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Does Sub-Barrier Bremsstrahlung in α-Decay of $^{210}$Po Exist?

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A quantum mechanical model for the description of the α-decay of heavy nuclei with accompanying photons emission is presented. The model is based on a quantum mechanical one-particle model of α-decay through a decay barrier. The bremsstrahlung spectrum calculation employs multipole expansion of the vector potential of the Coulomb field of the daughter nucleus and takes into account the dependence on the angle between the directions of the α-particle propagation and the photon emission. Spectra of $^{210}$Po are obtained for the angles 25° and 90°, and the best agreement with the experimental data in the 90° case in a comparison with other existing models is achieved. From the angular analysis, the model gives monotonic behavior of the bremsstrahlung spectrum for any value of the angle. We find that sub-barrier photon emission (i.e., bremsstrahlung photon emission during α-particle tunneling through decay barrier) exists but gives a small contribution to the total bremsstrahlung spectrum.

§1. Introduction

The existence of sub-barrier bremsstrahlung phenomena in α-decay (i.e., bremsstrahlung photon emission during tunneling of an α-particle through a decay barrier) has not yet been confirmed through the analysis of experimental data. In Ref. 1) we proposed a quantum mechanical approach to resolve this problem. With this approach, one can calculate the bremsstrahlung spectra both with and without sub-barrier photon emission for the nuclei $^{210}$Po, $^{214}$Po, $^{226}$Ra and $^{244}$Cm, for which experimental data2)–5) exist, and with a comparative analysis determine whether sub-barrier bremsstrahlung exists.

This approach can be realized only with such a model that yields sufficiently good agreement with experimental data. Note, that the present theoretical description of experimental data2) for $^{210}$Po is unsatisfactory, and there is a difficulty in explaining the existence of the “hole” in the bremsstrahlung spectrum. Note that these experimental spectrum carried out for a value of the angle between the direction of the photon emission and the direction of the α-particle propagation equal to 25°. New experimental data6) of the bremsstrahlung spectrum for $^{210}$Po have been obtained for the different value of the angle equal to 90°, exhibit monotonic behaviour and have no “hole” with such amplitude. We suppose that obtaining a description of the bremsstrahlung spectrum for $^{210}$Po with a sufficient degree of accuracy and clarification of the spectrum behaviour are important problems in the investigation of sub-barrier bremsstrahlung phenomena.

To make possible a comparative analysis of these two groups of experimental data

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for $^{210}$Po obtained for different angular values, we present here an angular quantum mechanical model of the $\alpha$-decay of heavy nuclei with accompanying (spontaneous) emission of photons. The foundation of this work was laid in Ref. 7, where we calculated the bremsstrahlung spectrum for the nucleus $^{214}$Po, taking into account $E_1$. This model is based on a quantum mechanical one-particle model of $\alpha$-decay through a decay barrier. A description of the electro-magnetic field of the daughter nucleus and the emission of photons in the $\alpha$-decay is constructed on the basis of quantum electrodynamics. In calculating the bremsstrahlung spectrum, we use an expansion of the vector potential of the electromagnetic field of the daughter nucleus in electric and magnetic multipoles, in contrast to Papenbrock and Bertsch quantum mechanical method, and we do not use the additional transformation used in Tkalya model. We first present the mathematical tools of this model.

§2. A non-stationary quantum mechanical model of $\alpha$-decay with bremsstrahlung

For investigation of the photon emission process, we consider the state of the system (the $\alpha$-particle and the daughter nucleus) before the emission as the initial ($i$-state) and the state of the system after the emission as the final ($f$-state). We calculate the bremsstrahlung spectrum using the transition matrix element

$$a_{fi} = \langle k_f, n_k + 1 | \tilde{V}(r, t) | k_i, n_k \rangle, \quad \tilde{V}(r, t) = -i \int_0^t e^{i \hat{H}_0 t'} \hat{V}(r, t') e^{-i \hat{H}_0 t'} dt', \quad (2.1)$$

where $\psi_i(r) = | k_i \rangle$ and $\psi_f(r) = | k_f \rangle$ are the wave functions of the system in the initial $i$-state and the final $f$-state from the unperturbed operator $\hat{H}_0$ which describes scattering of the $\alpha$-particle upon the daughter nucleus, uses the barrier and does not take into account the photon emission; $w = E$; $n_k$ is the number of photons of one sort with impulse $k$ in the initial $i$-state. We choose units for which $\hbar = 1$, and $c = 1$. In the Coulomb calibration, the interaction operator has the form

$$\hat{V}(r, t) = -Z_{\text{eff}} \frac{e}{m} \mathbf{A} \mathbf{p}, \quad (2.2)$$

where $Z_{\text{eff}}$ is the effective charge, $m$ is the reduced mass of the system, and $\mathbf{A}$ is the vector potential of the electromagnetic field of the daughter nucleus.

We choose the vector potential $\mathbf{A}(r, t)$ to take the form

$$\mathbf{A}(r, t) = \sum_{k,\alpha} \left( \hat{c}_{k,\alpha} \mathbf{A}_{k,\alpha}^0 + \hat{c}_{k,\alpha}^+ \mathbf{A}_{k,\alpha}^* \right), \quad \mathbf{A}_{k,\alpha} = \sqrt{\frac{2\pi}{w}} e^{(\alpha)} e^{i(kr - wt)}, \quad (2.3)$$

where $e^{(\alpha)}$ is the unit polarization vector of the photon, $k$ is the wave vector of the photon, and $w = k = | \mathbf{k} |$. The vector $e^{(\alpha)}$ is perpendicular to $k$ in the Coulomb calibration. We have two independent polarizations, $e^{(1)}$ and $e^{(2)}$ for any photon with impulse $k$. 
Let us consider a wave packet of the form

\[ \Psi_{i,f}(r,t) = \int_0^{+\infty} g(k - k_{i,f}) \psi_{i,f}(r,k) e^{-i(w(k)t)} dk, \]  

(2.4)

where in the expressions \( \Psi_{i,f}(r,t) \) and \( k_{i,f} \) we use the index \( i \) or \( f \) only in dependence on consideration of the initial \( i \)- or final \( f \)-state of the system. We take the weight amplitude \( g(k - k_{i,f}) \) in the form of a Gaussian:

\[ g(k - k_{i,f}) = C \exp \left(-\frac{(k - k_{i,f})^2}{2(\Delta k)^2}\right), \]  

(2.5)

where \( C \) is a normalization factor and can be obtained from the normalization condition \( \int |g(k - k_{i,f})|^2 dk = 1 \).

The wave packets (2.4) can be used for analyzing in time the tunneling of the \( \alpha \)-particle through the decay barrier and allow to calculate the tunneling time of this particle in dependence on the energy level of such process. Preliminary calculations of these times of the \( \alpha \)-decay with taking into account a possibility to emit photons from the barrier region are presented in Ref. 7. Development of the model of the bremsstrahlung in the \( \alpha \)-decay on the basis of the wave functions in form (2.4) allows to calculate both the bremsstrahlung spectra and tunneling times. Therefore, we use the wave packets (2.4) for \( \psi_i(r) \) and \( \psi_f(r) \), with the time factor \( e^{-i\hat{H}_0 t} \).

Substituting this wave packet into (2.1) and taking account of the property

\[ e^{-i\hat{H}_0 t}\psi_i(r) = e^{-iw_1 t}\psi_i(r), \quad \psi_f^*(r)e^{i\hat{H}_0 t} = \psi_f^*(r)e^{i\hat{H}_0 t}, \]  

(2.6)

we obtain the matrix element in the form

\[ \tilde{a}_{f1} = \int_0^{+\infty} dk_1 \int_0^{+\infty} dk_2 g(k_1 - k_i)g(k_2 - k_f)^* \langle k_2, n_k + 1 | \hat{V}(t) | k_1, n_k \rangle, \]  

(2.7)

where

\[ \hat{V}(t) = -i \int_0^t e^{i w_2 t'} \hat{V} e^{-i w_1 t'} dt'. \]  

(2.8)

We study the spontaneous emission of one photon with an impulse \( k \). Therefore, \( n_k = 0 \), and the sum \( \sum_k \) can be omitted in the calculation of the matrix element (2.1).

For calculation of the bremsstrahlung spectra we use the stationary approximation: \( t \to +\infty \). Taking into account (2.2), (2.3), (2.7) and (2.8), and also using the definition of the \( \delta \)-function, we calculate the transition matrix element

\[ \langle k_2, 1 | \hat{V} | k_1, 0 \rangle = F_{21} 2\pi \delta(w_2 + w - w_1), \]  

(2.9)

where

\[ F_{21} = Z_{\text{eff}} \frac{e}{m} \sqrt{\frac{2\pi}{w}} p(k_1, k_2), \]
\[ p(k_1, k_2) = \sum_{\alpha=1,2} e^{(\alpha)*} \langle k_2 \bigg| e^{-i k r} \frac{\partial}{\partial r} \bigg| k_1 \rangle. \] (2.10)

For monochromatic particles, we obtain
\[ a_{fi} = 2\pi F_{fi} |C|^2 \delta(w + w_f - w_i). \] (2.11)

We define the transition probability from the initial \( i \)-state to the final \( f \)-state in a unit time with a single photon emission of impulse \( k \) and polarization \( e^{(\alpha)} \) as follows:
\[ dW = \frac{mk_f}{(2\pi)^3} \frac{w_{fi}^2 |F_{fi}|^2}{(2\pi)^2} = \frac{Z_{\text{eff}}^2 e^2 k_f w_{fi}}{(2\pi)^4 m} |p(k_i, k_f)|^2, \]
\[ w_{fi} = E_i - E_f. \] (2.12)

The most difficult step in calculating the bremsstrahlung spectrum is calculating \( p(k_i, k_f) \). We have
\[ p(k_i, k_f) = \sum_{\alpha=1,2} e^{(\alpha)*} \int_0^{+\infty} d\Omega_r d\Omega_r' \int_0^\infty r^2 \psi_f^*(r) e^{-i kr} \frac{\partial}{\partial r} \psi_i(r). \] (2.13)

In calculating \( p(k_i, k_f) \), various approaches can be used, and the various quantum mechanical methods for these approaches differ. Papenbrock and Bertsch used a dipole approximation in obtaining a description of the vector potential of the electromagnetic field. \(^{11}\) Tkalya used a multipole approach in calculating the bremsstrahlung spectrum, \(^{12}\) taking into account \( E_1 \) and \( E_2 \). However, in calculating the integral (2.13) both in Tkalya model and in Papenbrock and Bertsch model the transformation
\[ -\langle f | p | i \rangle = \langle f | [H, p] | i \rangle \frac{1}{w_{fi}} = i\hbar \left( \frac{\partial}{\partial r} U(r) \right) \right|_i \frac{1}{w_{fi}} \] (2.14)
is used. This transformation can be used only in a case that the approximation
\[ \exp(i kr) = 1 \] (2.15)
is applied (see Appendix A). Therefore, using the transformation (2.14), the accuracy of the calculation of the integral decreases, and the calculations of the spectra with the models \(^{11,12}\) are carried out not in the dipole approximation or the multipolar approximation (taking into account \( E_1 \) and \( E_2 \)) but in the approximation (2.15). For this reason, we do not use (2.14).

In computing the integral (2.13), we apply the multipolar expansion of the vector potential of the electromagnetic field without using the transformation (2.14). We write the polarization vectors \( e^{(\alpha)} \) in terms of circular polarization vectors \( \xi \) with opposite directions of rotation:\(^9\)
\[ \xi_{-1} = \frac{1}{\sqrt{2}}(e^1 - ie^2), \quad \xi_{+1} = -\frac{1}{\sqrt{2}}(e^1 + ie^2). \] (2.16)
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We obtain

$$p(k_i, k_f) = \sum_{\mu = -1, 1} h_\mu \xi_\mu \int_0^{+\infty} dr \int d\Omega r^2 \psi_{i}^{*}(r)e^{-i|kr|\frac{\partial}{\partial r}\psi_{i}(r)},$$  \hspace{1cm} (2.17)$$

where

$$h_{-1} = \frac{1}{\sqrt{2}}(1 - i), \quad h_1 = \frac{1}{\sqrt{2}}(-1 - i).$$  \hspace{1cm} (2.18)$$

For the initial $i$- and final $f$-states, we can write

$$\psi_{i,f}(r) = \psi_{i,f}(r)Y_{lm}(n_{i,f}r),$$  \hspace{1cm} (2.19)$$

where $Y_{lm}(n)_{r}$ are the normalized spherical functions. Following Ref. 10), we have

$$\frac{\partial}{\partial r}\psi_{i}(r) = -\frac{d\psi_{i}(r)}{dr}T_{01,0}(n_{i}r),$$

$$T_{01,0}(n_{i}r) = \frac{1}{1!}(110| - \mu\mu 0)Y_{1,-\mu}(n_{i}r)\xi_{\mu},$$  \hspace{1cm} (2.20)$$

where $T_{jl,m}$ are the vector spherical harmonics, and $(j_{a}1j| - \mu\mu 0)$ are the Clebsch-Gordan coefficients.

We expand the vector potential $A(r, t)$ in multipoles as$^{10}$

$$\xi_{\mu}e^{ikr} = \sqrt{2\pi\mu} \sum_{l=1}^{\infty} \left( \sqrt{2l + 1} i^l [A_{lm}(r, M) + i\mu A_{lm}(r, E)] \right),$$  \hspace{1cm} (2.21)$$

with

$$A_{lm}(r, M) = j_{l}(kr)T_{jl,\mu}(n_{r}),$$

$$A_{lm}(r, E) = \sqrt{\frac{l+1}{2l+1}} j_{l-1}(kr)T_{ll-1,\mu}(n_{r}) - \sqrt{\frac{l}{2l+1}} j_{l+1}(kr)T_{ll+1,\mu}(n_{r}),$$  \hspace{1cm} (2.22)$$

where $j_{l}(kr)$ is the spherical Bessel function of order $l$, and $A_{lm}(r, M)$ and $A_{lm}(r, E)$ are the magnetic and electrical multipoles, respectively.

From the calculations, we obtain $p(k_i, k_f)$ as

$$p(k_i, k_f) = \sqrt{2\pi} \sum_{l=1}^{\infty} \left( \sqrt{2l + 1} i^l [p^{MI}(k_i, k_f) - ip^{EI}(k_i, k_f)] \right),$$

$$p^{MI}(k_i, k_f) = I_1 J_l(l),$$

$$p^{EI}(k_i, k_f) = -\sqrt{\frac{l+1}{2l+1}} I_2 J_l(l-1) + \sqrt{\frac{l}{2l+1}} I_3 J_l(l+1),$$  \hspace{1cm} (2.23)$$

where

$$J_l(n) = \int_0^{+\infty} r^2 \psi_{f,l}^{*}(r)\frac{d\psi_{i}(r)}{dr} j_{a}(kr)dr,$$  \hspace{1cm} (2.24)$$
\[ I_1 = \sum_{\mu = -1}^{1} \mu \hbar \mu \int Y_{LM}^*(n_f^i) T_{01,0}(n_f^i) T_{L,\mu}(n_f^i) d\Omega, \]
\[ I_2 = \sum_{\mu = -1}^{1} \mu^2 \hbar \mu \int Y_{LM}^*(n_f^i) T_{01,0}(n_f^i) T_{L-1,\mu}(n_f^i) d\Omega, \]
\[ I_3 = \sum_{\mu = -1}^{1} \mu^2 \hbar \mu \int Y_{LM}^*(n_f^i) T_{01,0}(n_f^i) T_{L+1,\mu}(n_f^i) d\Omega, \] (2.25)

and \( n \in N \) (here \( N \) is the natural number set), \( l \) is the quantum number for the wave function of the final \( f \)-state. For the transitions \( M1 \) and \( E1 \) we obtain
\[ I_{1 M1}^{E1} = -i \sqrt{\frac{2\pi}{3}} \cos (\theta), \quad I_{2}^{E1} = -i \sqrt{\frac{2\pi}{3}} (1 - 3 \cos^2 (\theta)), \] (2.26)

\[ Z_{\text{eff}} = \frac{2A_d - 4Z_d}{A_d + 4}, \] (2.27)

where \( A_d \) and \( Z_d \) are the mass number and charge of the daughter nucleus, and \( \theta \) is the angle between the direction of the \( \alpha \)-particle propagation and the direction of the photon emission.

§3. Angular analysis of the bremsstrahlung spectrum in \(^{210}\text{Po} \) \( \alpha \)-decay

We calculated the bremsstrahlung spectra for the \( \alpha \)-decay of \(^{210}\text{Po} \) at angular values \( 25^\circ \) and \( 90^\circ \) using the model described above. Our results for \( 90^\circ \) are presented in Fig. 1. In the calculations, we used the parameter values of the Coulomb barrier from Ref. 11) and the wave function of the initial \( i \)-state as the decay state and the final \( f \)-state as the scattered state.

In Fig. 1, it can be seen that the line 3 obtained using our model for \( 90^\circ \) lies nearer the experimental data\(^6\) than both the line 5 representing the bremsstrahlung spectra obtained using the models of Papenbrock and Bertsch\(^{11}\) and Tkalya\(^{12}\) (taking into account \( E1 \), based on quantum electrodynamics, and line 1 representing the bremsstrahlung spectrum obtained using Tkalya model\(^{12}\) based on classical electrodynamics. Such an accurate description of the experimental data\(^6\) of the bremsstrahlung spectrum for the nucleus \(^{210}\text{Po} \) has been obtained using a multipole approach without the transformation (2.14).

The total bremsstrahlung spectrum for \(^{210}\text{Po} \) calculated for \( 90^\circ \) using our model is represented by line 3 in Fig. 1 and is seen to be monotonic. The spectrum calculated for \( 25^\circ \) lies close to it. With angular analyzing, our model is found to predict a monotonic dependence of spectra on photon energy for any angle value (in the region \( 5^\circ \) – \( 90^\circ \)) and does not yield a slope as large as that of the experimental data.\(^2\) Taking into account \( M1, E2 \) and \( M2 \) multipoles the monotonic behaviour of the bremsstrahlung spectra for any angular value exists also where there is no visible “hole” with such amplitude, as in the experimental data.\(^2\) As we have seen, the quantum mechanical models of Papenbrock, Bertsch\(^{11}\) and Tkalya\(^{12}\) also fail to explain the existence of the “hole” in the spectrum.
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Fig. 1. Bremsstrahlung in the α-decay of $^{210}$Po: 1 — the result of Tkalya model$^{12}$ based on classical electrodynamics; 2 — experimental data$^6$; 3 — the result of our model for 90° with but E1 and sub-barrier photon emission; 4 — the result of our model for 90° with E1 but without sub-barrier photon emission; 5 — the result of Tkalya quantum mechanical model taking into account E1$^{12}$ and that of Papenbrock and Bertsch model$^{11}$; 6 — experimental data$^2$.

Note that taking account of the mixed region$^{16}$ is useful for detailed analysis of photon emission from different regions and the contributions to the total bremsstrahlung spectrum. However, we find that the total bremsstrahlung spectrum, calculated on the basis of the quantum mechanical models$^{11,12}$ and our model, is not depended on the choice of the mixed region and its change.

In addition, we calculated the bremsstrahlung spectrum for $^{210}$Po without taking into account the sub-barrier emission of the photons. The resulting spectrum is represented by line 4 in Fig. 1 for 90°. Here, it can be seen that line 4 lies somewhat farther from the experimental data$^6$ (obtained for 90° also) than line 3 for the total bremsstrahlung spectrum. From this we can conclude that taking account of the photon emission from the barrier region in the α-decay of $^{210}$Po increases the accuracy of experimental spectra description. But sub-barrier bremsstrahlung gives a small contribution to the total bremsstrahlung spectrum.

§4. Conclusions and perspectives

We presented a quantum mechanical approach to the calculation of the bremsstrahlung spectrum in the α-decay of heavy nuclei, where the angle between the directions of the α-particle propagation and the photon emission is taken into account. On the basis of the spectrum calculation carried out with this approach, we conclude that the model gives a monotonic bremsstrahlung spectrum for the different angles. (For readers interesting in the comparative analysis of the theoretical or experimental...
tal bremsstrahlung spectra obtained for different angles, we can also propose a new method presented in 17), where the dependence of the bremsstrahlung spectra on the values of such angle is obtained in a simple analytical form and is more obvious and simple.)

According with analysis by our model, taking into account multipoles of larger orders does not cause the spectra to become non-monotonic and does not explain the existence of the “hole” in the experimental data 2). The “hole” in the total bremsstrahlung spectrum reported in Ref. 2) cannot be explained by the choice of \( R_2 \), its adjustment, the arrangement of the mixed region (which was introduced in Ref. 16)) nor use of the isotropic quantum mechanical model, where the potential starting from \( R_1 \) is pure Coulomb. (Here, \( R_1 \) and \( R_2 \) are the internal and external radii of the Coulomb barrier.7), 12), 16) ) We agree with Tkalya that it would be useful to obtain more detailed experimental data for the bremsstrahlung spectrum in the \( \alpha \)-decay of \( ^{210}\text{Po} \) for different angle values.

From a comparative analysis, it is found that taking into account the sub-barrier bremsstrahlung phenomena in \( \alpha \)-decay of the nucleus \( ^{210}\text{Po} \) increases the accuracy of experimental spectra description. But sub-barrier bremsstrahlung gives a small contribution to the total bremsstrahlung spectrum (see Fig. 1). From this result, we can explain the fact that the models constructed on the basis of classical electrodynamics and those employing a semiclassical approximation describe the experimental data for the bremsstrahlung spectra of other nuclei sufficiently well.

We would also like to note further matters regarding the investigation of sub-barrier bremsstrahlung phenomena. It would be useful to obtain more accurate experimental data for the nucleus \( ^{210}\text{Po} \) for photons energies \( w_f \geq 400 \text{ MeV} \) and for the different angle values. The accuracy of the description of the \( \alpha \)-decay with photon emission can be improved by accounting for the individual properties of the nucleus. According to our model, small oscillations exist in the bremsstrahlung spectrum for \( ^{210}\text{Po} \) and the period can be estimated. The existence of oscillation can be explained through use of the multipole approach without recourse to the transformation (2.14) in the calculation of the bremsstrahlung spectrum.

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**Appendix A**

*Calculation of Multipoles*

In calculating the bremsstrahlung spectrum, we use a matrix element of the form

\[
\langle f | p e^{i k r} | i \rangle. \tag{A.1}
\]
In the calculation of this quantity, Tkalya\textsuperscript{12} and Papenbrock and Bertsch\textsuperscript{11} used the following transformation:

\begin{equation}
-f|p|i = \frac{1}{w_{f_i}} \frac{i}{\hbar} \langle f| \frac{\partial}{\partial r} U(r)|i \rangle \frac{1}{w_{f_i}}.
\end{equation}

However, this transformation can be used only in case that approximation

\begin{equation}
e^{ikr} = 1
\end{equation}
is made. Indeed, taking into account the properties

\begin{equation}
H f(r)\psi(r) = \left( \frac{\hbar^2}{2m} \Delta + U(r) \right) f(r)\psi(r) \neq f(r) \left( \frac{\hbar^2}{2m} \Delta + U(r) \right) \psi(r) = f(r) H \psi(r),
\end{equation}

we have

\begin{equation}
\langle f|H p f(r)|i \rangle = \langle f|E_f p f(r)|i \rangle = E_f \langle f|p f(r)|i \rangle,
\end{equation}

\begin{equation}
\langle f|p H f(r)|i \rangle \neq \langle f|p f(r) H|i \rangle = \langle f|p f(r)|E_i|i \rangle = E_i \langle f|p f(r)|i \rangle.
\end{equation}

Only in the case \( f(r) = \text{const} \) does the inequality (A.4) become an equality, in which case the transformation (A.2) can be used to simplify the calculation of (A.1). In a multipole expansion, we have

\begin{equation}
f(r) \rightarrow e^{ikr} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} i^l j_l(kr) Y_{l m}^* \left( \frac{k}{r} \right) Y_{l m} \left( \frac{k}{r} \right),
\end{equation}

where \( j_l(kr) \) is the spherically symmetric Bessel function of order \( l \).

References