Weak\^* Properties of Weighted Convolution Algebras

Sandy Grabiner\^*
Pomona College

Abstract

Suppose that $L^1(\omega)$ is a weighted convolution algebra on $\mathbb{R}^+ = [0, \infty)$ with the weight $\omega(t)$ normalized so that the corresponding space $M(\omega)$ of measures is the dual space of the space $C_0(1/\omega)$ of continuous functions. Suppose that $\phi : L^1(\omega) \to L^1(\omega')$ is a continuous nonzero homomorphism, where $L^1(\omega')$ is also a convolution algebra. If $L^1(\omega)*f$ is norm dense in $L^1(\omega)$, we show that $L^1(\omega')*\phi(f)$ is (relatively) weak\^* dense in $L^1(\omega')$, and we identify the norm closure of $L^1(\omega')*\phi(f)$ with the convergence set for a particular semigroup. When $\phi$ is weak\^* continuous it is enough for $L^1(\omega)*f$ to be weak\^* dense in $L^1(\omega)$. We also give sufficient conditions and characterizations of weak\^* continuity of $\phi$. In addition, we show that, for all nonzero $f$ in $L^1(\omega)$, the sequence $f^n/||f^n||$ converges weak\^* to 0. When $\omega$ is regulated, $f^{n+1}/||f^n||$ converges to 0 in norm.

\^*The research for this paper was done while the author enjoyed the gracious hospitality of the Australian National University in Canberra.
Weak* Properties of Weighted Convolution Algebras

Sandy Grabiner
Pomona College

1 Introduction

Suppose that $\omega(x)$ is a positive Borel measurable function on $\mathbb{R}^+ = [0, \infty)$. When both $\omega$ and $1/\omega$ are locally bounded on $[0, \infty)$, we say that $\omega$ is a weight. When $\omega(x)$ is a weight, then $L^1(\omega)$ is the Banach space of (equivalence classes of) locally integrable functions $f$ for which $f \omega$ is integrable. We give $L^1(\omega)$ the inherited norm

$$||f|| = ||f||_\omega = ||f\omega||_1 = \int_0^\infty |f(t)|\omega(t)dt.$$ 

Similarly $M(\omega)$ is the analogous space of measures with the norm

$$||\mu|| = ||\mu||_\omega = \int_{\mathbb{R}^+} \omega(t)d|\mu|(t),$$

and $C_0(1/\omega)$ is the space of continuous functions $h$ on $[0, \infty)$ for which $\lim_{x \to \infty} h(x)/\omega(x) = 0$, with the norm $||h(x)|| = ||h/\omega||_\infty = \sup \{h(x)/\omega(x)\}_{x \geq 0}$.

We are particularly interested in the case in which the weight $\omega$ is an algebra weight; that is, $\omega$ is submultiplicative (i.e., $\omega(x+y) \leq \omega(x)\omega(y)$), is everywhere right continuous, and has $\omega(0) = 1$. The submultiplicativity implies that both $L^1(\omega)$ and $M(\omega)$ are Banach algebras under convolution, and that $L^1(\omega)$ is a closed ideal in $M(\omega)$ when we identify the function $f(t)$ with the measure $f(t)d\mu(t)$. The other conditions guarantee that $M(\omega)$ is the dual space of the separable Banach space $C_0(1/\omega)$ under the natural duality $<\mu, h> = \int h(t)d\mu(t)$, [Gr1, Th. 2.2, p. 592] so that $M(\omega)$, and its subspace $L^1(\omega)$, are equipped with a natural weak*-topology. Requiring that $\omega$ be an algebra weight in our sense is just a normalization. Whenever $L^1(\omega)$ is an algebra we can always replace $\omega$ with an equivalent algebra weight without changing the space $L^1(\omega)$ or its norm topology [Gr1, Th 2.1, p. 591].

\footnote{The research for this paper was done while the author enjoyed the gracious hospitality of the Australian National University in Canberra.}
In this paper we examine the structure of $L^1(\omega)$, and particularly homomorphisms between such algebras, in the weak* topology. The weak* topology is in many ways better behaved than the norm topology, and, as we shall see, can be used as a tool in proving results for normed topologies.

In section 2, we collect results characterizing weak* convergence of bounded nets in $M(\omega)$ and relating weak* convergence to convergence in various norms. In sections 3 and 4 we consider continuous non-zero homomorphisms $\phi : L^1(\omega) \to L^1(\omega')$. In previous papers, starting with [GGM], we considered characterizations of sufficient conditions for $\phi$ to be what we called a standard homomorphism; that is, $L^1(\omega) * f$ being norm dense in $L^1(\omega)$ implies $L^1(\omega') * \phi(f)$ is norm dense in $L^1(\omega')$. In section 3 we show that $L^1(\omega') * \phi(f)$ is always weak* dense, and we describe the norm closure of $L^1(\omega') * \phi(f)$. When $\phi$ is weak* continuous, we show that it is enough for $L^1(\omega) * f$ to be weak* dense. In section 4, we give characterizations and useful sufficient conditions for the weak* continuity of $\phi$. In section 5 we give conditions on $\omega$ which guarantee that the sequences $f^{(n+1)}/\|f^n\|$ converge to 0 in norm. This is done by first showing that $f^n/\|f^n\|$ always converges to zero weak*, and then applying results collected in section 2.

2 Weak* Convergent Nets

We first collect, mostly from earlier papers, a number of equivalent characterizations of weak* convergence of nets in $M(\omega)$.

**Theorem 2.1** Suppose that $\omega$ is an algebra weight on $\mathbb{R}^+$, and let $\{\lambda_n\}$ be a bounded net in $M(\omega)$. If $\{\lambda_n\}$ converges to $\lambda$ weak* in $M(\omega)$ then we have:

(a) For all $\nu$ in $M(\omega)$, weak*-lim $\lambda_n * \nu = \lambda * \nu$

(b) For all continuous functions $f$ on $\mathbb{R}^+$ with $f(0) = 0$, the net $\{\lambda_n * f\}$ converges pointwise to $\lambda * f$.

(c) If $\omega'$ is a weight with $\omega'/\omega$ bounded and integrable and if $f$ belongs to $L^1(\omega)$, then $\{\lambda_n * f\}$ converges $\lambda * f$ in the norm of $L^1(\omega')$.

(d) For all locally integrable $f$ and all $a > 0$, $\lim \int_0^a |\lambda_n * f(t) - \lambda * f(t)| dt = 0$. 


Conversely, if one of the conditions (a), (b), (c), or (d) holds for a single non-zero function (or measure), then \{\lambda_n\} converges weak* to \lambda in M(\omega).

The proof that (a) is equivalent to weak* convergence is in [Gr1, Lemma (2.2)]. Part (b) is proved in [GG1, Th. (3.1)(a), p. 511]. The proof is given for sequences, but the same proof works for nets. Part (c), for sequences, is [GG1, Th. (3.2), pg. 512]; but [GG3, Th. (1.3)] shows that if (c) holds for some \(f\) and all weak*-convergent sequences, then convolution by \(f\) is a compact operator from \(M(\omega)\) to \(L^1(\omega')\). This then implies that (c) also holds for all bounded weak*-convergent nets. Part (d) is an easy consequence of (c) [GG1, Cor. (3.3), p. 513].

Once we know that some type of convergence, as in (a), (b), (c), or (d), follows from weak* convergence, the proof of the converse follows from the fact that every bounded net has a weak* convergent subnet, together with the fact that the convolution of non-zero measures on \(\mathbb{R}^+\) can never be zero. For the details see [Gr1, Lemma 3.2, p. 595] or [Gr3, Th. (4.1), p. 183].

There are numerous other useful characterizations of weak* convergence. For instance [GG2, p. 52], it is enough for \(\lim_n <\lambda_n, h> = <\lambda, h>\) for all continuous \(h\) with compact support, since the set functions with compact support are dense in \(C_0(1/\omega)\).

The nicest results occur when all \(\lambda_n * f\) converge to \(\lambda * f\) in the norm of \(L^1(\omega)\). Recall that the algebra weight \(\omega(t)\) is regulated at \(b \geq 0\) if \(\lim_{x \to -\infty} \omega(x+a)/\omega(x) = 0\) for all \(a > b\). Recall also that if \(\lambda\) is a locally integrable function or a locally finite measure of \(\mathbb{R}^+\), then \(\alpha(\lambda)\) is the infimum of the support of \(\lambda\) (\(\alpha(0) = \infty\)). The basic result relating convergence to weak* convergence in \(M(\omega)\) is the following result, taken from [GGM, Th. (3.2), p. 284], [GG1, Th. (2.3), p. 509].

**Theorem 2.2** Suppose that \(\omega\) is an algebra weight on \(\mathbb{R}^+\) and that \(b \geq 0\). Then the following are equivalent.

(a) \(\omega\) is regulated at \(b\).

(b) Whenever \(\{\lambda_n\}\) is a bounded net converging weak* to \(\lambda\) in \(M(\omega)\) and \(g\) is a function in \(L^1(\omega)\) with \(\alpha(g) \geq b\), then \(\lambda_n * g\) converges to \(\lambda * g\) in the norm of \(L^1(\omega)\).

Condition (c) in Theorem (2.1) is the simplest condition on \(\omega'/\omega\) which guarantees convergence in norm in \(L^1(\omega')\). A determination of precisely
which \( \omega' / \omega \) work is given in [GG3]. The most important case, \( \omega' = \omega \), is Theorem (2.2) above.

Theorem (2.1)(a) says that multiplication is weak* separately continuous on bounded subsets of \( M(\omega) \). In fact it is not hard to show [Gr1, Lemma 3.1, p. 595] that multiplication is weak* separately continuous on all of \( M(\omega) \). The following result shows that on bounded subsets of \( M(\omega) \), multiplication is actually jointly continuous in the weak* topology. Since the weak* topology restricted to bounded subsets of \( M(\omega) = C_0(1/\omega)^* \) is metrizable [DS, Th. V.5.1, p. 426], we need only consider sequences in the following result.

**Theorem 2.3** Suppose that \( \omega \) is an algebra weight and that \( \{\lambda_n\} \) and \( \{\mu_n\} \) are sequences in \( M(\omega) \). If weak*\(-\lim \lambda_n = \lambda \) and weak*\(-\lim \mu_n = \mu \), then weak*\(-\lim \lambda_n * \mu_n = \lambda * \mu \).

**Proof:** Choose some nonzero \( f \) in \( L^1(\omega) \) and let \( \omega' \) be as in Theorem (4.9)(c); for instance, we could let \( \omega'(t) = e^{-t} \omega(t) \). Then \( \lambda_n * f \to \lambda * f \) and \( \mu_n * f \to \mu * f \) in norm in the Banach algebra \( L^1(\omega') \). Hence the \( L^1(\omega') \) norm limit of \( (\lambda_n * \mu_n) * (f * f) \) is \( (\lambda * \mu) * (f * f) \). It then follows from Theorem (2.1) that \( \lambda_n * \mu_n \to \lambda * \mu \) in the weak* topology on \( M(\omega) \).

### 3 Weak*-Standard Homomorphisms

Throughout this section, \( \omega \) and \( \omega' \) are algebra weights and \( \phi : L^1(\omega) \to L^1(\omega') \) is a continuous nonzero homomorphism. Then \( \phi \) has a unique extension to a homomorphism from \( M(\omega) \) to \( M(\omega') \) and this extension is continuous with the same norm [Gr1, Th. 3.4, p. 596]. Because of the uniqueness, we let \( \phi \) denote both the original map and its extension. The homomorphism \( \phi \) is said to be standard [GGM, p. 278] if \( L^1(\omega') * \phi(f) \) is norm dense in \( L^1(\omega') \) whenever \( L^1(\omega) * f \) is norm dense in \( L^1(\omega) \). In [GGM] we gave several equivalent characterizations of the standardness of homomorphisms and we showed [GGM, Th. (3.4), p. 284] that \( \phi \) is standard if \( \omega' \) is regulated at any \( b \geq 0 \); that is, if \( \lim_{x \to -\infty} \omega'(x + a)/\omega'(x) = 0 \) for any \( a > 0 \). In this section we show that \( L^1(\omega') * \phi(f) \) is always weak* dense when \( L^1(\omega) * f \) is norm dense.

Let \( \{\delta_t\}_{t \geq 0} \) be the convolution semigroup of point masses, so that \( \delta_t * f(x) = f(x - t) \), the right translation of \( f \); and let \( \mu_t = \phi(\delta_t) \). Following [GGM, p. 280] we call

\[
I = \{ g \in L^1(\omega') : \lim_{t \to 0} \mu_t * g = g \}
\]
the convergence ideal of $\phi$ (or of the semigroup $\{\mu_t\}$). Since $\{\mu_t\}$ is norm bounded near 0, the set $I$ is easily seen to be a closed ideal. One of the characterizations of standardness of $\phi$ is that $I = L^1(\omega')$; that is, that (convolution by) $\mu_t$ is a strongly continuous semigroup on $L^1(\omega')$ [GGM, Th. (2.2)(a), p. 280]. We are now ready for our main result.

**Theorem 3.1** Suppose that $\omega$ and $\omega'$ are algebra weights and that $\phi: L^1(\omega) \to L^1(\omega')$ is a continuous nonzero homomorphism. If $L^1(\omega) * f$ is norm dense in $L^1(\omega)$, then we have:

(a) The norm closure of $L^1(\omega') * \phi(f)$ is the convergence ideal of $\phi$.

(b) $L^1(\omega') * \phi(f)$ is weak* dense in $L^1(\omega')$.

**Proof:** Since we know [Gr2, Cor. (2.5), p. 162] that the convergence ideal of $\phi$ is weak* dense, it will be enough to prove (a). We also know [GGM, Th. (2.4), pp. 281-282] that there exist $f$, specifically $f(t) = e^{-rt}$, with $L^1(\omega) * f$ dense and the norm closure $L^1(\omega') * \phi(f)$ equalling the convergence ideal of $\phi$. It is also easy to see that all $L^1(\omega') * \phi(g)$ belong to the convergence ideal of $\phi$ [GGM, p. 282]. To complete the proof to the theorem, we just need the following lemma.

**Lemma 3.2** If $L^1(\omega) * f$ is norm dense in $L^1(\omega)$, then the norm closure of $L^1(\omega') * \phi(f)$ contains the norm closure of $L^1(\omega') * \phi(g)$ for all $g$ in $L^1(\omega)$.

**Proof:** Since $L^1(\omega) * f$ is dense, we can find a sequence $\{h_n\}$ in $L^1(\omega)$ with $\lim (f * h_n) = g$, with the limit taken in the norm topology. By the continuity of $\phi$, this implies that $\lim \phi(f) * \phi(h_n) = \phi(g)$. Hence $\phi(g)$, and therefore the norm closure of $L^1(\omega') * \phi(g)$ as well, belongs to the norm closure of $L^1(\omega') * \phi(f)$. This completes the proof of the lemma, and of Theorem (3.1).

Notice that not only the theorem, but also the lemma, show that the norm closure of $L^1(\omega') * \phi(f)$ is the same for all $f$ in $L^1(\omega)$ with $L^1(\omega) * f$ norm dense. This greatly simplifies the formulas we were able to obtain in [GGM, p. 282].

The natural weak* analogue of standardness of homomorphisms should only assume that $L^1(\omega) * f$ is weak* dense in $L^1(\omega)$ rather than norm dense. The next result shows that this natural analogue does hold when the homomorphism is weak* continuous rather than just norm continuous. In Section 4, we will study when homomorphisms are weak* continuous.
Theorem 3.3 Suppose that the nonzero homomorphism \( \phi : L^1(\omega) \to L^1(\omega') \) is weak* continuous. If \( L^1(\omega) * f \) is weak* dense in \( L^1(\omega) \), then \( L^1(\omega') * \phi(f) \) is weak* dense in \( L^1(\omega') \).

Proof: Choose some \( g \), say \( g = e^{-rt} \), with \( L^1(\omega) * g \) norm dense. Since we already know that \( L^1(\omega) * f \) is weak* dense, it will be enough to show that \( \phi(g) \) is the weak* limit of a sequence in \( L^1(\omega') * \phi(f) \). Since \( L^1(\omega) * f \) is weak*-dense, we can find a sequence \( \{h_n\} \) in \( L^1(\omega) \) with weak*-lim(\( f * h_n \)) = \( g \). Since \( \phi \) is a weak* continuous homomorphism, this shows that \( \phi(f * h_n) = \phi(f) * \phi(h_n) \) converges weak* to \( \phi(g) \) in \( L^1(\omega') \). This completes the proof.

The major unsolved question in the ideal theory of radical \( L^1(\omega) \) is the standard ideal problem, which asks if \( L^1(\omega) * f \) must be norm dense for all \( f \) in \( L^1(\omega) \) with \( \alpha(f) = 0 \) [D, p. 557] [GG1, Question 1, p. 507]. The results in this section suggest the following, presumably easier, weak* analogue.

Question 3.4 Suppose that \( L^1(\omega) \) is a radical algebra and that \( f \) in \( L^1(\omega) \) has \( \alpha(f) = 0 \). Must \( L^1(\omega) * f \) be weak* dense in \( L^1(\omega) \)?

When \( \omega \) is regulated at any \( b \geq 0 \), \( L^1(\omega) * f \) is norm dense if it is weak* dense [GG3, Th. (5.1)(b)] [BD, Prop. 1.9, p. 72]. Hence an affirmative answer to Question (3.4) would solve the standard ideal problem for regulated weights. Of course a negative answer to Question (3.4) would also be a negative answer to the standard ideal problem.

4 Weak*-Continuous Homomorphisms

As in the previous section, we let \( \phi : L^1(\omega) \to L^1(\omega') \) be a continuous nonzero homomorphism, where \( \omega \) and \( \omega' \) are algebra weights. In this section we give sufficient conditions and characterizations of \( \phi \) being weak* continuous. We start with some preliminary results. The following result is essentially a variant of the Krein-Smulian Theorem.

Lemma 4.1 Let \( E \) and \( F \) be Banach spaces and let \( T : F^* \to E^* \) be a linear map. Then \( T \) is weak* continuous if weak*-lim \( T(\lambda_n) = T(\lambda) \) whenever \( \{\lambda_n\} \) is a bounded net with weak* limit \( \lambda \). When \( F \) is separable, it is enough to consider only bounded sequences.
Proof: It follows from the Krein-Smulian Theorem [DS, Th. V.5.7, p. 429] that it is enough to show that the restriction of $T$ to closed balls is weak* continuous. But this translates to the statement about bounded nets in the theorem. When $F$ is separable, then the weak* topology on closed balls of $F^*$ is metrizable [DS, Th. V.5.1, p. 426], so one only needs to consider sequences to prove continuity.

As one application of the above lemma we show that if $\phi$ is weak* continuous, then so is its extension to the corresponding measure algebras, just as with norm continuity.

**Lemma 4.2** If $\phi : L^1(\omega) \to L^1(\omega')$ is a nonzero weak*-continuous homomorphism, then so is its extension to a homomorphism from $M(\omega)$ to $M(\omega')$.

**Proof:** Let $\{\lambda_n\}$ be a bounded sequence (or net) in $M(\omega)$ with weak* limit $\lambda$. Let $f$ be a nonzero element of $L^1(\omega)$ with $\phi(f) \neq 0$. Then, by Theorem (2.1), $\lambda_n \ast f$ converges weak* to $\lambda \ast f$ in $L^1(\omega)$. By the weak* continuity of $\phi$ on $L^1(\omega)$, this means that $\phi(\lambda_n \ast f) = \phi(\lambda_n) \ast \phi(f)$ converges weak* to $\phi(\lambda) \ast \phi(f)$. Since a weak* continuous map, like $\phi$, must also be norm continuous, the sequence $\{\phi(\lambda_n)\}$ is bounded. It then follows, from Theorem (2.1) again, that weak*-$\lim \phi(\lambda_n) = \phi(\lambda)$. By Lemma (4.1), this implies that $\phi : M(\omega) \to M(\omega')$ is weak* continuous, and thus completes the proof.

We will need the following simple result both in this section and the next section.

**Lemma 4.3** Suppose that $\{\lambda_n\}$ is a net in $M(\omega)$. If $\lim \alpha(\lambda_n) = \infty$, then we have

(a) $|\lambda_n|(\{0,a\}) \to 0$ for all $a > 0$.

(b) If $\{\lambda_n\}$ is bounded, then weak*-$\lim \lambda_n = 0$.

**Proof:** Part (a) is clear, since $|\lambda_n|(\{0,a\}) = 0$ for all sufficiently large $n$. Similarly, if $h$ is a continuous function with compact support, $\lim < \lambda_n, h > = 0$. When $\{\lambda_n\}$ is bounded, the weak*-convergence of $\{\lambda_n\}$ then follows from the remarks after the proof of Theorem (2.1).

The convergence in part (a) is much stronger than weak* convergence. For instance $\delta_{1/n} - \delta_0$ converges to 0 weak* in every $M(\omega)$, but $|\delta_{1/n} - \delta_0|(\{0,a\}) = 2$ for every $a > 1$.

We now give two different sufficient conditions for weak* continuity of $\phi : L^1(\omega) \to L^1(\omega')$. These results are in part motivated by Theorem (3.3)
above, which essentially says that weak*-continuous homomorphisms are weak*-standard. First we give a simple proof of our earlier result [GGM, Th. (3.5), p. 285], which says that $\phi$ is weak* continuous if $\omega$ is regulated at any $b \geq 0$. In this case Theorem (3.3) does not improve on Theorem (3.1) because, when $\omega$ is regulated, if $L^1(\omega) * f$ is weak* dense then it must also be norm dense [GG3, Th. (5.1)(b)].

**Theorem 4.4** Suppose that $\omega$ and $\omega'$ are algebra weights. If $\omega$ is regulated at some $b \geq 0$, then every continuous homomorphism $\phi : L^1(\omega) \to L^1(\omega')$ is weak* continuous.

**Proof:** Without loss of generality, we can assume that $\phi$ is not the zero homomorphism, so that we can apply Lemma (4.1). Let $\{\lambda_n\}$ be a bounded sequence (or net) in $M(\omega)$ with weak*-lim $\lambda_n = \lambda$. Choose $g$ in $L^1(\omega)$ with $\phi(g) \neq 0$ and $\alpha(g) \geq b$ (for instance, choose any $f$ with $\phi(f) \neq 0$ and let $g = \delta_b * f$). By Theorem (2.2), $\lambda_n * g \to \lambda * g$ in the norm of $L^1(\omega)$. Since $\phi$ is norm continuous, this implies that $\phi(\lambda_n * g) = \phi(\lambda_n) * \phi(g)$ converges to $\lambda * g$ in norm and hence weak*. It then follows from Theorem (2.1) that $\phi(\lambda_n) \to \phi(\lambda)$ weak*. The theorem now follows from Lemma (4.1).

For our other sufficient condition for weak* continuity, we will need to recall some terminology and results from [Gr1]. Suppose that $\phi : L^1(\omega) \to L^1(\omega')$ is a nonzero homomorphism and let $\mu_0 = \phi(\delta_t)$. Then there is a nonnegative number $A$ for which $\alpha(\mu_t) = A t$ [Gr1, Th. (4.3)(a), p. 605][Gh, Lem. 1, p. 344]. We call $A$ the character of $\phi$, and of $\{\mu_t\}$. When the character $A$ is strictly positive, one also has [Gr1, Th. 4.9, p. 607] $\alpha(\phi(\lambda)) = A \alpha(\lambda)$ for all $\lambda$ in $M(\omega)$.

In our proof of the weak* continuity of homomorphisms of positive character, we will use Theorem (2.1)(d). The following notation will be convenient for this purpose. For each $a > 0$, we define the seminorm $||f||_a = \int_0^a |f(t)| dt$ on $L^1_{loc}$, the space of locally integrable functions on $\mathbb{R}^+$. Thus Theorem (2.1)(d) says that $\lambda_n * f$ converges to $\lambda * f$ in each of these seminorms. Similarly, Lemma (4.3)(a) says that $\lambda_n \to 0$ in the analogous seminorms on $M_{loc}(\mathbb{R}^+)$, the space of locally finite Borel measures on $\mathbb{R}^+ = [0, \infty)$. We can now prove:

**Theorem 4.5** Let $\phi : L^1(\omega) \to L^1(\omega')$ be a continuous nonzero homomorphism. If the character $A$ of $\phi$ is strictly positive, then $\phi$ is weak* continuous.


**Proof:** We first show that for each \( a > 0 \) there is an \( M = M(a) > 0 \) for which

\[ ||\phi(f)||_{Aa} \leq M||f||_a \tag{4.1} \]

for each \( f \) in \( L^1(\omega) \). Let \( L^1(\omega)_a = \{ f \in L^1(\omega) : \alpha(f) \geq a \} = \{ f \in L^1(\omega) : ||f||_a = 0 \} \) and define \( L^1(\omega')_{Aa} \) analogously. Since \( \phi(L^1(\omega)_a) \subseteq L^1(\omega')_{Aa} \), it follows that \( \phi \) induces a continuous map from the quotient Banach algebra \( L^1(\omega)/L^1(\omega)_a \) to the quotient algebra \( L^1(\omega')/L^1(\omega')_{Aa} \). Since \( \omega \) is bounded and bounded below on \([0,a)\), the quotient norm is equivalent to the norm induced by the seminorm \( f \mapsto ||f||_a \), and similarly for the norm induced by the seminorm \( g \mapsto ||g||_{Aa} \) on the quotient of \( L^1(\omega') \). Formula (4.1) is now just the statement that the map induced by \( \phi \) between the quotient spaces is bounded.

Now suppose that \( \lambda_n \) is a bounded sequence or net in \( M(\omega) \) which converges weak* to \( \lambda \). Choose \( f \in L^1(\omega') \) with \( \phi(f) \neq 0 \) (actually \( \phi \) has kernel \{0\} [Gr1, Appendix, p. 613]). By Theorem (2.1)(d), each \( ||\lambda_n * f - \lambda * f||_a \) converges to 0. By formula (4.1), this implies that each \( ||\phi(\lambda_n) * \phi(f) - \phi(\lambda) * \phi(f)||_\mu \) converges to 0. By Theorem (2.1), this means \( \phi(\lambda_n) \to \phi(\lambda) \) weak* in \( M(\omega') \). By Lemma (4.1) this implies that \( \phi \) is weak* continuous, and therefore completes the proof of the theorem.

The algebra \( L^1_{loc}(\mathbb{R}^+) \) of locally integrable functions on \( \mathbb{R}^+ \) is a Fréchet algebra under the seminorms \( || \cdot ||_a \) for \( a > 0 \). Thus Theorem (2.1)(d) says that \( \lambda_n * f \) converges to \( \lambda * f \) in the Fréchet topology on \( L^1_{loc}(\mathbb{R}^+) \), and formula (4.1) says that \( \phi \) is continuous in this topology, relativized by \( L^1(\omega) \) and \( L^1(\omega') \). The algebra \( L^1_{loc}(\mathbb{R}^+) \), and particularly its automorphisms and derivations, is studied by Ghahramani and McClure in [GhM]. In a paper in preparation I will study the continuous homomorphisms of \( L^1_{loc}(\mathbb{R}^+) \).

Homomorphisms of positive character seem to be better behaved than homomorphisms of character 0. Thus for positive character \( A \) we have \( \alpha(\phi(\mu)) = A\alpha(\mu) \) which implies that \( \phi \) is one-to-one [Gr1, Appendix, p. 613]. By the previous theorem and its proof we also know that \( \phi \) is weak* continuous (so that Theorem (3.3) applies) and is continuous in the (relativized) Fréchet topology on \( L^1_{loc}(\mathbb{R}^+) \). While much is known for character 0 [Gr1], the following natural question is open (for partial results see [Gr1, Th. 4.11, p. 608]).

**Question 4.6** If \( \phi : L^1(\omega) \to L^1(\omega') \) has character 0, must \( \alpha(\phi(\mu)) = 0 \) for all \( \mu \) in \( M(\omega) \)?
If the answer to Question (4.6) is yes, then $\phi$ would be one-to-one, and hence all continuous nonzero homomorphisms $\phi : L^1(\omega) \to L^1(\omega')$ would be one-to-one. For a discussion of known results on when $\phi$ is one-to-one, see [Gr3, Section 5].

We now give a relatively simple characterization of weak* continuity of homomorphisms.

**Theorem 4.7** Suppose that $\omega$ and $\omega'$ are algebra weights and that $\phi : L^1(\omega) \to L^1(\omega')$ is weak* continuous, and let $\mu_t = \phi(\delta_t)$. Then $\phi$ is weak* continuous if and only if $\text{weak*}-\lim_{x \to \infty} \mu_x/\omega(\cdot) = 0$.

We first separate out the direction that assumes that $\phi$ is weak* continuous, and we determine the pre-adjoint of $\phi$ in this case.

**Theorem 4.8** Suppose that $\phi : L^1(\omega) \to L^1(\omega')$ is a weak*-continuous nonzero homomorphism. Then we have

(a) $\text{weak*}-\lim_{x \to \infty} \mu_x/\omega(\cdot) = 0$.

(b) $\phi$ is the adjoint of the map $T : C_0(1/\omega') \to C_0(1/\omega)$ given by $Th(x) = \langle \mu_x, h \rangle$.

**Proof:** We first observe that we always have $\text{weak*}-\lim_{x \to \infty} \delta_x/\omega(\cdot) = 0$ in $M(\omega)$. For suppose that $h$ belongs to the predual $C_0(1/\omega)$. Then $\langle \delta_x/\omega(\cdot), h \rangle = h(x)/\omega(x)$ approaches 0 as $x \to \infty$, by the definition of $C_0(1/\omega)$. Now when $\phi$ is weak* continuous we therefore have that $\phi(\delta_x/\omega(x)) = \mu_x/\omega(x)$ has weak* limit 0 as $x$ goes to $\infty$. This proves (a).

Since $\phi$ is weak* continuous, there is some bounded linear map $L : C_0(1/\omega') \to C_0(1/\omega)$ with $\phi = L^*$. We just need to show that $L$ equals $T$ as defined in (b). Choose an $h$ in $C_0(1/\omega')$. Then for all $x \geq 0$ we have

$$Lh(x) = \langle \delta_x, Lh \rangle = \langle L^* \delta_x, h \rangle = \langle \phi(\delta_x), h \rangle = Th(x),$$

as required. This completes the proof of Theorem (4.8).

**Proof of Theorem (4.7):** Suppose that $\mu_x/\omega(x)$ approaches 0 weak* as $x \to \infty$. We need to show that $\phi$ is weak* continuous. We do this by first showing that the map $T$ of Theorem (4.9)(b) is a bounded linear map and then showing that $\phi = T^*$.

For $h$ in $C_0(1/\omega')$ we define the function $Th$ on $\mathbb{R}^+$ by $Th(x) = \langle \mu_x, h \rangle$. Since $\mu_x$ is weak* continuous [Gr1, Th. 3.6(A), p. 599], $Th$ is a continuous
function. Since \(Th(x)/\omega(x) = \langle \mu_x/\omega(x), h \rangle\), we have \(Th\) in \(C_0(1/\omega)\) by our assumption that \(\mu_x/\omega(x) \to 0\) weak*. We now need to show that the linear map \(T : C_0(1/\omega') \to C_0(1/\omega)\) is bounded. Since all \(\delta_x/\omega(x)\) are unit vectors in \(L^1(\omega)\), we have, for each \(x \geq 0\), that \(|Th(x)/\omega(x)| \leq \|\mu_x\|/\omega(x)\|h\| \leq \|\phi\| \|h\|\), where the norm of \(h\) is taken in \(C_0(1/\omega')\). Thus \(|Th| = \sup |Th(x)/\omega(x)| \leq \|\phi\| \|h\|\), so that \(T\) is bounded.

Now, for each \(f\) in \(L^1(\omega)\), we have \(\phi(f) = \int_0^\infty f(t)\mu_0 dt\) as a weak* integral on \(L^1(\omega')\) [Gr1, form. (3.7), p. 599]. This means that for each \(h\) in \(C_0(1/\omega')\), we have

\[<\phi f, h> = \int_0^\infty f(t) <\mu_t, h> dt = \int_0^\infty f(t)Th(t) = <f, Th> = <T^* f, h>.\]

Thus \(\phi = T^*\), so \(\phi\) is weak* continuous. This completes the proof of Theorem (4.7).

Verifying the condition that \(\mu_x/\omega(x) \to 0\) weak* should usually be easier to do than directly proving that \(\phi\) is weak* continuous. For instance, if \(\phi : L^1(\omega) \to L^1(\omega')\) is a continuous homomorphism with positive character, then \(\mu_x/\omega(x)\) is a bounded net with \(\lim_{x \to \infty} \alpha(\mu_x/\omega(x)) = 0\). It then follows from Lemma (4.3) that \(\mu_x/\omega(x) \to 0\) weak* as \(x\) goes to \(\infty\).

## 5 Normalized Powers

There have been several papers which have considered the sequences \(\omega_n = \|f^n\|\) of norms of powers and the sequence \(f^n/\|f^n\|\) of normalized powers of elements \(f\) of radical Banach algebras [A][W][S][LRRW]. There are two extreme cases [LRRW, Cor. 2.5]. Solovej [S] showed that for \(f\) in the Volterra algebra \(L^1[0,1]\) with \(\alpha(f) = 0\), we always have \(\lim_{n \to \infty} \omega_n = 0\), so that the sequence \(\{\omega_n\}\) is regulated at 1 in the sense of [BDL]. Loy, et al. [LRRW] construct and study \(f\) for which \(f^n/\|f^n\|\) has a subsequence which is a bounded approximate identity. This is the key part of their construction of a weakly amenable commutative radical Banach algebra. We show that the situation in \(L^1(\omega)\) is closer to the first extreme. We always have weak*-lim \((f^n/\|f^n\|) = 0\), and when \(\omega(t)\) is regulated we also have \(\lim(f^n/\|f^n\|) = 0\).

**Theorem 5.1** Suppose that \(\omega\) is an algebra weight. For all nonzero \(f\) in \(L^1(\omega)\), the sequence \(f^n/\|f^n\|\) converges to 0 in the weak* topology of \(L^1(\omega)\).
Proof: To simplify the notation, we let $g_n = f^n / ||f^n||$. If $\alpha(f) > 0$, then $\lim \alpha(g_n) = 0$; so $g_n \to 0$ weak* by Lemma (4.3)(b). Now suppose that $\alpha(f) = 0$. If $\omega(x) \geq C$ on $[0,a)$, then $||f^n||_\omega \geq C ||f^n||_a$. Hence it follows easily from Solovej’s result [S], and its obvious generalization to all $L^1[0,a)$, that $||g_n * f||_a = \int_0^a |g_n * f(t)| dt$ converges to 0 for all $a > 0$. Then $g_n \to 0$ weak*, by Theorem (2.1).

A direct application of Theorem (2.2) yields the following corollary.

Corollary 5.2 Suppose that $\omega(t)$ is an algebra weight which is regulated at $b \geq 0$. For all $f$ in $L^1(\omega)$ with $\alpha(f) \geq b$, we have $\lim \left(\frac{f^{n+1}}{||f^n||}\right) = 0$.

References


