'. \tag{8.5}$$

Taking Fourier transforms, for large ω' the coefficients α, β have the forms

$$\alpha_{\omega\omega'} \approx Ce^{i(\omega-\omega')v_0}\left(\frac{\omega'}{\omega}\right)^{1/2}\Gamma(1-i\omega/\kappa)\left[-i(\omega-\omega')\right]^{-1+i\omega/\kappa}, \tag{8.6}$$

$$\beta_{\omega\omega'} \approx Ce^{i(\omega+\omega')v_0}\left(\frac{\omega'}{\omega}\right)^{1/2}\Gamma(1-i\omega/\kappa)\left[-i(\omega+\omega')\right]^{-1+i\omega/\kappa}. \tag{8.7}$$

Note that there is a logarithmic singularity at $\omega' = \pm\omega$ in these expressions. For $\omega' \gg \omega$, analytically continuing $(\omega+\omega')$ around the upper half-plane in ω', we obtain

$$\begin{aligned} |\beta_{\omega\omega'}| &\approx |\alpha_{\omega\omega'}||(\omega+\omega')^{-1+i\omega/\kappa}|/|(\omega-\omega')^{-1+i\omega/\kappa}|, \\ &\approx e^{-\pi\omega/\kappa}|\alpha_{\omega\omega'}|. \end{aligned} \tag{8.8}$$

Now the total number of particles created in the frequency range ω to $\omega+d\omega$ is given by $d\omega\int_0^\infty|\beta_{\omega\omega'}|^2d\omega'$, and is infinite for all frequencies. This is a consequence of considering the total particle creation over all time, and neglecting the backreaction of the particle creation on the metric.

Hawking argues, however, that the *rate* of particle emission at late retarded times is finite. At late retarded times, the metric can be approximated by the Schwarzschild solution: in particular, the fraction of a back-propagated wavepacket entering the star (as opposed to scattering off the curvature of the metric outside the star) is similar to the fraction that would have crossed the past event horizon of the Schwarzschild black hole, if it had existed. This approximation allows us to relate the number of particles emitted at a given frequency to the number of particles that would have been absorbed from a similar wave packet incident on the black hole.

The probability flux in a wave packet peaked at ω is approximately proportional to

(Equation 8.2) $\int_{\omega_1}^{\omega_2} \left(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2\right) d\omega'$, where $\omega_2 \gg \omega_1 \gg 0$ and we have assumed that most of the contribution to the probability flux comes from high-frequency modes. Employing Equation 8.8, we can then write

$$\int |\beta_{\omega\omega'}|^2 d\omega' \approx \frac{1}{e^{2\pi\omega/\kappa} - 1} \int_{\omega_1}^{\omega_2} \left(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2\right) d\omega'. \tag{8.9}$$

Interpreting the left-hand side as particle creation in a mode of frequency ω, and the right-hand side as the number of particles absorbed from a mode of frequency ω, this is the relation between absorption and emission probabilities expected for a body with temperature $T = \kappa/2\pi$. Consequently a black hole is expected to emit particles in a thermal spectrum at a temperature given by its surface gravity κ.

8.3 Hawking Radiation with Modified High Wavenumber Dispersion

Corley [15] investigates particle production in a 1+1-dimensional geometry without an explicit collapse when a modified high-frequency dispersion relation is employed. In particular, he employs a superluminal dispersion relation identical to the Bogoliubov dispersion relation derived from the microscopic treatment of phase fluctuations in Bose-Einstein condensates (Equation 6.20, [12, 13, 25]).

Corley's model spacetime possesses the metric

$$ds^2 = -dt^2 + (dx - v(x)dt)^2. \tag{8.10}$$

The "velocity" $v(x)$ is taken to be non-positive, and units are employed where $c = 1$, so the horizon occurs at $v(x_h) = -1 = -c$. Corley considers equations of motion for the field ϕ of the form

$$\left(\partial_t + \partial_x v\right)\left(\partial_t + v\partial_x\right)\phi = \partial_x^2\phi \pm \frac{1}{k_0^2}\partial_x^4\phi. \tag{8.11}$$

In the case where $v(x)$ is constant or approximately so, this equation has plane wave solutions of the form

$$\phi(t,x) = e^{i(kx-\omega t)}. \tag{8.12}$$

Substituting these mode solutions into the equation of motion yields the dispersion relation

$$(\omega - vk)^2 = k^2 \mp k^4/k_0^2. \tag{8.13}$$

Differentiating gives the group velocity v_g,

$$|v_g - v| = \left|\frac{1 \mp 2(k/k_0)^2}{\sqrt{1 \mp (k/k_0)^2}}\right|. \tag{8.14}$$

In the first case (corresponding to the $-$ sign in the dispersion relation) the locally measured group velocity of a wavepacket centred about a wavenumber k is approximately unity when $k \ll k_0$, but decreases to zero with increasing k, and then increases again as k becomes very large: thus there exist both subluminal modes of high k and superluminal modes of extremely high k (i.e. $k \approx k_0$). In the second case, the locally measured group velocity is approximately unity for small k and then increases with increasing k, giving rise to superluminal modes of high wavenumber. It is this second case that is most relevant to Hawking radiation in BECs.

Solutions ϕ of the field equation satisfying $\partial_t\phi - i\omega\phi$ are said to be pure-Killing-frequency solutions with Killing frequency ω, while solutions satisfying $(\partial_t + v\partial_x)\phi = -i\omega'\phi$ are said to be pure-free-fall-frequency solutions with free-fall frequency ω'. *Positive free fall frequency* wavepackets are superpositions of pure-free-fall-frequency solutions with positive free-fall frequency, while *positive Killing frequency* wavepackets are superpositions of pure-Killing-frequency solutions with positive Killing frequency.

Corley defines an inner product $(\cdot, \cdot)$ that is conserved on solutions to the equation of motion,

$$(F, G) = i \int dx\, (F^*(\partial_t + v\partial_x)G - G(\partial_t + v\partial_x)F^*)\,. \tag{8.15}$$

Note that a positive free fall frequency wavepacket will have positive norm when $v(x)$ is constant, and a positive Killing frequency wave packet will have positive norm in regions where $v = 0$ [15, 18, 52].

Now suppose $\hat{\phi}(x)$ is a self-adjoint operator satisfying Equation 8.11 and also the canonical position-momentum commutation relations. Defining the generalised momentum by

$$\hat{\pi} = (\partial_t + v\partial_x)\,\hat{\phi}, \tag{8.16}$$

the equal time commutation relations become $\left[\hat{\phi}(x), \hat{\pi}(y)\right] = i\delta(x, y)$.

For a wavepacket $f(x,t)$, Corley defines an "annihilation operator" $a(f)$ by

$$a(f) = (f, \hat{\phi}). \tag{8.17}$$

From the definition of the inner product and the assumption that $\hat{\phi}$ is self-adjoint, the Hermitian conjugate of a is given by

$$a^\dagger(f) = -(f^*, \hat{\phi}). \tag{8.18}$$

The commutators of the a and $a^\dagger$ operators can be computed from the canonical commutation relations and the definition of the inner product:

$$\begin{aligned}
\left[a(f), a^\dagger(g)\right] &= -\left[(f, \hat{\phi}), (g^*, \hat{\phi})\right] \\
&= \int dx\, dy \left[f^*(x)(\partial_t + v\partial_x)\hat{\phi}(x) - \hat{\phi}(x)(\partial_t + v\partial_x)f^*(x),\right. \\
&\quad \left. g(y)(\partial_t + v\partial_y)\hat{\phi}(y) - \hat{\phi}(y)(\partial_t + v\partial_y)g(y)\right] \\
&= \int dx\, dy \left[f^*(x)\hat{\pi}(x) - \hat{\phi}(x)(\partial_t + v\partial_x)f^*(x),\right. \\
&\quad \left. g(y)\hat{\pi}(y) - \hat{\phi}(y)(\partial_t + v\partial_y)g(y)\right] \\
&= -\int dx\, dy \left(f^*(x)(\partial_t + v\partial_y)g(y)\left[\hat{\pi}(x), \hat{\phi}(y)\right]\right. \\
&\quad \left. + g(y)(\partial_t + v\partial_x)f^*(x)\left[\hat{\phi}(x), \hat{\pi}(y)\right]\right) \\
&= -\int dx\, dy\, (-if^*(x)(\partial_t + v\partial_y)g(y)\delta(x,y) \\
&\quad + ig(y)(\partial_t + v\partial_x)f^*(x)\delta(x,y)) \\
&= i\int dx\, (f^*(x)(\partial_t + v\partial_x)g(x) - g(x)(\partial_t + v\partial_x)f^*(x)) \\
&= (f, g).
\end{aligned} \tag{8.19}$$

The commutators $[a(f), a(g)] = -(f, g^*)$ and $[a^\dagger(f), a^\dagger(g)] = -(f^*, g)$ can be computed similarly.

It now follows that if (f, f) is positive, the operator $a(f)$ behaves as an annihilation operator. If (f, f) is negative, on the other hand, then $a(f)$ satisfies the commutation relations of a creation operator (up to a normalisation constant), and we write $a(f) = -a^\dagger(f^*)$. From the definition of the inner product (Equation 8.15), it follows immediately that $(f^*, f^*) = -(f, f)$, so if f has negative norm then that is equivalent to f^* having positive norm.

This analysis is rather similar to our earlier discussion of negative norm solutions to the BdG equations (Section 6.5). However in this case the field operator does not naturally contain creation operators: the presence of creation operators is a consequence of a solution f having negative norm.

The particle creation in a wavepacket ψ_{out} is given by the expectation value of the number operator defined by the corresponding annihilation operators,

$$N(\psi_{out}) = \langle f|a^\dagger(\psi_{out})a(\psi_{out})|f\rangle, \tag{8.20}$$

where $|f\rangle$ is the state of the field. It is usually assumed [15, 18] that the state of the field at early times is the free-fall vacuum, with the property that $a(p_{in})|f\rangle = 0$ for any wavepacket p_{in} of positive free-fall frequency.

Associating ψ_{out} to a solution of the equation of motion (by imposing the condition that the late-time solution is given by ψ_{out}) and propagating this solution backwards in time, we obtain an early-time wavepacket ψ_{in}. Let us write ψ_{in} as a superposition of two wavepackets of positive and negative free-fall frequency respectively,

$$\psi_{in} = \psi_- + \psi_+. \tag{8.21}$$

Then by the invariance of the inner product with time, and the definition of the annihilation operators, it follows that

$$\begin{aligned} a(\psi_{out}) &= a(\psi_{in}) \\ &= (\psi_-, \hat{\phi}) + (\psi_+, \hat{\phi}) \\ &= -a^\dagger(\psi_-^*) + a(\psi_+). \end{aligned} \tag{8.22}$$

We justify this rewriting by noting that as ψ_- is a wavepacket of negative free-fall frequency, in a region where $v(x)$ is constant or approximately so, it must have negative norm. Thus it is appropriate to rewrite the operator $a(\psi_-)$ as a creation operator associated with the conjugate solution ψ_-^*. Now ψ_-^* and ψ_+ are wave packets of positive free-fall frequency, and the particle production can be computed, noting that $a(\psi_+)|f\rangle = \langle f|a^\dagger(\psi_+) = 0$,

$$\begin{aligned} N(\psi_{out}) &= \langle f| \left(-a(\psi_-^*) + a^\dagger(\psi_+)\right) \left(-a^\dagger(\psi_-^*) + a(\psi_+)\right) |f\rangle \\ &= \langle f|a(\psi_-^*)a^\dagger(\psi_-^*)|f\rangle \\ &= \langle f|a^\dagger(\psi_-^*)a(\psi_-^*) + (\psi_-^*, \psi_-^*)|f\rangle \\ &= -(\psi_-, \psi_-). \end{aligned} \tag{8.23}$$

So to compute the particle production in a late-time wavepacket, that wavepacket must be propagated back to the early time at which the state of the field is known, and the negative free-fall frequency part of the resulting wavepacket must be extracted.

Figure 8.1: Free-fall frequency of the WKB solutions as a function of frequency, at a point inside the horizon. The parameters are set to $\hbar = m = 1$, $c = 0.5$, $v = 0.8$. The frequency range is chosen so that the wavenumbers of the different modes are well separated.

8.3.1 Free-Fall Frequency for WKB Solutions

A wavepacket localised either inside or outside the horizon, in a region where the WKB approximation is valid, can be written as a superposition of WKB solutions of the form $e^{i(\int_z k(z)\,dz - \omega t)}$, where the momenta k are given by the superluminal dispersion relation (Equation 6.20 or 8.13). To extract the negative free-fall frequency part, therefore, it is sufficient to determine which branches of the dispersion relation correspond to negative free-fall frequency.

The free-fall frequency of the WKB mode solutions - or in the BEC case, the frequency of the solutions in the frame comoving with the fluid - is given by

$$\omega' = \omega - vk. \tag{8.24}$$

For v negative, clearly ω' may be of opposite sign to ω only if k is of opposite sign to ω. In the case of the ordinary wave equation, where the dispersion relation is simply $\omega = (v+c)k$ for the counterpropagating modes of interest, this condition is satisfied exactly where $|v| > c$, that is inside the sonic horizon. In fact in this case $\omega' = ck = c\omega/(v+c)$, so the free-fall frequency is of opposite sign to the frequency ω everywhere inside the horizon.

The same argument applies to the asymptotically "acoustic" or small wavenumber branches of the Bogoliubov dispersion relation (Equation 6.20). To investigate the "trans-acoustic" or large wavenumber branches, $\omega - vk$ was plotted as a function of ω at a point inside the sonic horizon (Figure 8.1).

For both positive and negative values of ω, the large wavenumber mode that escapes across the horizon has free-fall frequency of the same sign as ω inside the horizon, that is the sign of its free-fall frequency does not change as it crosses the horizon. In the case of positive (negative) ω this is the large wavenumber mode with large positive (negative) wavenumber inside the horizon. In contrast, the large wavenumber mode that does not propagate across the horizon within the WKB approximation (instead joining onto the acoustic mode) has free-fall frequency of opposite sign to ω.

Thus inside the horizon and for positive Killing frequency ω, the large positive (negative) wavenumber branch of the dispersion relation (Figure 6.1) corresponds to outgoing modes of positive (negative) free-fall frequency. The small wavenumber branch of the acoustic relation corresponds to infalling modes of negative free-fall frequency.

8.3.2 Boundary Conditions

Corley and Jacobson [15, 18] investigate mode solutions of the dispersion relation (Equation 8.13), i.e. separated single-frequency solutions of the form $e^{-i\omega t}f(x)$. This reduces the equation of motion (Equation 8.11) to an ordinary differential equation, which can be solved once appropriate boundary conditions are specified.

The boundary conditions depend on whether the dispersion relation is subluminal or superluminal. In the case of the ordinary wave equation, the usual condition is that the positive Killing frequency solutions which are combined to form the late-time Hawking radiation wavepacket must vanish inside the horizon [18]. The justification for this boundary condition is that wavepackets initially inside the horizon cannot escape and contribute to the Hawking radiation directly.

The subluminal dispersion relation also does not allow the escape of waves from behind the horizon, but the solutions are well behaved at the horizon, so such a discontinuous boundary condition is inappropriate. Instead the boundary condition becomes that inside the horizon the wave packet solution decays exponentially with distance from the horizon [18].

In contrast, in the superluminal dispersion relation the origin of the late-time outgoing wavepacket is a pair of high-wavenumber modes within the event horizon [15]. The appropriate boundary condition is then that inside the event horizon, there is no counter-propagating small-wavenumber mode, as such a mode is trapped inside the event horizon [15].

8.3.3 Mode Conversion at the Horizon

Consider an outgoing wavepacket located far outside the horizon, strongly peaked in both ω and k. In both the superluminal and subluminal cases, outside the event horizon the only branch of the dispersion relation with positive group velocity is the outgoing acoustic branch: the trans-acoustic branches have negative group velocity and therefore correspond to ingoing modes. Thus the high-wavenumber modifications to the dispersion relation are irrelevant to the modes in which the particle production associated with Hawking radiation is measured.

Propagating this outgoing wavepacket backwards in time, back towards the horizon, its behaviour varies markedly between the subluminal and superluminal cases. The subluminal case is investigated in detail by Corley and Jacobson [18]: we will only discuss the superluminal case here, as that is the case relevant to BECs.

Following the superluminal dispersion relation (Figure 6.1) across the horizon, it is clear that the acoustic branch of the dispersion relation smoothly evolves to a trans-acoustic branch of high positive wavenumber as the wavepacket is propagated backwards across the horizon. However, upon crossing the horizon the wavepacket also acquires a component on the trans-acoustic branch with large negative wavenumber. This "mode conversion" can occur because the negative wavenumber branch of the dispersion relation has a turning point near the horizon, and the WKB approximation breaks down there. The wavepacket might also gain a component on the acoustic branch of the dispersion relation inside the horizon. In the present case, this possibility is forbidden because we have imposed the boundary condition that the late-time wavepacket exists only outside the horizon and is outgoing there. The acoustic branch of the dispersion relation has negative group velocity and thus represents ingoing modes, so mixing-in of the acoustic branch would result in a late-time ingoing wavepacket inside the horizon.

This mode conversion is studied by Corley [15] and Schutzhold and Unruh [62]. Their approach is to match up the WKB solutions on either side of the horizon by solving the equation of motion exactly in the vicinity of the horizon for pure-Killing-frequency mode solutions.

Insertion of the trial solution $e^{-i\omega t}f(x)$ into the equation of motion yields an ordinary differential equation, as discussed previously. Close to the horizon, the velocity profile is approximated as $v(x) \approx -1 + \kappa x$, where κ is the surface gravity of the black hole and $\kappa|x| \ll 1$. The differential equation can then be simplified by retaining only terms of leading order in x in the coefficients of the derivatives. The method of Laplace transforms is used to solve the resulting second-order differential equation, with the boundary conditions discussed earlier.

Physically, this resembles the problem of free-particle scattering from a potential barrier, where the WKB solutions on each side of the barrier are patched together by exact solutions of an approximate form of the Schrodinger equation (obtained by linearising in x) [63]. In the superluminal case, where the wavepacket is initially localised inside the horizon, the final Hawking radiation packet may be considered as a consequence of tunnelling through the horizon. The particles undergoing tunnelling arise from the non-zero value of the number expectation operator for a positive Killing frequency wavepacket localised inside the horizon (Equation 8.23).

8.3.4 Computing the Quantum State

An interesting extension is to compute the full quantum state of the system, as described by Corley [15] - or to put it another way, to re-express the assumed vacuum state (the free-fall vacuum) in terms of a vacuum state defined by late time observers.

In the time dependent picture, this is accomplished by propagating wavepackets of purely positive and purely negative free-fall frequency, and positive Killing frequency, forward in time. In the presence of superluminal dispersion, the original wavepackets are localised behind the event horizon, and as discussed previously the positive (negative) free-fall frequency outgoing wavepacket is composed of modes lying on the large positive (negative) wavenumber branch of the dispersion relation (Figure 6.1). The small wavenumber branch of the acoustic relation has negative group velocity, and so does not contribute to the outgoing wavepackets.

Close to the horizon, these outward-propagating wavepackets undergo scattering, giving rise to both a transmitted and a reflected wavepacket. Propagating both these wavepackets forward to regions where the WKB approximation holds, the transmitted wavepacket is described by the small positive wavenumber branch of the dispersion relation outside the horizon. The reflected wavepacket is described by the small negative wavenumber branch inside the horizon, as this is the only branch of the dispersion relation inside the horizon with negative group velocity (except the copropagating mode, which does not enter into this calculation).

In the vacuum state defined by the modes of positive free-fall frequency, the wavepacket of positive free-fall frequency is unpopulated, but as discussed earlier the wavepacket of negative free-fall frequency is associated with particle production. The transmitted component corresponds to the Hawking radiation wavepacket studied earlier. Now, however, there is also a reflected component. We say that this infalling wavepacket represents the "partner particles" of the Hawking radiation.

Chapter 9

Hawking Radiation as a Tunnelling Effect

Hawking radiation is often described heuristically as a consequence of pair production close to the horizon combined with tunnelling. In this picture, a particle-antiparticle pair is created just outside the horizon, and one member of the pair tunnels inside the horizon to a negative energy trajectory (as measured by an observer far from the black hole). The locally measured energy of the particle may change sign as it crosses the horizon, so the particle can propagate freely on this negative energy trajectory. In this way energy is conserved and the remaining member of the pair can escape to infinity, with a positive energy as measured by the observer far from the black hole. The same reasoning holds for particle production just inside the horizon: here one member of the created pair tunnels out of the horizon onto a positive-energy trajectory, while the other member falls into the black hole on a negative energy path (all energies measured far from the event horizon).

Parikh and Wilczek [19] quantify this idea by taking into account the backreaction due to the Hawking radiation on the mass of a Schwarzschild black hole. Energy conservation requires that if an outgoing particle is produced with positive energy, the energy of the black hole must decrease correspondingly. A Schwarzschild black hole is completely defined by its mass M, so Parikh and Wilczek examine the tunnelling of a particle of positive energy ω out through the event horizon under the constraint that $M + \omega$ is constant. The calculation of the emission rate by Parikh and Wilczek is done in terms of the semiclassical momentum and energy of a particle propagating on a radial null geodesic.

9.1 Application to Hydrodynamic Analogues

This line of argument cannot be directly extended to the hydrodynamic analogues. Even simple one-dimensional sonic horizons, unlike Schwarzschild black holes, are characterised by a large number of parameters (for example, the flow velocity and density profiles close to the horizon). Consequently it is not simple to determine the qualitative effect that analogue Hawking radiation will have on the background flow. This is an example of the point that the analogy applies only to a fluctuation field propagating on a fixed background: the dynamics of the background itself are goverened by the Euler and continuity equations in the hydrodynamic case, but the Einstein field equations in the case of an astrophysical black hole.

However, this does not mean that the tunnelling description is not applicable to fluids, only that it cannot be deduced directly from the analogy with the astrophysical case. Several authors have given semiclassical descriptions of analogue Hawking radiation as a tunnelling effect in fermionic superfluid systems [26, 64].

9.2 Hawking Radiation in a Non-Interacting Fermi Gas

Giovanazzi [64] considers a one-dimensional Fermi-degenerate non-interacting gas flowing past a smooth potential barrier $V_{ext}(z)$. The potential is assumed to be sharply peaked around $z = 0$ and negligible elsewhere, with $V_{ext}(0) = V_{max}$ being the potential maximum. The assumption of Fermi degeneracy means that all states are occupied up to some energy ϵ_{max}. As the particles are assumed to be non-interacting, the initial many-body wavefunction can be written in the form

$$|f\rangle = \prod_{0 \leq \epsilon \leq \epsilon_{max}} a_\epsilon^\dagger |0\rangle, \tag{9.1}$$

where $|0\rangle$ is the vacuum of particles and $a_\epsilon^\dagger$ is the creation operator for the single-particle eigenstate with energy ϵ.

To investigate scattering off the potential barrier, the single-particle eigenfunctions are taken to have the asymptotic free-particle form

$$\psi_\epsilon \propto \left[e^{ikx} + r(\epsilon)e^{-ikx}\right]\Theta(-x) + t(\epsilon)e^{ikx}\Theta(x). \tag{9.2}$$

The momentum k is related to the energy ϵ by the single-particle dispersion relation, $\epsilon = \hbar^2 k^2/2m$.

As the barrier is smooth, to a first approximation the particle propagation is classical and $|t(\epsilon)|^2 \approx \Theta(\epsilon - V_{max})$. Thus on both sides of the potential barrier, the Fermi-degenerate many-particle state (Equation 9.1) can be viewed as the analogue of the astrophysical "vacuum state", and treated as a background flow on which perturbations may propagate. Setting the initial state far from the horizon to the Fermi-degenerate state described by Equation 9.1 amounts to assuming there are no pre-existing excitations.

The momenta of the particles in this background flow will be distributed over an interval (v_L, v_R), as a consequence of the uniform Fermi-degenerate distribution of the particles over the energy eigenstates with $\epsilon \leq \epsilon_{max}$. The maximum energy of a left-moving particle is ϵ_{max}, so the maximum velocity of a left-moving particle far to the left of the potential barrier is given by $v_{max} = \sqrt{2\epsilon_{max}/m}$. By conservation of energy, it follows that the maximum velocity of a left-moving particle at any point in the potential will be given by

$$v_R = \sqrt{v_{\max}^2 - 2\frac{V_{ext}(z)}{m}}. \tag{9.3}$$

Far to the left (right) of the horizon, the "vacuum" particle flow propagating away from the horizon represents those particles that classically would have been reflected (transmitted) at the potential barrier, that is particles with $\epsilon \leq V_{max}$ ($\geq V_{max}$). Defining $v_{esc} = \sqrt{2V_{max}/m}$, conservation of energy implies that

$$v_L = \begin{cases} -\sqrt{v_{esc}^2 - 2V_{ext}(z)/m} & z < 0, \\ \sqrt{v_{esc}^2 - 2V_{ext}(z)/m} & z > 0. \end{cases} \tag{9.4}$$

In the local reference frame moving with velocity $v = (v_L + v_R)/2$, the total momentum of the particles is zero, and the fluid appears to be in equilibrium, so this reference frame defines the background flow velocity. Giovanazzi [64] then shows that the speed of sound can be self-consistently defined as the Fermi velocity,

$$c = v_F = \frac{v_R - v_L}{2}. \tag{9.5}$$

Thus the sonic horizon occurs where $v_L = 0$, that is at $z = 0$ (since at $z = 0$, $V_{ext}(z) = V_{max}$). This is what would be expected if Hawking radiation is to ensue from scattering at the potential barrier.

Tunnelling of particles with $\epsilon < V_{max}$ through the barrier will give rise to particles on the right-hand side of the sonic horizon that are not accounted for in the classical background flow. This tunnelling also means that the reflection probability $|r(\epsilon)|^2$ is less than would be expected classically, giving rise to "holes" in the classical flow on the left-hand side of the barrier. For $\epsilon > V_{max}$ the opposite effect occurs, with reflected particles and transmitted holes occurring in addition to the classical solution.

9.3 Hawking Radiation from a Sonic Horizon in a Fermi Superfluid

Volovik [26] reviews in detail the behaviour of superfluid ^{4}He and ^{3}He-A as analogue gravity systems. The superfluid component constitutes the "vacuum", while the excitations that make up the normal component correspond to particles.

The quasiparticle dispersion relation in ^{3}He-A admits both positive and negative energy solutions. Because the excitations are fermionic, the negative energy modes are assumed to be filled in the vacuum state. However the sign of the energy depends on the choice of reference frame, as changing to a reference frame moving with velocity v adds a term kv to the locally measured energy of excitations with momentum k. In particular, the measured energy of a mode will differ between the laboratory frame and the frame comoving with the fluid (assuming the fluid velocity is non-zero), and may even be of opposite sign in these two important frames.

It turns out that in the absence of a sonic horizon (that is, in regions where $v < c$) the sign of the energy is the same for the laboratory frame and the frame comoving with the fluid. In regions where $v > c$, however, modes with negative energy in the frame comoving with the fluid may have positive energy in the laboratory frame. Such states are occupied in the superfluid-comoving vacuum, but are empty in the laboratory frame vacuum (defined as the frame where any external potential is time-independent). Thus in the laboratory frame, the superfluid-comoving vacuum appears as an excited state with a non-equilibrium quasiparticle distribution. Volovik [26] asserts that the system reaches equilibrium by tunnelling from the occupied quasiparticle modes to modes of the same energy (in the laboratory frame) outside the horizon.

This mechanism for Hawking radiation is very similar to that described by Corley [15] in the previous chapter. The modes with negative energy in the frame comoving with the fluid correspond to the negative free-fall frequency modes in the astrophysical case. Volovik states that in the superfluid-comoving vacuum, the states with locally defined negative energy are filled: in Corley's case, this corresponds to the non-zero expectation value (in the free-fall vacuum) of the number operator associated with modes of negative free-fall frequency. In both cases, Hawking radiation is a consequence of tunnelling by modes with negative free-fall frequency and positive laboratory-frame frequency out from inside the sonic horizon, just as described at the beginning of this chapter.

Chapter 10

Hawking Radiation in Bose-Einstein Condensates

10.1 Derivation of Bogoliubov Modes Corresponding to Acoustic Phase Fluctuations

Suppose $(u_\omega(x), v_\omega(x))$ is a solution to the BdG equations, with $\psi_0 = |\psi_0|e^{i\theta_0}$ being the condensate field. Then noting that Equation 6.1 is identical in form to the linearised Gross-Pitaevskii equation for fluctuations in ψ_0 (Equation 4.29), it follows that so long as terms of second order and higher in the fluctuations are neglected,

$$\psi = \psi_0 + \left(u_\omega(x)e^{-i\omega t} + v_\omega^*(x)e^{i\omega^* t}\right), \tag{10.1}$$

is a solution to the linearised Gross-Pitaevskii equation. Our goal is to associate the fluctuation $\psi - \psi_0$ with fluctuations in the density and phase (obtained from solving the hydrodynamic wave equation) and thus to obtain an expression for the solution $(u_\omega(x), v_\omega(x))$ associated with a pure-frequency phase fluctuation in the hydrodynamic approximation.

As previously, in terms of the phase and density fluctuations $\hat{\theta}, \hat{\rho}$, the fluctuation in the field is given by Equation 4.25,

$$\hat{\psi} = e^{i\theta_0}\left(\frac{\hat{\rho}}{2\sqrt{\rho_0}} + i\hat{\theta}\sqrt{\rho_0}\right). \tag{10.2}$$

Furthermore, in the hydrodynamic approximation, the density fluctuation $\hat{\rho}$ is related to the phase fluctuations by

$$\hat{\rho} = -\frac{\hbar\rho_0}{mc^2}\left(\frac{\partial\hat{\theta}}{\partial t} + \vec{v}\cdot\nabla\hat{\theta}\right). \tag{10.3}$$

Therefore the field fluctuation can be written in terms of the phase alone,

$$\hat{\psi} = \sqrt{\rho_0}e^{i\theta_0}\left(\frac{-\hbar}{2mc^2}\left(\partial_t + \vec{v}\cdot\nabla\right) + i\right)\hat{\theta}. \tag{10.4}$$

Now consider a single frequency phase fluctuation, of the general form

$$\hat{\theta} = \sigma e^{-i\omega t} + \sigma^* e^{i\omega^* t}. \tag{10.5}$$

Substituting into Equation 10.4 and comparing to Equation 10.1 gives the relation

$$\begin{aligned} u_\omega(x)e^{-i\omega t} + v^*_\omega(x)e^{i\omega^* t} &= \sqrt{\rho_0}e^{i\theta_0}\left(\frac{-\hbar}{2mc^2}\left(-i\omega + \vec{v}\cdot\nabla\right) + i\right)\sigma e^{-i\omega t} + \\ &\quad \sqrt{\rho_0}e^{i\theta_0}\left(\frac{-\hbar}{2mc^2}\left(i\omega^* + \vec{v}\cdot\nabla\right) + i\right)\sigma^* e^{i\omega^* t}. \end{aligned} \tag{10.6}$$

Comparing the coefficients of the exponentials yields the relations

$$\begin{pmatrix} u_\omega \\ v_\omega \end{pmatrix} \sqrt{\rho_0} \begin{pmatrix} e^{i\theta_0}\left[(\hbar/2mc^2)\left(\omega + i\vec{v}\cdot\nabla\right) + 1\right] \\ e^{-i\theta_0}\left[(\hbar/2mc^2)\left(\omega + i\vec{v}\cdot\nabla\right) - 1\right] \end{pmatrix} i\sigma. \tag{10.7}$$

10.2 Acoustic Modes in a Simple System

Leonhardt et al [13] investigate the behaviour of the Bogoliubov modes close to the sonic horizon. They assume that the density and interaction strength are constants, independent of both position and time. Furthermore, they assume that the flow can be approximated as one-dimensional near the horizon, and that the flow velocity v can be approximated by a linear correction to the speed of sound, that is $v = -c + \alpha z$.

Under these conditions, in the hydrodynamic approximation the equation of motion for the fluctuations has the general solution (Equation 4.48)

$$\hat{\theta} = f\left(t - \int^z \frac{1}{v \pm c}dz\right). \tag{10.8}$$

The solutions of interest are counterpropagating to the flow, so for $v < 0$ we examine solutions with the + sign in the denominator. This solution class is given by

$$\hat{\theta} = f\left(t - \int^z \frac{1}{\alpha z}dz\right) = f\left(t - \frac{1}{\alpha}\ln z\right). \tag{10.9}$$

In particular, modes of a single frequency ω have the form

$$\hat{\theta} = z^{i\omega/\alpha}e^{-i\omega t}. \tag{10.10}$$

Writing $\sigma = Az^{i\omega/\alpha}$ and substituting into Equation 10.7 yields an expression for the Bogoliubov modes corresponding to single-frequency phase fluctuations far from the horizon,

$$\begin{aligned} \begin{pmatrix} u_\omega \\ v_\omega \end{pmatrix} &= A\sqrt{\rho_0}\begin{pmatrix} e^{i\theta_0}\left[(\hbar/2mc^2)\left(\omega + i(-c+\alpha z)(i\omega/\alpha z)\right) + 1\right] \\ e^{-i\theta_0}\left[(\hbar/2mc^2)\left(\omega + i(-c+\alpha z)(i\omega/\alpha z)\right) - 1\right] \end{pmatrix} i z^{i\omega/\alpha}, \\ &= A\sqrt{\rho_0}\begin{pmatrix} e^{i\theta_0}\left[(\hbar\omega/2mc\alpha z) + 1\right] \\ e^{-i\theta_0}\left[(\hbar\omega/2mc\alpha z) - 1\right] \end{pmatrix} i z^{i\omega/\alpha}. \end{aligned} \tag{10.11}$$

Defining $A_\omega = iA\sqrt{\rho_0}\hbar/mc$, in order to agree with the notation used by Leonhardt et al [13], we obtain the mode solutions,

$$\begin{aligned} u_\omega(z) &= A_\omega e^{i\theta_0}\left(\frac{\omega}{2\alpha z} + \frac{mc}{\hbar}\right) z^{i\omega/\alpha}, \\ v_\omega(z) &= A_\omega e^{-i\theta_0}\left(\frac{\omega}{2\alpha z} - \frac{mc}{\hbar}\right) z^{i\omega/\alpha}. \end{aligned} \tag{10.12}$$

The relationship between these modes and the WKB solutions considered earlier (Section 6.3-6.4) is not obvious. Employing the approximate acoustic solution to the Bogoliubov dispersion relation (Equation 6.29), $k\omega/v + c = \omega/\alpha z$, the single frequency Bogoliubov solutions in this background flow become

$$\begin{aligned} u_\omega(z) &= U_\omega(z) e^{i\theta_0} z^{i\omega/\alpha}, \\ v_\omega(z) &= V_\omega(z) e^{-i\theta_0} z^{i\omega/\alpha}. \end{aligned} \tag{10.13}$$

Inserting the solution $V_\omega(z) = A_\omega(\omega/2\alpha z - mc/\hbar)$ into the equations constraining the coefficients of the exponential Bogoliubov solutions (Equation 6.18), we obtain $U_\omega(z) = A_\omega(\omega/2\alpha z + mc/\hbar - \hbar\omega^3/2m^2c^2\alpha^3z^3)$. So long as the last term is negligible (which is reasonable as we have made a small-frequency approximation to obtain the expression we used for k), the expressions for the asymptotically acoustic modes obtained from the hydrodynamic wave equation can also be recovered from the BdG formalism.

Recall that these are only approximate forms of the Bogoliubov modes in regions where the hydrodynamic approximation is valid. Clearly this approximation breaks down close to the sonic horizon at $z = 0$, as there these "acoustic" modes become singular.

Furthermore, the expression $z^{i\omega/\alpha}$ is multivalued. More precisely, we can write

$$z^{i\omega/\alpha} = e^{i\omega \ln z/\alpha} = e^{i\omega(\mathrm{Ln} z + 2in\pi)/\alpha}, \tag{10.14}$$

where n is any integer and $\mathrm{Ln}z$ is the principle value of the logarithm, defined by $\mathrm{Ln}z = \ln|z| + i\arg(z)$.

Thus to define the acoustic modes simultaneously on both sides of the sonic horizon, it is necessary to specify which sheet of the logarithm function is to be used on each side of the horizon.

On the positive real line, we will follow Leonhardt et al in choosing the principal branch of the function, so $z^{i\omega/\alpha}|z|^{i\omega/\alpha}$. Analytically continuing the function around the singularity, there are then two choices for the sheet on the negative real axis, corresponding to continuing the function around the upper or lower half-plane. The value of $z^{i\omega/\alpha}$ on the negative real line can be written as,

$$z^{i\omega/\alpha} = (e^{\pm i\pi})^{i\omega/\alpha}|z|^{i\omega/\alpha} = e^{\mp\pi\omega/\alpha}|z|^{i\omega/\alpha}, \tag{10.15}$$

where the upper (lower) sign corresponds to analytic continuation through the positive (negative) half-plane. Note that while on the positive real line $\left(z^{i\omega/\alpha}\right)^* = z^{-i\omega/\alpha}$, on the negative real line

$$\begin{aligned} \left(z^{i\omega/\alpha}\right)^* &= e^{\mp\pi\omega/\alpha}|z|^{-i\omega/\alpha} \\ &= e^{\mp 2\pi\omega/\alpha} z^{-i\omega/\alpha}. \end{aligned} \tag{10.16}$$

Leonhardt et al [13] argue that it is the analytic quasiparticle modes which are acoustic on both sides of the event horizon that determine the vacuum state of the system. Their rationale is that in a condensate without a horizon, the vacuum state will be the quasiparticle vacuum defined by the continuous acoustic modes which extend over the entire condensate. Provided that the process of formation of the sonic horizon is sufficiently smooth, Leonhardt et al argue that the initial acoustic modes will evolve to the modes described by Equation 10.12 on both sides of the horizon. For purposes of comparison, the analogue of this step in the astrophysical case is Corley's definition of the vacuum as the locally defined free-fall vacuum.

10.3 Normalisation of the Acoustic Modes

The imposition of the normalisation conditions on $u_{\omega_k}, v_{\omega_k}$ is equivalent to the definition of a conserved inner product in Corley's work [15]. It is straightforward to compute the normalisation integral (Equation 6.7) for the analytic acoustic solution given by Equations 10.12 and 10.15.

$$
\begin{aligned}
\int dz\,(u^*_\omega u_{\omega'} - v^*_\omega v_{\omega'}) &= A^*_\omega A_{\omega'} \int dz \left[\left(\frac{\omega}{2\alpha z} + \frac{mc}{\hbar} \right) \left(\frac{\omega'}{2\alpha z} + \frac{mc}{\hbar} \right) \right. \\
&\quad \left. - \left(\frac{\omega}{2\alpha z} - \frac{mc}{\hbar} \right) \left(\frac{\omega'}{2\alpha z} - \frac{mc}{\hbar} \right) \right] \left(z^{i\omega/\alpha} \right)^* \left(z^{i\omega'/\alpha} \right) \\
&= A^*_\omega A_{\omega'} \frac{mc}{\hbar\alpha} (\omega + \omega') \left[\int_0^\infty \frac{1}{z} z^{i(\omega'-\omega)/\alpha} dz \right. \\
&\quad \left. + \int_{-\infty}^0 \frac{1}{z} e^{\mp\pi(\omega+\omega')/\alpha} |z|^{i(\omega'-\omega')/\alpha} dz \right] \\
&= A^*_\omega A_{\omega'} \frac{mc}{\hbar\alpha} (\omega + \omega') \left(1 - e^{\mp\pi(\omega+\omega')/\alpha} \right) \int_0^\infty \frac{1}{z} z^{i(\omega'-\omega)/\alpha} dz \\
&= A^*_\omega A_{\omega'} \frac{mc}{\hbar} (\omega + \omega') \left(1 - e^{\mp\pi(\omega+\omega')/\alpha} \right) \int_{-\infty}^\infty e^{i(\omega'-\omega)y} dy \\
&= |A_\omega|^2 \frac{4\pi mc\omega}{\hbar} \left(1 - e^{\mp 2\pi\omega/\alpha} \right) \delta(\omega' - \omega), \qquad (10.17)
\end{aligned}
$$

substituting $y = \ln z/\alpha$ in the second-last line.

This calculation demonstrates that the norm is positive (and hence the solution is normalisable) if and only if the function is analytically continued around the upper half-plane rather than the lower half-plane. If this condition is met, however, then modes of negative frequency also have positive norm: this can be seen by replacing ω with $-\omega$.

These negative energy modes give rise to negative energies in the part of the Hamiltonian describing the fluctuations, meaning that there is no natural ground state of the elementary excitations. However in reality this treatment breaks down as the negative energy states become too highly occupied, as we have assumed throughout that the fluctuations are small.

10.4 Modes Exhibiting the Horizon

Following Leonhardt et al [13] we note that the part of the Hamiltonian describing the excitations,

$$\hat{H} = \int \hbar\omega \left(a^\dagger_\omega a_\omega - a^\dagger_{-\omega} a_{-\omega} \right) N d\omega, \qquad (10.18)$$

is invariant under transformations of the form

$$
\begin{aligned}
a'_{\pm\omega} &= a_{\pm\omega} \cosh\xi + a^\dagger_{\mp\omega} \sinh\xi, && (10.19) \\
u'_{\pm\omega} &= u_{\pm\omega} \cosh\xi + v^*_{\mp\omega} \sinh\xi, && (10.20) \\
v'_{\pm\omega} &= v_{\pm\omega} \cosh\xi + u^*_{\mp\omega} \sinh\xi. && (10.21)
\end{aligned}
$$

We are now interested in positive frequency modes that are zero inside the horizon, as such modes can be combined to form purely outgoing positive frequency wavepackets outside the horizon, corresponding to the Hawking radiation wavepackets in the astrophysical case. To obtain such modes requires finding ξ such that $u'_\omega(z) = v'_\omega(z) = 0 \,\forall\, z < 0$.

For $z < 0$, Equation 10.12 gives

$$\begin{aligned} u_\omega(z) &= A_\omega e^{i\theta_0}\left(\frac{\omega}{2\alpha z} + \frac{mc}{\hbar}\right) z^{i\omega/\alpha}, & (10.22)\\ v^*_{-\omega}(z) &= A^*_{-\omega} e^{i\theta_0}\left(\frac{-\omega}{2\alpha z} - \frac{mc}{\hbar}\right) e^{\pm 2\pi\omega/\alpha} z^{i\omega/\alpha}. & (10.23)\end{aligned}$$

Taking A_ω to be real,

$$A_\omega = \left[\frac{4\pi mc\omega}{\hbar}\left(1 - e^{\mp 2\pi\omega/\alpha}\right)\right]^{-1/2}, \tag{10.24}$$

and consequently,

$$A_{-\omega} = \left[\frac{4\pi mc\omega}{\hbar}\left(e^{\pm 2\pi\omega/\alpha} - 1\right)\right]^{-1/2} = e^{\mp\pi\omega/\alpha} A_\omega. \tag{10.25}$$

Then setting $u'_\omega(z) = 0$, it follows that

$$\tanh\xi = -\frac{u_\omega}{v^*_{-\omega}} = e^{\mp\pi\omega/\alpha}. \tag{10.26}$$

The coefficients in Equations 10.19 - 10.21 then become

$$\begin{aligned} \cosh\xi &= \frac{1}{\sqrt{1 - e^{\mp 2\pi\omega/\alpha}}},\\ \sinh\xi &= \frac{e^{\mp\pi\omega/\alpha}}{\sqrt{1 - e^{\mp 2\pi\omega/\alpha}}}. \end{aligned} \tag{10.27}$$

In terms of these coefficients, and choosing $A_\omega, A_{-\omega}$ so that the normalisation condition (Equation 6.7) is satisfied, the asymptotic expressions for the positive-norm analytic modes with acoustic asymptotics on both sides of the horizon (Equation 10.12) can be rewritten in the form,

$$\begin{aligned} u_\omega(z) &= \sqrt{\frac{\hbar}{4\pi mc\omega}} e^{i\theta_0}|z|^{i\omega/\alpha} \begin{cases} \cosh\xi\,(\omega/2\alpha|z| + mc/\hbar) & z > 0\\ \sinh\xi\,(-\omega/2\alpha|z| + mc/\hbar) & z < 0 \end{cases},\\ u_{-\omega}(z) &= \sqrt{\frac{\hbar}{4\pi mc\omega}} e^{i\theta_0}|z|^{-i\omega/\alpha} \begin{cases} \sinh\xi\,(-\omega/2\alpha|z| + mc/\hbar) & z > 0\\ \cosh\xi\,(\omega/2\alpha|z| + mc/\hbar) & z < 0 \end{cases},\\ v_\omega(z) &= \sqrt{\frac{\hbar}{4\pi mc\omega}} e^{-i\theta_0}|z|^{i\omega/\alpha} \begin{cases} \cosh\xi\,(\omega/2\alpha|z| - mc/\hbar) & z > 0\\ \sinh\xi\,(-\omega/2\alpha|z| - mc/\hbar) & z < 0 \end{cases},\\ v_{-\omega}(z) &= \sqrt{\frac{\hbar}{4\pi mc\omega}} e^{-i\theta_0}|z|^{-i\omega/\alpha} \begin{cases} \sinh\xi\,(-\omega/2\alpha|z| - mc/\hbar) & z > 0\\ \cosh\xi\,(\omega/2\alpha|z| - mc/\hbar) & z < 0 \end{cases}. \end{aligned} \tag{10.28}$$

Substituting these expressions back into Equations 10.19-10.21 yields transformed modes that are acoustic in form (Equation 10.12) but vanish inside the horizon for positive frequency ω, and vanish outside the horizon for negative frequency.

$$\begin{aligned}
u'_{\omega}(z) &= \Theta(z)\sqrt{\frac{\hbar}{4\pi mc\omega}}e^{i\theta_0}\left(\frac{\omega}{2\alpha|z|}+\frac{mc}{\hbar}\right)|z|^{i\omega/\alpha}, \\
u'_{-\omega}(z) &= \Theta(-z)\sqrt{\frac{\hbar}{4\pi mc\omega}}e^{i\theta_0}\left(\frac{\omega}{2\alpha|z|}+\frac{mc}{\hbar}\right)|z|^{-i\omega/\alpha}, \\
v'_{\omega}(z) &= \Theta(z)\sqrt{\frac{\hbar}{4\pi mc\omega}}e^{-i\theta_0}\left(\frac{\omega}{2\alpha|z|}-\frac{mc}{\hbar}\right)|z|^{i\omega/\alpha}, \\
v'_{-\omega}(z) &= \Theta(-z)\sqrt{\frac{\hbar}{4\pi mc\omega}}e^{-i\theta_0}\left(\frac{\omega}{2\alpha|z|}-\frac{mc}{\hbar}\right)|z|^{-i\omega/\alpha}. \qquad (10.29)
\end{aligned}$$

The fact that modes restricted to the subsonic region have positive norm only if their frequency is positive, while modes existing in the supersonic region must have negative frequency in order to have positive norm, is related to the earlier discussion of free-fall frequency inside the sonic horizon (Section 8.3.1). Just as in the astrophysical case studied by Corley [15], the modes of negative norm are exactly those with negative free-fall frequency, and for acoustic modes inside the horizon, the signs of the frequency and the free-fall frequency are always opposite (Figure 8.1).

10.5 Hawking Radiation as Depletion of the Condensate

The assumption that the system is in the quasiparticle vacuum defined by the analytic acoustic modes transcending the horizon (Equation 10.12) simply means that there are no excitations in these quasiparticle modes. The expectation values of the quasiparticle number operators $a^{\dagger}_{\pm\omega}a_{\pm\omega}$ in the vacuum state are zero for all ω.

However, it does not follow that the expectation value of the fluctuation field operator $\hat{\Psi}'$ is zero in the quasiparticle vacuum state, because $\hat{\Psi}$ contains terms proportional to creation operators as well as annihilation operators (Equation 6.2). In fact, if $|0\rangle$ denotes the quasiparticle vacuum state defined by the analytic modes (Equation 10.12), then eliminating all terms of the form $a_{\pm\omega}|0\rangle$ or $\langle 0|a^{\dagger}_{\pm\omega}$ we obtain

$$\begin{aligned}
\langle 0|\hat{\Psi}'^{\dagger}\hat{\Psi}'|0\rangle &= \sum_{\omega}\left(|v_{\omega}|^2\langle 0|a_{\omega}a^{\dagger}_{\omega}|0\rangle + |v_{-\omega}|^2\langle 0|a_{-\omega}a^{\dagger}_{-\omega}|0\rangle\right) \\
&= \sum_{\omega}\left(|v_{\omega}|^2+|v_{-\omega}|^2\right). \qquad (10.30)
\end{aligned}$$

Note that the non-zero terms arise from the commutators $[a_{\pm\omega}, a^{\dagger}_{\pm\omega}]$. This is similar to the situation in the astrophysical case, where the non-zero expectation value of the number operator in the vacuum state arises from the commutator associated with the conjugate of a mode with negative free-fall frequency (Equation 8.23).

This non-zero expectation value of the number operator in the quasi-particle vacuum state is termed the *quantum depletion* of the condensate [5, 6]. Its physical meaning is that in the presence of interparticle interactions, the state with all particles in the condensate is not an eigenstate of the full Hamiltonian (describing both the condensate and the fluctuations). There is a certain amount of mixing into non-condensate states, giving rise to a cloud of non-condensed atoms (but not quasiparticles described by the modes transcending the horizon).

To see how this cloud of non-condensed atoms is related to Hawking radiation, we rewrite the expression for the non-condensed atoms in terms of the purely positive (negative) frequency modes that are zero inside (outside) the horizon (Equation 10.29).

The analytic modes $v_\omega, v_{-\omega}$ are given by

$$v_\omega(z) \quad = \quad \begin{cases} \cosh\xi\, v'_\omega(z) & z > 0, \\ -\sinh\xi\,(u'_{-\omega}(z))^* & z < 0, \end{cases} \tag{10.31}$$

$$v_{-\omega}(z) \quad = \quad \begin{cases} -\sinh\xi\,(u'_\omega(z))^* & z > 0, \\ \cosh\xi\, v'_{-\omega}(z) & z < 0. \end{cases} \tag{10.32}$$

Thus writing the depletion in terms of the primed modes and recalling that $\cosh^2\xi = 1 + \sinh^2\xi, \sinh^2\xi = 1/\left(e^{2\pi\omega/\alpha} - 1\right)$, for $z > 0$ we obtain

$$|v_\omega|^2 + |v_{-\omega}|^2 = |v'_\omega|^2 + \left(|u'_\omega|^2 + |v'_\omega|^2\right)\frac{1}{e^{2\pi\omega/\alpha} - 1}. \tag{10.33}$$

Now if the expectation value of the fluctuation field number operator, $\hat{\Psi}'^\dagger\hat{\Psi}'$, is computed in some state $|f\rangle$ in terms of the quasiparticle mode functions and annihilation and creation operators, we obtain, in addition to the usual depletion term, a term of the form,

$$\sum_\omega \left(|u_\omega|^2 + |v_\omega|^2\right)\langle f|a^\dagger_\omega a_\omega|f\rangle + \left(|u_{-\omega}|^2 + |v_{-\omega}|^2\right)\langle f|a^\dagger_{-\omega}a_{-\omega}|f\rangle. \tag{10.34}$$

This is the term representing the quasiparticle excitations, with the number operators $a^\dagger_{\pm\omega}a_{\pm\omega}$ giving the number of excitations [5].

In particular, let us consider the expansion of the field operator in terms of the primed quasiparticle modes that exist only on one side of the horizon, and choose $|f\rangle$ to be the quasiparticle vacuum $|0\rangle$ defined by the unprimed modes. The expectation value of the number operator for the fluctuation field is unchanged, and for $z > 0$ is given by Equation 10.33. We can now identify the terms in Equation 10.33 as the quantum depletion of the condensate plus a term due to the quasiparticle excitations, and it follows that the number of quasiparticle excitations measured in the modes restricted to the subsonic side of the horizon is given by the Hawking radiation distribution,

$$n(\omega) = \frac{1}{e^{2\pi\omega/\alpha} - 1}. \tag{10.35}$$

Thus in BECs, the analogue of Hawking radiation is the depletion of atoms from the condensate due to the interaction, in combination with the interpretation of this depletion as quasiparticle excitations in modes existing on one side of the sonic horizon only.

Conclusion

We have given a complete and detailed derivation of the theory needed to describe dilute gas Bose-Einstein condensates in the mean-field approximation, assuming that the Bose field operator can be well approximated by the superposition of a classical field and a small perturbative quantum field operator. We have consistently expressed the theory in terms of the hydrodynamic variables $\vec{v}$, ρ and c, thus facilitating its application to configurations containing sonic horizons. The key results here are scattered through the literature and often described only briefly: hopefully our collation of these results will be of use in future investigations into analogue gravity in Bose-Einstein condensates.

In the same vein, we have completed a wide-ranging review of the literature concerning the creation of analogue black holes. We have considered not only Bose-Einstein condensates but also general irrotational inviscid fluids, as well as some other specific fluid systems. We have reviewed a number of detailed proposals, with a particular emphasis on investigations of instability at sonic horizons or ergospheres.

Based on this work, BECs appear to be an ideal system for testing predictions of analogue Hawking radiation. Unlike many other systems, quantum effects such as Hawking radiation are expected to possess analogues in BECs. There have been a number of proposals for the creation of sonic horizons in BECs, and several suggestions on how to increase the Hawking radiation temperature. Instability of black and white hole horizons in BECs has been examined using the Bogoliubov-deGennes equations, with the conclusion that there are wide regions of stability.

We have demonstrated that Unruh's analogy [4] between fields propagating in curved space-time and sound waves in inviscid irrotational fluids also applies to phase perturbations in BECs, provided that the length scale of the density variations is large compared to the coherence length. We have collected the essential results in the field of analogue gravity, including the form of the effective metric and the appropriate analogues of event horizons and ergoregions. Considering a vortex in a BEC as an analogue of an ultrarelativistic rotating body with an ergoregion but no event horizon, we have presented an original investigation of analogue superradiance within the hydrodynamic approximation, and discussed the breakdown of the hydrodynamic approximation and its consequences for our model.

Moving on to the quantum theory of the fluctuation field, we have discussed the inclusion of modes of negative norm in the field operator, following Garay et al [12]. We have shown how to apply these ideas to the treatment of negative frequency solutions, and discussed the significance of anomalous modes and the conditions under which they occur. We were unable to find any source in the literature which elucidated and linked these concepts in this way.

The later chapters of this thesis constitute a detailed review and analysis of several different derivations of Hawking radiation. We have attempted to link these different methods together by elucidating the common physics behind them. In particular, we have showed that in Corley's model [15], Hawking radiation can be interpreted as a consequence of spontaneous population of the negative free-fall frequency, positive Killing frequency modes behind the horizon, combined with tunnelling out through the horizon. This mechanism is identical to that proposed by Volovik for analogue Hawking radiation in Fermi superfluids [26].

Finally, we have given a derivation of Hawking radiation in BECs as a consequence of depletion of the condensate, and related the methods employed in the BEC case to the techniques used in the astrophysical case. In particular, depletion of the condensate in anomalous modes is the analogue of particle creation in modes of negative free-fall frequency in Corley's work [15]. While this chapter is based on work by Leonhardt et al [13], our analysis is considerably more detailed than the originally published version. There are also errors in the original paper that have been corrected in the analysis we have presented (this was confirmed by private communication with Professor Leonhardt).

11.1 Future Directions

The methods developed in Section 4.3 allow the wave equation for the fluctuations to be solved exactly on certain time-dependent background flows. This suggests that it might be feasible to study the behaviour of the fluctuations during the formation of the sonic horizon without resorting to the adiabatic approximation, which in turn might allow the original derivation of the Hawking effect - which relies on a collapse geometry - to be adapted to analogue systems. An exact description of the evolution of the fluctuation field during the horizon formation process would also allow us to determine the late-time vacuum state. This would provide a test of the argument that the late-time vacuum state must be the quasiparticle vacuum state defined by the quasiparticle modes acoustic on both sides of the horizon and analytic on the upper complex plane.

In Section 4.3 we also developed an approximate solution for the case where the interaction strength varies with position, giving rise to a sonic horizon in a flow of constant velocity and density. It would be interesting to develop this idea further, as the interaction strength in BECs may be tuned by application of a magnetic field [5]. It may be easier experimentally to impose a varying magnetic field on a condensate than to accelerate the condensate to the speed of sound, especially if the goal of the experiment is to measure the quasiparticle production associated with analogue Hawking radiation. We note that in the case of constant background density ρ_0, the BdG equations reduce to Equations 6.14 and 6.15 regardless of the speed of sound profile, so the WKB solutions we have used repeatedly in the latter half of this thesis may also be employed in this case.

The theoretical model for Hawking radiation in Bose-Einstein condensates, as discussed in Chapter 10, could be improved by explicitly including the large-wavenumber modes in the analysis. However simply extending the analysis of Chapter 10 to the large-wavenumber modes is problematic, as there is no way to obtain asymptotic forms of the large-wavenumber BdG solutions from the ordinary wave equation for the fluctuations. It is tempting to apply the methods outlined in Section 8.3 to describe the solutions to the BdG equations across the sonic horizon, but it is not obvious how to apply these techniques to a pair of coupled differential equations (for u_ω and v_ω) as opposed to a single ordinary differential equation.

It is possible that our perturbative approach to the quantum fluctuation field (expanding to lowest order in the fluctuations) is insufficient to accurately describe the BEC in the presence of a sonic horizon. However at present there seems to be little reason to extend the analysis to terms of higher order in the fluctuations, given the increase in complexity of the calculations such an extension normally entails.

Salazar et al [6] discuss the possibility of *interacting* quasiparticles. Interactions between the Hawking radiation quasiparticles might significantly alter the measured spectrum, so this possibility might be worth investigation. It is likely however that any such investigation would need to be numerical rather than analytical. Similarly, our current model does not allow for backreaction of the Hawking particles on the bulk flow of the condensate, which will become important at long timescales.

Chapter 12

Glossary

This glossary is intended as an aid to understanding rather than a compendium of rigorous definitions. Due to the interdisciplinary nature of this thesis it is anticipated that the reader will not be expert in all aspects of the relevant theory, so I have attempted to make these definitions accessible to a non-expert. I apologise for any resulting lapses in rigour.

Event horizon The boundary of a region of spacetime from which no particle can escape at any time in its future.

Sonic horizon The hydrodynamic analogue of an event horizon; the boundary of a region in a moving fluid from which no sound wave can escape, because that would require the wave to exceed the speed of sound.

Black hole sonic horizon A sonic horizon where the flow velocity is directed towards the region of supersonic flow.

White hole sonic horizon A sonic horizon where the flow velocity is directed away from the region of supersonic flow.

Ergoregion In the astrophysical context, a region of spacetime (associated with dense rapidly rotating bodies) where it is not possible for a particle to remain stationary relative to a distant observer. In the hydrodynamic sense, a region in a moving fluid where the magnitude of the fluid flow velocity exceeds the local speed of sound.

Hawking radiation / Hawking effect Emission of particles from near the event horizon of a black hole. The spectrum of the particles is thermal at a temperature $T = \kappa/2\pi$, where κ is the surface gravity of the black hole.

Superradiance The amplification of low-frequency waves scattered from the ergoregion of a rotating black hole.

Metric In general relativity, a 4×4 matrix describing the Lorentz-invariant distance element in terms of locally defined coordinates. The metric determines the trajectories of particles and fields (in the absence of external forces) in the given background spacetime.

Einstein field equations A system of six independent partial differential equations that relate the metric to the energy-momentum tensor describing the distribution of mass and energy.

Schwarzschild solution The exact spherically symmetric solution to the Einstein field equations describing the structure of spacetime outside a spherically symmetric mass M. In particular, in the case of a point mass the Schwarzschild solution is valid everywhere except at the location of the point mass, and we have the Schwarzschild black hole.

Killing energy The energy of a particle defined far from the origin in an asymptotically flat spacetime, and conserved on the particle trajectory.

Null geodesic The trajectory of a massless particle, such as a photon, in the absence of external forces.

Timelike geodesic The trajectory of a massive particle, such as a neutron, in the absence of external forces.

Advanced time coordinate Time as recorded along null geodesics travelling in towards the centre of a gravitating system.

Retarded time coordinate Time as recorded along null geodesics travelling out from the centre of a gravitating system.

Bose-Einstein condensate A cold gas of indistinguishable bosons in which a single quantum state is macroscopically occupied.

Condensate wavefunction A commonly used term for the expectation value of the Bose field operator for a Bose-Einstein condensate.

Gross-Pitaevskii equation The equation of motion for the condensate wavefunction. The Gross-Pitaevskii equation governs the dynamics of the condensate within the semiclassical approximation where the field operator is approximated by its expectation value.

Bogoliubov-deGennes equations The perturbative equations of motion for the quantum fluctuation field given by the difference between the Bose field operator and its expectation value.

Hydrodynamic approximation The approximation that the Gross-Pitaevskii equation is equivalent to the continuity equation and the zero-viscosity Euler equation; holds when the length scale of density variation exceeds the coherence length.

Coherence length The distance over which the condensate wavefunction tends to its bulk value when subjected to a localised perturbation. Also called the *healing length*.

Bibliography

[1] B. F. Schutz, *A first course in general relativity* (Cambridge University Press, United Kingdom, 1985).

[2] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time* (Cambridge University Press, United States of America, 1973).

[3] S. W. Hawking, Nature **248**, 30 (1974).

[4] W. G. Unruh, Phys. Rev. Lett **46**, 1351 (1981).

[5] C.J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, United Kingdom, 2002).

[6] J. Rogel-Salazar *et al*, J. Opt. B: Quantum Semiclass. Opt. **6** R33-R59 (2004).

[7] M. H. Anderson *et al*, Science **269**, 198 (1995).

[8] C. C. Bradley *et al*, Phys. Rev. Lett. **75**, 1687 (1995).

[9] K. B. Davis *et al*, Phys. Rev. Lett. **75**, 3969 (1995).

[10] C. Barcelo, S. Liberati and M. Visser, Phys. Rev. A **68** 053613 (2003).

[11] L. J. Garay *et al*, Phys. Rev. Lett. **85**, 4643 (2000).

[12] L. J. Garay *et al*, Phys. Rev. A **63**, 023611 (2001).

[13] U. Leonhardt, T. Kiss and P. Ohberg, J. Opt. B: Quantum Semiclass. Opt. **5** S42-S49 (2003).

[14] T. Slatyer and C. M. Savage, arXiv:cond-mat/0501182 (2005).

[15] S. Corley, Phys. Rev. D **57** no. 10, 6280 (1998).

[16] W. G. Unruh, Phys. Rev. D **51** no. 6, 2827 (1995).

[17] T. Jacobson, Phys. Rev. D **44** no. 6, 1731 (1991).

[18] S. Corley and T. Jacobson, Phys. Rev. D **54**, 1568 (1996).

[19] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85** no. 24, 5042 (2000).

[20] M. Visser, in *Artificial Black Holes* edited by M. Novello, M.Visser and G. Volovik (World Scientific, Singapore, 2002).

[21] M. Visser, Class. Quant. Grav. **15** 1767-1791 (1998).

[22] S. Basak and P. Majumdar, Class. Quantum Grav. **20**, 2929 (2003); ibid **20**, 3907 (2003).

[23] G. E. Volovik, J. Low. Temp. Phys. **24** 127-129 (1998).

[24] T. Vachaspati, arXiv:cond-mat/0404480 (2004).

[25] C. Barcelo, S. Liberati and M. Visser, Class. Quant. Grav. **18** 1137 (2001).

[26] G. E. Volovik, Physics Reports **351** 195-348 (2001).

[27] G. E. Volovik, JETP Lett. **69** 705-713 (1999).

[28] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[29] A. Vilenkin, Physics Letters **78B**, 301 (1978).

[30] R. Parentani, Int. J. Mod. Phys. **A17** 2721-2726 (2002).

[31] R. Schutzhold and W. G. Unruh, Phys. Rev. D **66**, 044019 (2002).

[32] W. G. Unruh and R. Schutzhold, Phys. Rev. D **68**, 024008 (2003).

[33] M. Sakagami and A. Ohashi, Prog. Theor. Phys. **107** 1267-1272 (2002).

[34] C. Barcelo, S. Liberati and M. Visser, Int. J. Mod. Phys. **A18** 3735 (2003).

[35] J. L. Friedman, Commun. Math. Phys. **63**, 243 (1978).

[36] N. Comins and F. Schutz, Proc. R. Soc. Lon. A **364**, 211 (1978).

[37] S. Liberati, S. Sonego and M. Visser, Class. Quantum Grav. **17** 2903-2923 (2000).

[38] G. E. Volovik, Grav. Cosmol. Suppl. **6** 187-203 (2000).

[39] R. Schutzhold, G. Plunien and G. Soff, Phys. Rev. Lett. **88** no. 6, 061101 (2002).

[40] Novello *et al*, Class. Quantum Grav. **20**, 859 (2003).

[41] S. Bose, Z. Phys. **26**, 178 (1924).

[42] A. Einstein, Preussische Akad. Wissenschaften **3**, 261 (1924).

[43] U. Leonhardt, T. Kiss, and P. Ohberg, Phys. Rev. A **67**, 033602 (2003).

[44] C. Barcelo, S. Liberati and M. Visser, arXiv: gr-qc/030506

[45] C. Barcelo, S. Liberati and M. Visser, arXiv:cond-mat/0307491 v2 14 Aug 2003.

[46] W. A. Strauss, *Partial Differential Equations: An Introduction* (John Wiley and Sons, Inc., United States of America, 1992).

[47] A.A. Svidzinsky and A.L. Fetter, Phys. Rev. A **58**, 3168 (1998).

[48] T.P Simula, S.M.M. Virtanen, and M.M. Salomaa, Phys. Rev. A **65**, 033614 (2002).

[49] D.M. Stamper-Kurn *et al.*, Phys. Rev. Lett. **81**, 2194 (1998).

[50] A description of the XMDS package may be found at: http://www.xmds.org

[51] N. N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).

[52] S. Corley and T. Jacobson, Phys. Rev. D **59**, 124011 (1999).

[53] I. Bloch, T. W. Hansch and T. Esslinger, *Atom lasers and phase coherence of atomic Bose gases*, RIKEN Review No. 33 (March, 2001).

[54] I. Bloch, T.W. Hansch, and T. Esslinger, Phys. Rev. Lett. **82**, 3008 (1999).

[55] N.P. Robins *et al.*, Phys. Rev. A **69**, 051602(R) (2004).

[56] N.P. Robins *et al.*, cond-mat/0501747 (2005).

[57] S. Giovanazzi *et al*, Phys. Rev. A **70**, 063602 (2004).

[58] D. L. Feder et al, Phys. Rev. Lett. **86**, 564 (2001).

[59] D. L. Feder et al, Phys. Rev. A **62**, 053606 (2000).

[60] C. K. Law, P. T. Leung and M-C Chu, J. Phys. B: At. Mol. Opt. Phys. **35** 3583 (2002).

[61] J. D. Bekenstein and M. Schiffer, Phys. Rev. D **58**, 064014 (1998).

[62] W. G. Unruh and R. Schutzhold, Phys. Rev. D **71** 024028 (2005).

[63] D. J. Griffiths, *Introduction to Quantum Mechanics* (Prentice Hall, Inc., United States of America, 1995).

[64] S. Giovanazzi, Phys. Rev. Lett. **94** 061302 (2005).