Cost-Benefit Analysis and the Shadow Value of Government Revenue

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Abstract

This paper proves the Hatta (1977) coefficient is the shadow value of government revenue - it is a scaling coefficient that converts efficiency effects from marginal policy changes into dollar changes in utility. The decomposition is generalised to economies with heterogenous consumers and variable producer prices to show (a) the Foster and Sonnenschein (1970) effect, where extra income reduces consumer utility, makes the shadow value of government revenue negative; and (b) when Bruce and Harris (1982) and Dievert (1983) isolate Pareto improvements they choose patterns of revenue transfers to make the shadow value of government revenue positive for every consumer. We use the decomposition to extend the welfare test in Bruce-Harris by allowing revenue transfers with distorting taxes, and generalise the welfare decomposition of tax inefficiency in Diamond and Mirrlees (1971) by allowing variable producer prices.
1. Introduction

Policy analysts must decide how to aggregate welfare changes over consumers in a cost benefit analysis. Dreze and Stern (1985) argue they should assign distributional weights to the dollar changes in utility for each consumer and then sum them to report a final welfare change. And, as a way to indicate how important the subjectively chosen weights are to the overall analysis, the efficiency and equity effects can be reported separately. Another popular approach relies on the compensation principle to convert aggregate welfare gains into Pareto improvements. Harberger (1971), for example, sums unweighted dollar changes in consumer utilities and assumes implicitly they can be converted into Pareto improvements through lump-sum redistributions of income. Bruce and Harris (1982) and Diewert (1983) actually invoke the compensation principle and test for Pareto improvements. That is, they make revenue transfers between consumers to see whether the efficiency gains for a policy change can be converted into dollar gains in utility for every consumer.

This paper isolates the welfare effects of transfer policy choices in cost benefit analysis by generalising a decomposition in Hatta (1977). For any marginal policy change (k) we decompose the dollar change in utility for each consumer (h), as:

\[ w_k^h = w_R^h \hat{W}_k \forall h, \]

where \( w_R^h \) is the personal shadow value of government revenue, and \( \hat{W}_k \) the aggregate efficiency effect. This decomposition is obtained from an individualistic social welfare function, where the change in social welfare, is:

\[ \sum_h \beta^h w_k^h = \sum_h \beta^h w_R^h \hat{W}_k, \]

with \( \beta^h \) being the distributional weight for each consumer. All the income effects are isolated in the shadow values of government revenue which are scaling coefficients on the aggregate efficiency effect. In essence, efficiency gains (\( \hat{W}_k > 0 \)) are surplus revenue the government can collect by implementing policy k and, at the same time, making lump-sum transfers to hold the utility of every consumer constant; it is foreign aid the economy could pay at no cost to domestic utility. When this surplus revenue is transferred to domestic consumers, each dollar raises their utility by their personal shadow value of government revenue. Thus, socially profitable projects endow surplus revenue on the economy (\( \hat{W}_k > 0 \)), and \( w_R^h \) converts it into utility (\( w_k^h \)). For Pareto improvements (with \( w_R^h > 0 \ \forall h \)), the revenue transfers must be chosen to make \( w_R^h \) positive for every consumer.

A number of useful properties follow from this decomposition. First, the welfare effects of transfer policies are conveniently isolated in the shadow values of government revenue, which allows us to test for potential Pareto improvements by choosing transfers to make \( w_R^h > 0 \ \forall h \). Also, the decomposition finds income effects are irrelevant in cost-benefit analysis if the government uses the same pattern of revenue transfers to balance its budget.

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1. The final change in social welfare is \( \sum_h \beta^h w_k^h \), and the aggregate shadow value of government revenue \( \sum_h \beta^h w_R^h \) with h. Since policy k does not discriminate between consumers it cannot be a lump-sum redistribution of income. We consider income redistribution in section 4 of the paper.

2. When making this analogy it must be assumed domestic consumers do not benefit from aid payments to foreign consumers.
where at a social optimum we must have: $w^h_k = 0 \forall_h$ whenever $\hat{W}_k = 0$. Indeed, Ballard and Fullerton (1992) examine the revised Samuelson condition in a single consumer economy and conjecture: “To be relevant to cost-benefit analysis, the Pigou-Harberger-Browning approach must assume that the public good itself essentially compensates the consumer so that income effects wash out and only substitution effects remain.”\(^3\) By the Hatta decomposition income effects do wash out when policies are optimally chosen in this setting.

When Foster and Sonnenschein (1970) find a consumer is made worse off by extra real income the shadow value of government revenue is negative. It can occur if there are multiple equilibrium outcomes, which are possible in economies with tax-distorted markets. And this makes the compensation principle unreliable because there are efficiency gains when income is transferred from consumers with $w^h_k < 0$ to consumers with $w^h_k > 0$.\(^4\) Thus, the welfare test in Bruce and Harris that finds efficiency gains are a necessary and sufficient condition for Pareto improvements requires $w^h_k > 0 \forall_h$. And it does this by ruling out Foster-Sonnenschein effects and choosing lump-sum revenue transfers to raise the real income of every consumer. But, as Diebert demonstrates, this may not be possible when governments use distorting taxes. In policy evaluation it is necessary to determine if there are sufficient tax instruments to make $w^h_k > 0 \forall_h$ when testing for Pareto improvements. Otherwise subjective judgements will have to me made about the distributional weights to assign to consumers when there are winners and losers. In this paper we derive computable expressions of the shadow value of government revenue that can be estimated to test for Foster-Sonnenschein effects, and to determine whether or not Pareto improvements are possible using the available tax instruments.

In a conventional Harberger analysis where governments make lump-sum transfers to balance their budgets, the welfare effects of these transfers are rarely examined. For example, when a marginal policy change drives the government budget into surplus there are an infinite number of ways the revenue can be transferred to consumers, and the equilibrium outcome will depend on the transfer policy chosen when they have different marginal propensities to consume income. Giving them an equal share of the surplus revenue will not guarantee they have higher real income when prices vary endogenously. If, for example, the revenue transfers drive up prices for goods that are net consumed by individuals they are made worse, and this offsets benefits they get from receiving a share in the surplus revenue. To raise the real income of every consumer, personalised revenue transfers are required in the presence of these distributional effects. And when the government does so it makes the shadow value of government revenue positive for every consumer, in which case, aggregate dollar gains in utility will also be Pareto improvements. Furthermore, if Foster-Sonnenschein effects can be ruled out, these dollar gains in utility will signal efficiency gains. Thus, in cost benefit analysis we need to know what transfer policy the government will choose, or to recommend one if Pareto improvements are being sought, in order to solve actual equilibrium outcomes.

The formal analysis commences in the next section of the paper by demonstrating the Hatta decomposition of the welfare effects from a marginal tax change for a single consumer. In

\(^3\) Ballard and Fullerton (1992) page 124.

\(^4\) There are also well known intransitivities and reversals using the compensation principle when utility possibility frontiers cross, which is likely for comparisons between second-best policies. Bruce and Harris (1982) show these problems do not arise for marginal policy changes, which are what we examine in this paper.
section 3 we prove the Hatta coefficient, which isolates income effects from marginal tax changes, is the shadow value of government revenue, and we generalise the decomposition to any marginal policy change in economies with heterogeneous consumers and non-linear production frontiers. When governments distribute surplus revenue using distorting taxes the welfare effects from the transfers are isolated in the “revised” shadow value of government revenue. We derive it in an economy with heterogenous consumers in section 4 and use it to extend the welfare tests for Pareto improvements in Bruce and Harris. The paper concludes in section 5 with a brief summary of the main findings.

2. The Hatta Decomposition - An Intuitive Explanation

Much of the intuition for the Hatta decomposition can be obtained by marginally raising a commodity tax (t) for a single price-taking consumer who purchases two goods to maximise utility, \( u = u(x_o, x_1) \). The budget constraint is: \( M + L = q_0 x_0 + (p_1 + t)x_1 \), with \( M \) being a fixed endowment of money income, \( L \) a lump-sum transfer from the government, \( q_0 \) the consumer price of good 0, and \( q_1 = p_1 + t \) the consumer price of good 1. The welfare loss from marginally raising the tax, is:

\[
W_t = \frac{du}{dt} \lambda = \frac{\partial x_0}{\partial t} q_0 + \frac{\partial x_1}{\partial t} (p_1 + t) q_1
\]

where \( \lambda \) the marginal utility of income. In a conventional Harberger analysis tax revenue is returned to the consumer as a lump-sum transfer (\( L = tx \)) to balance the government budget. A conventional measure of the change in tax inefficiency is obtained by using the consumer's budget constraint to write the welfare change in (1), as:

\[
W_t = t \frac{\partial x_1}{\partial t},
\]

It is the welfare loss from driving down demand for good 1 in the presence of the tax distortion. Since demand is a function of consumer prices and money income, this loss can be decomposed as:

\[
\frac{\partial x_1}{\partial t} = \frac{\partial x_1}{\partial q_1} \frac{\partial q_1}{\partial t} + \frac{\partial x_1}{\partial I} \frac{\partial I}{\partial t}
\]

With constant producer prices we have \( \partial q_1 / \partial t = 1 \) and \( \partial q_2 / \partial t = 0 \). There are two ways of solving the income effect in (3), and we consider each in turn.

(a) A Slutsky decomposition holds money income constant. By substituting \( \partial I / \partial t = 0 \) into (3) and applying the Slutsky decomposition, the welfare loss in (2), becomes:

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5 This is obtained using the first order conditions for an interior solution to the consumer problem, where \( \partial u / \partial x_0 = \lambda q_0 \) and \( \partial u / \partial x_1 = \lambda (p_1 + t) \).

6 It is the measure obtained by Harberger (1964), and can also be found in Diamond and Mirrlees (1971) and Hatta (1977) who compute it in a single consumer economy with a linear production frontier. They generalise the welfare change by including additional taxed goods.

7 When money income changes endogenously with market prices we are using Bailey (1954) demand schedules.
This is the approach adopted by Diamond and Mirrlees (1971) who rule out lump-sum transfers.

(b) A Hatta decomposition allows money income to change by returning tax revenue to the consumer in a conventional manner as a lump-sum transfer. It is obtained by computing the change in money income in (3) using the budget constraint, as: \( \partial I / \partial t = t(\partial x_i / \partial t) + x_1 \), and applying the Slutsky decomposition, where the welfare loss in (2), becomes:

\[
W_t = \frac{\hat{W}_t}{1 - \theta}.
\]

Hatta isolates the income effect in the coefficient \( AIM = 1 - \theta \), and writes the welfare loss in (5), as: \( W_t = \hat{W}_t / AIM \), where the dollar change in utility originates from the substitution effect. There is good economic intuition for this result. The compensated welfare loss (\( \hat{W}_t \)) is the amount of foreign aid the government would need to receive to balance its budget after making lump-sum transfers to hold utility constant. When this loss is financed from within the economy by imposing a lump-sum tax on the consumer, each dollar reduces utility by one over the Hatta coefficient AIM. In the next section we show that \( 1 / AIM \) is the shadow value of government revenue (\( W_R \)), which is the amount utility rises when a dollar of revenue is endowed on the economy (as foreign aid). This, thus, the smaller the substitution effect from the tax change, the smaller the dollar fall in utility.

This is illustrated in Figure 1 where the two goods are perfect complements to eliminate the substitution effect. After the tax rises, and before the revenue is returned, the consumer moves from A to B. When the revenue is returned as a lump-sum transfer the consumer moves back along the income expansion path (IEP) to the initial bundle A. Since no substitution has occurred there is no welfare loss. With substitution the new consumption bundle would lie to the left of A along the old budget line (\( q^\theta \)) on an indifference curve below \( u_0 \). This is illustrated in Figure 2 for homothetic preferences where the tax moves the consumption bundle from A to B before the

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8 This is the approach adopted by Diamond and Mirrlees (1971) who rule out lump-sum transfers.

9 This allows us to write the Hatta decomposition as: \( W_k = W_R \hat{W}_k \).
tax revenue is returned. When it is returned as a lump-sum transfer, consumption moves along the new income expansion path $\text{IEP}(q')$ to point C, which lies on the original budget constraint on a lower indifference curve (not illustrated). Due to the substitution effect, outside compensation is required to move the consumer back onto $u_n$; it is DE (in units of good $x_n$). This tax inefficiency increases with the substitution effect from the tax change. When the loss DE falls on the consumer, the income effects move the consumption bundle to point C where utility falls. This change in utility is determined by the Hatta coefficient, which is positive when both goods are normal.

Foster and Sonnenschein effects make the Hatta coefficient negative, and they occur when there a multiple equilibria, which requires one of the two goods to be inferior. An example is illustrated in Figure 3 where the income expansion path for the initial tax distortion $q^0 - p^0$ cuts the consumption frontier more than once due to good 0 being inferior; and with non-satiation it must cut an odd number of times. The equilibria at A and C are for the same tax $q^0 - p^0$, while B is associated with a higher tax $q^1 - p^0$. As consumption moves along the frontier from A to C utility rises. Thus, if we start at B and reduce the tax to $q^0 - p^0$ the new equilibrium could be at A where utility is lower. If this happens, the increase in real income drives down utility, and we have $\Delta IM < 0$.

Hatta, and Foster and Sonnenschein look at equilibrium adjustment mechanisms to rule this

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10 The deadweight loss in DE is the foreign aid needed to fund the compensating variation net of the tax revenue raised at E.

11 There is no suggestion by this demonstration of the tax change that anything is wrong with the Slutsky decomposition. On the contrary, it is used in the Hatta decomposition. What the analysis does is extend the Slutsky decomposition by allowing consumer income to change endogenously with the transfers of surplus that accompany price changes. Thus, it is a decomposition that arises naturally in a general equilibrium analysis where surplus transfers from policy changes are included through the private and public sector budget constraints. We take this up in the next section of the paper.
out. Clearly, homothetic preferences will rule it out because there is a unique equilibrium associated with each tax when the goods are normal. But, as Foster and Sonnenschein show, this will not rule out multiple equilibria when the production frontier is non-linear and there are more than two goods.

In summary, income effects for incremental policy changes are a scaling coefficient when the surplus transfers that accompany price changes are taken into account. It arises naturally in general equilibrium where surplus transfers are automatically included through the private and public sector budget constraints. This is now demonstrated in a single consumer economy.

3. Welfare Measures in a Single Consumer Economy

Consider a single consumer who chooses a vector of \((N+1)\) non-traded goods \(x\) to maximise utility \(u(x)\) with expenditure \((qx)\) constrained by income: \(I = q\bar{x} + py + L = qx\); it is the market value of the endowment vector \((\bar{x})\) plus profits from private production \((py)\) and lump-sum transfers from the government \((L)\). The private net output vector \(y\) has \(y > 0\) for outputs and \(y < 0\) for inputs. Good 0 is chosen as numeraire, and the only distortions are the vector of specific taxes \(t\) on final production that drive wedges between the consumer \((q)\) and producer \((p)\) prices, with \(q = p_i + t_i\). Since the taxes are fixed in dollar terms we have \(dq = dp\) when prices change endogenously, and the price changes clear the goods markets, which are competitive, with: \(x_i = \bar{x}_i + y_i + z_i\) for all \(i \in N\). Finally, the public sector budget constraint is: \(T + pz = L + R\) where \(T = t(y + z)\) is production tax revenue, \(pz\) profit from public production, and \(R\) foreign aid payments.

In this economy dollar changes in utility for marginal changes in the three exogenous policy variables \((z, t\) and \(R))\) are solved using:

\[
\frac{du}{dx} = -zdp - (x - \bar{x})dt + dL, \tag{6}
\]

where the first two terms are standard changes in private surplus to the left of demand and supply schedules. Higher producer prices and taxes lower private surplus, while additional lump-sum transfers raise it by \(dL\). By using the market clearing conditions, we can write the first term in (6), as: \(-zdp - (x - \bar{x})dp\); it is the fall in consumer surplus \(-(x - \bar{x})dq\) plus the increase in producer surplus \(ydp\). When the government balances its budget, the

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12 Hatta uses a Marshallian adjustment process to show the points A and B in Figure 3 are stable equilibria because both goods are normal. It is a process, where:

(i) consumer prices are determined by a bidding process to clear any excess demand for output which cannot change instantaneously; and,

(ii) producers adjust output in the next period if there are differences between the relative producer prices and relative marginal production costs.

13 The welfare measures can be extended to include internationally traded goods, but they are omitted here to simplify the analysis.

14 By totally differentiating the private budget constraint, we have:

\[(p + t)dx + x(dp + dt) = \bar{x}(dp + dt) + pdy + ydp + dL.
\]

Using the first order conditions for consumers the change in utility becomes: \(du = u, dx = \lambda q dx\), and for competitive profit maximising firms \(pdy = 0\). After substitution, we obtain (6) by using the market clearing conditions: \(x_i = \bar{x}_i + y_i + z_i\) for all \(i \in N\).

15 Recall that \(dp = dq_i\) \(\forall i\) with specific tax wedges.
Using the market clearing conditions we can write tax revenue, as:

\[ T = t(y + z) = t(x - \bar{x}). \]

It will be assumed the domestic consumer does not benefit from foreign aid payments. They are hypothetical payments used to isolate potential gains, which in most cases, are returned to the domestic economy.

Harberger also includes changes in foreign prices to account for terms of trade effects when there are internationally traded goods.
output coefficient for private firms. It is a “substitution effect only” problem when the public sector uses the same production technology as private firms (with \( a_{j,k} = a_{j,\lambda} \)). If the public sector is less efficient due to soft budget constraints or principal agent problems (with \( -a_{j,k} > -a_{\lambda,j} \)), the first term in (10) represents an exogenous fall in real income; it makes the project a combined “income and substitution effect” problem. In fact, many policy choices are accompanied by exogenous reductions in real income. Rent controls are a good example because they usually involve rationing and rent seeking costs that can reduce private surplus by even more than the standard allocative inefficiency under the scheme.

Our next task is to isolate the income effects for the welfare changes in (9) and (10).

### 3.1 Welfare Decompositions with Constant Producer Prices

Hatta (1977) decomposes the welfare effects for marginal tax changes in a C-economy where goods are supplied at constant aggregate marginal cost. We replicate the analysis here for the tax change and the small public project.

**Proposition 1:** With constant producer prices, the welfare effects in (9) and (10) can be decomposed, as:

\[
W_k = \frac{\hat{W}_k}{1-\theta} \quad \text{for} \quad k \in \{x, z\},
\]

with: \( \hat{W}_k = \sum_{i \in I} \frac{\partial x_i}{\partial q_k} \), \( \hat{W}_z = p_j(a_{j,k} - a_{\lambda,j}) \) and \( \theta = \sum_{i \in I} \frac{\hat{h}_i(q)}{\hat{h}_i} \).

**Proof:** See section 1 of the Appendix.

All the income effects are isolated in the Hatta coefficient \((1-\theta)\), and we examine its sign in section 3.3. The compensated welfare effects may be positive or negative. With Ramsey optimal commodity taxes there is a compensated welfare loss from the tax change (with \( \hat{W}_k < 0 \)), but with non-optimal taxes, second best effects may result in a welfare gain if taxed activities expand sufficiently to make \( \hat{W}_k > 0 \). When the government produces less efficiently than private firms (with \( -p_j a_{j,k} > -p_j a_{\lambda,j} \)) there is a compensated welfare loss from the project, but if the public sector is equally efficient (with \( p_j a_{j,k} = p_j a_{\lambda,j} \)), then by proposition 1, there can be no dollar change in utility from the project (since \( \hat{W}_z = 0 \)). This confirms a result in Diamond and Mirrlees (1976) where shadow prices for goods are equal to their producer prices in C-economies when the public and private firms use the same technologies. There is no welfare gain from extra public output because it crowds out the same private output. However, the result no longer applies when the public sector has higher production costs.

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20 We use the fact that profit maximizing firms produce until at the margin:

\[ p_k = p_j a_{j,k} = 0. \]

21 In the compensated equilibrium the conventional welfare equation is obtained from (8) by setting \( du/k = 0 \), with \( dR = p dx + t dx \). Thus, the compensated welfare effects for the policy changes measure the changes in foreign aid payments \((dR/d_k \text{ and } dR/dz)\) to hold utility constant.

22 Diamond and Mirrlees (1971) also prove the relative producer prices of goods are equal to their relative shadow prices when producer prices vary endogenously if taxes are Ramsey optimal. Dasgupta and Stiglitz (1972) obtain the same result in economies with variable producer prices when there is a 100 percent profits tax and Ramsey optimal commodity taxes.
3.2 Welfare Decompositions with Variable Producer Prices

Even though the welfare decompositions become more cumbersome when producer prices change endogenously in economies with non-linear production frontiers, the income effects continue to be a scaling coefficient on the compensated welfare effects. This is confirmed by:

**Proposition 2:** With variable producer prices, the welfare effects in (9) and (10) can be decomposed, as:

\[ W_k = \frac{\hat{W}_k}{1 - \theta} \text{ for } k \in t_z, \]

with: \( \hat{W}_k = \sum_i \sum_s \frac{\partial \hat{e}(q)}{\partial q_s} a_{sp_k} \); \( \hat{W}_s = p_j (a_{j,k} - ay_{j,k}) \); and \( \theta = \sum_i \frac{\partial \hat{e}(q,I)}{\partial I} + \sum_s \frac{\partial \hat{e}(q)}{\partial q_s} a_{sp} \).

We define: \( a_{sp_k} = -\sum_i \delta_{i,t} \frac{\partial y_i(p)}{\partial p_k} \) and \( a_{sp} = -\sum_i \delta_{i,t} \frac{\partial y_i(q,I)}{\partial I} \), where \( \delta_{i,t} \) is an element of the matrix \( \delta_s = \left| \begin{array}{cc} \frac{\partial \hat{e}(q)}{\partial q_s} & \frac{\partial y_i(p)}{\partial p_s} \end{array} \right| \).

**Proof:** See section 2 of the Appendix.

These decompositions have exactly the same interpretation as those obtained with constant producer prices in Proposition 1. Differences arise through changes in taxed activities which determine the resource movements across distorted activities. In particular, prices now change endogenously with the income effects.

Before establishing necessary and sufficient conditions for welfare improvements in single consumer economies we examine the sign of the Hatta coefficient (1-\( \theta \)). If it is positive, incremental policy changes will raise utility whenever there are efficiency gains.

3.3 The Shadow Value of Government Revenue

It is clear from Propositions 1 and 2 that the Hatta coefficient (1-\( \theta \)) converts compensated welfare effects into utility, and this leads to:

**Proposition 3:** The inverse of the Hatta coefficient is the shadow value of government revenue, with:

\[ W_R = \frac{1}{1 - \theta} \]

where: \( \theta = \sum_i \frac{\partial \hat{e}(q,I)}{\partial I} - \text{ for constant producer prices; and,} \)

\( \theta = \sum_i \frac{\partial \hat{e}(q,I)}{\partial I} + \sum_s \frac{\partial \hat{e}(q)}{\partial q_s} a_{sp} - \text{ for variable producer prices.} \)

**Proof:** If a dollar of revenue is endowed on the economy as foreign aid, the welfare change is obtained using (8), as:

\[ \theta \]
\[ W_R = \frac{du}{dR} \mathcal{K} = 1 - \frac{\partial T}{\partial R}, \]

where the decomposition in Proposition 3 is obtained by solving the endogenous changes in prices and income to write the change in tax revenue, as: \( \partial T/\partial R = -\theta/(1-\theta). \)

When Foster and Sonnenschein find additional income lowers utility in a single consumer economy the shadow value of government revenue is negative. From (11) we can see that this occurs when income effects reduce tax revenue by more than the initial dollar of revenue endowed on the economy (with \( -\partial T/\partial R < 0 \)). In C-economies where income effects do not affect producer prices, we can use the budget constraint to write the Hatta coefficient as:

\[ 1 - \theta = \sum p_i \frac{\partial \chi(q,I)}{\partial I}, \]

which can only be negative with inferior goods. Thus, local Foster-Sonnenschein effects (LFSE) can be ruled out here by adopting a homogenous utility function. And with all goods normal the shadow value of government revenue will be larger than unity because income effects generate additional tax revenue (with \( -\partial T/\partial R > 0 \)). But this will not rule out LFSE in economies with non-linear production frontiers. Once producer prices vary endogenously the Hatta coefficient becomes:

\[ 1 - \theta = \sum p_i \frac{\partial \chi(q,I)}{\partial I} - \sum_{i=1}^{n} \sum \frac{\partial \chi(q_i)}{\partial q_i} \alpha_{it}, \]

where the first term is positive when goods are normal, but that will not stop the second term being larger and negative if there are more then two goods. Its sign depends on the price changes which are determined by the shape of the production frontier. Thus, to rule out LFSE we will need to rely on equilibrium adjustment mechanisms that choose outcomes where utility is highest. Foster and Sonnenschein consider one such mechanism - it involves making Slutsky compensation to overcompensate consumers for a policy change and then withdraw income until full equilibrium is achieved at the highest possible utility. For the example illustrated in Figure 3, a reduction in the tax at B would move the economy to equilibrium C where utility is higher than it would be at A. In practice we can test for LFSE empirically by computing the shadow value of government revenue using estimates of the own and cross price elasticities for the demands and supplies of goods to solve the price changes. At this point we follow Bruce and Harris and rule out LFSE to establish the following welfare test.

**Theorem 1:** In the absence of LFSE, the necessary and sufficient condition for a welfare gain from any small policy change \( k \) in a single consumer economy where the government balances its budget using lump-sum transfers is \( W_k > 0 \).

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24 This decomposition is derived in section 3 of the Appendix.

25 With two goods \( (x_0, x_1) \) we have:

\[ 1 - \theta = \sum p_i \frac{\partial \chi(q_i, I)}{\partial I} - t_i \frac{\partial \gamma(p_i)}{\partial p_i} \alpha_{it}, \]

with: \( \alpha_{it} = -\frac{\partial \chi(q_i, I)}{\partial I} \left\{ t_i \frac{\partial \gamma(p_i)}{\partial p_i} + \frac{\partial \chi(q_i)}{\partial q_i} \right\} \). When both goods are normal the relative price of good 1 increases with income, with \( \alpha_{it} > 0 \), so that \( 1 - \theta > 0 \).
Proof: Since consumer demands are functions of the exogenous policy variables \( z, t \) and \( R \), we can write indirect utility as \( v(z,t,R) \). This function is a unique mapping in the absence of multiple equilibria (which is the case with no LFSE). For incremental policy change \( k \), the outside compensation \((dR)\) to hold utility constant, solves:

\[
d v = \frac{\partial v}{\partial k} dk + \frac{\partial v}{\partial R} dR = 0 \quad \text{for } k \in t_{\cdot z}.
\]

After multiplying this expression by one over the marginal utility of income \((\lambda)\), we have:

\[
W_k = W_k W_k \quad \text{for } k \in t_{\cdot z},
\]

where in the absence of LFSE, \( W_k > 0 \) in the single consumer economy. Thus, utility rises \((W_k > 0)\) whenever there are efficiency gains \((W_k > 0)\).\(^{26}\)

Theorem 1 holds with constant and variable producer prices, and it follows directly from the welfare decompositions in Propositions 1, 2 and 3. We now generalise this theorem to economies where the government balances its budget using distorting taxes instead of lump-sum transfers.

### 3.4 Revised Welfare Changes

A number of studies acknowledge the fact that governments balance their budgets with distorting taxes. Diamond and Mirrlees (1971, 1976) and Dasgupta and Stiglitz (1972) obtain welfare measures when lump-sum taxes are ruled out. We follow Pigou (1947) by referring to them as revised welfare measures. They can be decomposed as conventional (lump-sum) welfare effects plus changes in tax inefficiency when distorting taxes are used to balance the government budget, where:

**Proposition 4:** The revised welfare effects for the incremental policy choices examined in (9) and (10), are:

\[
(W_k)_D = W_k + \text{meb}_d \frac{\partial L}{\partial d} \quad \text{for } k = t_{\cdot z}, \quad \text{and } d = t.
\]

with \( \text{meb}_d = - (W_k)_D \frac{\partial L}{\partial d} \) being the marginal excess burden for tax \( d \); it is the welfare loss per dollar change in government revenue \((\partial L/\partial d)\), where the revenue transfers are solved using (7).

**Proof:** See section 4 in the Appendix.

In project evaluation, changes in tax inefficiency are determined by how much revenue the government must transfer to balance its budget. Thus, the revised and conventional welfare measures are equal if policy changes do not affect the government budget (with \( \partial L/\partial d = 0 \)). The revised welfare change from marginally increasing tax \( k \) in Proposition 4 provides a convenient way to identify the Ramsey rule for optimal commodity taxes, where:

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\(^{26}\) The shadow value of government revenue \( W_R \) is not the marginal social cost of public funds (MCF). Instead, it measures the welfare gain from an extra dollar of revenue as foreign aid, while the MCF measures the welfare gain from transferring a dollar of revenue between the government and the private sector of the economy. Thus, based on the way Harberger distinguishes between problems, \( W_R \) is the welfare change for a “income and substitution effect” problem, while MCF is the welfare change for a “substitution effect only” problem.
\begin{equation}
(W_{D})_{D} = \left\{ \text{meb}_{d} - \text{meb}_{k} \right\} \frac{dL}{d_{k}}.
\end{equation}

Since the extra revenue collected by increasing tax \( k \) is returned to the consumer by lowering tax \( d \), it represents a marginal change in the tax mix to raise a given amount of revenue. Thus, when taxes are Ramsey optimal with \((W_{D})_{D} = 0\), they have the same marginal excess burden. Diamond and Mirrlees (1971) prove the income effects play no role in choosing these optimal taxes in single consumer economies with constant producer prices. We extend this to economies with variable producer prices by decomposing the marginal excess burden for tax \( d \) as: \( \text{meb}_{d} = (W_{D})_{d} \text{meb}_{d} \), where \((W_{D})_{D} \) is the revised shadow value of government revenue. After substitution the welfare change in (13) for Ramsey optimal taxes, becomes:
\begin{equation}
(W_{D})_{D} = (W_{D})_{d} \left\{ \text{meb}_{d} - \text{meb}_{k} \right\} \frac{dL}{d_{k}} = 0,
\end{equation}
with \( \text{meb}_{d} = \sum t_{j}(\partial \tilde{x}_{j}/\partial t_{j})/(\partial L/\partial t_{d}) \) being the compensated tax inefficiency per dollar of compensating transfer using tax \( d \). Since the income effects are isolated in \((W_{D})_{D} \), they will not play a role in setting the optimal taxes.

**Proposition 5:** With variable producer prices, the revised welfare effects for the policy changes in Proposition 4 can be decomposed, as:
\begin{equation}
(W_{D})_{D} = (W_{D})_{d}(\hat{W}_{D})_{D} \text{ for } k = t_{k}, z_{k},
\end{equation}
where the revised shadow value of government revenue, is:
\begin{equation}
(W_{D})_{D} = W_{D} \left\{ \text{mcf}_{d} + \text{meb}_{d} \sum z_{i}a_{i} \right\},
\end{equation}
with \( \text{mcf}_{d} = 1 - \text{meh}_{d} \) being the marginal social cost of public funds; it is the change in private surplus when a dollar of revenue is transferred from the private economy to balance the government budget using tax \( d \).

**Proof:** When the government balances its budget using tax \( d \), we can write indirect utility as: \( v(z,t_{D},R) \), where \( t_{D} \) is the vector of taxes excluding tax \( d \) which changes endogenously. For marginal policy changes, the outside compensation \((dR)\) to hold utility constant, solves:
\begin{equation}
\frac{\partial v}{\partial R} = \frac{\partial v}{\partial d} d_{k} + \frac{\partial v}{\partial k} dR = 0 \text{ for } k = t_{k}, z_{k},
\end{equation}
where the decomposition in Proposition 5 is obtained by multiplying this expression by one over the marginal utility of income \((\lambda)\) and rearranging terms.]

Now when a dollar of revenue is endowed on the economy as foreign aid, the government balances its budget by lowering tax \( d \). Utility will rise in the absence of LFSE (with

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27 This decomposition is derived in section 5 of the Appendix. \((W_{D})_{D} \) measures the change in utility when a dollar of foreign aid is endowed on the government who transfers it to the private sector by lowering tax \( d \).

28 The compensated welfare changes, are:
\begin{equation}
(W_{D})_{D} = \hat{W}_{D} + \text{meh}_{d}(\partial L/\partial t_{k}) \text{ and } (\hat{W}_{D})_{D} = \hat{W}_{D} + \text{meh}_{d}(\partial L/\partial z_{k}),
\end{equation}
where the compensating transfers are solved using (6) with \( du/\lambda = 0 \).
(W_rh_o > 0) if there are efficiency gains from lowering the tax (with meb_q> 0). Foster and Sonnenschein find this to be the case for radial tax reductions, and it also applies when taxes are Ramsey optimal. While these are sufficient conditions for the revised shadow of value of government revenue in (14) to be positive, they are not necessary. Thus, we have:

**Theorem 2:** In the absence of LFSE, the necessary and sufficient condition for a welfare gain from any small policy change k in a single consumer economy where the government balances its budget using distorting taxes to make the revised shadow value of government revenue positive is (W_kh_o > 0).

**Proof:** From Proposition 5, if (W_rh_o > 0) then (W_kh_o > 0 whenever (W_kh_o > 0).]

Thus, if there are efficiency gains from a policy change it will generate surplus revenue that allows the government to lower distorting taxes. The resulting fall in tax inefficiency will raise the revised welfare change above the conventional (lump-sum) welfare change.

### 4. Welfare Measures with Multiple Consumers

In economies with heterogenous consumers there are distributional effects to account for in project evaluation. Indeed, for most policy changes some consumers gain and others lose. Three factors determine the welfare changes - the sign and size of the compensated welfare effects, the distributional weights assigned to consumers, and the pattern of revenue transfers the government uses to balance its budget. Harberger assigns the same distributional weights to consumers and identifies a welfare gain when the sum of the dollar changes in utility is positive. This approach relies on two implicit assumptions - first, the government can make personalised lump-sum revenue transfers to convert compensated gains into Pareto improvements i.e., to raise the utility of every consumer, and second, the shadow value of government revenue is positive. In practice, governments use distorting taxes to make revenue transfers, and unless they have sufficient tax instruments, or can personalise tax changes, it may not be possible to convert compensated gains into Pareto improvements.

Diewert (1983) recognises this by including transfer policy choices in welfare tests to determine whether there are Pareto improvements for small policy changes. The tests pass if governments can personalise the transfers using the available set of taxes to raise the utility of every consumer, and they do not rely on the choice of distributional weights. Rather, they rely on social welfare rising with Pareto improvements. Bruce and Harris (1982) refine the tests with lump-sum transfers by ruling out LFSE. And by doing so they make the shadow value of government revenue positive for every consumer so that Pareto improvements occur whenever there are efficiency gains. We now extend this to allow revenue transfers with distorting taxes, and demonstrate the role played by the shadow of government revenue. Throughout the remaining analysis producer prices vary endogenously, and aggregate welfare is measured using the individualistic social welfare function:

\[ W = W[v^h(z,t,R,g), \ldots \ldots , v^H(z,t,R,g)], \]

where: \(v^h(z,t,R,g)\) is indirect utility for each consumer \(h = 1, \ldots , H\), and \(g\) the vector of

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29 Each consumer maximises \(u^h(x)\) subject to the budget constraint: \(q x^h = q x^h + \rho^h p_y + g^h L\) where \(\rho^h\) is h's share of the profits from private production with \(\sum \rho^h = 1\), and \(g^h\) is h's share of lump-sum transfers the government makes to balance its budget with \(\sum g^h = 1\).
shares in the lump-sum transfers made by the government to balance its budget with \( \sum_k g^h = 1 \). In a conventional Harberger analysis lump-sum transfers change endogenously, but the shares in these transfers are a policy variable the government can choose, and this makes the final welfare change a function of these transfer shares.

Compensated welfare effects for any policy change \( k \) are isolated as the foreign aid payments \((dR)\) that would hold the utility of every consumer constant, where from (15) we have:

\[
dW = \sum_h \beta_h W_k^h dk - \sum_h \beta_h W_R^h dR = 0 \quad \text{for } k \in t_1, z_1,
\]

with \( \beta_h = (\partial W / \partial u^h) \lambda^h \) being the distributional weight. Since the personal shadow value of government converts each consumer's share of the aggregate efficiency gain \((\hat{W}_k = dR/dk)\) into utility, we have:

\[
W_k^h = W_R^h \hat{W}_k \quad \text{for } k \in t_1, z_1.
\]

The final welfare changes \((W_k^h)\) are ultimately determined by each consumer's share of the aggregate efficiency gain \((\hat{W}_k)\) plus any distributional effects when prices change endogenously with the revenue transfers. This is confirmed by decomposing the personal shadow value of government revenue, as:

\[
W_R^h = W_R^h \left\{ g^h + \sum_s \left( DE_s^h + g^h z_s \right) a_{js} \right\}, \quad 30
\]

where \( DE_s^h = (x_s^h - x_s^h + \rho_s^h) \) is the distributional effect for each \( h \) from a rise in the price of good \( s \). When these distributional effects are combined with \( h \)'s share of the change in profit on public production, it isolates the net change in private surplus due to price changes, which are purely distributional since \( \sum_h (DE_s^h + g^h z_s) = 0 \quad \forall s \). Giving each consumer an equal share in the surplus revenue (with \( g^h = \bar{g} \ \forall h \)) will not guarantee they have \( W_R^h > 0 \). If the endogenous price changes reduces a consumer's surplus (with \( \sum_s (DE_s^h + g^h z_s) a_{js} < 0 \)) it is possible for \( W_R^h < 0 \). Thus, the transfer shares will need to be personalised to offset any adverse distributional effects.

There are two sets of personalised lump-sum transfers in the decomposition in (16); one set isolates the efficiency gains \((\hat{W}_k)\), while the other set distributes these gains to consumers \((W_k^h)\). Since there are different substitution effects for different policy changes, the compensating transfers are a function of \( k \), but the revenue transfers are not because they distribute surplus revenue to consumers for any policy change. If the transfers are chosen to make the personal shadow value of government revenue in (18) positive for every consumer (with \( W_R^h > 0 \quad \forall h \)), there will be Pareto improvements (in the absence of LFSE, which makes \( W_R^h > 0 \), whenever there are efficiency gains (with \( W_R^h > 0 \)). Thus, final changes in social welfare \((\hat{W}_k)\) are accompanied by the sum of these two sets of personalised transfers. This leads to:

**Theorem 3 (Theorem 5 - Bruce and Harris (1982)):** In the absence of LFSE, the necessary and sufficient condition for a welfare gain from any small policy change \( k \) in a multiple consumer economy where the government balances its budget using lump-sum transfers is \( \hat{W}_k > 0 \).

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30 The Hatta coefficient is defined in Proposition 3 for variable producer prices.
Proof: By ruling out LFSE we have $W_R > 0$, and by choosing personalised lump-sum transfers to make $g^b + \sum_k (DE_k^b + g^b)z_k > 0 \forall_k$, the shadow value of government revenue in (18) is positive for all consumers. Thus, from (17), there are Pareto improvements (with $W_k^b > 0 \forall_k$) whenever $\hat{W}_h > 0$.]

This is the welfare test in Bruce and Harris that relies on the compensation principle to convert compensated gains into Pareto improvements. The decompositions in (17) and (18) make it clear how the transfer and other policy changes are combined in this test. Dievert obtains sufficient conditions for Pareto improvements which do not require efficiency gains. If Foster-Sonnenschein effects make the shadow value of government revenue negative for every consumer, there can be Pareto improvements from efficiency losses. For example, radial tax increases can move consumers from low to high utility outcomes in the presence of multiple equilibria. However, once Foster-Sonnenschein effects are ruled out, which is the basis for the Bruce-Harris test, Pareto improvements require efficiency gains.

Before extending the welfare test in Theorem 3 to transfers with distorting taxes, we consider the welfare effects of a lump-sum redistribution of income between consumers, where:

$$dW = \sum_k W_k^b dg^b - \sum_k W_R^b d\hat{R} = 0,$$

with $\sum_k dg^b = 0$.

In the absence of Foster-Sonnenschein effects there are no efficiency effects, with $\hat{W}_h = \sum_k d\hat{R}/dg^b = 0$. Thus, when consumers have the same distributional weights ($\hat{\beta}$) there can be no change in social welfare. Any welfare gains must exploit differences in the distributional weights across consumers in these circumstances.

When governments use distorting taxes they can make the revised shadow value of government revenue positive for every consumer if there are sufficient tax instruments to personalise the transfers. And this extends the Bruce-Harris test to distorting taxes when Foster-Sonnenschein effects are ruled out. However, if the transfers cannot be personalised, or if there are Foster-Sonnenschein effects, the shadow value of government revenue may not be positive for every consumer. In this case, efficiency gains cannot be converted into Pareto improvements.

For small policy changes, the foreign aid payments that would hold utility constant for every consumer $h$ when the government makes revenue transfers with distorting taxes, is obtained using (15), as:

$$(dW)_h = \sum_k \hat{\beta}_h (W_k^b)_h dk - \sum_k \hat{\beta}_h (W_R^b)_h (d\hat{R})_h = 0 \text{ for } k \in t_h, z_h.$$  

Since the personalised transfers convert surplus revenue into utility, we have:

$$(W_k^b)_h = (W_R^b)_h (\hat{W}_h)_h \text{ for } k \in t_h, z_h,$$

where: $(\hat{W}_h)_h = (d\hat{R}/dk)_h$ is the compensated revised welfare change; it is the surplus revenue the government can raise when it uses distorting taxes to hold the utility of every consumer constant.

31 By summing the dollar changes in utility over consumer using (6) with $dl = 0$, we can write the change in social welfare, as:

$$(dW)_D = \sum_k \hat{\beta}_h \left\{ \sum_i DE_i^b dp_i - \sum_i (x_i^b - \bar{x}_i) dt \right\}.$$
constant. If tax d is used to balance the government budget, the personalised shadow value of government revenue becomes:

$$(W_R^h)_{d} = W_R \{ mcfd_d^h + \sum \{DE^h + mcfd_d z_i\} a_{d,i}\},$$

where mcfd_d^h is the personal marginal cost of public funds; it replaces the lump-sum transfer share g^h in (18) and is the direct increase in the private surplus of consumer h when a dollar of surplus revenue is transferred from the government budget using tax d. There are additional distributional effects from the tax changes that are included in mcfd_d^h, and we confirm the distributional effects will wash out by summing the (unweighted) personal shadow value of government revenue in (21) over consumers, where:

$$\Sigma_h(W_R^h)_{d} = W_R \{ mcfd_d + mcfd_d \sum z_i a_{d,i}\}. $$

This is the revised shadow value of government revenue in (14) for the single (aggregated) consumer. Whenever marginal tax increases increase the marginal excess burden of taxation, we have mcfd_d^h > 1. And, as noted previously, this occurs for radial tax increases, and for increases in Ramsey optimal taxes. Diewert makes the observation that it may be necessary to change a potentially large number of taxes to realise Pareto improvements for a policy change, whereas in other circumstances it may be achieved by using a single tax. Ultimately, the tax changes that make the shadow value of government revenue positive for every consumer will depend on the distributional effects in (21), which are determined by the net demands for taxed goods across consumers. If there are insufficient commodity taxes to achieve this, the government must be able to personalise the tax changes on one or a number of commodities. That is, it must be able to set different taxes for different consumers. This leads to:

**Theorem 4:** If, in the absence of LSFE, the government can make the shadow value of government revenue positive for every consumer using distorting taxes, then the necessary and sufficient condition for a welfare gain from any small policy change k in a multiple consumer economy is $$(W_k^h)_{d} > 0.$$  

**Proof:** By ruling out LFSE we have $W_R > 0$, and by choosing personalised lump-sum transfers to make $mcfd_d^h + \sum (DE^h + mcfd_d z_i) a_{d,i} > 0$, the revised shadow value of government revenue in (21) is positive for all h. Thus, from (20), there are Pareto improvements (with $W_k^h > 0 \forall h$) whenever $(W_k^h)_{d} > 0.$]

5. Concluding Remarks

This paper has isolated income effects for marginal policy changes in the shadow value of government revenue. It provides a convenient way to determine the welfare effects of government transfer policy choices in cost-benefit analysis. In a conventional analysis there are lump-sum revenue transfers, and little mention is made about them because they are

32 For tax d we have: $mcfd_d^h = \left\{ (x_d^h - \overline{x_d^h}) + \sum \{DE^h + mcfd_d z_i\} a_{d,i} \right\} / \delta L / \delta d.$

When multiple taxes are used to balance the budget, the revised shadow value of government revenue will be:

$$(W_R^h)_{d} = W_R \sum \{DE^h + mcfd_d a_{d,i}\} \frac{dt_i}{dR} \right\},$$

where the tax changes required to balance the government budget, solve:

$$\left\{ \frac{dL}{dR} \right\}_d = \frac{\delta L}{dR} + \sum \left\{ \frac{\delta L}{dt_i} \right\}_d = 0.$$
non-distorting. However, final welfare changes are a function of the way revenue is transferred across consumers. And if the pattern of transfers is chosen to make the shadow value of government revenue positive for every consumer there will be Pareto improvements whenever aggregate dollar changes in utility are positive. Also, if Foster-Sonnenschein effects are ruled out, these utility gains will signal efficiency gains. We derived computable expressions of the personal shadow value of government revenue for lump-sum and distorting taxes that can be used in applied work to estimate the welfare effects of government transfer policy choices, and to test for Foster-Sonnenschein effects.
Appendix

1. **Proof to Proposition 1:** Since demand is a function of consumer prices and money income, we can write the change in tax revenue in (9), as:

\[ W_i = \sum_t \frac{\partial x_i}{\partial t_k} + \sum_t \frac{\partial x_i (q, I)}{\partial q} \frac{\partial q}{\partial t_k} + \sum_t \frac{\partial x_i (q, I)}{\partial I} \frac{\partial I}{\partial t_k} \]

and the changes in tax revenue in (10), as:

\[ \frac{\partial T}{\partial z_v} = \sum_t \frac{\partial x_i (q, I)}{\partial q} \frac{\partial q}{\partial z_v} + \sum_t \frac{\partial x_i (q, I)}{\partial I} \frac{\partial I}{\partial z_v} \text{ for } v \in k, j. \]

With **constant producer prices**, we have \( \frac{\partial q}{\partial t_k} = 1, \frac{\partial q}{\partial t_s} = 0 \quad \forall s \neq k, \) and \( \frac{\partial q}{\partial z_v} = 0 \quad \forall v. \) The changes in income are solved by combining the private and public sector budget constraints, where:

\[ \frac{\partial I}{\partial t_k} = \frac{\partial I}{\partial t_s} = \sum_t \frac{\partial x_i (q, I)}{\partial q} + \sum_t \frac{\partial x_i (q, I)}{\partial I} \text{ for } v \in k, j. \]

After substitution, we have:

\[ \sum_t \frac{\partial x_i}{\partial t_k} = \frac{\sum_t \frac{\partial x_i (q, I)}{\partial q}}{1 - \theta}, \quad \sum_t \frac{\partial x_i}{\partial z_v} = \frac{-\theta x_i}{1 - \theta} \text{ and } \sum_t \frac{\partial x_i (q, I)}{\partial z_v} = \frac{\theta p_{av_{j,k}}}{1 - \theta} \]

The welfare decompositions in Proposition 1 are obtained by substituting these changes in tax revenue into the welfare changes in (9) and (10), and using the first order conditions for profit maximising private firms (with: \( p_k = p_j a_{v_{j,k}} = 0).\]

2. **Proof of Proposition 2:** We use welfare decompositions in Sieper (1981). With **variable producer prices** the change in tax revenue in (9) can be decomposed, as:

\[ \sum_t \frac{\partial x_i}{\partial t_k} = \sum_t \frac{\partial x_i (q, I)}{\partial q} \frac{\partial q}{\partial t_k} + \sum_t \frac{\partial x_i (q, I)}{\partial I} \frac{\partial I}{\partial t_k} \]

and the changes in tax revenue in (10), as:

\[ \frac{\partial T}{\partial z_v} = \sum_t \frac{\partial x_i (q, I)}{\partial q} \frac{\partial q}{\partial z_v} + \sum_t \frac{\partial x_i (q, I)}{\partial I} \frac{\partial I}{\partial z_v} \text{ for } v \in k, j. \]

The changes in income are solved using the budget constraints, where:

\[ \frac{\partial I}{\partial t_k} = \sum_s \frac{\partial q_s}{\partial t_k} + \frac{\partial T}{\partial t_k} \text{ and } \frac{\partial I}{\partial z_v} = \sum_s \frac{\partial q_s}{\partial z_v} + \frac{\partial T}{\partial z_v} \text{ for } v \in k, j. \]

---

33 By combining the private and public sector budget constraints, we have:

\[ I = \left( p + i \right)x + \left( p + i \right)\tilde{x} + \left( p + i \right)(y + z) - R. \]

The income effect is based on profit maximising production choices in competitive markets, with:

\[ p(\partial x_i/\partial q) = p(\partial y_i/\partial q) = 0 \quad \text{and} \quad p(\partial x_i/\partial z_v) = p(\partial y_i/\partial z_v) + 1 = 1. \]
The price changes are determined by the goods market clearing conditions for each good $i$, and are solved using the following system of equations which are stacked over $N$ goods:

\[
\frac{\partial x_i(q,I)}{\partial q_s} - \frac{\partial y_i(p)}{\partial p_s} \bigg|_{d_0} + \frac{\partial x_i(q,I)}{\partial I} \bigg|_{d_0} = 0 \bigg|_{dR} + \frac{\partial y_i(p)}{\partial p_s} \bigg|_{d_0} + \frac{\partial I}{\partial z_v}, \quad v \in k,j,
\]

where:

\[
\frac{\partial q_j}{\partial t_k} = \alpha \sum_{i} \frac{\partial x_i}{\partial t_k} - \beta \frac{\partial y_j}{\partial p_k} \quad \text{and} \quad \frac{\partial q_j}{\partial z_v} = \delta_v \cdot \alpha \left( \rho_v - \frac{\partial T}{\partial p_s} \right) \quad \text{for} \ v \in k,j, \tag{34}
\]

with: $\alpha_d = -\sum_{i} \frac{\partial x_i(q,I)}{\partial t} \bigg|_{d} \quad \text{and} \quad \delta_{ij} \text{ an element of the matrix } b \bigg|_{b_{ij}} = \frac{\partial x_j(q)}{\partial q_s} - \frac{\partial y_j(p)}{\partial p_s} \bigg|_{d}^{-1}.

After substituting the income effects and price changes into the changes in tax revenue, we have:

\[
\sum_{i} \frac{\partial x_i}{\partial t_k} = \frac{\sum_{i} \frac{\partial x_i}{\partial q_s} \alpha_{v_i}}{1 - \theta} \text{ and } \frac{\partial T}{\partial z_v} = \frac{\theta_p}{1 - \theta} \quad \text{for} \ v \in k,j,
\]

with: $\alpha_{v_i} = \sum_{i} \delta_{ij} \frac{\partial y_j(p)}{\partial p_s} \bigg|_{d}.

The welfare decompositions in Proposition 2 are obtained by substituting these changes in tax revenue into the welfare changes in (9) and (10).

3. With variable producer prices the change in tax revenue in Proposition 3 can be decomposed, as:

\[
\frac{\partial T}{\partial R} = \sum_{i} \frac{\partial x_i(q,I)}{\partial q_s} \frac{\partial q_s}{\partial R} + \sum_{i} \frac{\partial \alpha_s}{\partial R} \bigg|_{d} + \sum_{i} \frac{\partial x_i(q,I)}{\partial I} \frac{\partial I}{\partial R}
\]

The price changes are solved using the market clearing conditions for the $N$ goods whose prices change, where for each good $s$, we have:

\[
\frac{\partial q_s}{\partial R} = \alpha \left( \frac{\partial T}{\partial R} - 1 \right).
\]

The change in consumer income is solved using the budget constraints, as:

\[
\frac{\partial I}{\partial R} = \sum_{i} \frac{\partial q_i}{\partial R} + \frac{\partial T}{\partial R} - 1.
\]

After substitution, we have: $\frac{\partial T}{\partial R} = -\theta/(1 - \theta)$, with: $\theta = \sum_{i} \left( \frac{\partial x_i(q,I)}{\partial I} + \sum_{i} \frac{\partial x_i(q,I)}{\partial q_s} \alpha_{v_i} \right)$

for variable producer prices. With fixed producer prices, we have: $\alpha_{v_i} = 0 \quad \forall i$. 

4. **Proof of Proposition 4:** The revised welfare effect for representative policy $k$ solves:

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34 When solving the price changes we use the Slutsky decomposition.
\[ (W_k)_D = \left( \frac{\partial u}{\partial k} \right) \frac{1}{\partial \lambda} = \frac{\partial u}{\partial k} \frac{1}{\partial \lambda} + \frac{\partial u}{\partial \lambda} \frac{1}{\partial \lambda}. \]

It is a combination of two conventional Harberger terms that are isolated by “notional” lump-sum transfers to balance the government budget. The first is the conventional welfare change \( W_k \), and the second the change in tax inefficiency \( W_{th} \). When tax \( h \) changes to balance the government budget it offsets the “notional” transfers, with:

\[ \left( \frac{\partial L}{\partial k} \right)_D = 0 = \frac{\partial L}{\partial k} \frac{\partial \left( \frac{\partial L}{\partial d} \right)}{\partial k}. \]

The welfare change in Proposition 4 is obtained by solving the tax change.

5. Using the conventional welfare equation in (8) and the revenue transfer equation in (7), we can write the marginal excess burden for tax \( d \) with variable producer prices, as:

\[ mb_d = \frac{-\sum t \frac{\partial x_i}{\partial t_d}}{x_d - \tilde{x}_d + \sum t \frac{\partial x_i}{\partial t_d}}. \]

Following the approach in sections 2 and 3 above, this can be decomposed, as:

\[ mb_d = \frac{1 + \sum t \left( \frac{\partial x_i}{\partial t_d} \right)}{1 - \theta} mb_d, \]

with: \( mb_d = \frac{-\sum t \sum s \frac{\partial x_i}{\partial z s}}{x_d - \tilde{x}_d - \sum z \frac{\partial x_i}{\partial z s}} \) and \( \theta = \sum t \sum s \frac{\partial x_i}{\partial p_s} + \sum t \frac{\partial x_i}{\partial d}. \)

The revised shadow value of government revenue is obtained from (8), as:

\[ (W_R)_D = W_R - mb_d \frac{\partial L}{\partial R}. \]

This is decomposed by solving the revenue transfers using (7), where:

\[ (W_R)_D = \frac{1 + \sum t \left( \frac{\partial x_i}{\partial t_d} \right)}{1 - \theta} \]

After substitution we have: \( mb_d = (W_R)_D mb_d. \)

---

35 This is obtained using the price-tax relationships, where:

\[ \frac{\partial q_d}{\partial d} = \frac{\partial p_d}{\partial d} + 1 \quad \text{and} \quad \frac{\partial q_d}{\partial d} = \frac{\partial p_d}{\partial d} \]

for \( s \neq d \).

36 From (7) we have: \( \frac{\partial L}{\partial R} = -W_k + \sum s \frac{\partial p_s}{\partial R} \), with price changes: \( \frac{\partial p_s}{\partial R} = \frac{\partial q_s}{\partial R} = -a_d W_k \).
References


