Unusual properties - mathematical and physical - of the *a*-boundary construction.

Ingrid Irmer

THE AUSTRALIAN
NATIONAL UNIVERSITY

A thesis submitted for the degree of
Honours in Theoretical Physics at
The Department of Physics and Theoretical Physics
The Faculty of Science
The Australian National University

Honours Thesis submitted June, 2003

Declaration

This thesis is an account of research undertaken in 2002-2003 at the Department of Physics and Theoretical Physics, the Australian National University, Canberra, Australia. Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Ingrid Irmer
June, 2003

Acknowledgements

I would like to thank my supervisor, Dr Susan Scott, for suggesting a project that has turned out to be so full of surprises. Thanks also for four years of mentoring, defending the rights of all "serious mathematical physicists", and for convincing me, in the end, that the universe is not really illogical, just more subtle than I had thought.
Thanks are also due to Mike Ashley for getting me started on the abstract boundary, and for supplying me with Figures 1.1, 1.2, 1.3, 29 and 3.1, originally out of [24]. The cartoon on this page was very kindly drawn for me by Aska Dolinska.
Writing this thesis has necessitated a certain amount of interaction with computers. Many people have helped to reduce the intrinsic painfullness of this process. In particular, thanks to the "gravity guys" : Andrew, Antony, Benedict, Geoff and Mike, and also to Phil, Simon and anyone else who came to the rescue on hearing my distressed bleatings. Especial thanks to Antony, for spending several days resuscitating the computer after I "broke" it.

Abstract

This thesis is written within the framework of the abstract boundary (or a-boundary) of Scott and Szekeres, [24]. The a-boundary provides a concept of "boundary" for any n-dimensional, paracompact, connected, Hausdorff manifold, defined in such a way that the boundary is independant of the particular embedding used to display the manifold. This makes it possible to define various types of boundary points of space-time such as "singularities" and "points at infinity".
The original research that will be presented in this thesis can be roughly divided up into two categories; results relating to the existence of optimal embeddings of solutions to Einstein's Field Equations and a-boundary singularity theorems. In addition, the implications of the "finite connected neighbourhood region property" and the bounded "acceleration" property are explored. It is also shown that not all space-times are maximally extendable.

Contents

Chapter 1

Introduction

General Relativity describes structures on a 4-dimensional, differentiable manifold with a Lorentzian metric, i.e. a space-time. In general terms, singularities are boundary sets of the manifold on which this structure breaks down. In order to describe them, it is necessary to have some method of connecting boundary points to the space-time. This is known as a boundary construction. It has become clear, largely due to a collection of results known as "The Singularity Theorems" of Hawking and Penrose, that "singularities" occur in a wide variety of space-times, and are an inescapable feature of our universe. Surprisingly, these conclusions were reached without a generally accepted definition of a singularity. A singularity was thought to be "a point where something goes wrong" or "something to do with geodesic incompleteness". Unlike in field theories such as Electromagnetism, there is no background metric space in which to describe singular behaviour of the metric. As a result, there is a greater subtlety and diversity of singularities in General Relativity. Before the a-boundary was developed by Scott and Szekeres, various boundary constructions were devised, each of which captured some aspect of singular behaviour. However, none of these boundary constructions could be considered to be the final solution because of various associated problems. Schmidt's b-boundary and Geroch's g-boundary appear most often in the literature, and will be described briefly here. For a more detailed description, see [12], [13], [15] and [21].

A central concept in the study of singularities is that of an incomplete curve.

Definition 1 (Complete and incomplete curves)
A *complete curve* is a curve which can be extended in both directions for arbitrarily large values of a specified parameter. A curve which is not complete is *incomplete.*

The b and g-boundaries are based on the idea that an incomplete curve (the class of curves varies from one boundary construction to another) is an indication of a singularity sitting at the end of it. The singularity theorems, for example, show the existence of incomplete curves. For various reasons, it seems to have been generally assumed that in a "physical" space-time, an incomplete geodesic gives rise to some kind of curvature singularity. A proof of a result along these lines is a topic of ongoing research.

The g-boundary associates a boundary point with every geodesic which is incomplete with respect to its affine parameter. A set of neighbourhoods of each geodesic is defined. To ensure that each boundary point is only attached once, if a second geodesic enters and stays within every element of this set of neighbourhoods of the first geodesic, the associated boundary points are identified.

The b-boundary is a generalization of this idea. It involves forming the bundle of orthonormal frames $\mathcal{L}(\mathcal{M})$ over the manifold $\mathcal{M}$. Any curve in $\mathcal{L}(\mathcal{M})$ can be projected down to a curve in $\mathcal{M}$ and every C^1 curve in $\mathcal{M}$ can be lifted to a curve in $\mathcal{L}(\mathcal{M})$, unique modulo the action of the Lorentz group $Gl(n,R)$ on the fibre. The horizontal vector fields $\mathbf{B}_i$, $i = 1,..,n$, are defined by $\pi_*(\mathbf{B}_i) = \mathbf{X}_i$, where π is the projection map, and $\{\mathbf{X}_i\}$ forms an ordered basis of $T_x(\mathcal{M})$. Horizontal vector fields are identified with vector fields tangent to $\mathcal{M}$. Vertical vector fields are tangent to the fibre, and orthogonal to the horizontal vector fields in the metric that will be defined. The vertical vector fields are isomorphic to the Lie algebra $gl(n,R)$ corresponding to $Gl(n,R)$. (Since the fibre is isomorphic to $Gl(n,R)$, this is just saying that their tangents are also isomorphic.) Let $\{\tilde{\mathbf{E}}^i_k\}$ be the basis of vertical vector fields corresponding to the basis of $gl(n,R)$, $\{\mathbf{E}_k{}^i\}$. Let $\omega_k{}^i$ be dual to $\{\tilde{\mathbf{E}}^i_k\}$, and θ^k be dual to $\mathbf{B}^k$. Then

$$g(\mathbf{X},\mathbf{Y}) = \sum_i \theta^i(\mathbf{X})\theta^i(\mathbf{Y}) + \sum_{i,k} \omega_k{}^i(\mathbf{X})\omega_k{}^i(\mathbf{Y}) \tag{1.1}$$

is a positive definite metric on $\mathcal{L}(\mathcal{M})$, and can be used to find the closure $\overline{\mathcal{L}(\mathcal{M})}$ of $\mathcal{L}(\mathcal{M})$. The action of $Gl(n,R)$ can be shown to be uniformly continuous on $\mathcal{L}(\mathcal{M})$ with respect to g, and can therefore be extended to $\overline{\mathcal{L}(\mathcal{M})}$. Let π be the projection of $\overline{\mathcal{L}(\mathcal{M})}$ onto $\overline{\mathcal{M}}$, i.e., $\overline{\mathcal{M}} = \frac{\overline{\mathcal{L}(\mathcal{M})}}{Gl(n,R)}$. The b-boundary of $\mathcal{M}$ is defined as the set $\dot{\mathcal{M}} = \overline{\mathcal{M}} - \mathcal{M}$. Despite the fact that g depends on the choice of basis and is used in the definition of $\dot{\mathcal{M}}$, the properties of g used in the definitions are unaffected by a uniformly bounded transformation of the metric that results from a different choice of basis.

Since $\dot{\mathcal{M}}$ contains all the limit points of Cauchy sequences in $\mathcal{M}$, in particular, it contains all the endpoints of incomplete geodesics. However, as Geroch pointed out [13], an observer could also travel along curves of bounded acceleration, and so a definition of singularities should perhaps focus on a more general class of curves than geodesics. The b-boundary provides endpoints for all C^1 curves which are incomplete with respect to their path length as measured by g. This path length is called a generalized affine parameter because it agrees with a choice of affine parameter when restricted to geodesics.

Definition 2 (*b*-boundary singularity)
A *b-boundary singularity* is a point in the b-boundary $\dot{\mathcal{M}}$ which is contained in the b-boundary for every extension of the space-time.

A point which seems to have been ignored in the (pre a-boundary) boundary constructions is that there is a certain degree of freedom in describing the boundary of a space-time. The following definitions are out of [24] or [1].

Definition 3 (Enveloped manifold)

An *enveloped manifold* is a triple $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ where $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are differentiable manifolds of the same dimension and ϕ is a C^∞ embedding $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$. The enveloped manifold is also called an envelopment, where $\widehat{\mathcal{M}}$ is the enveloping manifold.

One of the major difficulties in proving results in pseudo-Riemannian geometry is that, in general, there is no positive definite metric related to the metric g in some way, which can be used to define convergence properties and lengths of curves. For example, the b-boundary metric is a metric on the tangent bundle, and does not give rise to a metric on the base space $\mathcal{M}$ unless $\mathcal{M}$ is flat. It will be assumed throughout this thesis that all manifolds are paracompact. An auxilliary metric h is a positive definite metric defined on $\widehat{\mathcal{M}}$ whose existence is gauranteed by the assumption of paracompactness. Since an auxilliary metric is regular (in particular, its components are bounded in every valid coordinate system) everywhere on $\widehat{\mathcal{M}}$, and since $\phi(\mathcal{M}) \cup \partial_\phi \mathcal{M}$ is contained in $\widehat{\mathcal{M}}$, then all auxilliary metrics on $\phi(\mathcal{M})$ give the same results when discussing local properties such as convergence.

Definition 4 (Extension)
An *extension* of a pseudo-Riemannian manifold $(\mathcal{M},g)$ is an envelopment of it by a pseudo-Riemannian manifold $(\widehat{\mathcal{M}}, \hat{g})$ such that $\hat{g}|_{\phi(\mathcal{M})} = g$.

While preserving the geometry of a space-time, it is possible to display the boundary points in very different ways by choosing different envelopments of the manifold. For example, an a-boundary point could be identified with other boundary points, blown up into a bubble, line segment or disconnected set. A well-known example is the singularity of Schwarzschild solution. In the usual spherical coordinates, the singularity at any given time is a point, however in the Penrose diagram shown in Figure 1.1 it is represented by the dark black line (a four dimensional object).

In particular, since singularities occur on the boundary of space-time, there is often no real sense in which a singularity can be represented by a point as opposed to some particular wierd set. Also, due to the fact that whether or not two boundary points are identified or even “nearby” is partly a matter of choice, difficulties occur when singularities are thought of as an inherently local aspect of a space-time, as an analogy with electromagnetism would imply. This is probably the most significant problem faced by earlier boundary constructions.

Whether or not a space-time contains singularities depends on the choice of curves used to define them. In the a-boundary scheme, this choice is made to suit the aim of the investigation, and needs to be decided on before any conclusions can be drawn. The only restriction is that the family of curves have a parameter satisfying the bounded parameter property.

Definition 5 (bounded parameter property (b.p.p.))
A family $\mathcal{C}$ of parametrized curves in $\mathcal{M}$ satisfies the b.p.p. if:

1. for any point $p \in \mathcal{M}$ there is at least one curve of the family passing through p

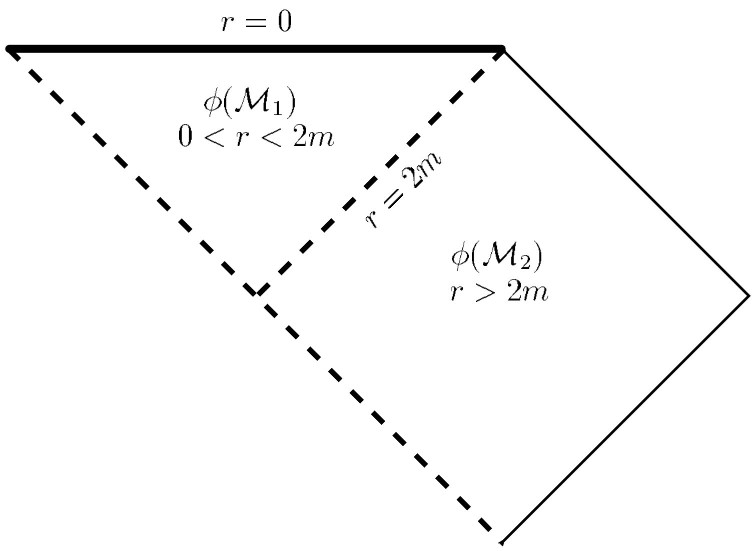


Figure 1.1: Penrose Diagram of the Schwarzschild Solution. The θ and ϕ coordinates are suppressed

2. if $\gamma(t)$ is a curve of the family then so is any connected subset of it

3. if γ and γ' are in $\mathcal{C}$ and γ' is obtained from γ by a change of parameter then either the parameter is bounded or unbounded on both curves.

Curves satisfying the b.p.p. are a generalization of geodesics with affine parameter.

In order to define the a-boundary, it is necessary to make precise what is meant by two boundary sets representing the same abstract set in different envelopments. This is done by an equivalence relation.

Definition 6 (Covering relation)
If B is a boundary set of $\phi(\mathcal{M})$ and B' is a boundary set of $\phi'(\mathcal{M})$ then B *covers* B' (denoted $B \rhd B'$) if for every open neighbourhood $\mathcal{U}$ of B in $\widehat{\mathcal{M}}$ there exists an open neighbourhood $\mathcal{U}'$ of B' in $\widehat{\mathcal{M}}'$ such that

$$\phi \circ {\phi'}^{-1}(\mathcal{U}' \cap \phi'(\mathcal{M})) \subset \mathcal{U}. \tag{1.2}$$

Two boundary sets B and B' are equivalent if they cover each other. This defines an equivalence relation. An equivalence class is denoted by a representative element with a square bracket around it, for example $[B]$. In the a-boundary formulation, a boundary set is an equivalence class, and an a-boundary point is an equivalence class with a point on the boundary of some envelopment as a representative element.

Definition 7 (Abstract boundary $B(\mathcal{M})$)
$B(\mathcal{M}) := \{[p] \mid p \in \partial_\phi(\mathcal{M})$ for some envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)\}$

Points of the a-boundary are then divided into various categories.

Definition 8 (Regular point)
A boundary point p of an envelopment $(\mathcal{M},\ g,\ \widehat{\mathcal{M}},\ \phi)$ is *regular* if there exists a manifold $(\overline{\mathcal{M}}, \overline{g})$ such that $\phi(\mathcal{M}) \cup \{p\} \subseteq \overline{\mathcal{M}} \subseteq \widehat{\mathcal{M}}$ and $(\mathcal{M}, g, \overline{\mathcal{M}}, \overline{g}, \phi)$ is an extension of $(\mathcal{M},\ g)$.

Regularity does not "pass to the a-boundary" (i.e. it is not invariant under the equivalence relation used to define the a-boundary). Several examples which illustrate this can be found in Chapter Three or [24]. A regular a-boundary point is defined as follows:

Definition 9 (Regular *a*-boundary point)
A regular a-boundary point is an equivalence class with a regular point as a representative element.

A maximally extended space-time is essentially one whose a-boundary does not contain any regular a-boundary points. It would seem that a non-maximally extendable space-time would have to be very artificial. Chapter Four contains an example of a space-time that cannot be maximally extended.

Definition 10 (Maximally extended)
A C^k pseudo-Riemannian manifold $(\mathcal{M},\ g)$ is termed C^l maximally extended $(1 \leqslant l \leqslant k)$ if there does not exist a C^l extension $(\mathcal{M},\ g, \widehat{\mathcal{M}}, \widehat{g}, \phi)$ of $(\mathcal{M},\ g)$ such that $\phi(\mathcal{M})$ is a proper open submanifold of $\widehat{\mathcal{M}}$

In the a-boundary, irregularity replaces the concept of curve incompleteness in the other boundary constructions. The two concepts are very closely related, however. Scott and Ashley [1] have proven a result linking curve incompleteness with the existence of a-boundary essential singularities, under various conditions. Results like these demonstrate that the rule of thumb that "an incomplete curve is an indication of singular behaviour", which occurs throughout the literature, gives a result in agreement with the a-boundary under very general conditions. As is usual in proofs of this sort, one of the so-called "causality conditions" was assumed, the most common of which is probably the Strong Causality condition.

Definition 11 (Strongly causal)
A space-time $(\mathcal{M}$,g) is *strongly causal* at p if every neighbourhood of p contains a neighbourhood of p which no non-spacelike curve intersects in a disconnected set.

If $\mathcal{C}$ is chosen to be the set of affinely parametrised causal geodesics, strong causality is a useful condition, because it rules out the possibility that curves in $\mathcal{C}$ could enter a compact set and be "imprisoned" within it and have more than one limit point. The Misner example [15] & [18] is a typical example of this sort of behaviour. It is a two dimensional manifold with metric

$$ds^2 = 2dtd\psi + td\psi^2, \text{where t} \in \mathbb{R}, 0 \leqq \psi < 2\pi. \tag{1.3}$$

The curve $t = 0$ (which is generally referred to as the "waist") is a null geodesic, and is approached by incomplete geodesics which become imprisoned in a compact set and spiral around infinitely many times, as shown in Figure 1.2. These incomplete, infinitely spiralling geodesics have every point on the waist as a limit point. The Misner example does not contain any a-boundary singularities despite the fact that it contains incomplete curves which cannot be extended. It is necessary to rule out the sort of behaviour that occurs in this example in order to prove many different results. The Misner example is a simplification of "Taub-NUT" space, which was published in a paper entitled "Taub-NUT spaces as a counterexample to almost everything" [18], and the extent to which it is a near universal counterexample is incredible. Although causality conditions have traditionally been used to rule out this sort of pathological behaviour, there are other alternatives. When proving results relating the existence of incomplete curves to a-boundary essential singularities, it turns out to be important to know under what conditions an incomplete curve can have a regular boundary point as a limit point as opposed to an endpoint.

Definition 12 (Limit point of a curve)
We say that $p \in \phi(\mathcal{M}) \cup \partial_\phi \mathcal{M}$ is a *limit point* of a curve $\gamma : [a, b) \to \phi(\mathcal{M})$ if there exists an increasing infinite sequence of real numbers $t_i \to b$ such that $\gamma(t_i) \to p$.

Definition 13 (Endpoint of a curve)
We say that p is an *endpoint* of the curve γ if $\gamma(t) \to p$ as $t \to b$.

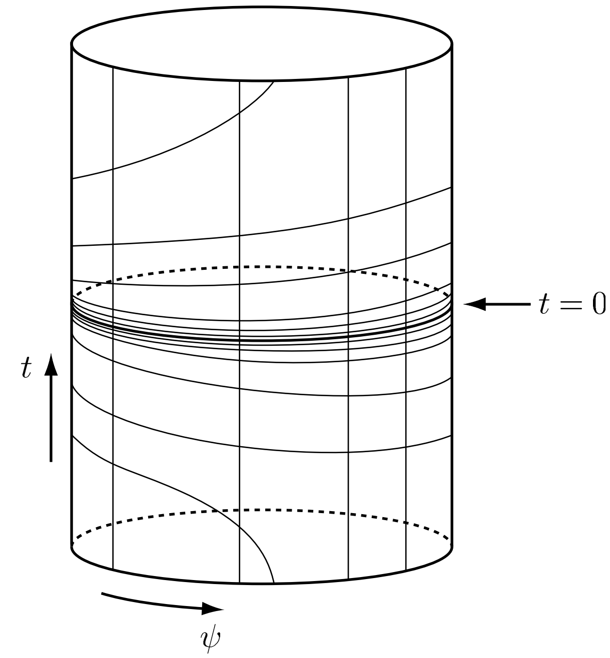
Definition 14 (Approachable boundary point)
A parametrised curve $\gamma : I \to \mathcal{M}$ *approaches* the boundary set B if the curve $\phi \circ \gamma$ has a limit point lying in B. A point $p \in \partial_\phi \mathcal{M}$ is *approachable* if it is approached by a curve from the family $\mathcal{C}$.

Chapter Two contains a proof of a result that rules out the possibility of a regular boundary point being a limit point (as opposed to an endpoint) of an incomplete curve. This result further clarifies the connection between incomplete geodesics and a-boundary singularities.

Irregular boundary points consist of singularities, points at infinity and irregular unapproachable boundary points.

Definition 15 (Point at infinity)
A boundary point p of the envelopment $(\mathcal{M}, g, \widehat{\mathcal{M}}, \mathcal{C}, \phi)$ is a *point at infinity* if

1. p is not a regular boundary point
2. p is approachable by an element of $\mathcal{C}$, and
3. no curve of $\mathcal{C}$ approaches p with bounded parameter.



$$ds^2 = 2dtd\psi + td\psi^2,\ t \in \mathbb{R},\ \psi \in [0, 2\pi)$$

Figure 1.2: Misner Example

Definition 16 (Removable point at infinity)
A boundary point p at infinity is termed a *removable point at infinity* if there is a boundary set, $B \subset \partial_\phi \mathcal{M}$ composed purely of regular boundary points such that $B \rhd p$

Definition 17 (Essential point at infinity)
A point p at infinity is an *essential point at infinity* if it is not removable.

Definition 18 (Mixed point at infinity)
An essential point p at infinity is a *mixed point at infinity* if it covers a regular boundary point.

Definition 19 (Pure point at infinity)
An essential point at infinity is a *pure point at infinity* if it does not cover any regular boundary points.

Definition 20 (Singular boundary points)
A boundary point p of an envelopment $(\mathcal{M}, g, \widehat{\mathcal{M}}, \mathcal{C}, \phi)$ is called *singular* or a *singularity* if

1. p is not a regular boundary point,

2. p is approachable by a curve γ, where γ is an element of $\mathcal{C}$ and has finite parameter.

Definition 21 (Removable singularity)

A singular boundary point p will be called *removable* if it can be covered by a non-singular boundary set B of another embedding.

Definition 22 (Essential singularity)
A singular boundary point p is called *essential* if it is not removable.

Definition 23 (Directional and pure singularities)
An essential singularity p is called a *directional singularity* if it covers a boundary point of another embedding which is either regular or a point at infinity. Otherwise p is called a *pure singularity.*

The definitions of directional and pure singularities pass to the a-boundary, as shown in [24].

The a-boundaryclassification scheme is summarised in Figure 1.3.

Fama and Scott [11] have investigated topological invariants of boundary sets. In particular, they have shown that properties such as compactness and isolation pass to the a-boundary. Local topological properties (according to the following definitions) were also shown to be properties of a-boundary equivalence classes.

Definition 24 (T-niceness)
Let B be a boundary set of an envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ and let T be a topological property. An open neighbourhood of B, $\mathcal{U}(B) \subset \widehat{\mathcal{M}}$, will be called T-nice if $T(\mathcal{U}(B) \cap \phi(\mathcal{M}))$ is true in the relative topology of $\mathcal{U}(B) \cap \phi(\mathcal{M})$.

Definition 25 (Topological neighbourhood property (TNP))
A boundary set B satisfies the *topological neighbourhood property* (TNP) if every open neighbourhood, $\mathcal{U}(B)$, in $\widehat{\mathcal{M}}$, contains a T-nice, open neighbourhood, $\mathcal{V}(B) \subset \widehat{\mathcal{M}}$.

Theorem 26 (Theorem 4.3 of [11])
Let T be a topological property. If B satisfies the TNP and $B \sim B'$ then B' also satisfies the TNP.

It is also illustrative to know what types of a-boundary points can cover other types. A table appears in [24] with this information, and is reproduced in Chapter Three. However, a couple of the entries had not been verified since no explicit example had been found. Chapter Three provides examples of all the possibilities of one type of boundary point covering another type. In the process, an algorithm was developed for re-enveloping a manifold such that given directions of approach to a boundary point are separated. This procedure turns out to be very useful later on in Chapter Four when discussing regular boundary points in optimal embeddings.

It will become clear in Chapter Four that regularity is not a very stable property under re-envelopment. Since regular boundary points are effectively interior points of a larger space-time, a regular point p has many properties in common with interior

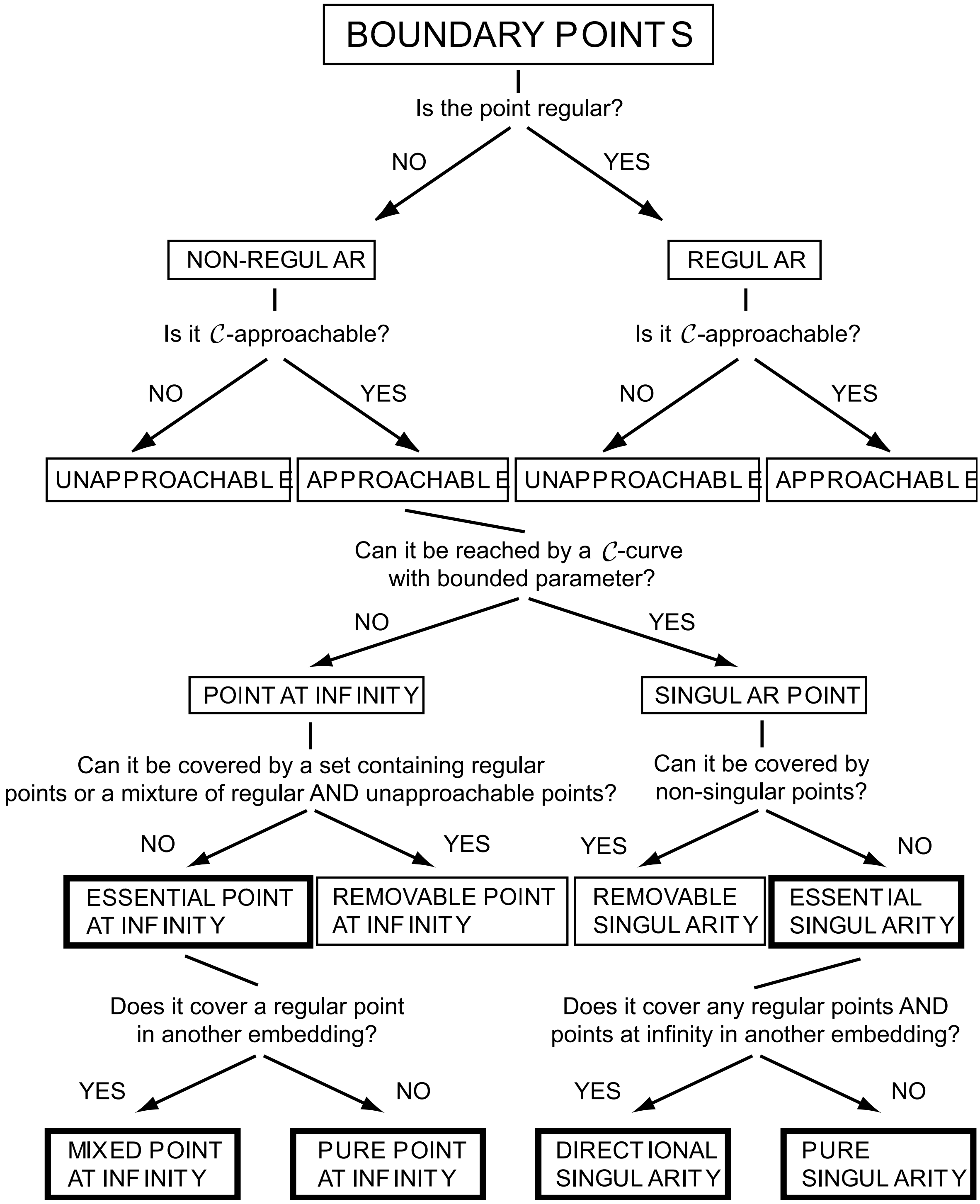


Figure 1.3: Summary of the a-boundary point classification scheme. Categories that pass to the a-boundary are in bold.

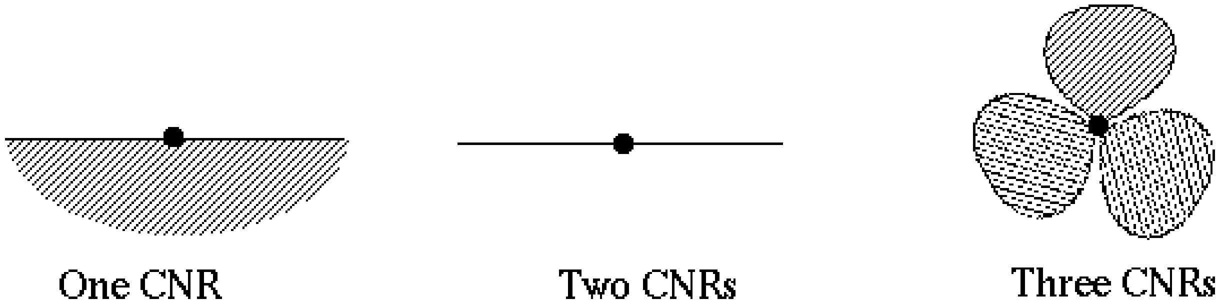


Figure 1.4: In this diagram, $\phi(\mathcal{M})$is represented by the unshaded region

points, for example, the metric, curvature tensor and Christoffel symbols are all well behaved near p, the existence of a neighbourhood of p in $\widehat{\mathcal{M}}$ whose intersection with the boundary contains only regular points and the fact that every vector in the tangent space $T_p\widehat{\mathcal{M}}$ pointing into $\phi(\mathcal{M})$ generates a unique geodesic.

A condition that was developed in [1] and which has turned out to be extremely useful in proving theorems about regular a-boundary points is the finite connected neighbourhood region property.

Definition 27 (Connected Neighbourhood Region (CNR))
Suppose $p \in \partial_\phi \mathcal{M}$ and $\mathcal{N}$ is a neighbourhood of p in $\widehat{\mathcal{M}}$. Then a connected component of $\mathcal{N} \cap \phi(\mathcal{M})$ is called a *connected neighbourhood region of* p.

Definition 28 (The finite connected neighbourhood region property (FCNR property))
We say that p has n *connected neighbourhood regions* if for any open neighbourhood $\mathcal{N}(p)$ there exists a sub-neighbourhood $\mathcal{U}(p) \subset \mathcal{N}(p)$ for which $\mathcal{U}(p) \cap \phi(\mathcal{M})$ is composed of exactly n connected components, and n is the smallest natural number for which this is true. The boundary point p satisfies the *finite connected neighbourhood region property* if it has only finitely many connected neighbourhood regions.

It is a consequence of Theorem 4.3 of [11] that the FCNR property passes to the a-boundary. The following is an example of a point that does not satisfy the FCNR property.

Example 29 (Example 13 of [24])
Let $\mathcal{M}$ be the open submanifold of $\mathbb{R}^2$ defined by $\{(x, y)|0 < x \text{ and } y < \sin(\frac{1}{x})\}$. None of the points in the set $B := \{(0, y)| -1 < y \leq 1\}$ satisfy the FCNR property.

Some of the consequences of the finite connected neighbourhood region property are investigated in section 4.1.

For many applications it is important to know under what conditions regularity is preserved under re-envelopment. In particular, it seems intuitively clear that that if a regular point is re-enveloped so as to be no longer pointlike, then

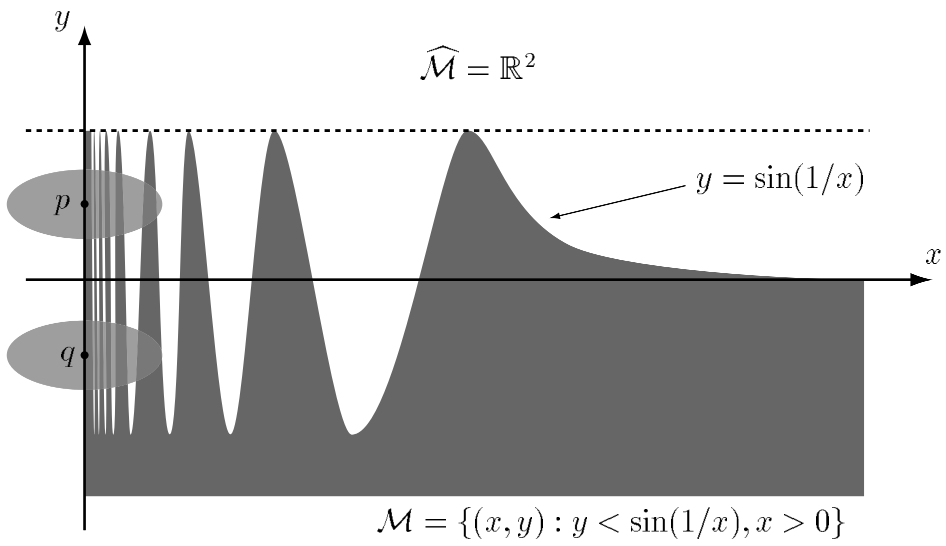


Figure 1.5: The boundary points p and q do not satisfy the FCNR property

this re-envelopment destroys the regularity of that point. This is actually not true without several assumptions. A more general version of this question and its relevance to the search for optimal embeddings will be dealt with in Chapter Four.

Given that the boundary of a manifold can be presented in different ways, depending on the choice of envelopment, some envelopments are more suited to analysing various properties of the manifold. The Schwarzschild solution is generally used to illustrate the properties of an optimal embedding. In the spherical coordinates used to reflect the spherical symmetry of the solution, there is an artificial barrier at $r = 2m$. The solution can be extended past $r = 2m$ by using Kruskal-Szekeres coordinates for $r > 2m$, which can then be extended past $r = 2m$ and are well behaved everywhere but at the singularity. Points at infinity are analysed by compacting the space-time. All of the regular points are displayed (i.e. there are no removable singularities or regular points mixed up in essential singularities) and therefore it is clear how the manifold can be extended. The result is represented in 1.1. A precise definition of what constitutes an optimal embedding can be given in terms of cross sections. A cross section generalizes the concept of an embedding as a means of portraying the boundary of a manifold. Some definitions are required at this point. For a discussion of why some of these particular variants of the definitions were chosen, see [1].

Definition 30 (p is in contact with q)
Let $p \in \partial_\phi \mathcal{M} \subset \widehat{\mathcal{M}}$ and $q \in \partial_{\phi'} \mathcal{M} \subset \widehat{\mathcal{M}}'$. Then p and q are said to be *in contact* (denoted $p \dashv q$) if for all neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of p and q respectively

$$\phi^{-1}(\mathcal{U} \cap \phi(\mathcal{M})) \cap \phi'^{-1}(\mathcal{V} \cap \phi'(\mathcal{M})) \neq \emptyset \tag{1.4}$$

There is an equivalent definition of the contact relation in terms of sequences.

Definition 31 (In contact, sequence definition)
Two boundary points $p \in \partial_\phi \mathcal{M} \subset \widehat{\mathcal{M}}$ and $q \in \partial_{\phi'} \mathcal{M} \subset \widehat{\mathcal{M}}'$ are *in contact* if there exists a sequence $\{p_i\} \subset \mathcal{M}$ such that $\{\phi(p_i)\}$ has p as an endpoint and $\{\phi'(p_i)\}$ has q as an endpoint.

By theorem 19 of [24], the contact properties of a-boundary points depend only on the equivalence classes from which the two points come, hence

Definition 32 ($[p] \dashv [q]$)
Let $[p]$ and $[q]$ be a-boundary points. We say that $[p]$ and $[q]$ are *in contact*, denoted $[p] \dashv [q]$ if $p \dashv q$ for representatives p and q.

Definition 33 (Separation of boundary points)
Two boundary points, $p \in \partial_\phi \mathcal{M} \subset \widehat{\mathcal{M}}$ and $q \in \partial_{\phi'} \mathcal{M} \subset \widehat{\mathcal{M}}'$, are termed *separate* (denoted by $p \parallel q$) if they are not in contact. Similarly for a-boundary equivalence classes.

Definition 34 (Partial Cross Section σ)
Let $\sigma \subset \mathcal{B}(\mathcal{M})$. σ is termed a *partial cross section* if for every $[p]$, $[q] \in \sigma$, $[p] \parallel [q]$ or $[p] = [q]$.

Since $\widehat{\mathcal{M}}$ is Hausdorff, any two distinct points $p \in \partial_\phi \mathcal{M}$ and $q \in \partial_\phi \mathcal{M}$ are separate, and therefore $[p] \parallel [q]$. Therefore an envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ defines a partial cross section

$$\sigma_\phi := \{[p] | p \in \partial_\phi \mathcal{M}\}. \tag{1.5}$$

Definition 35 (Regular Partial Cross Section)
Let σ_r be a set of regular a-boundary points. σ_r will be said to be a *regular partial cross section* if for each $[p] \in \sigma_r$ there is a representative boundary point $q \in [p]$, $q \in \partial_\phi \mathcal{M}$ such that

1. q is regular and,
2. there is an open neighbourhood, $\mathcal{U}(q) \subset \widehat{\mathcal{M}}$ such that

$$\{[r] | r \in \mathcal{U} \cap \partial_\phi \mathcal{M}\} \subset \sigma_r \tag{1.6}$$

In a given envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$, regular boundary points only occur in open sets of the topology induced on $\partial_\phi \mathcal{M}$. The second part of the definition ensures that a regular partial cross section locally resembles the boundary of an extension. A partial cross section consisting only of regular a-boundary points which does not satisfy condition 2 above is not much use when investigating how the manifold can be extended. The next example can be used to illustrate several properties relating to partial cross sections and is particularly relevant to the discussion of maximal regular partial cross sections in Chapter Four.

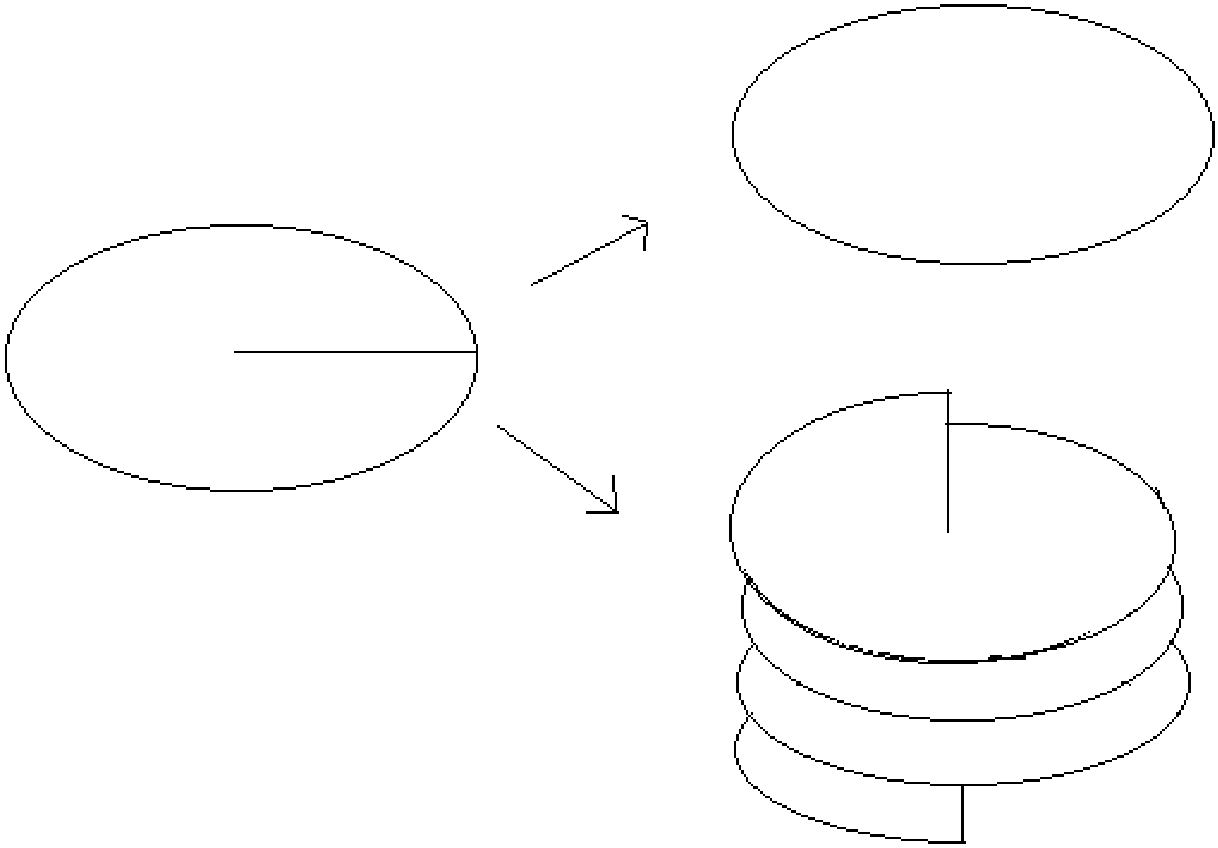


Figure 1.6: Two Envelopments of $\mathcal{M}$

Example 36 (A partial cross section consisting of only regular a-boundary points which can't be augmented to satisfy condition (2))
Consider the manifold

$$0 < r < \infty \ , \ 0 < \theta < 2\pi \text{ and metric } ds^2 = dr^2 + r^2 d\theta^2 \tag{1.7}$$

The envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ is the envelopment with $\phi =$ identity and $\widehat{\mathcal{M}}$ formed by including the line $\theta = 0$ or 2π. In this envelopment, sufficiently small neighbourhoods of the boundary points with $r \neq 0$ all contain a neighbourhood in $\widehat{\mathcal{M}}$ whose intersection with $\phi(\mathcal{M})$ has two or more connected components. A second envelopment $(\mathcal{M}, \widehat{\mathcal{M}}', \phi')$ embeds $\mathcal{M}$ in a spiral. One side of the edge $\theta = 0$ or 2π is identified with the opposite side of the edge of a copy of $\mathcal{M}$. The other side is identified with the opposite side of the edge of a third copy of $\mathcal{M}$, as shown in Figure 1.6.

Every $p \in \partial_{\phi'}\mathcal{M}$ has arbitrarily small neighbourhoods in $\widehat{\mathcal{M}}'$ whose intersection with $\phi'(\mathcal{M})$ has only one connected component. Every point $q \in \partial_{\phi}\mathcal{M}$ except the origin is equivalent to two boundary points in the other envelopment.

The partial cross section σ where

$$\sigma := \{p | p \in \partial_{\phi}\mathcal{M} \text{ and the r coordinate of } p \neq 1 \text{ or } p \in \partial_{\phi'}\mathcal{M} \text{ with } r = 1\} \tag{1.8}$$

has the property that the point $p \in \partial_{\phi'}\mathcal{M}$ with $r = 1$ does not satisfy the second condition of regular partial cross sections.

Definition 37 (Total Regular Partial Cross Section)
A regular partial cross section σ is termed *total* if it covers every regular abstract boundary point.

The next definition uses the concept of a maximal regular partial cross section. There is some flexibility in the definition of "maximal", and an alternative definition will be used in Chapter Three. The definition given here is out of [1].

Definition 38 (Maximal Regular Partial Cross Section)
A regular partial cross section will be termed *maximal* if no regular partial cross section contains it as a proper subset.

Definition 39 (Admissable Partial Cross Section)
σ is an *admissable partial cross section* if

1. there is a maximal regular partial cross section contained in σ
2. for each $[p] \in \sigma$ there is a representative boundary point $q \in [p]$, $q \in \partial_\phi \mathcal{M} \subset \widehat{\mathcal{M}}$, and an open neighbourhood, $\mathcal{U}(q) \subset \widehat{\mathcal{M}}$ such that

$$\{[r] | r \in \mathcal{U} \cap \partial_\phi \mathcal{M}\} \subset \sigma \tag{1.9}$$

Definition 40 (Maximal Admissable Partial Cross Section)
An admissable partial cross section is said to be maximal if no admissable partial cross section contains it as a proper subset.

As discussed in [1], an optimal embedding is not a maximally extended version of the manifold. The extensions are not generally unique, and so to maximally extend a manifold it is necessary to favour a particular extension above other equally plausible extensions. The definition of an optimal embedding is designed such that if $(\phi, \widehat{\mathcal{M}})$ is an optimal embedding then all regular extension hypersurfaces are explicitly displayed.

Definition 41 (Optimal Embedding (minimal definition))
Let $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$ be an envelopment. To be an *optimal embedding* at the very least ϕ has to satisfy the condition that the partial cross section

$$\sigma_\phi = \{[p] | p \in \partial_\phi \mathcal{M}\} \tag{1.10}$$

is a maximal admissable partial cross section.

This definition ensures that the regular points are not tangled up in irregular boundary sets. An optimal embedding maximises regularity in the sense that the partial cross section associated with it contains a maximal regular partial cross section. Another property that would be very desirable is totality.

Definition 42 (Totality)
A partial cross section σ is total if for every a-boundary point $[p]$ there is a set of points $[B] \in \sigma$ such that $[B] \triangleright [p]$. An envelopment is total if its corresponding partial cross section is total.

If an envelopment is not total it lacks information about the boundary. It could be the case that some feature of interest, for example a singularity, has been sent off to infinity. It was concluded in [3] that for most space-times it is possible (although in general not easy) to find a total envelopment. It would be very interesting to know under what conditions it is possible to find a total optimal embedding. In Chapter Four progress has been made towards a related problem, namely under what conditions a maximal regular partial cross section can be found that covers every regular a-boundary point. Unfortunately, the existence of certain counterexamples indicate that results along these lines can't be as general as one might hope.

Chapter 2

a-boundary Singularity Theorems

In order to be able to translate the singularity theorems of Hawking, Penrose *et. al.* into results for the a-boundary, it is necessary to relate the existence of incomplete geodesics with essential a-boundary singularities. This was done in Chapter Four of [1], and the following theorem was the result.

Theorem 43 (Theorem 4.12 of [1])
Let $\mathcal{M}$ be a strongly causal, C^l maximally extended, C^k space-time ($1 \leq l \leq k$) and C be the family of affinely parametrised causal geodesics in ($\mathcal{M}$, g). Then the a-boundary contains a C^l essential singularity iff there is an incomplete causal geodesic in ($\mathcal{M}$, g).

The proof of this theorem only used strong causality to rule out the possibility that an incomplete causal geodesic could have two or more distinct limit points in $\mathcal{M}$. It could therefore be replaced with any other condition that prevents this. As pointed out in [1], Theorem 8.5.2 of [15] shows that a certain restriction placed on the Ricci tensor will do the same job. The proof is included here with a couple of details that weren't present in the original proof. Theorem 46 investigates conditions under which a regular boundary point can only be the endpoint (as opposed to a limit point) of an incomplete geodesic.

The next two theorems both use the following lemma.

Lemma 44
If $\lambda \in \phi(\mathcal{M})$ is a C^1 curve with limit point but not endpoint p, the part of λ contained within any neighbourhood $\mathcal{N} \subset \widehat{\mathcal{M}}$ of p will have infinite arc-length parameter t with respect to an auxilliary Riemannian metric, h.

Proof. Let $\mathcal{U}_n$ be the set of all points which are a distance (with respect to h) less than $\frac{1}{n}$ from p.
Suppose there is no n^* such that for all $n \geq n^*$, λ enters and leaves $\mathcal{U}_n$ an infinite number of times. Then there is a subsequence $\mathcal{U}_{n'}$ of $\mathcal{U}_n$, such that λ enters and leaves each $\mathcal{U}_{n'}$ only finitely many times. Let s^* be the largest value of λ's parameter s corresponding to an intersection of λ with the boundary of some given $\mathcal{U}_{n'}$. Then if $s^* < s, \lambda \subset \mathcal{U}_{n'}$. This implies that λ approaches p as an endpoint, since p is the

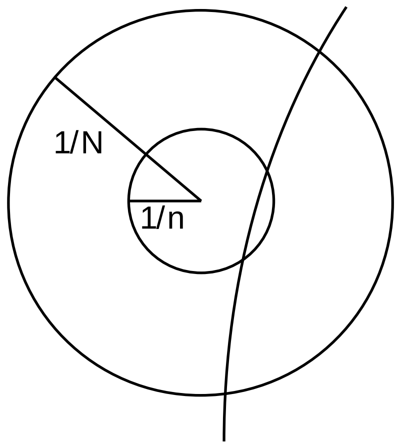


Figure 2.1:

only point in the intersection of all the $\mathcal{U}_{n'}$.
For $n \geq n^*$ denote by λ_n a connected component of $\lambda \cap \mathcal{U}_{n^*}$ which enters $\mathcal{U}_n$. For every $\mathcal{U}_n$, $n \geq n^*$, the infinite number of components of $\lambda \cap \mathcal{U}_n$ correspond to an infinite number of components of $\lambda \cap \mathcal{U}_{n^*}$, since if this were not so, there is a component of $\lambda \cap \mathcal{U}_{n^*}$ corresponding to all allowed values of the parameter greater than some fixed number, contradicting the assumption that λ is not trapped within $\mathcal{U}_{n^*}$. Therefore λ_n can always be chosen such that $\lambda_n \neq \lambda_m$, for $m < n$.
Suppose also that n^* is chosen to be large enough such that $\mathcal{U}_{n^*} \subset \mathcal{N}$. Then the arc-length of λ_n is greater than $2(1/N - 1/n)$, see figure 2.1. Therefore the arc-length of the part of λ contained in $\mathcal{N}$ is bounded below by:

$$\sum_{n=N+1}^{\infty} 2(1/N - 1/n) = \frac{2}{N} \sum_{n=N+1}^{\infty} (n - N)/n,$$

which diverges since $n - N \geq 1$.

$\square$

The reasoning used in the previous lemma can not be used to show that an incomplete geodesic can only approach a regular boundary point as an endpoint (as opposed to a limit point). This is because if a geodesic λ enters and leaves a neighbourhood $\mathcal{N}$, there is in general no way to compare the affine parameter of the disconnected segments of $\lambda \cap \mathcal{N}$.

An incomplete curve λ is said to correspond to a parallelly propagated curvature singularity (p.p. curvature singularity) if any of the components of the curvature tensor become unbounded in a parallelly propagated basis along λ.

Theorem 45 (Proposition 8.5.2 of Hawking and Ellis)
If $p \in \mathcal{M}$ is a limit point of a b-incomplete curve $\lambda(s)$ and if at p, $\mathbf{R}_{ab}\mathbf{K}^a\mathbf{K}^b \neq 0$ for all non-spacelike vectors $\mathbf{K}$, then λ corresponds to a p.p. curvature singularity

Proof. Let $\mathcal{U}$ be a convex normal coordinate neighbourhood of p with compact closure, and let $\{\mathbf{Y}_i\}$, $\{\mathbf{Y}^i\}$ be a field of dual orthonormal bases on $\mathcal{U}$. Let $\{\mathbf{E}_a\}$, $\{\mathbf{E}^a\}$

be a parallelly propagated dual orthonormal basis on the curve $\lambda(s)$. Let t be a parameter on λ such that in $\mathcal{U}$,

$$\frac{dt}{ds} = \left(\sum_i X^i X^i\right)^{\frac{1}{2}}, \tag{2.1}$$

where X^i are the components of the tangent vector $\partial/\partial s$ in the basis $\{\mathbf{Y}_i\}$. Then t measures arc-length in the positive definite metric on $\mathcal{U}$ in which the bases $\{\mathbf{Y}_i\}, \{\mathbf{Y}^i\}$ are orthonormal. Let $\mathbf{Z}_a$ be a unit time-like vector. Since $\mathbf{R}_{ab}\mathbf{K}^a\mathbf{K}^b \neq 0$ at p for all non-spacelike vectors $\mathbf{K}$, then in any frame $\mid R_{11} \mid > R_{ab} (a \neq 1, b \neq 1)$, therefore it is possible to subtract a constant C from R_{11} such that $\mathbf{R}_{ij}(p) = C\mathbf{Z}_i\mathbf{Z}_j + \tilde{\mathbf{R}}_{ij}(p)$ and $C\tilde{\mathbf{R}}_{ab}\mathbf{K}^a\mathbf{K}^b \neq 0$ for all non-spacelike $\mathbf{K}$ (ie C has the same sign as $R_{11}(p)$ and has magnitude less than the minimum difference between $\mid \mathbf{R}_{11}(p) \mid$ and any other component of the Ricci tensor at p.) By continuity, these conditions hold on a neighbourhood $\mathcal{V} \subset \mathcal{U}$ with the same value of C.

Suppose that after some value t_o of t the curve λ intersects $\mathcal{V}$. Since λ has no endpoint and p is a limit point of λ, the part of λ in $\mathcal{V}$ will have infinite length as measured by t (see lemma). However, the generalized affine parameter u is given by

$$\frac{du}{dt} = \left(\sum_a (E_i^a \tilde{\mathbf{X}}^i)^2\right)^{\frac{1}{2}}$$

where $\tilde{X}^i$ are the components of the tangent vector $(\partial/\partial t)_\lambda$, i.e. $\sum_i \tilde{X}^i\tilde{X}^i = 1$, and E_i^a are the components of the basis $\{\mathbf{E}^a\}$ in the basis $\{\mathbf{Y}^i\}$. Since u is finite on the curve, the modulus of the vector $E_i^a\tilde{\mathbf{X}}^i$ must go to zero, and so the lorentz transformation represented by the components E_i^a must become unboundedly large. Since $\mathbf{Z}$ is a unit timelike vector, the components of $\mathbf{Z}$ in the basis $\{\mathbf{E}_a\}$ will therefore become unboundedly large and hence some component of the Ricci tensor in the basis $\{\mathbf{E}_a\}$ will become unboundedly large. □

Theorem 46
Suppose $\overline{\phi(\mathcal{M})}$ does not contain a null surface with the regular point $p \in \partial_\phi\mathcal{M}$ in its interior. Then p is not approached by any geodesic which does not approach p as an endpoint.

Proof. Suppose there is a geodesic λ with finite affine parameter that approaches p but not as an endpoint. λ can't be a null geodesic. To show this, suppose for the moment that λ is null. Let $\mathcal{N}$ be a normal neighbourhood of p in $\widehat{\mathcal{M}}$, chosen small enough such that $\lambda \cap \mathcal{N}$ has infinitely many disconnected segments.
To see that $\mathcal{N}$ can always be chosen this way, suppose there is no $\mathcal{N}$ for which this is true. Then there is a sequence $\{\mathcal{U}_n\}$ of convex normal neighbourhoods such that $\overline{\mathcal{U}}_n \subset \mathcal{U}_{n+1}$ and $\mathcal{U}_n$ contract around p as $n \to \infty$. By assumption λ enters and leaves each $\mathcal{U}_n$ only a finite number of times. Let s^* be the largest value of λ's parameter corresponding to an intersection of λ with the boundary of some given $\mathcal{U}_n$. Then if $s^* < s, \lambda \subset \mathcal{U}_n$. This implies that λ approaches p as an endpoint, since p is the only

point in the intersection of all the $\mathcal{U}_n$. This contradicts the assumption that p is not an endpoint of λ.
Let the disconnected segments of $\lambda \cap \mathcal{N}$ be denoted by λ_n, where the values of the affine parameter on λ_n increase with n. Let h be an auxilliary metric on $\mathcal{N}$. Choose a subsequence $\lambda_{n'}$ of λ_n with the property that the sequence $\{a_{n'}\}$, where $a_{n'}$ is the point on $\lambda_{n'}$ closest (with respect to h) to p, form a sequence with limit p. If $a_{n'}$ is close to p then the components (expressed in the normal coordinates around p) of λ's tangent vector at the point $a_{n'}$ are close to the components of the tangent vector at p of one of the null generators of p's lightcone. Due to the continuous dependance of ODEs on their initial conditions, for all ϵ there is an n' such that $\lambda_{n'}$ stays within ϵ of some null generator $\gamma_{n'}$ of p's lightcone. (This is where the assumption of regularity becomes important. For this argument to work the Christoffel symbols need to be continuous and bounded on $\mathcal{N}$). Since $\overline{\phi(\mathcal{M})}$ does not contain any null surfaces with p on their interior, in order to asymptope to p's lightcone in this way, λ has to exit $\phi(\mathcal{M})$, which is a contradiction. The same argument applies to an incomplete geodesic $\lambda(s)$, $0 \leq s \leq a < \infty$ whose tangent vector becomes null in the limit as $s \to a$.
Since λ's tangent vector $\mathbf{V}$ can't be null, it's affine parameter τ can be chosen as follows

$$\tau := \int_\lambda \sqrt{|g_{ij}dx^i dx^j|} \qquad t := \int_\lambda \sqrt{h_{ij}dx^i dx^j} \tag{2.2}$$

$$= \int_\lambda \sqrt{h(\mathbf{V},\mathbf{V})}d\tau \tag{2.3}$$

Setting $|g(\mathbf{V},\mathbf{V})| = 1$ gives

$$\frac{dt}{d\tau} = \sqrt{h(\mathbf{V},\mathbf{V})} \tag{2.4}$$

Since $\mathbf{V}$ does not approach a null vector, the constraint that $|g(\mathbf{V},\mathbf{V})| = 1$ prevents the components of $\mathbf{V}$ in the normal coordinates on $\mathcal{N}$ around p from becoming unbounded, therefore $h(\mathbf{V},\mathbf{V})$ remains bounded inside $\mathcal{N}$. Therefore $\int_{\lambda\cap\mathcal{N}} \frac{dt}{d\tau} d\tau$ is finite since $\int_{\lambda\cap\mathcal{N}} d\tau$ is finite. However, according to the lemma, the length of the part of λ contained in $\mathcal{N}$ is infinite, which is a contradiction. □

The Misner example (example 1.2 of the introduction) shows that neither of the last two theorems are true without the given assumptions. The region $t > 0$ contains incomplete null, space-like and time-like geodesics that have every point on the waist (i.e. the null hypersurface $t = 0$) as a limit point. Since the space-time can be extended past $t = 0$, all the boundary points with $t = 0$ are regular. The incomplete time-like and space-like geodesics have tangent vectors which become increasingly null as they approach the boundary. Since the misner example is flat, Theorem 45 does not apply either. This sort of behaviour would seem counterintuitive because a particle moving along the geodesic would seem to "accelerate" without any apparent reason.

The previous Theorem does not apply to interior points of the boundary, since every interior point is contained in a null surface in $\phi(\mathcal{M})$. This condition was only used to show that $\frac{dt}{d\tau}$ does not approach infinity, where t is the length of the incomplete geodesic with respect to h and τ is its affine parameter. Therefore if a point $p \in \phi(\mathcal{M})$ has a neighbourhood $\mathcal{N}$ on which $\frac{dt}{d\tau}$ remains bounded along all geodesics in $\mathcal{N}$ then p can't be a limit point of an incomplete geodesic. This condition could be thought of as a requirement that the "acceleration" is bounded along curves that go near p, and is not unreasonable from a physical point of view. It could, for example, be used as an alternative to strong causality in Theorem 43.

Definition 47 (Bounded "acceleration" Property)
Let τ be a choice of generalized affine parameter along the C^1 curve $\lambda \in \phi(\mathcal{M})$, and let t be the length of λ with respect to an auxilliary metric h on $\widehat{\mathcal{M}}$. Then a point p satisfies the *bounded "acceleration" property* if there is a neighbourhood $\mathcal{N} \subset \widehat{\mathcal{M}}$ of p such that for every curve λ that intersects $\mathcal{N}$ there is a positive constant a such that $0 < \frac{dt}{d\tau} < a$ everywhere on $\mathcal{N} \cap \lambda$.

Note that the choice of generalized affine parameter is not important in this definition, since they are all uniformly equivalent, i.e., if τ' is another choice of generalized affine parameter along λ, then $\frac{d\tau}{d\tau'}$ is uniformly bounded. The same applies to the choice of auxilliary metric.

The bounded "acceleration" property does not rule out the possibility that $\frac{dt}{d\tau}$ approaches zero. This is not necessary, because if p is a regular boundary point, then there is automatically a neighbourhood $\mathcal{N}$ of p such that if λ intersects $\mathcal{N}$, then $\frac{dt}{d\tau}$ does not approach zero anywhere along $\mathcal{N} \cap \lambda$. This can be seen as follows. Construct a specific choice of generalized affine parameter along λ as in section 8.1 of [15]. Let $\{\mathbf{E}_i\} = \{\mathbf{X}_1, \mathbf{X}_2 ... \mathbf{X}_n\}$ be a basis of $\mathbf{T}_{\lambda(0)}$ consisting only of time-like and space-like vectors and with the normalization $g(\mathbf{X}_i, \mathbf{X}_j) = \pm\delta_{ij}$. $\{\mathbf{E}_i\}$ is parallelly propagated along $\lambda(t)$ to obtain a basis of $\mathbf{T}_{\lambda(t)}$. Express the tangent vector $\mathbf{V}$ of λ in terms of this basis as $\mathbf{V} = V^i(t)\mathbf{E}_i$. A choice of generalized affine parameter on $\lambda(t)$ is then given by

$$\tau = \int (\sum_i V^i V^i)^{\frac{1}{2}} dt. \tag{2.5}$$

Therefore if $\frac{dt}{d\tau}$ approaches zero somewhere along $\mathcal{N} \cap \lambda$, then for at least one i, $h(\mathbf{X}_i, \mathbf{X}_i) \to 0$ while $g(\mathbf{X}_i, \mathbf{X}_i) = \pm 1$. If $\mathcal{N}$ is chosen such that g is regular on $\overline{\mathcal{N} \cap \phi(\mathcal{M})}$, then in $\mathcal{N} \cap \phi(\mathcal{M})$ if $g(\mathbf{X}_i, \mathbf{X}_i) = \pm 1$, the components of $\mathbf{X}_i$ expressed in some coordinate system around p can't approach zero, and therefore by regularity of h on $\overline{\phi(\mathcal{M})}$, $h(\mathbf{X}_i, \mathbf{X}_i)$ does not approach zero.

Since the bounded "acceleration" property is defined in terms of an auxilliary metric, it does not pass to the a-boundary and will in general be destroyed by re-envelopments that "blow up" a boundary point. It is intended more as an indication that the manifold is enveloped in such a way as to express regular a-boundary points as regular points as opposed to removable singularities. A breakdown in the bounded "acceleration" property is very closely related to non-Hausdorffness of the projection onto $\overline{\phi(\mathcal{M})}$ of the metric space topology central to the b-boundary construction, see for example section 8.3 of [15]. Despite its formulation in terms of neighbourhoods, the bounded "acceleration" property is not in general a local property of an envelopment.

Clarke and Scott have shown in [7] that for any curve $\lambda \subset \mathcal{M}$ there is an envelopment in which λ has a limit point.

Corollary 48 (Corollary to Theorem 46)
Let $\mathcal{C}$ be the class of geodesics with affine parameter. If an incomplete geodesic λ does not have an a-boundary endpoint but has a limit point $p \in \partial_\phi\mathcal{M}$ which satisfies the condition in Theorem 46, p is an essential singularity.

Proof. Theorem 46 rules out the possibility of p being a regular point. By Theorem 43 of [24] p can't be covered by a non-singular boundary set B because B would have to contain a regular boundary point q which is a limit point (by assumption not an endpoint) of λ. Since $\overline{\phi(\mathcal{M})}$ does not contain a null surface with the regular point $p \in \partial_\phi\mathcal{M}$ in its interior, as shown in Theorem 46, λ's tangent vector can't approach a null vector as it approaches the boundary point p, and therefore λ can't approach a regular point q, where $q \triangleright p$ with bounded affine parameter. Since p is approachable and is not regular or removable, p is an essential singularity. $\square$

This corollary has the advantage that it can be used to predict the existence of essential singularities without requiring the space-time to be maximally extended. However, it might often be difficult in practice to show that λ has no endpoint in any envelopment. The results proven in [11] could come in useful here, since they can be used to show that boundary sets with certain properties can't be a-boundary points.

Chapter 3

Examples

If a boundary point p has a finite number k of connected neighbourhood regions, it is sometimes useful to have a procedure to split the boundary point up into a set of k a-boundary points, each point of which has only one connected neighbourhood region which is locally indistinguishable from one of the connected neighbourhood regions of p. This procedure, outlined below, will be used throughout this Chapter and also in Chapter Four.

Suppose $\phi(\mathcal{M})$ is an n dimensional manifold and $\mathcal{N}_1$ is a connected neighbourhood region of the point $p \in \partial_\phi \mathcal{M}$. Let $S \in \widehat{\mathcal{M}} \backslash \phi(\mathcal{M})$ be an $n-1$ dimensional closed surface with p in its interior, chosen such that $\mathcal{N}_1$ is on one side of S and the other connected neighbourhood regions are on the other. (If p is regular, there is no loss of generality in assuming that the enveloping manifold extends out past p in this way.) Remove S from $\widehat{\mathcal{M}}$ and identify the lower edge of the slit with the upper edge of the slit in a second copy of $\widehat{\mathcal{M}} \backslash S$. Identify the upper edge of the slit with the lower edge of the slit in a third copy of $\widehat{\mathcal{M}} \backslash S$. If S can be chosen in such a way that it does not have any boundary points in common with $\phi(\mathcal{M})$, (as will always be the case in this chapter) then this process does not destroy the regularity properties of any a-boundary points. This will be referred to as "unidentifying" points in the remainder of this Chapter.

Figure 3.1 is a summary of what types of a-boundary points can cover other types. This Chapter confirms the entries in the Figure by providing examples wherever possible. It turns out that if it is not immediately apparent that a specific type of a-boundary point can't cover another type, then an example can be found to show that it is possible.

In all the following examples, let $\mathcal{C}$ be the set of geodesics with affine parameter.

Example 49 (An irregular unapproachable point that covers a regular point)
Put coordinates (r,θ) on $\mathbb{R}^2$ and let $\mathcal{M}$ be the manifold satisfying $r > 1$, $r\cos(\theta) < 1$, $0 < \theta < 2\pi$ and

$$ds^2 = \frac{-1}{\theta} dr^2 + r^2 d\theta^2$$

Let p be the boundary point with coordinates $r = 1$ and $\theta = 0$, as shown in Figure 3.2. Then p is irregular. (The curvature scalar becomes unbounded near p). Curves approaching p with increasing θ have a finite limit of the curvature scalar,

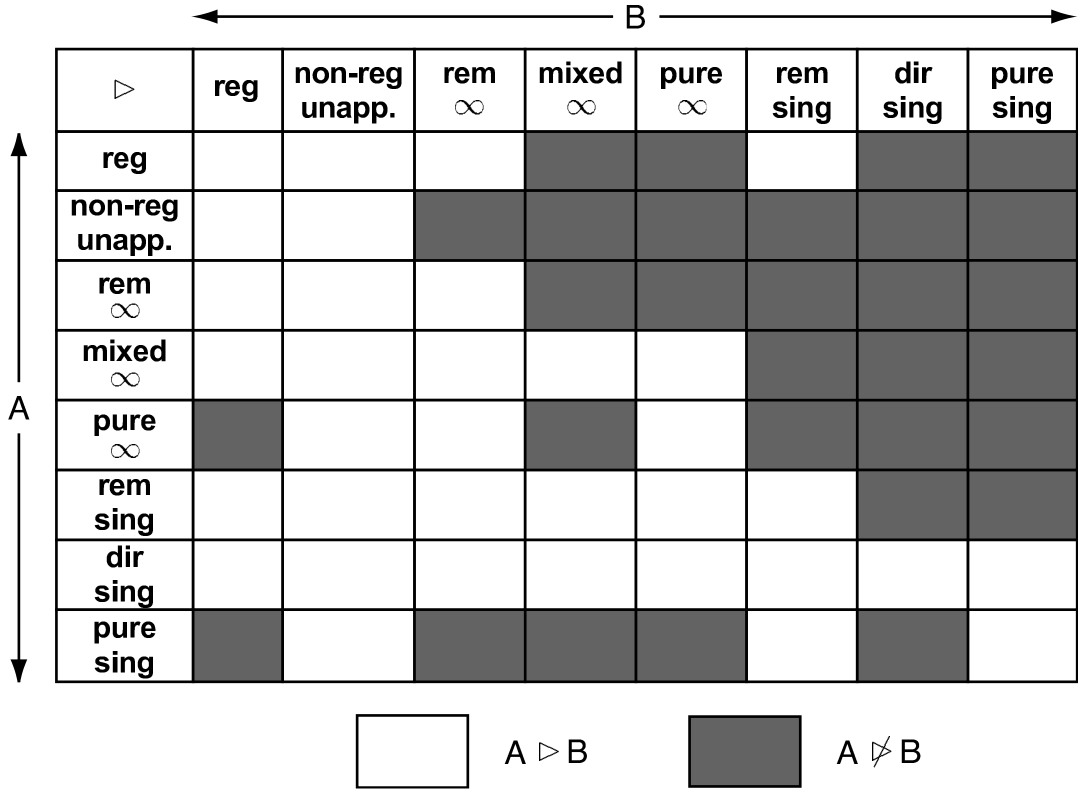


Figure 3.1:

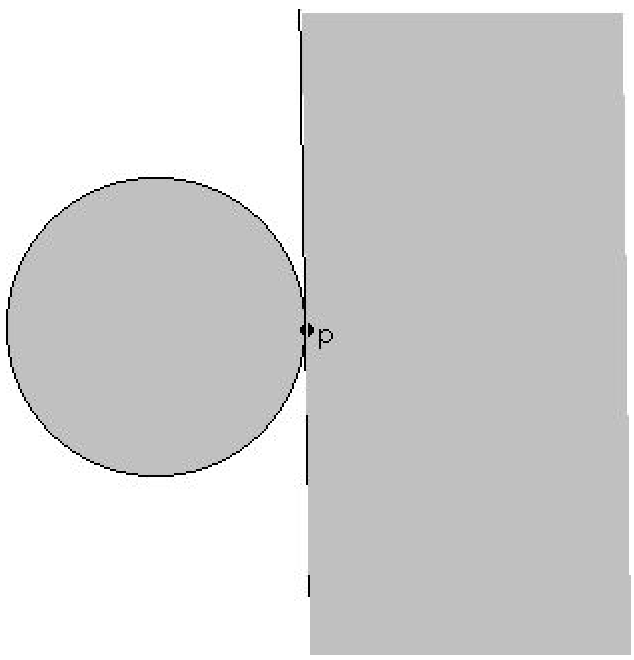


Figure 3.2: An irregular unapproachable point

while curves approaching p with decreasing θ have unbounded curvature scalar. The removal of the circle from the space-time makes p unapproachable. Unidentifying the boundary points $(1,0)$ and $(1,2\pi)$ reveals a regular point $(1,2\pi)$ and an irregular point covered by p.

Example 50 (A removable point at infinity that covers a regular point) Consider the subset of $\mathbb{R}^2$ satisfying $y<-1$, $y>\frac{1}{x^4}$ and with metric

$$ds^2 = -dy^2 + dx^2$$

The re-envelopment

$$x \to x' = x,\; y \to y' = \arctan(y).$$

compactifies the manifold, and reveals the point p at infinity with coordinates $(x', y') = (0, -\pi/2)$. Call this envelopment ϕ'. To see that p is removable, consider a second re-envelopment

$$x \to x'' = x + \frac{4}{3y^3},\; y \to y'' = y \bmod 1$$

The action of this re-envelopment is depicted in Figure 3.4. Call this envelopment $\phi''(\mathcal{M})$. $\phi''(\mathcal{M})$ is contained in the compact subset

$$\{(x'', y'') | -7/3 \leqslant x'' \leqslant 2/3, 0 \leqslant y'' \leqslant 1\}$$

of $\widehat{\mathcal{M}}''$. p is equivalent to the set of regular points

$$\{(x'', y'') | x'' = 2/3, 0 \leqslant y'' \leqslant 1\}.$$

Therefore p is removable and covers a regular boundary point.

Example 51 (A regular point that covers a removable point at infinity) Example 33 of [24] contains a regular boundary point that is only approachable by geodesics with infinite affine parameter. In this example, $\widehat{\mathcal{M}}$ is the unit torus with metric $ds^2 = dx^2 + dy^2$. On the central line $L = \{(x, 1/2) | 0 \leqslant x < 1\}$ choose the points

$$p_{\pm i} = \left(\frac{1}{2}\left(1 \pm \frac{1}{2^i}\right), \frac{1}{2}\right), \qquad i = 1, 2, 3, \ldots$$

For each $i = \pm 1, \pm 2, \ldots$ let L_i be the closed line segment of length $1/2$ and slope $\sqrt{2}$ centered on the point p_i and let L_0 be a similar line segment with center p. $\phi(\mathcal{M})$ is the open submanifold of $\widehat{\mathcal{M}}$ consisting of the complement in $\widehat{\mathcal{M}}$ of this infinite set of closed line segments. p is clearly a regular boundary point and is approached only by infinite geodesics with $\frac{dy}{dx} = \sqrt{2}$. The re-envelopment

$$x \to x' = x,\; y \to y' = \arctan\left(\frac{y}{x^2}\right)$$

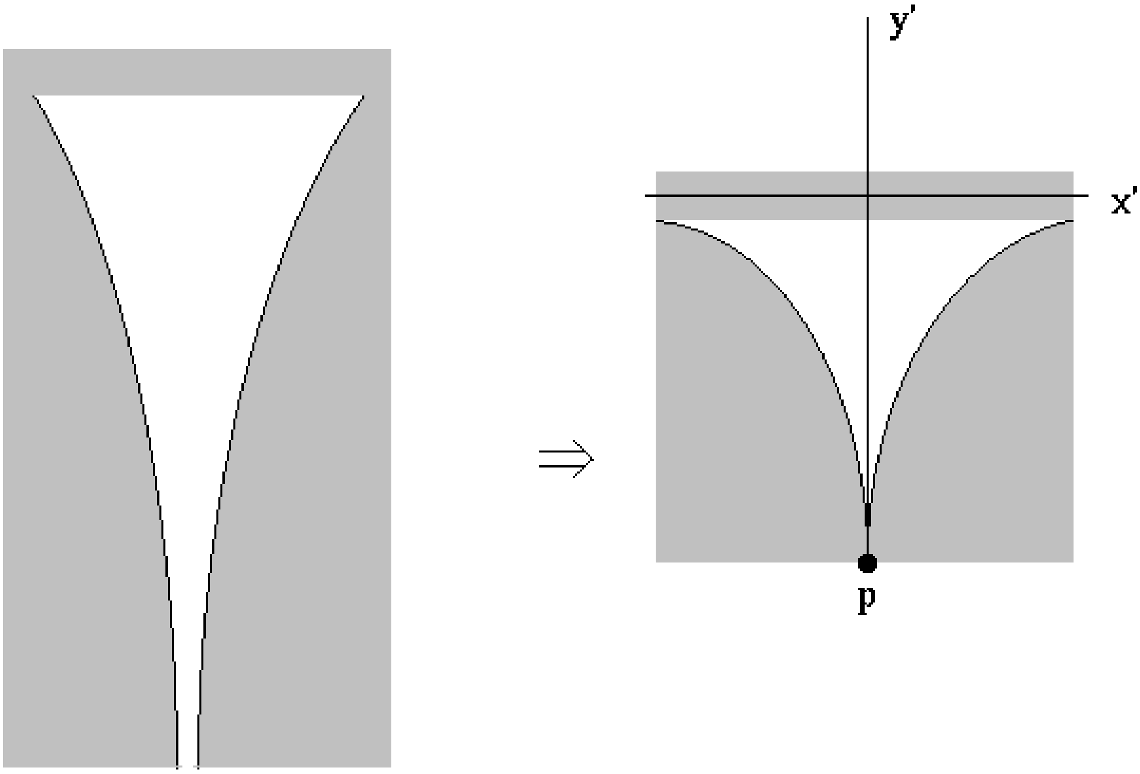


Figure 3.3: Removable point at infinity

blows up the point p into a compact interval I of irregular boundary points. By theorem 19 of [24], $[p] = [I]$, therefore there is an irregular point covered by p that is only approachable by infinite geodesics, i.e. p covers a point at infinity.

Example 39 of [24] shows a removable singularity covering a regular point, Example 45 shows a directional singularity covering a regular point, and Example 25 shows a regular boundary point covering an irregular unapproachable point.

Example 52 (A removable point at infinity covering an irregular unapproachable point)
Consider the removable point at infinity from Example 50. The width $w(y')$ of the manifold around the y' axis is given by

$$w(y') = \frac{2}{y'^4}.$$

The envelopment $x' \to x'' = x' e^{\frac{1}{w(y')}}$, $y' \to y'' = y'$ reveals unapproachable irregular boundary points covered by p, see Figure 3.6.

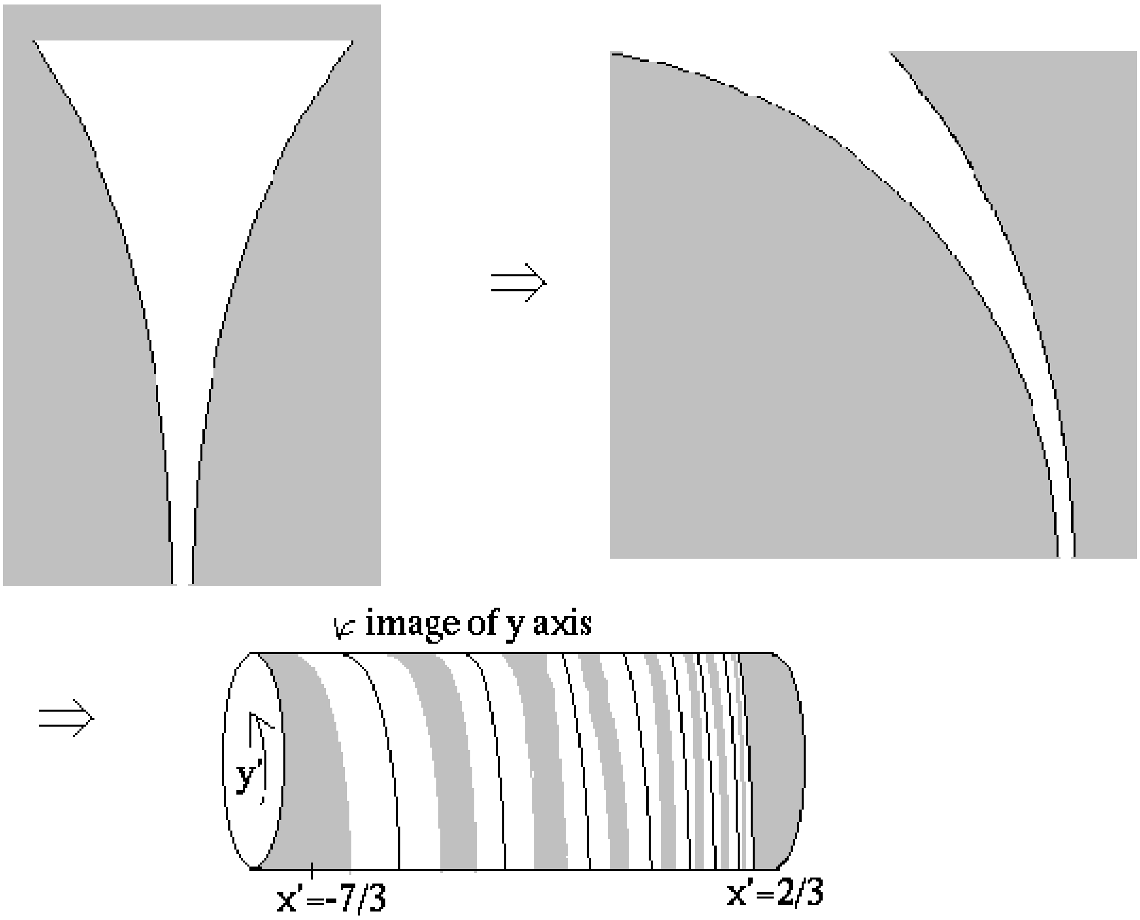


Figure 3.4:

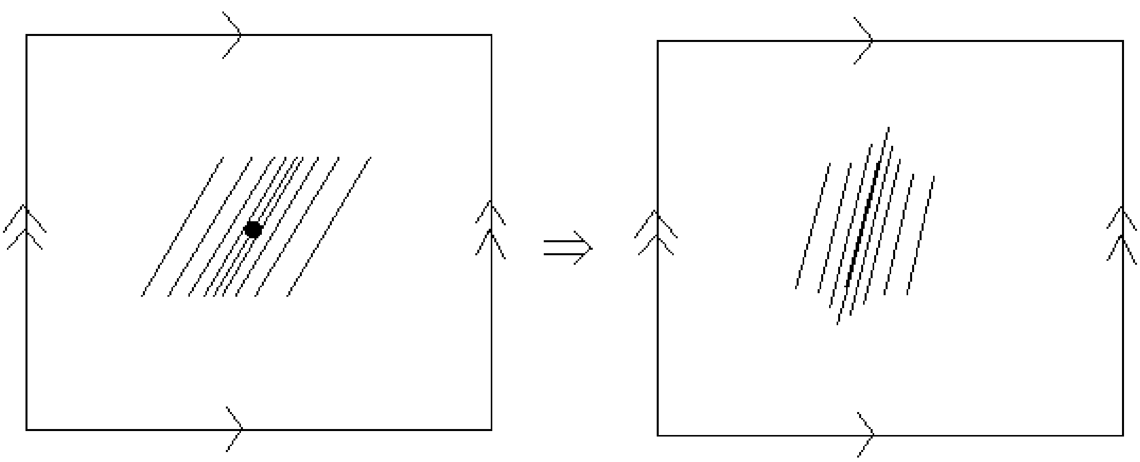


Figure 3.5:

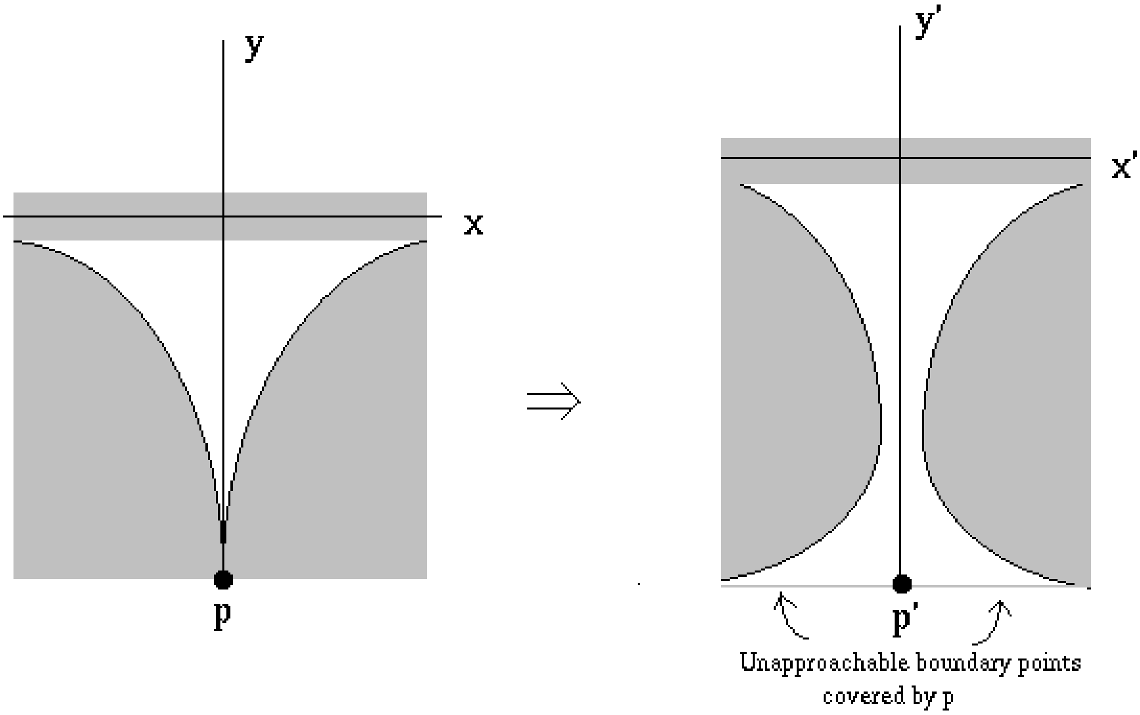


Figure 3.6:

Example 53 (A mixed point at infinity covering a removable point at infinity and an irregular unapproachable point)
Start with the manifold containing a point at infinity given in example 50. Multiplying the metric by $\frac{1}{(x'^2+(y+\pi/2)'^2)^2}$ makes the point $(0,-\pi/2)$ a pure point at infinity. Attach the manifold from example 50, as shown in Figure 3.7.

Let g_1 be the metric on the upper half of the diagram, and let g_2 be the metric on the lower part of the diagram. In order to make this manifold connected, attach a "bridge" from the top part to the lower part, as shown in the diagram. Let l be a parameter along the connecting piece, such that $l = 0$ on the boundary of the upper part of the example, and increases smoothly to $l = 1$ on the boundary of the other part. Then $\mathcal{M}$ is the manifold consisting of the two components + "bridge", where the metric g is given by

$$g|_{\text{region1}} = g_1,\ g|_{\text{region2}} = g_2 \text{ and } g|_{\text{bridge}} = (1-l)g_1' + lg_2'$$

(Because the two metrics are both simultaneously diagonalizable, it is easy to verify that the metric obtained in this way is nonsingular everywhere on $\mathcal{M}$.)

The origin is a mixed point at infinity, since it is not removable, but covers the removable point at infinity from example 50 that covers regular points. As shown in the previous example, this removable point at infinity covers irregular unapproachable points.

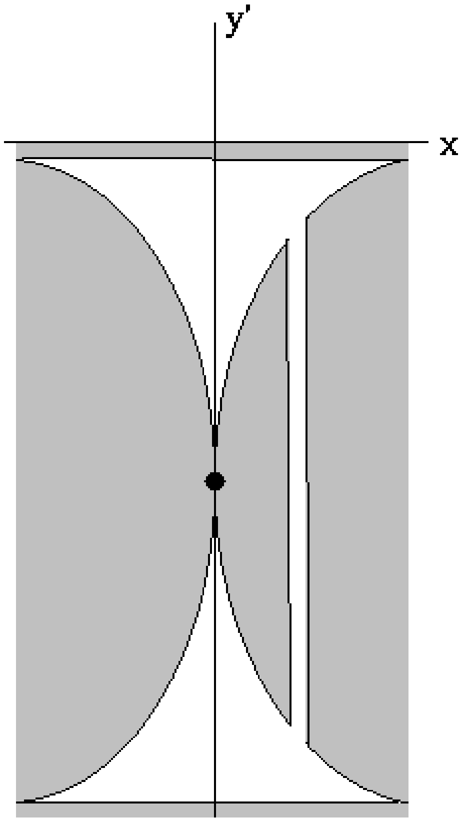


Figure 3.7:

Example 54 (A pure point at infinity covering an irregular unapproachable point)
This example is the same as example 52 only the removable point at infinity is made into a pure point at infinity by multiplying the metric by the conformal factor $\frac{1}{(x'^2+(y+\pi/2)'^2)^2}$.

Example 55 (A removable singularity covering an irregular unapproachable point)
Start with the region $y > 0$ of two dimensional space with metric $ds^2 = -dy^2 + dx^2$. The re-envelopment

$$x \to x' = \frac{x}{y^2},\ y \to y' = y$$

makes the origin, p, in these new coordinates a removable singularity, approached only by the geodesic $x' = 0$. Repeating this process (only with x' instead of x and y' instead of y) reveals irregular unapproachable points covered by p.

Example 56 (A directional singularity covering an irregular unapproachable point and a removable singularity)
Example 45 of [24] is a directional singularity. The details of this example are not required here. To show that it covers an irregular unapproachable point, choose a regular point p covered by the singularity. Put normal coordinates (x, y) around p.

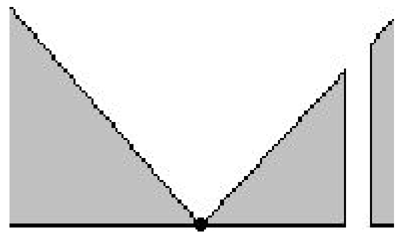


Figure 3.8: A pure singularity covering a removable singularity

The re-envelopment

$$x \to x' = \frac{x}{y^2},\ y \to y' = y$$

fixes the geodesic locally given by x=0, and sends all the other geodesics approaching p off to infinity. Any point other than the origin on the x' axis is an irregular unapproachable point covered by the directional singularity. The origin of the x' axis is a removable singularity covered by a directional singularity.

Example 25 of [24] is a pure singularity (To be more specific it is a cone singularity, see [10]) which covers irregular unapproachable points.

Example 57 (A pure singularity covering a removable singularity, and a pure point at infinity covering a removable point at infinity)
Start with the two dimensional manifold with metric

$$ds^2 = \frac{1}{\sqrt{x^2+y^2}}(-dy^2 + dx^2), \qquad\qquad \frac{\pi}{4} < \theta < \frac{3\pi}{4}.$$

Then take the section of the two dimensional manifold

$$\{(x,y)|y>0\},\text{ with metric } ds^2 = -dy^2 + dx^2$$

and re-envelop it as follows:

$$x \to x' = \arctan\left(\frac{x}{y}\right),\ y \to y' = y$$

The point $(x', y') = (0, 0)$ is a removable singularity. The manifold $\mathcal{M}$ formed by identifying the origins of these two manifolds and inserting a connecting piece has a pure singularity at the origin which covers a removable singularity.

An example of a pure point at infinity covering a removable point at infinity is formed from the previous example by sending the singularity off to infinity.

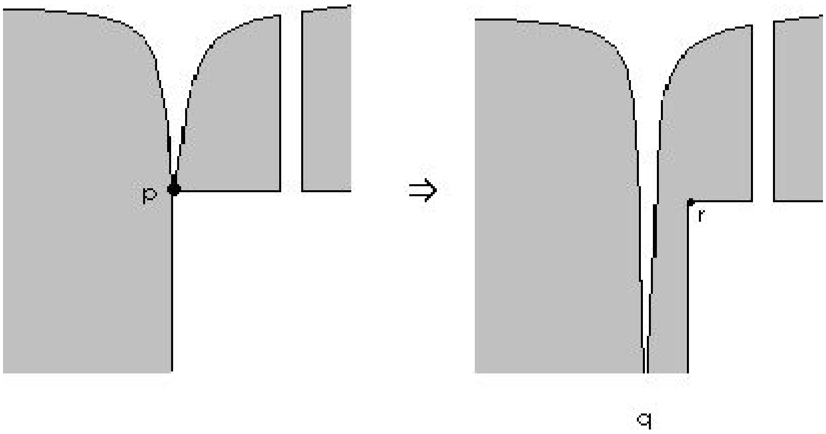


Figure 3.9: A removable singularity covering a removable point at infinity

Example 58 (A removable singularity covering a removable point at infinity)
$\mathcal{M}$ consists of the manifold in example 50 containing a removable point at infinity, connected to a quadrant of flat space, with a connecting piece, as shown in Figure 3.9. The point p is singular because it is irregular and approachable by finite geodesics. By theorem 19 of [24] it is equivalent to the a-boundary set consisting of $[q]$ - a removable point at infinity, and $[r]$ - a regular point. Therefore p is a removable singularity which covers a removable point at infinity.

Example 59 (A directional singularity covering a removable point at infinity)
This example is the same as the previous example, except that the metric on the quadrant is replaced by

$$ds^2 = \frac{1}{\sqrt{x^2 + y^2}}(-dy^2 + dx^2).$$

This turns p into a directional singularity but does not influence the classification of the point q in the re-envelopment.

Example 60 (A removable singularity covering a mixed point at infinity and a pure point at infinity)
Consider the manifold

$$\{(x, y)|y < 0\} \text{ with metric } ds^2 = dx^2 + dy^2.$$

Remove a semicircle from this space as shown in Figure 3.10. This is done in such a way that the boundary point $(0, 0)$ has two connected neighbourhood regions, one associated with a regular unapproachable point and the other associated with a regular point approached by finite geodesics. Take the top half of example 53 and identify the pure point at infinity with the point $(0, 0)$ of the manifold just

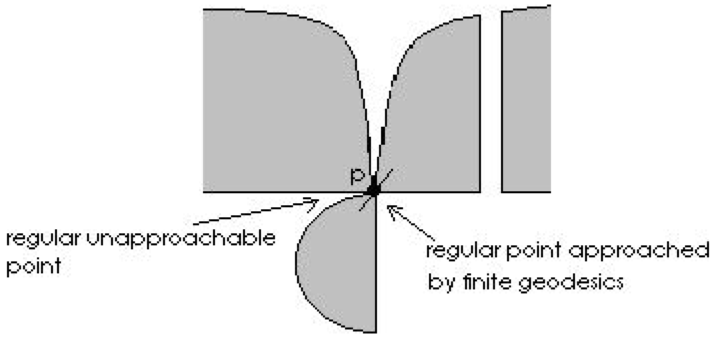


Figure 3.10: A removable singularity covering a mixed point at infinity and a pure point at infinity

constructed, and call this boundary point p. Make this manifold connected as in the earlier examples. p is irregular, approachable by geodesics with finite affine parameter, and is equivalent to a set consisting of regular points and points at infinity. Therefore p is a removable singularity. Unidentifying the regular approachable part of p reveals a mixed point at infinity covered by p. This mixed point at infinity covers the pure point at infinity from Example 53.

Example 61 (A directional singularity covering a mixed point at infinity and a pure point at infinity)
This is the same as the previous example except with the following alteration. Break the point p up into its connected neighbourhood regions. Let q be the regular approachable point covered by p. Locally alter the metric around the point q so that q becomes a curvature singularity. Putting the three connected neighbourhood regions back together again results in a directional singularity that covers a mixed point at infinity (i.e. the boundary point with the neighbourhood region around which the metric has been altered removed) and a pure point at infinity.

The Curzon solution [22] & [23] contains an example of a directional singularity that covers a pure singularity.

Chapter 4

Optimal Embeddings

4.1 The Finite Connected Neighbourhood Region Property

The FCNR property turns out to be very useful when proving results about regular boundary points. It is not a particularly strong assuption to make about a regular point, because if a regular point does not satisfy the FCNR property, the manifold can be extended by an arbitrarily small amount so that it does. The next few theorems provide some examples of how it can be used.

Theorem 62
If $p \in \partial_\phi \mathcal{M}$ satifies the FCNR property, then p is the endpoint of a smooth curve in $\phi(\mathcal{M})$.

Proof. Since p has a finite number, call it n, of CNRs, there is a neighbourhood $\mathcal{N}$ of p in $\widehat{\mathcal{M}}$ whose intersection with $\phi(\mathcal{M})$ has n connected components, $\mathcal{N}_1, \mathcal{N}_2 ... \mathcal{N}_n$. Let $\{x_i\}$ be a sequence of points in $\mathcal{N} \cap \phi(\mathcal{M})$ with limit p. Since there are only n components of $\mathcal{N} \cap \phi(\mathcal{M})$, there is a subsequence $\{y_i\}$ of $\{x_i\}$ contained within a connected component of $\mathcal{N} \cap \phi(\mathcal{M})$, which can be called $\mathcal{N}_1$ without loss of generality. Since is a subsequence of $\{x_i\}$, $\{y_i\}$ also has limit p. Therefore $p \in \partial \mathcal{N}_1$. Let $\mathcal{U}_{n'}$ be a sequence of neighbourhoods of p in $\widehat{\mathcal{M}}$ that contract around p as $n \to \infty$ and whose intersection with $\phi(\mathcal{M})$ have n connected neighbourhood regions. (The existence of such a sequence is gauranteed by the FCNR property.) Choose a subsequence $\mathcal{U}_n$ of $\mathcal{U}_{n'}$ with the property that $\mathcal{U}_1 \subset \mathcal{N}$ and $\overline{\mathcal{U}}_{n+1} \subset \mathcal{U}_n$. Suppose $\mathcal{U}_1 \cap \mathcal{N}_1$ has more than one connected component. Since $\mathcal{U}_1 \cap \mathcal{N}_1$ can't have more than n connected components, by the previous argument there is a connected component (call it $\mathcal{U}_{1p}$) of $\mathcal{U}_1 \cap \mathcal{N}_1$ with p on its boundary. $\mathcal{U}_2 \cap \mathcal{N}_1$ has at most n components. Since $\overline{\mathcal{U}}_2 \subset \mathcal{U}_1$, any component of $\mathcal{U}_2 \cap \mathcal{N}_1$ must be contained in exactly one component of $\mathcal{U}_1 \cap \mathcal{N}_1$. Therefore $\mathcal{U}_2 \cap \mathcal{U}_{1p}$ has finitely many connected components, and by the previous argument has a connected component $\mathcal{U}_{2p} \subset \mathcal{U}_{1p}$ with p on its boundary. $\mathcal{U}_{3p} \subset \mathcal{U}_{2p}$ is defined similarly, etc. Choose a sequence of points $\{z_i\}$, where $z_i \in \mathcal{U}_{ip}$. Since $\mathcal{U}_{ip}$ is connected, there is a smooth curve $\lambda_i \in \mathcal{U}_{ip}$ going from z_i to z_{i+1}. Let $c(t)$ be the path formed by traversing λ_1 then λ_2...Then $c(t)$ has endpoint p since it eventually becomes trapped within every neighbourhood of p. Smoothing off $c(t)$ results in a curve with endpoint p as required. □

Example 29 demonstrates that the previous result is not true if p does not satisfy the FCNR property.

Theorem 63
If p satisfies the FCNR property then every neighbourhood of p in $\widehat{\mathcal{M}}$ contains an open neighbourhood whose complement in $\phi(\mathcal{M})$ is connected.

Proof. Suppose there is a neighbourhood $\mathcal{N}$ of p in $\widehat{\mathcal{M}}$ which does not contain an open neighbourhood whose complement in $\phi(\mathcal{M})$ is connected. Let $\mathcal{B}_{\frac{1}{n}}(p,h)$ be the open ball with center p and radius $\frac{1}{n}$ with respect to an auxilliary metric h on $\widehat{\mathcal{M}}$. For large enough n, $\mathcal{B}_{\frac{1}{n}}(p,h)$ is contained in $\mathcal{N}$, so if $\mathcal{N}$ does not contain an open neighbourhood whose complement in $\phi(\mathcal{M})$ is connected, neither will $\mathcal{B}_{\frac{1}{n}}(p,h)$. Let $\mathcal{U}_n$ be a connected open neighbourhood of p contained in $\mathcal{B}_{\frac{1}{n}}(p,h)$ with the property that $\mathcal{U}_n \cap \phi(\mathcal{M})$ has the minimum number of connected neighbourhood regions, and also such that $\overline{\mathcal{U}}_{n+1} \subset \mathcal{U}_n$. Since $\phi(\mathcal{M})\backslash\mathcal{U}_n$ is by assumption not connected, it can be covered by disjoint open sets V_n and W_n, chosen such that

$$V_n \cap \phi(\mathcal{M}) \subset V_{n+1} \cap \phi(\mathcal{M}) \text{ and } W_n \cap \phi(\mathcal{M}) \subset W_{n+1} \cap \phi(\mathcal{M}) \tag{4.1}$$

(To see that V_n and W_n can always be chosen in this way, given V_{n+1} and W_{n+1}, construct V_n and W_n as follows

$$V'_n := V_{n+1} \cap (\phi(\mathcal{M})\backslash\mathcal{B}_{\frac{1}{n}}(p,h)), \qquad W'_n := W_{n+1} \cap (\phi(\mathcal{M})\backslash\mathcal{B}_{\frac{1}{n}}(p,h)) \text{ and} \tag{4.2}$$

$$B_{V_n} := \partial V'_n \cap V'_n, \qquad B_{W_n} := \partial W'_n \cap W'_n \tag{4.3}$$

Since B_{V_n} and B_{W_n} are disjoint and closed in $\phi(\mathcal{M})$, they can be covered by disjoint open sets $C_{V_n} \subset \phi(\mathcal{M})$ and $C_{W_n} \subset \phi(\mathcal{M})$ respectively. Then let $V_n = V'_n \cup C_{V_n}$ and $W_n = W'_n \cup C_{W_n}$. Similarly for V_{n-1}, W_{n-1}, etc.)
Suppose that for sufficiently large n, V_n and W_n both intersect a given neighbourhood region, call it X. Then define

$$X_{V_n} = V_n \cap X \text{ and } X_{W_n} = W_n \cap X \tag{4.4}$$

Then $X_{V_n} \subset X_{V_{n+1}}$, etc, and X_{V_n} and X_{W_n} are disjoint. Therefore X_{V_i} is disjoint with X_{W_j} for $i \leqslant j$ and X_{W_j} is disjoint with X_{V_i} for $j \leqslant i$. Therefore $\lim_{n\to\infty} \bigcup_{i=1...n} X_{V_i}$ and $\lim_{n\to\infty} \bigcup_{i=1...n} X_{W_i}$ are disjoint, open and cover X. This contradicts connectedness of X. Therefore V_n and W_n intersect distinct connected neighbourhood regions of p. Let the connected neighbourhood regions be labelled $X_1, ...X_k$, where p has k connected neighbourhood regions. Since p has only finitely many connected neighbourhood regions, for large enough n, V_n will intersect some subset A of $\{A_1, ...A_k\}$ while W_n intersects its complement. (This is where the assumption of the FCNR property is necessary). Therefore $V_n \cup A$ and $W_n \cup A^c$ are disjoint, non-empty open sets which cover $\phi(\mathcal{M})$, which contradicts the connectedness of $\phi(\mathcal{M})$. $\square$

In Example 29 of the introduction, none of the boundary points in the set

$\{(x,y)|x=0 \text{ and } |y|<1\}$ have arbitrarily small neighbourhoods in $\widehat{\mathcal{M}}$ whose complement in $\phi(\mathcal{M})$ is connected.

Theorem 64 (Theorem 18 of [24])
If every curve in $\mathcal{M}$ which approaches a boundary set B' also approaches a boundary set B, and if every neighbourhood of B in $\widehat{\mathcal{M}}$ contains an open neighbourhood $\mathcal{U}$ of B whose complement in $\phi(\mathcal{M})$ is connected, then B covers B'.

Corollary 65
If $p \in \partial_\phi \mathcal{M}$ satisfies the FCNR property and every curve in $\mathcal{M}$ that approaches a boundary set S also approaches p, then p covers S.

Proof. This is a direct consequence of the last two theorems. □

Theorem 66
If $p \in \partial_\phi \mathcal{M}$ is a regular boundary point which satisfies the FCNR property and p is approached by a geodesic in $\phi(\mathcal{M})$ which does not have p as an endpoint, then p is the endpoint of some other geodesic in $\phi(\mathcal{M})$.

Proof. Let λ be a geodesic in $\phi(\mathcal{M})$ which approaches p as a limit point but not as an endpoint. Suppose there is no geodesic in $\phi(\mathcal{M})$ with p as an endpoint. Let $\mathcal{V}$ be a coordinate neighbourhood of p in $\widehat{\mathcal{M}}$ on which a set of coordinates $(x_1 ... x_n)$ valid everywhere on $\mathcal{V}$ are defined. Let $\mathcal{U}(p) \subset \mathcal{V}$ be a subneighbourhood such that $\mathcal{U}(p) \cap \phi(\mathcal{M})$ has a finite number of connected components, and small enough such that λ enters and leaves $\mathcal{U}(p)$ infinitely many times. (It was shown in Theorem 46 that a curve with limit point p but not endpoint p enters and leaves all sufficiently small neighbourhoods of p infinitely many times.) Since $\mathcal{U}(p) \cap \phi(\mathcal{M})$ has only finitely many connected components, there is a component $\mathcal{W}$ that contains an infinite number of disconnected segments of $\lambda \cap \phi(\mathcal{M})$ and a sequence $\{y_i\}$ of points in $\lambda \cap \mathcal{W}$ that approach p. Therefore $p \in \partial \mathcal{W}$.
No tangent vector at p points into $\phi(\mathcal{M})$, otherwise, since p is a regular point, this vector would correspond to a geodesic in $\phi(\mathcal{M})$ with endpoint p. Let $(x_1, .., x_n)$ be coordinates on $\mathcal{U}(p)$, and let $\mathbf{V}(t)$ be the tangent vector to $\lambda(t)$. Since $\mathcal{W}$ becomes arbitrarily narrow, if λ is to come arbitrarily close to p, turn around and come back again, then for some i and j,

$$\frac{\partial V^j}{\partial x_i} \to \infty.$$

If τ is a choice of affine parameter, then the geodesic equation

$$\frac{\partial x_i}{\partial \tau}\frac{\partial V^j}{\partial x_i} = -\Gamma^j_{ab}\frac{dV^a}{d\tau}\frac{dV^b}{d\tau}$$

implies that at least one of the Christoffel symbols becomes unbounded near p, which contradicts the assumption of regularity.

□

Example 33 of [24] shows that this result is not true if the FCNR property is dropped.

In Chapter Two, it was shown that a regular a-boundary point $[p]$ with n CNRs is equivalent to a set of n regular boundary points $[p_1], [p_2] ... [p_n]$ each with only one connected neighbourhood region. In a proof, it is sometimes useful to be able to show that a result is true for each of the points $[p_1], [p_2] ... [p_n]$ and then put all the pieces together to show that the result is true for p. This trick can't be used on a boundary point $[q]$ that does not satisfy the FCNR condition, because $[q]$ can be approached by a sequence with one point in every CNR and is therefore not equivalent to the "sum of its parts".

4.2 Regular Points in Optimal Embeddings

Recall that an envelopment $\phi(\mathcal{M})$ is an optimal embedding if its corresponding partial cross section

$$\sigma_\phi = \{[p] | p \in \partial_\phi \mathcal{M}\} \tag{4.5}$$

is a maximal admissable partial cross section. A maximal admissable partial cross section contains a maximal regular partial cross section. It would therefore be useful to know that

1. The set of all regular partial cross sections has (at least one) maximal element
2. A maximal regular partial cross section covers all regular boundary points.

Some definitions are useful at this point, for details see [16]

Definition 67 (Partial Order)
A *Partial Order* on a set A is a relation denoted by $\leqslant$ that satisfies the following properties for all a, b and $c \in A$

1. **Reflexivity** $a \leqslant a$
2. **Antisymmetry** If $a \leqslant b$ and $b \leqslant a$ then $a = b$
3. **Transitivity** If $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$.

Definition 68 (Chain)
$B \subset A$ is a *chain* if for any two elements $a, b \in B$ either $a \leqslant b$ or $b \leqslant a$.

Definition 69 ($\sigma \triangleright p$, $\sigma \dashv p$ and $\sigma \parallel p$)
The partial cross section σ is said to cover the boundary point p if there is a set of points in σ that covers p. σ is in contact with p if there is an a-boundary point in σ that is in contact with p, and σ is separate to p if it is not in contact with p.

Two regular partial cross sections σ_a and σ_b will be said to be equivalent if they cover the same set of a-boundary points. A partial order can then be defined on the set of all equivalence classes of regular partial cross sections as follows

Definition 70 ($\sigma_a \leqslant \sigma_b$)
Let A and B be the sets of a-boundary points covered by σ_a and σ_b respectively. Then $\sigma_a \leqslant \sigma_b$ if $A \subset B$.

In the previous definition, it would be undesirable to replace "covered by" with "in contact with". To see why this is so, let p and q be a-boundary points and σ be a regular partial cross section such that $\sigma \dashv p$ but $\sigma \parallel q$. An a-boundary point can often be formed by identifying $[p]$ and $[q]$, and this point would be in contact with σ but not covered by σ. If the "in contact" version of the definition were adopted, this would make it necessary to look at many irrelevant boundary points when comparing two regular partial cross sections. However, it will be shown in Theorem 74 that for regular boundary points, " in contact with" often implies "equivalent to".

The definition of maximal regular partial cross section given in the introduction can therefore be restated as follows

Definition 71 (Maximal Regular Partial Cross Section, (previous definition))
A regular partial cross section is *maximal* if it is the maximal element of a chain.

The definition of "maximal regular partial cross section" given in the introduction does not always turn out to be the most convenient one to use. According to this definition, if there is a regular partial cross section σ_1 that covers every regular a-boundary point, then σ_1 will also be maximal. However, it will generally also be possible to choose a maximal regular partial cross section σ_2 which does not cover every regular a-boundary point. A maximal regular partial cross section like σ_2 is only maximal with respect to a chain of regular partial cross sections that were not necessarily constructed in an optimal way. Take for example the manifold in Example 36 of the introduction. The maximal partial cross section

$$\sigma_1 = \{p | p \in \partial_\phi \mathcal{M}\} \tag{4.6}$$

is total, while another maximal partial cross section

$$\sigma := \{p | p \in \partial_\phi \mathcal{M} \text{ and the r coordinate of } p \notin (1,2) \text{ or } p \in \partial_{\phi'} \mathcal{M} \text{ with } r \in (1,2)\} \tag{4.7}$$

does not cover the boundary points with $r = 1$ or $r = 2$. It would seem that σ_1 is more "maximal" that σ_2. For the remainder of this thesis, the following definition of maximal will be used unless stated otherwise.

Definition 72 (Maximal Regular Partial Cross Section)
A regular partial cross section σ_m is *maximal* if $\sigma_a \leqslant \sigma_m$ for all regular partial cross sections σ_a.

With this new definition, it becomes trivial to show that a maximal regular partial cross section covers every regular a-boundary point. The disadvantage is that not every manifold has a maximal regular partial cross section according to this definition. However, the existence problems of the two different types of regular partial cross sections are not unrelated. According to Zorn's lemma, if every regular partial cross section is contained in a maximal regular partial cross section (according to the first definition), then there exists a maximal regular partial cross section.

Theorem 73
A maximal regular partial cross section covers every regular a-boundary point.

Proof. Suppose that σ_m does not cover the regular boundary point $p \in \partial_\phi \mathcal{M}$. Since p is regular, it has a neighbourhood $\mathcal{N}$ in $\widehat{\mathcal{M}}$ such that $\mathcal{N} \cap \partial_\phi \mathcal{M}$ contains only regular boundary points. Let $\sigma_a = \{q | q \in \mathcal{N} \cap \partial_\phi \mathcal{M}\}$. Then $\sigma_a \nleq \sigma_m$, and therefore σ_m is not maximal. $\square$

In order to contruct a maximal regular partial cross section it would seem to be necessary to posess detailed information about the a-boundary of the manifold. Also, it would seem that if there is a maximal regular partial cross section, it is not in general unique. An algorithm for constructing a maximal regular partial cross section would then have to favour some extensions over others. For example, the region $t > 0$ of the Misner example [15] & [18], contains two classes of null geodesics. No extension of this example preserves the symmetry between these two families of null geodesics; i.e. an extension can only extend one class of null geodesics at once. An algorithm would therefore either have to make arbitrary decisions, or perhaps give infinitely many solutions. Therefore it would seem that the best that can be done is to prove that maximal regular partial cross sections actually exist for most space-times.

A non constructive proof of the existence of maximal regular partial cross sections requires a result along the lines that if two regular boundary points $[p]$ and $[q]$ are in contact, then they are equivalent. Clearly, this result requires both $[p]$ and $[q]$ to be regular, otherwise a "blowup" map could be used to reveal many boundary points covered by $[p]$ (and hence in contact with $[p]$) that are not equivalent to $[p]$. Also, the result will not be true in general if $[p]$ and $[q]$ each have more than one connected neighbourhood region. This is because if a regular boundary point has n connected neighbourhood regions, the process outlined in Chapter Two could be used to re-envelop the manifold in such a way that $[p]$ is expressed as n regular boundary points, each of which is in contact with, but not equivalent to, $[p]$. Another necessary assumption in the following proof is the bounded "acceleration" property. The boundary of the region $t > 0$ of the Misner example, when enveloped as outlined in the introduction, consists of regular boundary points each with only one connected neighbourhood region. These regular boundary points do not satisfy the bounded "acceleration" property, and are limit points of incomplete curves that spiral around the space-time infinitely many times. There is a second envelopment that unwinds the spiralling geodesics and winds up the curves that had endpoints on the boundary of the first envelopment. A regular boundary point in one envelopment is in contact with every regular boundary point in the second envelopment but is not equivalent to any of them.

Theorem 74
Suppose $p \in \partial_\phi \mathcal{M}$ and $q \in \partial_{\phi'} \mathcal{M}$ are regular boundary points that satisfy the bounded "acceleration" property and that each have only one connected neighbourhood region. Suppose p is approached by a curve $\lambda_1(t)$ in $\phi(\mathcal{M})$ with finite generalized affine parameter. Then if p is in contact with q then $[p] = [q]$.

Proof. Since p is regular and satisfies the bounded "acceleration" property, λ_1 approaches p as an endpoint. (This was discussed in Chapter Two.) By the same reasoning, $\phi' \circ \phi^{-1}(\lambda_1(t))$ either approaches q as an endpoint or not at all.

Let $\mathcal{U}_{\lambda_1} \subset \phi(\mathcal{M})$ be a covering of λ_1 by open balls of the form $\mathcal{B}_{x_i}(r_i)$, where $\mathcal{B}_{x_i}(r_i) \in \phi(\mathcal{M})$ is the ball of radius r_i (defined with respect to an auxilliary metric h on $\widehat{\mathcal{M}}$) centered around the point x_i on λ_1. Since each ball is contained in $\phi(\mathcal{M})$, the radii approach zero as the curve approaches p. In this proof it is necessary to show that if $\phi' \circ \phi^{-1}(\lambda_1)$ has endpoint p then the neighbourhood $\phi' \circ \phi^{-1}(\mathcal{U}_{\lambda_1})$ contracts around $\phi' \circ \phi^{-1}(\lambda_1)$. The bounded "acceleration" property will be used to do this. This property relates lengths with respect to h to a specified choice of generalized affine parameter, (an intrinsic property of the manifold). The choice of generalized affine parameter is then related back to lengths with respect to an auxilliary metric h' on $\widehat{\mathcal{M}}'$ in order to show that the radius of the neighbourhood approaches zero near q. Let $\mathcal{V} \in \widehat{\mathcal{M}}$ be the neighbourhood of p gauranteed by the bounded "acceleration" condition. Since λ_1 approaches p as an endpoint, it eventually becomes trapped within $\mathcal{V}$. Also, if $\phi' \circ \phi^{-1}(\lambda_1)$ approaches q as an endpoint, then for large enough t, $\phi' \circ \phi^{-1}(\lambda_1(t)) \subset \mathcal{V}'$, where $\mathcal{V}' \in \widehat{\mathcal{M}}'$ is a neighbourhood of q whose existence is gauranteed by the bounded "acceleration" property. Therefore, there is an i^* such that $i^* < i$ implies that $\mathcal{B}_{x_i}(r_i) \subset \mathcal{V}$ and $\phi' \circ \phi^{-1}(\mathcal{B}_{x_i}(r_i)) \subset \mathcal{V}'$. To simplify the notation, let $i^* = 1$. For every $\mathcal{B}_{x_i}(r_i)$ form a smooth curve $c_i \subset \mathcal{B}_{x_i}(r_i)$ from x_i to a point on the boundary of $\mathcal{B}_{x_i}(r_i)$. Let $c(t)$ be a curve in $\mathcal{V} \cap \phi(\mathcal{M})$ formed by smoothly joining up all the curves c_i, as shown in Figure 4.2. Let τ be a choice of generalized affine parameter along $c(t)$, let s be the length with respect to h and let s' be the length of $\phi' \circ \phi^{-1}(c(t))$ with respect to an auxilliary metric h' on $\widehat{\mathcal{M}}'$. Then by the bounded "acceleration" property, $\frac{ds'}{d\tau}$ is uniformly bounded in $\phi' \circ \phi^{-1}(\mathcal{U}_{\lambda_1}) \subset \mathcal{V}'$. As discussed in Chapter Two, since p is regular, $\frac{d\tau}{ds}$ is bounded above near p. Therefore, taking the pullback of all metrics and curves into $\mathcal{M}$, $\frac{ds'}{d\tau}\frac{d\tau}{ds} = \frac{ds'}{ds}$ is bounded everywhere along $\phi^{-1}(c(t) \cap \mathcal{U}_{\lambda_1})$. Since $c(t)$ could be chosen to include any sequence of radial curves c_i, this shows that $\phi' \circ \phi^{-1}(\mathcal{U}_{\lambda_1})$ contracts around $\phi' \circ \phi^{-1}(\lambda_1)$.

Therefore, if λ_2 is a curve contained in $\mathcal{U}_{\lambda_1}$ with endpoint p, then since the neighbourhood contracts in $\phi'(\mathcal{M})$, $\phi' \circ \phi^{-1}(\lambda_2)$ approaches q as an endpoint if $\phi' \circ \phi^{-1}(\lambda_1)$ does and does not approach q otherwise. Define $\mathcal{U}_{\lambda_2}$ in the same way as $\mathcal{U}_{\lambda_1}$. If there is a third curve λ_3 with endpoint p contained in $\mathcal{U}_{\lambda_2}$, then $\phi' \circ \phi^{-1}(\lambda_3)$ approaches q as an endpoint if $\phi' \circ \phi^{-1}(\lambda_1)$ does, etc. Therefore any curve λ_n homotopic to λ_1 (homotopic in the sense that the endpoint, p, is fixed while the starting point is free to vary) has the property that $\phi' \circ \phi^{-1}(\lambda_n)$ approaches q as an endpoint if $\phi' \circ \phi^{-1}(\lambda_1)$ does and does not approach q otherwise.

It could be the case that there are infinitely many holes in $\phi(\mathcal{M})$ near p, in which case there could be a curve γ with endpoint p that is not homotopic to λ_1. Then as shown in Figure 4.1, there is a curve λ_m with endpoint p homotopic to λ_1 and a curve γ_m homotopic to γ such that every point on γ_m is contained in $\mathcal{U}_{\lambda_m}$. Therefore $\phi' \circ \phi^{-1}(\gamma)$ approaches q as an endpoint if $\phi' \circ \phi^{-1}(\lambda_1)$ does and does not approach q otherwise.

Since $p \dashv q$, there is a sequence $\{z_i\}$ of points in $\phi(\mathcal{M})$ with limit p such that $\{\phi' \circ \phi^{-1}(z_i)\}$ has limit q. Since p has only one connected neighbourhood region, the points of the sequence $\{z_i\}$ can be joined up to make a smooth curve $z(t) \subset \phi(\mathcal{M})$ with endpoint p. (The proof of this is the same as the proof of Theorem 62). Since

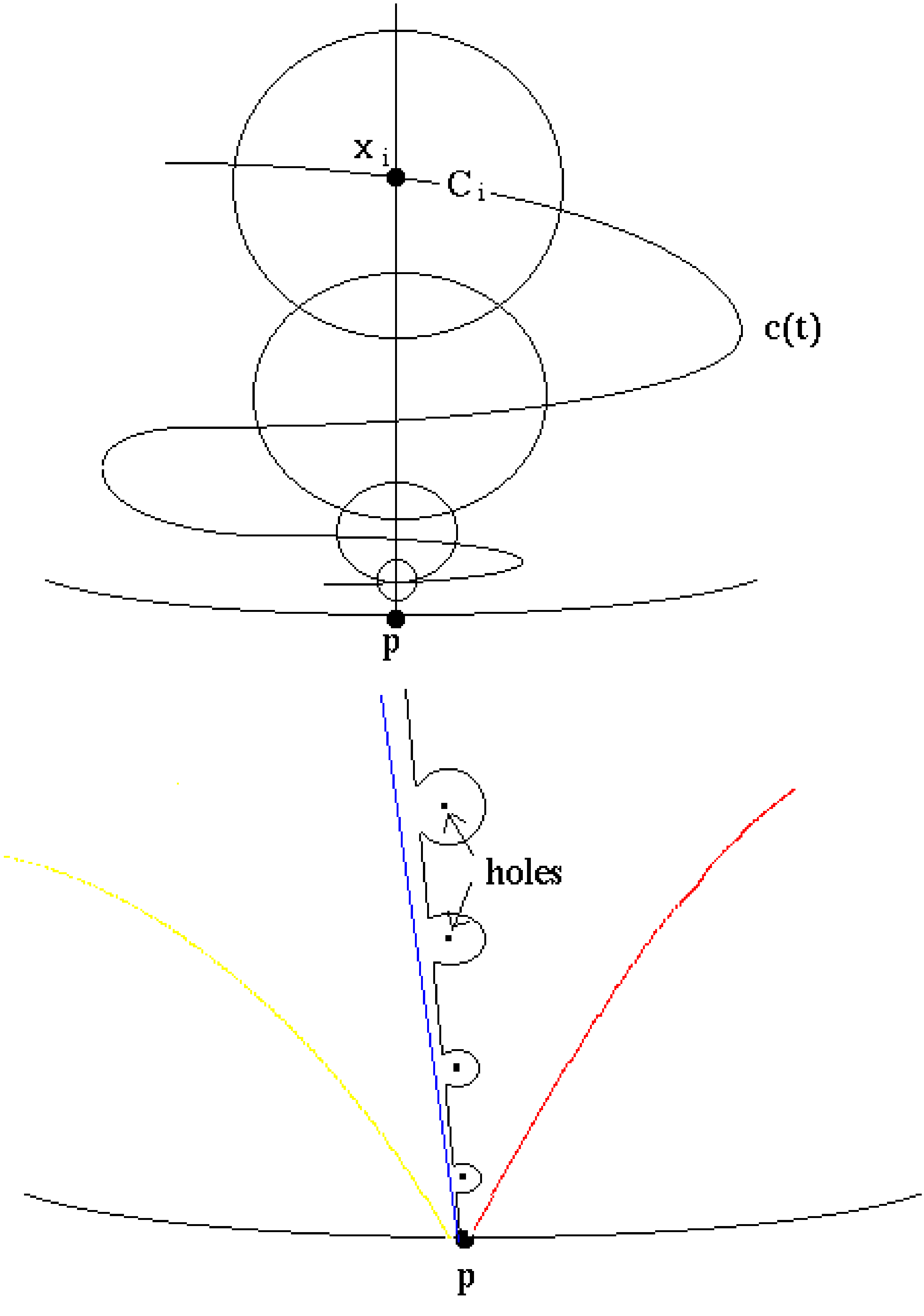


Figure 4.1: In this diagram, the red curve is λ_1, the black curve is λ_m, the blue curve is γ_m and the yellow curve is γ

$\phi' \circ \phi^{-1}(z(t))$ approaches q, by the previous argument, $\phi' \circ \phi^{-1}(\lambda_1)$ approaches q, therefore $\phi' \circ \phi^{-1}(\lambda_1)$ has endpoint q. Therefore every curve with endpoint p is mapped to a curve with endpoint q. As a result of Corollary 65, q covers p. An identical argument shows that p covers q. Therefore $[p] = [q]$.

□

This theorem effectively says that if two regular boundary points are in contact, they have a connected neighbourhood region in common. This formalises the intuitive notion that regularity is an unstable property of boundary points.

Remark 75
Although, as mentioned earlier, the bounded "acceleration" property is necessary in Theorem 74, it was only convenient to use it to show that certain neighbourhoods converge around curves. The pseudo-Riemannian metric g contains enough information to show that "nearby" curves that approach a regular boundary point as an endpoint stay "nearby" if regularity is preserved. To motivate this statement, suppose that γ_r is a family of geodesics in $\phi(\mathcal{M})$ with endpoint p, some of which have the regular point $q \in \partial_{\phi'}(\mathcal{M})$ as an endpoint when re-enveloped. The index r can be chosen such that γ_0 is mapped to a curve that has q as an endpoint. Let $\mathbf{E}(t) = \{\mathbf{X}_i(t)\}$, where $i = 1 \ldots n$ and $g(\mathbf{X}_i, \mathbf{X}_j) = \pm\delta_{ij}$ be a parallelly propagated frame along $\gamma_0(t)$, and let $\mathbf{J} = j_i \mathbf{X}_i$ be a jacobi field along γ_0. Under re-envelopment, $\phi' \circ \phi^{-1*}(\mathbf{E}(t))$ is a parallelly propagated frame along the geodesic $\phi' \circ \phi^{-1}(\gamma_0)$ with the same normalization as before. $\phi' \circ \phi^{-1*}(\mathbf{J}(\mathbf{t})) = j_i(t)\phi' \circ \phi^{-1*}(\mathbf{X_i}(\mathbf{t}))$. Therefore $\phi' \circ \phi^{-1*}(\mathbf{J}(\mathbf{t}))$ has a zero as t approaches infinity or its upper bound. By regularity, $\lim \phi' \circ \phi^{-1*}(\mathbf{E}(t))$ defines a normalized basis at q. Therefore the fact that the jacobi field has a zero in the limit as p is approached implies that the entire family of geodesics have endpoint q.

Example 76
Identify the edges EF and AB of the model attached to this page. The resulting manifold does not posess a maximal regular partial cross section. To see this, suppose a regular partial cross section contains the regular boundary points labelled as B and D. To satisfy the second condition of regular partial cross sections, a maximal regular partial cross section σ_m has to identify all the regular boundary points along the edge AD with the points along CD. Similarly for the edges BC and BA. However the regular boundary points A and C can only be included in a regular partial cross section if the edges just mentioned are not identified.

While Example 76 does not have an optimal embedding, it does satisfy the desirable property that there are a finite number of envelopments whose boundaries collectively display all the regular extension hypersurfaces and cover all a-boundary points.

In order to prove the existence of a maximal regular partial cross section for a manifold, it is necessary to rule out the previous example. This is done by assuming that the boundary of the manifold is suffiently "convex" so that the process outlined in Chapter Two for breaking apart connected neighbourhood regions does not destroy the regularity of any boundary points. It is not unreasonable that the existence of maximal regular partial cross sections should depend on a property along these lines. The constraint that a particular mapping $f : \mathcal{M} \to \mathcal{A}$ be an envelopment is only a restriction of the behaviour of f on the points of $\mathcal{M}$. Therefore if $\partial_\phi \mathcal{M}$ is convex in some sense, there is more flexibility in how the boundary can be depicted.

Corollary 77 (Corollary of Theorem 74)
Suppose that the process outlined in Chapter Three can be used to express every regular boundary point of $\mathcal{M}$ as a finite set of regular boundary points each with only one connected neighbourhood region. Suppose also that the hypersurfaces that are removed from the enveloping manifold in order to do this can be chosen such that they do not have any boundary points in common with $\mathcal{M}$. Then if the conditions of Theorem 74 are satisfied, $\mathcal{M}$ has a maximal regular partial cross section.

Proof. Let p be a regular boundary point. By assumption, there is an envelopment $\phi(\mathcal{M})$ in which p is expressed as a finite number $p_1, p_2 ... p_n$ of regular points, each with only one connected neighbourhood region. Each of the points p_i has a neighbourhood $\mathcal{N}_i \subset \widehat{\mathcal{M}}$ whose intersection with $\partial_\phi \mathcal{M}$ contains only regular points with one connected neighbourhood region. (This is where the assumption about the hypersurfaces is necessary. Otherwise it could be the case that the regularity of a boundary point has to be destroyed in order to break up the connected neighbourhood regions of nearby regular points. The boundary points labelled B and D in Example 76 do not satisfy this "convexity" condition). Let

$$\sigma_p := \bigcup_i \mathcal{N}_i \cap \partial_\phi \mathcal{M}. \tag{4.8}$$

Then σ_p is a regular partial cross section. Let

$$\sigma := \bigcup \sigma_p \tag{4.9}$$

where the union is taken over the regular boundary points p_i corresponding to every regular boundary point p. By theorem 74, any two a-boundary points in σ are either separate or equivalent. By construction, the two necessary conditions for σ to be a regular partial cross section are satisfied. σ is also clearly maximal. □

If a regular boundary point p has infinitely many connected neighbourhood regions, it is not necessarily covered by a set of regular points each with only one connected neighbourhood region. That was why the condition that "the a-boundary does not contain any regular points with infinitely many connected neighbourhood regions" was needed in the preceeding proof. However, when the regular partial cross section consists of points on the boundary of a total envelopment, (as is generally the case when dealing with optimal embeddings), this requirement is not necessary.

4.3 Singular Points in Optimal Embeddings

To construct an optimal embedding, it would be convenient to be able to envelop $\mathcal{M}$ in such a way that an essential singularity is displayed as a set of singularities that do not cover regular boundary points and a separate set of non-singular points. The classic example of this is the maximally extended Curzon solution [22] & [23]. It has been seen to be possible (although by no means easy) to "unravel" essential singularities in this way in all the examples for which it has been attempted, hence the following conjecture, out of [24].

Conjecture 78
Every essential singularity covers a pure singularity.

However, when constructing a re-envelopment that does this, it is necessary to make use of the detailed structure of the singularity, and hence there is no general recipe for unravelling essential singularities. For this reason, the question of whether or not every essential singularity covers a pure singularity has never been settled, although a counterexample to this rule of thumb would seem at first glance to be very counterintuitive. It would consist of an essential singularity, from which, regardless of how much it has been "spread out", non-singular points could always be extracted from it, like rabbits out of a hat. However, after having given the problem a considerable amount of thought, it would seem that the conjecture might not be true in general. The following example highlights what might go wrong, and is presented as a suspected counterexample.

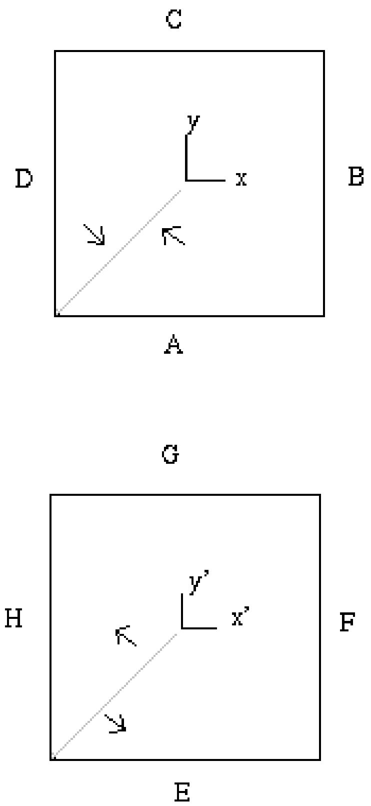
Example 79
Let $\mathcal{C}$ be the set of geodesics with affine parameter. Take the two submanifolds of $\mathbb{R}^2$ consisting of

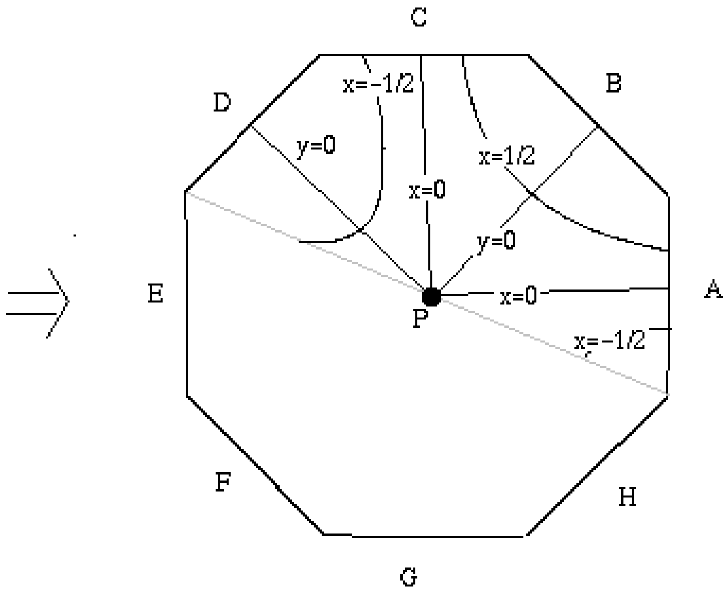
$$(x, y) \in (-1, 1) \times (-1, 1) \text{ with metric } ds^2 = dx^2 - dy^2 \tag{4.10}$$
$$(x', y') \in (-1, 1) \times (-1, 1) \text{ with metric } ds^2 = dx'^2 - dy'^2 \tag{4.11}$$

Remove the slits $x = y$ with $x \leq 0$ and $x' = y'$ with $x' \leq 0$. Identify one side of the slit with the opposite side of the slit on the other submanifold, and identify the remaining two sides of the slit, as shown in figure 79.

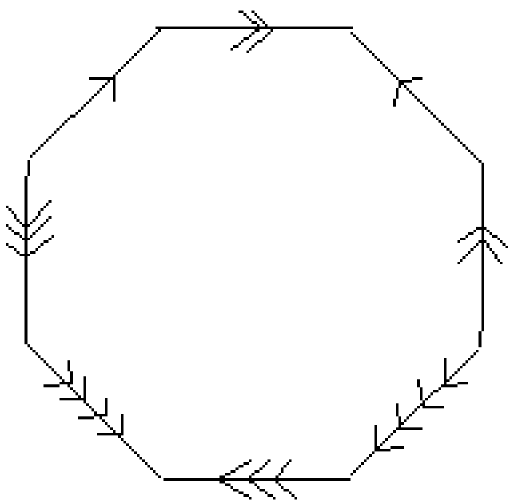
Identify the sides, as shown in figure 79 to make a double torus. (Because of the symmetries involved in this example, the metric matches up smoothly along the identifications.) The point marked P is an essential singularity. In particular, it is a quasiregular singularity, because it has too many directions of approach to be regular, see [10].

There is one point Q at which all the verticies come together. Q is also an essential singularity, for the same reason as P. Let $\phi(\mathcal{M})$ be the double torus minus the two singularities P and Q.





The point marked P is an essential singularity. The slit is represented by the grey line.



Let γ_1 be the geodesic with starting point $(x, y) = (\frac{1}{2}, \frac{1}{2})$ and initial tangent vector specified by $\frac{dy}{dx} = \sqrt{2}$. Let γ_2 be the geodesic with starting point $(x', y') = (\frac{1}{2}, \frac{1}{2})$ and initial tangent vector specified by $\frac{dy'}{dx'} = \sqrt{2}$. γ_1 and γ_2 are infinitely long geodesics. Every point of $\phi(\mathcal{M})$ is a limit point of one or both of the two timelike geodesics γ_1 and γ_2. Suppose $\phi'(\mathcal{M})$ is another envelopment of $\mathcal{M}$. Then since the re-envelopment $\phi' \circ \phi^{-1}$ is a homeomorphism, every point of $\phi'(\mathcal{M})$ is a limit point of one or both of the curves $\phi' \circ \phi^{-1}(\gamma_1)$ and $\phi' \circ \phi^{-1}(\gamma_2)$. Therefore every point in $\overline{\phi'(\mathcal{M})}$ as a limit point of one or both of the two geodesics. Suppose $[q]$ is an essential singularity covered by $[P]$, and let $\mathcal{F}$ be the set of geodesics with finite affine parameter that approach $[q]$. It seems reasonable that by choosing a re-envelopment that spreads $[q]$ out by a large enough amount, $[q]$ covers points not approachable by an element of $\mathcal{F}$. A boundary point covered by $[q]$ that is not approached by a finite geodesic is irregular and approachable only by infinite geodesics and is therefore a point at infinity. In which case, $[q]$ is not a pure singularity. To make this argument more plausible, a sequence of line segments could be removed from $\mathcal{M}$ as in Example 33 of [24] such that $[P]$ is only approached by one geodesic λ with finite affine parameter. Then if $[P]$ covers a pure singularity, this pure singularity can only cover points approachable by λ, since any other point would be a point at infinity.

If the previous example did turn out to be a counterexample, this would not really be a problem for the theory of optimal embeddings. To construct an optimal embedding, it is only necessary to find an envelopment in which no essential singularity covers a regular point.

Definition 80 (Quintessential Singularity)
An essential singularity will be called a *quintessential singularity* if it does not cover any regular a-boundary points.

This definition passes to the a-boundary.

It seems clear intuitively that if $[p]$ is an essential singularity and ϕ is an envelopment in which the curves approaching $[p]$ are sufficiently spread out, then an essential singularity $q \in \partial_\phi \mathcal{M}$, where $p \triangleright q$, will be a quintessential singularity.

Conjecture 81 (Modified Conjecture)
Every essential singularity covers a quintessential singularity

However, when constructing an optimal embedding, the aim is to find a re-envelopment that displays $[p]$ as a set of boundary points consisting of non-singular points and quintessential singularities. If the re-envelopments that spread out the curves approaching $[p]$ reveal infinitely many essential singularities covered by $[p]$, and if infinitely many of these essential singularities also cover regular points, examples such as the following present difficulties.

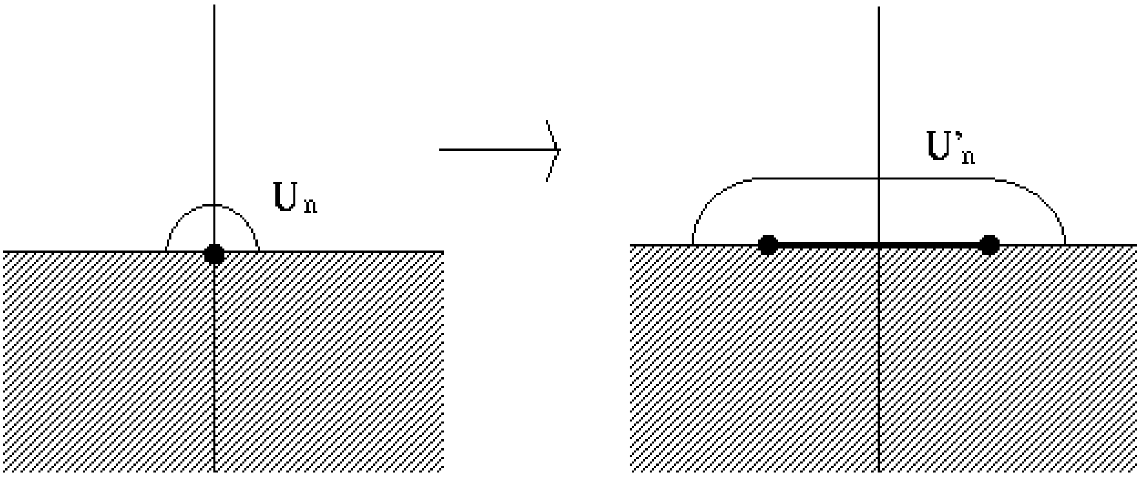


Figure 4.2:

Example 82 (A sequence of re-envelopments of $\mathcal{M}$ whose limit is not an envelopment)
Consider the manifold

$$\phi(\mathcal{M}) = \{(x,y)| -\infty < x < \infty \ , \ 0 < y < \infty\} \text{ and with metric } ds^2 = dx^2 + dy^2 \tag{4.12}$$

Let a second envelopment $\phi' : \mathcal{M} \to \mathbb{R}^2$ be defined by

$$(x', y') = \phi'(x,y) = (\arctan(\frac{x}{y}), \sqrt{x^2+y^2}) \tag{4.13}$$

This re-envelopment blows the boundary point $(x,y) = (0,0)$ up into the closed interval

$$I = \{(x,y)| -\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2} \ , \ y = 0\} \tag{4.14}$$

Define $\mathcal{U}_n = \{p \in \mathcal{M}|$ distance between p and the origin is less than $\frac{1}{n}\}$ and let $\mathcal{U}'_n$ be the image of $\mathcal{U}_n$ under re-envelopment. Since the image of the boundary point $(x,y) = (0,0)$ under this re-envelopment is compact and every sequence entering U_n is mapped to a sequence entering $\mathcal{U}'_n$, Theorem 19 of [24] says that the point $(x,y) = (0,0)$ of $\partial_\phi \mathcal{M}$ and the set

$$\{(x', y')| -\frac{\pi}{2} \leqq x' \leqq \frac{\pi}{2} \ , \ y' = 0\}$$

are equivalent.
Repeating this process around the point $(x', y') = (0,0)$ of $\phi'(\mathcal{M})$ results in a manifold $\phi''(\mathcal{M})$ with a boundary set

$$\{(x'', y'')| -\pi \leqq x'' \leqq \pi \ , \ y'' = 0\}$$

equivalent to the boundary point $(x,y) = (0,0)$ of the original envelopment. If in the limit as this process is repeated infinitely many times the resulting manifold is a re-envelopment of $\phi(\mathcal{M})$, then the point $(0,0) \in \partial_\phi \mathcal{M}$ is equivalent to a noncompact set, which contradicts Theorem 2.2 of [11]

Example 83 (A Manifold without an Optimal Embedding)

$$f(r) \quad := \quad C^{\infty} \text{ function such that} \begin{cases} f(r) = 1 & \text{for} \quad 0 \leqslant r \leqslant \frac{1}{3} \\ f(r) = 0 & \text{for} \quad \frac{1}{2} \leqslant r \end{cases} \tag{4.15}$$

Let $\phi'(\mathcal{M}_1)$ be the set $\{(r,\theta)|0 < \theta < \pi\}$ with metric

$$ds^2 = \left(1 + \frac{f(r)}{r^2}\right) dr^2 + r^2 d\theta^2.$$

In $\phi'(\mathcal{M}_1)$ the origin is a pure singularity.

Let (x', y') be cartesian coordinates on $\phi'(\mathcal{M}_1)$ with origin at $r = 0$. $\phi(\mathcal{M}_1)$ is the envelopment

$$\phi(x', y') = (x, y) = (y' \cos(x'), y' \sin(x')) \tag{4.16}$$

This re-envelopment compresses the interval

$$I = \{(x', y')| \frac{-\pi}{2} \leqslant x' \leqslant \frac{\pi}{2}, y' + 0\} \tag{4.17}$$

into the point $(x, y) = (0, 0)$. (This is the inverse of the re-envelopment defined in the previous example).

By Theorem 19 of [24], the point $(x, y) = (0, 0) \in \partial_{\phi}\mathcal{M}_1$ is equivalent to the interval $I \in \partial_{\phi'}\mathcal{M}_1$. Therefore the boundary point $(x, y) = (0, 0)$ of $\phi(\mathcal{M}_1)$ is an essential singularity that covers regular boundary points.

In the envelopment $\phi(\mathcal{M}_1)$, the metric takes the form

$$ds^2 = dx^2 + dy^2 + f(\sqrt{(arctan(x/y))^2 + x^2 + y^2})(g_{11}(x,y)dx^2 + g_{12}(x,y)dxdy + g_{22}(x,y)dy^2).$$

Let (x_2, y_2) be cartesian coordinates centered around the boundary point $(x', y') = (1, 0)$ of $\phi'(\mathcal{M}_1)$. Call this point p. $\mathcal{M}_2$ is the manifold formed by locally altering the metric around p to look like the directional singularity at the origin of $\phi(\mathcal{M})$. In particular, let $\mathcal{B}_{\frac{1}{2}}(p)$ be the set

$$\{(x_2, y_2) \in \phi'(\mathcal{M}_1)|\sqrt{x_2^2 + y_2^2} < 1/2\}.$$

On $\mathcal{B}_{\frac{1}{2}}(p)$ the metric takes the form $ds^2 = dx_2^2 + dy_2^2$. Let $\phi'(\mathcal{M}_2)$ be the manifold identical to $\phi'(\mathcal{M}_1)$ except with the metric

$$ds^2 = dx_2^2+dy_2^2+f(\sqrt{(arctan(x_2/y_2))^2 + x_2^2 + y_2^2})(g_{11}(x_2, y_2)dx_2^2+g_{12}(x_2, y_2)dx_2dy_2+g_{22}(x_2, y_2)$$

on $\mathcal{B}_{\frac{1}{2}}(p)$. $\phi(\mathcal{M}_2)$ is the envelopment with the interval I compressed into a point, where the re-envelopment that does this is defined in the same way as the re-envelopment $\phi \circ \phi'^{-1}$ of $\mathcal{M}_1$.

$\phi(\mathcal{M}_3)$ is constructed from $\phi(\mathcal{M}_2)$ similarly. The directional singularity is "unravelled", and the metric is locally altered such that the directional singularity in $\phi'(\mathcal{M}_2)$ covers a directional singularity, and then the interval is compressed to a point.

$\mathcal{M} := \lim_{n\to\infty} \phi(\mathcal{M}_n)$ has an essential singularity for which there is no envelopment that expresses the singularity as a set of quintessential singularities and non-singular points. This is because, as in the previous example, an envelopment that did this would contradict Theorem 2.2 of [11].

The re-envelopments that "open up" the essential singularity in $\mathcal{M}$ do not affect the regularity of any point except at the singularity. Therefore the previous argument also shows that $\mathcal{M}$ cannot be maximally extended, because it will always contain an essential singularity from which more regular points can be extracted.

The moral of this story is that although it would seem to be the case that an essential singularity $[p]$ is equivalent to a set consisting of non-singular a-boundary points and quintessential singularities, if the re-envelopments that spread out the curves approaching $[p]$ reveal infinitely many essential singularities covered by $[p]$, and if infinitely many of these essential singularities also cover regular points, then there will not in general be an envelopment of $\mathcal{M}$ whose boundary contains the a-boundary point $[p]$ expressed in such a way that the regular a-boundary points it covers are separate to the essential singularities. Therefore, to construct an optimal embedding, it is not in general enough that every essential singularity covers a quintessential singularity.

Chapter 5

Conclusion

The original research presented in this thesis can be roughly divided up into two categories; results relating to the existence of optimal embeddings of solutions to Einstein's field equations and results relating to a-boundary singularity theorems.

Chapter One looked at the connection between geodesic incompleteness and the existence of a-boundary essential singularities. Previous results, [1] and [2] have used causality conditions to rule out the possibility of an incomplete causal geodesic without an a-boundary endpoint, and then assumed maximal extendedness to show that the a-boundary endpoint of an incomplete curve is an essential singularity. The result proven in this thesis is a different approach, in that it shows that the existence of an incomplete curve without an a-boundary endpoint is itself an indication of singular behaviour. Unlike other singularity theorems, this result does not require the space-time to be maximally extended.

In Chapter two a procedure for re-enveloping a manifold was developed such that a given regular boundary point is expressed as a set of regular boundary points each with only one connected neighbourhood region. This procedure turns out to be convenient for constructing examples/counterexamples such as the ones outlined in that Chapter, and also in proving theorems.

Chapter Three contains most of the new results in this thesis. The finite connected neighbourhood region assumption was shown to ensure various convenient properties of boundary points, which were then used in later proofs. Examples were devised to show that not all space-times can be optimally embedded, either because there is no maximal regular partial cross section (Example 76), or because there is an essential singularity that no re-envelopment can separate into quintessential singularities and non-singular boundary points (Example 83). In the first instance, this is not a significant problem as long as a maximal regular partial cross section consisting of points on the boundary of only finitely many envelopments can be constructed. Examples that do not satisfy this condition do exist, however it would seem to be the case that they are all very contrived, and not good models of space-time. The same applies to Example 83. This sort of behaviour can only happen when there are an infinite number of singularities that "accumulate" in some region. Example 83 also demonstrates that not every manifold is maximally extendable, although it would still seem to be the case for every "reasonable" manifold. It was also shown that a maximal regular partial cross section covers every regular a-boundary point, and the existence of maximal regular partial cross sections under certain circumstances was proven. Finally, the issue of whether or not every essential singularity covers a pure singularity was discussed, and a suspected counterexample was given.

There remain many unresolved problems relating to this work, particularly in the area of optimal embeddings. While it is known that optimal embeddings do exist, there are not many examples in the literature. Much insight would probably be gained by actually constructing optimal embeddings for a wider variety of space-times. It would also be of interest to investigate conditions under which maximal extensions and compactifications of optimal embeddings exist, and to find some final, conclusive answer to whether or not essential singularities always cover pure or quintessential singularities.

Many of the results in this thesis have turned out to posess unexpected subtleties. This seems to be representative of much of the work done in this area. The two main sources of these unintuitive outcomes seem to be

1. space-time has a pseudo-Riemannian metric
2. the necessity of describing properties of the boundary by using only intrinsic properties of the manifold.

The a-boundary is particularly well suited to dealing with the type of problems addressed in this thesis, because unlike earlier boundary constructions, it deals directly with the second difficulty and does not require any elaborate constructions to achieve sensible answers. However, the generality of this approach has the effect that there are many desirable results are not true because of obscure, "unphysical" counterexamples. One of the major challenges is therefore to devise assumptions that rule out pathological examples such as the Misner example and Example 83 but that would also seem to be prerequisite properties of all "reasonable" space-times.

Appendix A

$\boldsymbol{a}$-boundary Definitions

Definition 84 (Abstract boundary $B(\mathcal{M})$)
$B(\mathcal{M}) := \{[p] \mid p \in \partial_\phi(\mathcal{M})$ for some envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)\}$.

Definition 85 (Admissable partial cross section)
σ is an *admissable partial cross section* if

1. there is a maximal regular partial cross section contained in σ
2. for each $[p] \in \sigma$ there is a representative boundary point $q \in [p]$, $q \in \partial_\phi\mathcal{M} \subset \widehat{\mathcal{M}}$, and an open neighbourhood, $\mathcal{U}(q) \subset \widehat{\mathcal{M}}$ such that

$$\{[r] | r \in \mathcal{U} \cap \partial_\phi\mathcal{M}\} \subset \sigma.$$

Definition 86 (Approachable boundary point)
A parametrised curve $\gamma : I \to \mathcal{M}$ *approaches* the boundary set B if the curve $\phi \circ \gamma$ has a limit point lying in B. A point $p \in \partial_\phi\mathcal{M}$ is *approachable* if it is approached by a curve from the family $\mathcal{C}$.

Definition 87 (bounded parameter property (b.p.p.))
A family C of parametrized curves in $\mathcal{M}$ satisfies the b.p.p. if:

1. for any point $p \in \mathcal{M}$ there is at least one curve of the family passing through p
2. if $\gamma(t)$ is a curve of the family then so is any connected subset of it
3. if γ and γ' are in C and γ' is obtained from γ by a change of parameter then either the parameter is bounded or unbounded on both curves.

Definition 88 (Complete and incomplete curves)
A *complete curve* is a curve which can be extended in both directions for arbitrarily large values of a specified parameter. A curve which is not complete is *incomplete*.

Definition 89 (Connected neighbourhood region (CNR))
Suppose $p \in \partial_\phi\mathcal{M}$ and $\mathcal{N}$ is a neighbourhood of p in $\widehat{\mathcal{M}}$. Then a connected component of $\mathcal{N} \cap \phi(\mathcal{M})$ is called a *connected neighbourhood region of p*.

Definition 90 (Covering relation)
If B' is a set of points in the boundary of a second envelopment $(\mathcal{M}, \widehat{\mathcal{M}}', \phi')$ of $\mathcal{M}$ then B *covers* B' (denoted $B \rhd B'$) if for every open neighbourhood $\mathcal{U}$ of B in $\widehat{\mathcal{M}}$ there exists an open neighbourhood $\mathcal{U}'$ of B' in $\widehat{\mathcal{M}}'$ such that

$$\phi \circ \phi'^{-1}(\mathcal{U}' \cap \phi'(\mathcal{M})) \subset \mathcal{U}.$$

Definition 91 (p is in contact with q)
Let $p \in \partial_\phi \mathcal{M} \subset \widehat{\mathcal{M}}$ and $q \in \partial_{\phi'} \mathcal{M} \subset \widehat{\mathcal{M}}'$ be two envelopped boundary points of $\mathcal{M}$. Then p and q are said to be *in contact* (denoted $p \dashv q$) if for all neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of p and q respectively

$$\phi^{-1}(\mathcal{U} \cap \phi(\mathcal{M})) \cap \phi'^{-1}(\mathcal{V} \cap \phi'(\mathcal{M})) \neq \emptyset.$$

Definition 92 (In contact, sequence definition)
Two boundary points $p \in \partial_\phi \mathcal{M} \subset \widehat{\mathcal{M}}$ and $q \in \partial_{\phi'} \mathcal{M} \subset \widehat{\mathcal{M}}'$ are *in contact* if there exists a sequence $\{p_i\} \subset \mathcal{M}$ such that$\{\phi(p_i)\}$ has p as an endpoint and $\{\phi'(p_i)\}$ has q as an endpoint.

Definition 93 ($[p] \dashv [q]$)
Let $[p]$ and $[q]$ be a-boundary points. We say that $[p]$ and $[q]$ are *in contact*, denoted $[p] \dashv [q]$ if $p \dashv q$ for representatives p and q.

Definition 94 ($\sigma \rhd p$, $\sigma \dashv p$ and $\sigma \parallel p$)
A partial cross section σ is said to cover the boundary point p if there is a set of points in σ that covers p. σ is in contact with p if there is an a-boundary point in σ that is in contact with p, and σ is separate to p if it is not in contact with p.

Definition 95 (Directional and pure singularities)
An essential singularity p is called a *directional singularity* if it covers a boundary point which is either regular or a point at infinity. Otherwise p is called a *pure singularity*.

Definition 96 (Endpoint of a curve)
We say that p is an *endpoint* of the curve γ if $\gamma(t) \to p$ as $t \to b$.

Definition 97 (Enveloped manifold)
An *enveloped manifold* is a triple $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ where $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are differentiable manifolds of the same dimension and ϕ is a C^∞ embedding $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$. The enveloped manifold is also called an envelopment, where $\widehat{\mathcal{M}}$ is the enveloping manifold.

Definition 98 (Essential singularity)
A singular boundary point p is called *essential* if it is not removable.

Definition 99 (Essential point at infinity)
A point p at infinity is an *essential point at infinity* if it is not removable.

Definition 100 (Extension)
An *extension* of a pseudo-Riemannian manifold $(\mathcal{M}, g)$ is an envelopment of it by a pseudo-Riemannian manifold $(\widehat{\mathcal{M}}, \hat{g})$ such that $\hat{g}|_{\phi(\mathcal{M})} = g$.

Definition 101 (Finite connected neighbourhood region property (FCNR property))
We say that p has n *connected neighbourhood regions* if for any open neighbourhood $\mathcal{N}(p)$ there exists a sub-neighbourhood $\mathcal{U}(p) \subset \mathcal{N}(p)$ for which $\mathcal{U}(p) \cap \phi(\mathcal{M})$ is composed of exactly n connected components, and n is the smallest natural number for which this is true. The boundary point p satisfies the *finite connected neighbourhood region property* if it has only finitely many connected neighbourhood regions.

Definition 102 (Limit point of a curve)
We say that $p \in \overline{\phi(\mathcal{M})}$ is a *limit point* of a curve $\gamma : [a, b) \to \phi(\mathcal{M})$ if there exists an increasing infinite sequence of real numbers $t_i \to b$ such that $\gamma(t_i) \to p$.

Definition 103 (Maximal admissable partial cross section)
An admissable partial cross section is said to be maximal if no admissable partial cross section contains it as a proper subset.

Definition 104 (Maximally extended)
A C^k pseudo-Riemannian manifold $(\mathcal{M}, g)$ is termed C^l maximally extended ($1 \leqslant l \leqslant k$) if there does not exist a C^l extension $(\mathcal{M}, g, \widehat{\mathcal{M}}, \widehat{g}, \phi)$ of $(\mathcal{M}, g)$ such that $\phi(\mathcal{M})$ is an open submanifold of $\widehat{\mathcal{M}}$.

Definition 105 (Mixed point at infinity)
An essential point p at infinity is a *mixed point at infinity* if it covers a regular boundary point.

Definition 106 (Optimal embedding (minimal definition))
Let $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$ be an envelopment. To be an *optimal embedding* at the very least ϕ has to satisfy the condition that the partial cross section

$$\sigma_\phi = \{[p] | p \in \partial_\phi \mathcal{M}\}$$

is a maximal admissable partial cross section.

Definition 107 (Partial cross section σ)
Let $\sigma \subset \mathcal{B}(\mathcal{M})$. σ is termed a *partial cross section* if for every $[p]$, $[q] \in \sigma, [p] \parallel [q]$ or $[p] = [q]$.

Definition 108 (Point at infinity)
A boundary point p of the envelopment $(\mathcal{M}, g, \widehat{\mathcal{M}}, C, \phi)$ is a *point at infinity* if

1. p is not a regular boundary point
2. p is approachable by an element of C, and
3. no curve of C approaches p with bounded parameter.

Definition 109 (Pure point at infinity)
An essential point at infinity is a *pure point at infinity* if it does not cover any regular boundary points.

Definition 110 (Regular *a*-boundary point)
A regular a-boundary point is an equivalence class with a regular point as a representative element.

Definition 111 (Regular partial cross section)
Let σ_r be a set of regular a-boundary points. σ_r will be said to be a *regular partial cross section* if for each $[p] \in \sigma_r$ there is a representative boundary point $q \in [p]$, $q \in \partial_\phi \mathcal{M}$ such that

1. q is regular and,
2. there is an open neighbourhood, $\mathcal{U}(q) \subset \widehat{\mathcal{M}}$ such that

$$\{[r] | r \in \mathcal{U} \cap \partial_\phi \mathcal{M}\} \subset \sigma_r.$$

Definition 112 (Regular point)
A boundary point p of an envelopment $(\mathcal{M},\ g,\ \widehat{\mathcal{M}}, \phi)$ is *regular* if there exists a manifold $(\overline{\mathcal{M}}, \overline{g})$ such that $\phi(\mathcal{M}) \cup \{p\} \subseteq \overline{\mathcal{M}} \subseteq \widehat{\mathcal{M}}$ and $(\mathcal{M}, g, \overline{\mathcal{M}}, \overline{g}, \phi)$ is an extension of $(\mathcal{M},\ g)$.

Definition 113 (Removable point at infinity)
A boundary point p at infinity is termed a *removable point at infinity* if there is a boundary set, $B \subset \partial_\phi \mathcal{M}$ composed purely of regular boundary points such that $B \triangleright p$.

Definition 114 (Removable singularity)
A singular boundary point p will be called *removable* if it can be covered by a non-singular boundary set B.

Definition 115 (Separation of boundary points)
Two boundary points, $p \in \partial_\phi \mathcal{M} \subset \widehat{\mathcal{M}}$ and $q \in \partial_{\phi'} \mathcal{M} \subset \widehat{\mathcal{M}}'$, are termed *separate* (denoted by $p \parallel q$) if they are not in contact. Similarly for a-boundary equivalence classes.

Definition 116 (Singular boundary points)
A boundary point p of an envelopment $(\mathcal{M},\ g, \widehat{\mathcal{M}}, C, \phi)$ is called *singular* or a *singularity* if

1. p is not a regular boundary point,
2. p is a approachable by a curve γ, where γ is an element of C and has finite parameter.

Definition 117 (Strongly causal)
A space-time ($\mathcal{M}$,g) is *strongly causal* at p if every neighbourhood of p contains a neighbourhood of p which no non-spacelike curve intersects in a disconnected set.

Definition 118 (T-niceness)
Let B be a boundary set of an envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ and let T be a topological property. An open neighbourhood of B, $\mathcal{U}(B) \subset \widehat{\mathcal{M}}$, will be called T-nice if $T(\mathcal{U}(B) \cap \phi(\mathcal{M}))$ is true in the relative topology of $\mathcal{U}(B) \cap \phi(\mathcal{M})$.

Definition 119 (Topological neighbourhood property (TNP))
A boundary set B satisfies the *topological neighbourhood property* (TNP) if every open neighbourhood, $\mathcal{U}(B)$, in $\widehat{\mathcal{M}}$, contains a T-nice, open neighbourhood, $\mathcal{V}(B) \subset \widehat{\mathcal{M}}$.

Definition 120 (Total regular partial cross section)
A regular partial cross section σ is termed *total* if it covers every regular abstract boundary point.

Definition 121 (Totality)
A partial cross section σ is total if for every a-boundary point $[p]$ there is a set of points $[B] \in \sigma$ such that $[B] \triangleright [p]$. An envelopment is total if its corresponding partial cross section is total.

Bibliography

[1] Ashley, M. J., (2002) *PhD Thesis.*

[2] Ashley, M. J., Scott, S. M.(2003) 'Curvature singularities and abstract boundary singularity theorems for space-time', to be published in *Contemporary Mathematics.*

[3] Ashley, M. J., Scott, S. M. and Szekeres, P. preprint, unpublished.

[4] Beem, J. K., Ehrlich, P.E. and Easley, K.L. (1996), *Global Lorentzian Geometry*, Marcel Dekker Inc., New York.

[5] C.J.S. Clarke, (1973), 'Local extensions in Singular Space-times', *Commun. Math. Phys.* **32**, 205-214.

[6] Clarke, C. J. S. and Schmidt, B. G. (1977), 'Singularities: The State of the Art', *Gen. Rel. Grav.* **8**, 129-137.

[7] Clarke, C. J. S. and Scott, S. M. (n.d.), preprint. Unpublished.

[8] Dieudonné, (1960), *Foundations of Modern Analysis*, Academic Press Inc., New York.

[9] Dodson, C. T. J. (1978), 'Space-Time Edge Geometry', *Inter. J. Theor. Phys.* **17**(6), 389-504.

[10] Ellis, G. F. R. and Schmidt, B. G. (1979), 'Classification of Singular Space-times', *Gen. Rel. Grav.* **10**(12), 989-997.

[11] Fama, C. J. and Scott, S. M. (1994), 'Invariance properties of boundary sets of open embeddings of manifolds and their application to the abstract boundary',*Contem. Math.* **170**, 79-111.

[12] Geroch, R. (1968a), 'Local characterization of singularities in general relativity', *J. Math. Phys.* **9**(3) 405-465.

[13] Geroch, R. (1968c), 'What is a singularity in general relativity?', *Ann. Phys.* **48**526-540.

[14] Georch, R., Kronheimer, E. H. and Penrose, R. (1972), 'Ideal points in space-time', *Proc. R. Soc. Lond.* **A327**, 545-567.

[15] Hawking, S. W. and Ellis, G. F. R. (1973), *The Large Scale Structure of Space-time*, Cambridge University Press.

[16] Hutchinson, J. and Loy, R (1994), *Introduction to Mathematical Analysis B21H* Lecture notes (unpublished).

[17] Kobayashi, S. and Nomizu, K. (1963), *Foundations of Differential Geometry, Volume I*, Interscience Publishers-Wiley and Sons, New York.

[18] Misner, C. W. (1967), 'Taub-NUT spaces as a counterexample to almost everything', *Relativity Theory and Astrophysics I: Relativity and Cosmology*, ed. J. Ehlers, Lectures in Applied Mathematics, Vol. 8, pp 160-169

[19] Misner, C. W., Thorne, K. S. and Wheeler, J. A. (1973), *Gravitation*, W. H. Freeman and Company, New York: New York.

[20] Penrose, R. (1972), *Techniques of Differential Topology in Relativity*, Society for Industrial and Applied Mathematics, Philadelphia.

[21] Schmidt, B. G. (1971), 'A new definition of singular points in general relativity', *Gen. Rel. Grav.***1**(30), 269-280.

[22] Scott, S. M. and Szekeres, P. (1986a), 'The Curzon singularity. I: Spatial sections', *Gen. Rel. Grav.***18**, 557-570.

[23] Scott, S. M. and Szekeres, P. (1986b), 'The Curzon singularity. II: Global picture', *Gen. Rel. Grav.* **18**, 571-583.

[24] Scott, S. M. and Szekeres, P. (1994), 'The abstract boundary - a new approach to singularities of manifolds', *J. Geom. Phys* **13**, 223-253.

[25] von Westenholz (1978), *Differential Forms in Mathematical Physics*, orth-H lland Publishing Company: New York.