Effective Actions of SU_n on Complex n-dimensional Manifolds*†

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For $n \geq 2$ we classify all connected n-dimensional complex manifolds admitting effective actions of the special unitary group SU_n by biholomorphic transformations.

0 Introduction

We are interested in the classification all connected complex manifolds M of dimension $n \geq 2$ admitting effective actions of the special unitary group SU_n by biholomorphic transformations. For the full unitary group U_n such a classification was obtained in [IKruzh]. One motivation for our study there was the following characterization of the complex space $\mathbb{C}^n$ obtained as a result of the classification. Let M be a connected complex manifold of dimension n and assume that the group $\mathrm{Aut}(M)$ of all holomorphic automorphisms of M equipped with the compact open topology is isomorphic to $\mathrm{Aut}(\mathbb{C}^n)$ as a topological group, then M is biholomorphically equivalent to $\mathbb{C}^n$. In fact, the above characterization holds if $\mathrm{Aut}(M)$ merely contains a subgroup isomorphic to the group of affine isometries of $\mathrm{Aut}(\mathbb{C}^n)$.

It appears, however, to be of independent interest to determine all 'rotationally symmetric' complex manifolds. From this point of view it is more natural to consider SU_n-actions rather than U_n-actions. On the other hand the existence of a 1-dimensional center in U_n was of considerable help in [IKruzh]. The center of SU_n is discrete and $\dim SU_n = n^2 - 1$. Due to this, our arguments in the case of SU_n are more complicated than for U_n.

By an argument similar to that in [IKruzh] we find all dimensions that orbits of an SU_n-action on M can *a priori* have. It turns out (see Proposition 1.1) that an orbit can be a point (and therefore SU_n has a fixed point in M), a real hypersurface in M, a complex hypersurface in M, or the whole of M (in which case M is homogeneous).

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Manifolds admitting actions with fixed point are the easiest to describe. In Section 2 we carry this out using the results of [GK] and [BDK] (Theorem 2.1).

In Section 3 we classify manifolds with SU_n-actions such that all their orbits are real hypersurfaces. We show that for $n \geq 3$ such a manifold is either a spherical layer in $\mathbb{C}^n$, or a Hopf manifold, or the quotient of one of these manifolds by the action of a discrete subgroup of the center of U_n (Theorem 3.7). This result is similar to the case of U_n. For $n = 2$, however, the situation is more interesting. Apart from the above manifolds the classification in this case also includes spherical layers in $\mathbb{C}^2$ with a non-standard complex structure inherited from the non-standard complex structure on $\mathbb{CP}^2 \setminus \{0\}$ introduced in [R1] (Theorem 3.9). This structure is still invariant under the ordinary action of SU_2, but the induced CR-structure on the orbits (spheres with center at the origin) is not spherical.

In Section 4 we consider the situation when each orbit is a real or a complex hypersurface in M and show that there can exist at most two orbits that are complex hypersurfaces. As in the case of U_n, such orbits turn out to be biholomorphically equivalent to $\mathbb{CP}^{n-1}$ and, for $n \geq 3$, can only arise either as a result of blowing up $\mathbb{C}^n$ or a ball in $\mathbb{C}^n$ at the origin, or adding the hyperplane $\infty \in \mathbb{CP}^n$ to the exterior of a ball in $\mathbb{C}^n$, or blowing up $\mathbb{CP}^n$ at one point, or taking the quotient of any of these examples by the action of a discrete subgroup of the center of U_n (Theorem 4.5). For $n = 2$ the classification also includes the exterior of a ball in $\mathbb{CP}^2 \setminus \{0\}$ with non-standard complex structure to which the hyperplane $\infty \in \mathbb{CP}^2$ is added (Theorem 4.6).

Finally, in Section 5 we consider the homogeneous case. Analysis more detailed than in the proof of Proposition 1.1 shows that there exist in fact no n-dimensional complex manifolds admitting transitive actions of SU_n.

Thus, Theorem 2.1, Theorem 3.7, Theorem 3.9, Theorem 4.5, and Theorem 4.6 provide a complete list of connected manifolds of dimension $n \geq 2$ admitting effective actions of SU_n by biholomorphic transformations.

Our proofs involve algebraic arguments. In particular, an important issue is the description of all connected closed subgroups of SU_n of dimension $\geq n^2 - 2n - 1$ (see [IKruzh] for the corresponding statements in the case of U_n). All connected closed subgroups of SU_n of dimension $\geq n^2 - 4n + 8$ for $n \geq 7$ have been already determined in [U]. This was used in [U] to classify actions of SU_n on compact orientable real manifolds of dimension $2n$

for $n \geq 5$. That classification yields, in particular, Theorem 5.2 for $n \geq 5$. All our arguments here are independent of [U]. We also remark that the literature on SU_n-actions on real manifolds is quite extensive, and we do not attempt to survey it in this paper.

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1 The Dimensions of Orbits

In this section we prove the following proposition, which is similar to Proposition 1.1 from [IKruzh].

Proposition 1.1 *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations. Let $p \in M$ and let $O(p)$ be the SU_n-orbit of p. Then $O(p)$ is either*
(i) the whole of M (and therefore M is compact), or
(ii) a single point, or
(iii) a compact complex hypersurface in M, or
(iv) a compact real hypersurface in M.

Proof: For $p \in M$ let I_p be the isotropy subgroup of SU_n at p, i.e., $I_p := \{g \in SU_n : gp = p\}$. We denote by Ψ the continuous homomorphism of SU_n into $\mathrm{Aut}(M)$ (the group of biholomorphic automorphisms of M) induced by the action of SU_n on M. Let $L_p := \{d_p(\Psi(g)) : g \in I_p\}$ be the linear isotropy subgroup, where $d_p f$ is the differential of a map f at p. Clearly, L_p is a compact subgroup of $GL(T_p(M), \mathbb{C})$. Since the action of SU_n is effective, L_p is isomorphic to I_p. The isomorphism is given by the map

$$\alpha : I_p \to L_p, \qquad \alpha(g) := d_p(\Psi(g)). \tag{1.1}$$

Let $V \subset T_p(M)$ be the tangent space to $O(p)$ at p. Clearly, V is L_p-invariant. We assume now that $O(p) \neq M$ (and therefore $V \neq T_p(M)$) and consider the following three cases.

Case 1. $d := \dim_{\mathbb{C}}(V + iV) < n$.

Since L_p is compact, one can choose coordinates in $T_p(M)$ such that $L_p \subset U_n$. Further, the action of L_p on $T_p(M)$ is completely reducible and the subspace $V + iV$ is invariant under this action. Hence L_p can in fact be embedded in $U_d \times U_{n-d} \subset GL(T_p(M), \mathbb{C})$. Since $\dim O(p) \le 2d$, it follows that

$$n^2 - 1 \le d^2 + (n-d)^2 + \dim O(p) \le d^2 + (n-d)^2 + 2d,$$

and therefore either $d = 0$ or $d = n-1$. If $d = 0$, then we obtain (ii). If $d = n-1$, then, in addition, either $\dim O(p) = 2n-2$ or $\dim O(p) = 2n-3$.

Assume first that $\dim O(p) = 2n-2$. Then we have $iV = V$, which yields (iii). Suppose now that $\dim O(p) = 2n-3$. In this case $\dim I_p = n^2 - 2n + 2$. Since L_p can be embedded in $U_1 \times U_{n-1}$, it follows that L_p — and hence I_p — are isomorphic to $U_1 \times U_{n-1}$. It is now clear from Lemma 2.1 of [IKran] that I_p is conjugate to $U_1 \times U_{n-1}$ (realized in the block-diagonal form in the obvious way). But this is impossible since I_p is not contained in SU_n in this case. Hence, in fact, $\dim O(p) \ne 2n-3$.

Case 2. $T_p(M) = V + iV$ and $r := \dim_{\mathbb{C}}(V \cap iV) > 0$.

As above, L_p can be embedded in $U_r \times U_{n-r}$ (clearly, we have $r < n$). Moreover, $V \cap iV \ne V$ and since L_p preserves V, it follows that $\dim L_p < r^2 + (n-r)^2$. We have $\dim O(p) \le 2n-1$, and therefore

$$n^2 - 1 < r^2 + (n-r)^2 + \dim O(p) \le r^2 + (n-r)^2 + 2n - 1,$$

which shows that either $\dim O(p) = 2n-1$ or $\dim O(p) = 2n-2$.

The case $\dim O(p) = 2n-1$ yields (iv).

Assume now that $\dim O(p) = 2n-2$. Then $\dim I_p = (n-1)^2$ and by Lemma 2.1 of [IKruzh], I_p^c, the connected component of the identity in I_p, is conjugate in SU_n to the group H^n of all matrices of the form

$$\begin{pmatrix} 1/\det B & 0 \\ 0 & B \end{pmatrix}, \tag{1.2}$$

where $B \in U_{n-1}$. Therefore, I_p contains the center of SU_n. Let $g \ne \mathrm{id}$ be an element of this center. Then g acts trivially on $O(p)$, i.e., $gq = q$ for all $q \in O(p)$. Therefore, $\alpha(g)(v) = v$ for all $v \in V$, where α is the isomorphism defined in (1.1). Since $T_p(M) = V + iV$ and $\alpha(g)$ is complex-linear on $T_p(M)$, it follows that $\alpha(g) = \mathrm{id}$ and $g = \mathrm{id}$, which is a contradiction. Hence

$\dim O(p) \neq 2n-2$.

Case 3. $T_p(M) = V \oplus iV$.

In this case $\dim V = n$ and L_p can be embedded in the real orthogonal group $O_n(\mathbb{R})$, therefore

$$\dim L_p + \dim O(p) \leq \frac{n(n-1)}{2} + n.$$

Thus, for $n \geq 3$ we have $\dim L_p + \dim O(p) < n^2 - 1$ which is a contradiction. Assume now that $n = 2$. In this case $\dim I_p = 1 = (n-1)^2$, and we arrive at a contradiction by arguing as in Case 2 above.

The proof of the proposition is complete. □

2 The Case of Fixed Point

We start with the case when the action of SU_n has a fixed point in the manifold. Here we establish the following result.

THEOREM 2.1 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations that has a fixed point in M. Then M is biholomorphically equivalent to either*
(i) the unit ball $B^n \subset \mathbb{C}^n$, or
(ii) $\mathbb{C}^n$, or
(iii) $\mathbb{CP}^n$.
The biholomorphic equivalence f can be chosen to satisfy either the relation

$$f(gq) = gf(q), \tag{2.1}$$

or, if $n \geq 3$,

$$f(gq) = \overline{g}f(q), \tag{2.2}$$

for all $g \in SU_n$ and $q \in M$ (here B^n, $\mathbb{C}^n$ and $\mathbb{CP}^n$ are considered with the standard action of SU_n).

Proof: Let p be a fixed point of the action of SU_n on M. Then $I_p = SU_n$. Let L_p be as above the linear isotropy subgroup. Clearly, L_p is also isomorphic to SU_n. Since L_p is a compact subgroup of $GL(T_p(M), \mathbb{C})$, one can find coordinates in $T_p(M)$ such that $L_p \subset U_n$. In these coordinates $L_p = SU_n$ (note that SU_n can be embedded in U_n in the unique way). The group SU_n acts transitively on the unit sphere in $T_p(M)$.

Assume first that M is non-compact. Then by [GK] the manifold M is biholomorphically equivalent to either B^n or $\mathbb{C}^n$, and a biholomorphism F may be chosen so as to satisfy $F(gq) = \gamma(g)F(q)$ for all $g \in SU_n$ and $q \in M$, and some automorphism γ of SU_n, where the action of SU_n on $\mathbb{C}^n$ in the right-hand side is standard. Every automorphism of SU_n has either the form

$$g \mapsto hgh^{-1}, \tag{2.3}$$

or, for $n \geq 3$, the form

$$g \mapsto h\overline{g}h^{-1}, \tag{2.4}$$

for a fixed $h \in SU_n$ (see, e.g., [VO]). Thus, setting $f = \hat{h}^{-1} \circ F$, where $\hat{h}$ is the automorphism of $\mathbb{CP}^n$ induced by h, we obtain either (2.1) or (2.2), respectively.

Assume now that M is compact. Then, by [BDK], M is biholomorphically equivalent to $\mathbb{CP}^n$. It is not clear from the argument in [BDK] whether a biholomorphism $F : M \to \mathbb{CP}^n$ constructed there can be chosen to satisfy (2.1) or (2.2), and therefore we prove this fact below.

The action of SU_n on M induces an embedding $\tau : SU_n \to \mathrm{Aut}(\mathbb{CP}^n)$, and $\tau(SU_n)$ has a fixed point in $\mathbb{CP}^n$. Therefore, $\tau(SU_n)$ is conjugate in $\mathrm{Aut}(\mathbb{CP}^n)$ to SU_n embedded in $\mathrm{Aut}(\mathbb{CP}^n)$ in the standard way. Hence there exists an automorphism γ of SU_n such that for some $s \in \mathrm{Aut}(\mathbb{CP}^n)$ we have $s \circ F(gq) = \gamma(g)s \circ F(q)$ for all $g \in SU_n$ and $q \in M$, where the action of SU_n on $\mathbb{CP}^n$ in the right-hand side is standard. We again use that γ has an explicit expression as in (2.3) or (2.4) and setting $f = \hat{h}^{-1} \circ s \circ F$ obtain either (2.1) or (2.2), respectively.

The proof is complete. □

Remark 2.2 In [IKruzh] we used the results of [Ka] in the case when the action of U_n on M had a fixed point, but in fact the above proof works for U_n as well.

3 The Case of Real Hypersurface Orbits

We shall now consider orbits in M that are real hypersurfaces. We classify such orbits up to equivariant diffeomorphisms first.

Proposition 3.1 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations. Let $p \in M$ and assume that the orbit $O(p)$ is a real hypersurface in M. Then $O(p)$ is equivariantly diffeomorphic to a lense manifold $\mathcal{L}_m^{2n-1} := S^{2n-1}/\mathbb{Z}_m$ obtained by the identification of each point $x \in S^{2n-1}$ with $e^{\frac{2\pi i}{m}}x$, where $(m,n) = 1$ (here $\mathcal{L}_m^{2n-1}$ is considered with the standard action of SU_n).*

Proof: We show first that I_p^c is conjugate in SU_n to SU_{n-1} embedded in SU_n in the standard way. Obviously, $\dim I_p^c = n^2 - 2n$, and therefore the assertion is trivial for $n = 2$. Assume that $n \geq 3$. We now apply Lemma 4.2 from [IKruzh]. Since I_p^c lies in SU_n, it clearly cannot be conjugate to $U_1 \times U_1 \times U_1$ for $n = 3$ or to $U_2 \times U_2$ for $n = 4$. Therefore, to show that I_p^c is conjugate to SU_{n-1}, we must only show that I_p^c is not an irreducible subgroup of $GL_n(\mathbb{C})$.

Let I_p^c be irreducible. Then we proceed as in the last part of the proof of Lemma 2.1 in [IKruzh]. Let $\mathfrak{g} \subset \mathfrak{su}_n \subset \mathfrak{sl}_n$ be the Lie algebra of I_p^c and $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} + i\mathfrak{g} \subset \mathfrak{sl}_n$ its complexification. Then $\mathfrak{g}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and by a theorem of É. Cartan (see, e.g., [GG]), $\mathfrak{g}^{\mathbb{C}}$ is semisimple.

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into the direct sum of simple ideals. Then (see, e.g., [GG]) the irreducible n-dimensional representation of $\mathfrak{g}^{\mathbb{C}}$ given by the embedding of $\mathfrak{g}^{\mathbb{C}}$ in $\mathfrak{gl}_n$ is the tensor product of some irreducible faithful representations of the $\mathfrak{g}_j$. Let n_j be the dimension of the corresponding representation of $\mathfrak{g}_j$, $j = 1, \ldots, k$. Then $n_j \geq 2$, $\dim_{\mathbb{C}} \mathfrak{g}_j \leq n_j^2 - 1$, and $n = n_1 \cdot \ldots \cdot n_k$. The following observation is simple.

> **Claim:** *If $n = n_1 \cdot \ldots \cdot n_k$, $k \geq 2$, $n_j \geq 2$ for $j = 1, \ldots, k$, then $\sum_{j=1}^k n_j^2 \leq n^2 - 2n$.*

Since $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = n^2 - 2n$, it follows from the above claim that $k = 1$, i.e., $\mathfrak{g}^{\mathbb{C}}$ is simple. The lowest dimensions of irreducible faithful representations of complex simple Lie algebras are well-known (see, e.g., [VO]). In the table below V denotes representations of the lowest dimension.

$\mathfrak{g}$	$\dim V$	$\dim \mathfrak{g}$
$\mathfrak{sl}_k\ k \geq 2$	k	$k^2 - 1$
$\mathfrak{o}_k\ \ k \geq 7$	k	$\frac{k(k-1)}{2}$
$\mathfrak{sp}_{2k}\ k \geq 2$	$2k$	$2k^2 + k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

(3.1)

Since $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = n^2 - 2n$, it follows that $\mathfrak{g}^{\mathbb{C}}$ can only be isomorphic to $\mathfrak{sl}_{n-1}$. But $\mathfrak{sl}_{n-1}$ has no irreducible n-dimensional representations unless $n = 3$ (and the only 3-dimensional irreducible representation of $\mathfrak{sl}_2$ is – up to isomorphism – its adjoint representation). Hence $n = 3$ and $\mathfrak{g}$ is isomorphic to $\mathfrak{su}_2$.

We now apply Lemma 4.2 in [IKruzh] to $L_p^c \subset GL(T_p(M), \mathbb{C})$. We choose coordinates in $T_p(M)$ such that $L_p^c \subset U_3$. As noted in Case 2 of the proof of Proposition 1.1, L_p^c can be embedded in $U_1 \times U_2$ and therefore cannot be irreducible. Further, the Lie algebra of L_p^c is isomorphic to $\mathfrak{g}$ and hence to $\mathfrak{su}_2$; this shows that L_p^c cannot be conjugate in U_3 to $U_1 \times U_1 \times U_1$. Therefore, L_p^c is conjugate in U_3 to SU_2, and hence I_p^c is isomorphic to SU_2. This isomorphism gives rise to a faithful irreducible representation $\phi : SU_2 \to GL_3(\mathbb{C})$. It can be extended to a complex irreducible representation $\phi^{\mathbb{C}} : SL_2(\mathbb{C}) \to GL_3(\mathbb{C})$. However, the only 3-dimensional irreducible representation of $SL_2(\mathbb{C})$ (up to isomorphism) is its adjoint representation, and it is not faithful, for $Ad(-\mathrm{id}) = \mathrm{id}$. Hence $\phi(-\mathrm{id}) = \phi^{\mathbb{C}}(-\mathrm{id}) = \mathrm{id}$, which is a contradiction. Hence I_p^c is conjugate to SU_{n-1}. We assume now that $n \geq 3$ and use Lemma 4.4 from [IKruzh]. It shows that if m is the number of connected components of I_p, then I_p is conjugate in SU_n to $G_m^n \cdot SU_{n-1}$, where G_m^n is the group of all matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \cdot \mathrm{id} \end{pmatrix}, \tag{3.2}$$

with $\alpha^m = 1$ and $\alpha\beta^{n-1} = 1$. Hence $O(p)$ is equivariantly diffeomorphic to $\mathcal{L}_m^{2n-1}$. Clearly, the SU_n-action can only be effective on M (and hence on $O(p)$) if $(m, n) = 1$.

In the case $n = 2$ we require the following lemma, which in fact holds for all $n \geq 2$.

Lemma 3.2 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations. Let $p \in M$ and suppose that $O(p)$ is a real hypersurface in M. Then $O(p)$ is strongly pseudoconvex at every point.*

Proof: We show first that $O(p)$ is either Levi-flat or strongly pseudoconvex. This is obvious for $n = 2$ since $O(p)$ is a homogeneous real hypersurface and the corresponding Levi form has only one eigenvalue.

Assume now that $n \geq 3$. Since $O(p)$ is a real hypersurface in M, it arises in Case 2 of the proof of Proposition 1.1. Let W be the orthogonal complement to $V \cap iV$ in $T_p(M)$. Clearly, $\dim_{\mathbb{C}} V \cap iV = n-1$ and $\dim_{\mathbb{C}} W = 1$. The group L_p is a subgroup of U_n and preserves both $V \cap iV$ and W. In addition, it preserves V and hence the line $W \cap V$. Therefore, it can only act as $\pm\mathrm{id}$ on W. Thus, the identity component L_p^c of L_p is a subgroup of the group of unitary transformations preserving $V \cap iV$ and acting trivially on W. Since $\dim L_p^c = n^2 - 2n$, L_p^c is isomorphic to SU_{n-1} and acts transitively on $V \cap iV$. Therefore, either all eigenvalues of the Levi form vanish or they all are of the same sign, which means that $O(p)$ is either Levi-flat, or strongly pseudoconvex.

Assume that $O(p)$ is Levi-flat. Then it is foliated by complex hypersurfaces in M. Let $\mathfrak{m}$ be the Lie algebra of all holomorphic vector fields on $O(p)$ corresponding to the automorphisms of $O(p)$ generated by our action of SU_n. Clearly, $\mathfrak{m}$ is isomorphic to $\mathfrak{su}_n$. Let M_p be the leaf of the foliation passing through p, and consider the subspace $\mathfrak{l} \subset \mathfrak{m}$ of vector fields tangent to M_p at p. The vector fields in $\mathfrak{l}$ remain tangent to M_p at each point $q \in M_p$, and therefore $\mathfrak{l}$ is in fact a Lie subalgebra of $\mathfrak{m}$. However, $\dim \mathfrak{l} = n^2 - 2$ and $\mathfrak{su}_n$ has no subalgebras of codimension 1.

Hence $O(p)$ must be strongly pseudoconvex, as required. □

We now proceed with the case $n = 2$. Following the argument in the proof of Lemma 3.2 we see that, since $O(p)$ is strongly pseudoconvex, L_p can in fact act only trivially on W and therefore L_p — and hence I_p — are isomorphic to a subgroup of U_1. This means that I_p is a finite cyclic group, i.e., $I_p = \{A^l, 0 \leq l < m\}$ for some $A \in SU_2$ and $m \in \mathbb{N}$ such that $A^m = \mathrm{id}$.

Choosing new coordinates in which A is in the diagonal form we see that I_p is conjugate in SU_2 to the group G_m^2 of matrices of the following form:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \qquad \alpha^m = 1. \tag{3.3}$$

Hence $O(p)$ is SU_2-equivariantly diffeomorphic to the lense manifold $\mathcal{L}_m^3$. Clearly, the action of SU_2 is effective on M (and therefore on $O(p)$) only if m is odd.

The proof of the proposition is now complete. □

Recall next that a strictly pseudoconvex real hypersurface in M is said to be *spherical* if it is locally biholomorphically equivalent to the standard real sphere of the corresponding dimension. We require the following result.

Proposition 3.3 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations. Let $p \in M$ and suppose that $O(p)$ is a real hypersurface in M. Assume also that we have either*
(i) $n \geq 3$, or
(ii) $n = 2$ and $O(p)$ is equivariantly diffeomorphic to a non-trivial lense manifold $\mathcal{L}_m^3$, $m > 1$.
Then $O(p)$ is spherical at every point.

Proof: By Lemma 3.2, $O(p)$ is strongly pseudoconvex. Since $O(p)$ is real-analytic, there exist local coordinates $(z_1, \dots, z_{n-1}, w = u + iv)$ in a neighbourhood of p such that $O(p)$ can be described by an equation in the Chern-Moser normal form [CM]:

$$v = |z|^2 + \sum_{k\geq 2, l\geq 2} F_{k\bar{l}}(z, \overline{z}, u), \tag{3.4}$$

where by $F_{k\bar{l}}$ we denote terms of order k in z and order l in $\overline{z}$, and the following normalization conditions hold:

$$\operatorname{tr} F_{2\overline{2}} = 0, \qquad \operatorname{tr}^2 F_{2\overline{3}} = 0, \qquad \operatorname{tr}^3 F_{3\overline{3}} = 0, \tag{3.5}$$

with operator tr defined as follows:

$$\operatorname{tr} := \sum_{j=1}^{n-1} \frac{\partial^2}{\partial z_j \partial \overline{z_j}}.$$

Let $\mathrm{Aut}(O(p))$ be the Lie group of holomorphic automorphisms of $O(p)$ and $\mathrm{Aut}_p(O(p))$ the isotropy subgroup of p. Let Ψ the homomorphism of SU_n into $\mathrm{Aut}(O(p))$ induced by the action of SU_n on $O(p)$. Clearly, $K := \Psi(I_p)$ is a closed subgroup of $\mathrm{Aut}_p(O(p))$, and K^c is isomorphic to SU_{n-1}. Assume that $O(p)$ is not spherical at p. Then by [KL], in some normal coordinates all elements of $\mathrm{Aut}_p(O(p))$ can be written near p in the form

$$z \mapsto Uz, \qquad w \mapsto w \tag{3.6}$$

with $U \in U_{n-1}$.

Assume first that $n \geq 3$. Since $\mathrm{Aut}_p(O(p))$ contains K^c, all maps of the form (3.6) with $U \in SU_{n-1}$ are in $\mathrm{Aut}_p(O(p))$ and hence each $F_{k\bar{l}}$ in (3.4) depends only on $|z|$. In particular, $F_{2\bar{2}} = c|z|^4$, $c \in \mathbb{R}$, which in combination with the first condition in (3.5) shows that $F_{2\bar{2}} = 0$, that is, the point p is 'umbilic'. Since $O(p)$ is homogeneous, all its points are umbilic and therefore it is spherical at every point (see [CM]).

Assume now that $n = 2$ and $O(p)$ is equivariantly diffeomorphic to $\mathcal{L}^3_m$ with $m > 1$. In this case K is a finite cyclic subgroup of $\mathrm{Aut}_p(O(p))$ of order m. Therefore, every $F_{k\bar{l}}$ in (3.4) is invariant under multiplication of z by the roots of 1 of order m. Since $m > 1$ is odd, this shows that $F_{4\bar{2}} \equiv 0$. Therefore, the point p is umbilic, and since $O(p)$ is homogeneous, it is spherical at every point.

We have arrived at a contradiction, which proves the proposition. □

The next result shows that for $n \geq 3$ the equivariant diffeomorphism between $O(p)$ and $\mathcal{L}^{2n-1}_m$ constructed in Proposition 3.1 is a CR or an anti-CR-diffeomorphism.

Proposition 3.4 *Let M be a complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations. For $p \in M$ suppose that $O(p)$ is a real hypersurface in M equivariantly diffeomorphic to a lense manifold $\mathcal{L}^{2n-1}_m$. Assume that either*
(i) $n \geq 3$, or
(ii) $n = 2$ and $O(p)$ is equivariantly diffeomorphic to a non-trivial lense manifold $\mathcal{L}^3_m$, $m > 1$.
Then there exists a CR-diffeomorphism $\mathcal{F} : \mathcal{L}^{2n-1}_m \to O(p)$ that satisfies either relation (2.1) or, for $n \geq 3$, relation (2.2) for all $g \in SU_n$ and $q \in \mathcal{L}^{2n-1}_m$ (here $\mathcal{L}^{2n-1}_m$ is considered with the CR-structure inherited from S^{2n-1}).

Proof: Consider the standard covering map $\pi : S^{2n-1} \to \mathcal{L}_m^{2n-1}$ and the induced map $\tilde{\pi} := f \circ \pi : S^{2n-1} \to O(p)$, where $f : \mathcal{L}_m^{2n-1} \to O(p)$ is an equivariant diffeomorphism. The covering map $\tilde{\pi}$ satisfies the relation

$$\tilde{\pi}(gq) = g\tilde{\pi}(q), \tag{3.7}$$

for all $g \in SU_n$ and $q \in S^{2n-1}$.

Using $\tilde{\pi}$ we pull back the CR-structure from $O(p)$ to S^{2n-1}. We denote by $\tilde{S}^{2n-1}$ the sphere S^{2n-1} equipped with this new CR-structure. It follows from (3.7) that the CR-structure on $\tilde{S}^{2n-1}$ is invariant under the standard action of SU_n on S^{2n-1}. Further, Proposition 3.3 shows that $\tilde{S}^{2n-1}$ is spherical. Hence, by [VEK], $\tilde{S}^{2n-1}$ is CR-equivalent to S^{2n-1}, i.e., there exists a CR-isomorphism $F : \tilde{S}^{2n-1} \to S^{2n-1}$.

Using F we can push the action of SU_n on $\tilde{S}^{2n-1}$ to an action of SU_n on S^{2n-1} by CR-transformations. This action induces an embedding of SU_n into $\mathrm{Aut}(S^{2n-1})$, the group of all CR-automorphisms of S^{2n-1}. The group $\mathrm{Aut}(S^{2n-1})$ is isomorphic to $SU_{n,1}/Z$, where Z is the center of $SU_{n,1}$, therefore we obtain an embedding $\tau : SU_n \to SU_{n,1}/Z$. Since SU_n is compact, $\tau(SU_n)$ lies in a maximal compact subgroup of $SU_{n,1}/Z$, i.e., in a subgroup conjugate to U_n (here U_n is embedded into $SU_{n,1}/Z$ in the standard way). Since SU_n can be embedded in U_n in the unique way, this shows that $\tau(SU_n) = sSU_ns^{-1}$ for some $s \in SU_{n,1}/Z$. Therefore, if in place of F we consider now the map $F_s := \hat{s}^{-1} \circ F$, where $\hat{s}$ is the element of $\mathrm{Aut}(S^{2n-1})$ corresponding to s, and push the action of SU_n from $\tilde{S}^{2n-1}$ to S^{2n-1} by using F_s, then for the corresponding embedding τ_s we have $\tau_s(SU_n) = SU_n$. Now, for $g \in SU_n$ and $q \in S^{2n-1}$,

$$F_s(gq) = \gamma(g)F_s(q), \tag{3.8}$$

where γ is an automorphism of SU_n.

Assume now that $n \geq 3$. It follows from (3.8) that for $q \in S^{2n-1}$ we have

$$\gamma(J_q) = J_{F_s(q)}, \tag{3.9}$$

where J_q is the isotropy subgroup of $q \in S^{2n-1}$ with respect to the standard action of SU_n on S^{2n-1}. In addition, each automorphism of SU_n has either the form (2.3) or the form (2.4) for some $h \in SU_n$.

Assume first that γ has the form (2.3). Then (3.9) implies that

$$J_{hq} = J_{F_s(q)} \tag{3.10}$$

for every $q \in S^{2n-1}$. We choose $q = q' := h^{-1}q_0$, where $q_0 := (1, 0, \dots, 0)$. Then from (3.10) we obtain $F_s(q') = e^{i\alpha}hq'$ for some $\alpha \in \mathbb{R}$, and (3.8) yields the equality $F_s(gq') = e^{i\alpha}hgq'$ for all $g \in SU_n$, i.e., $F_s(p) = Ap$, with $A \in U_n$, for all $p \in S^{2n-1}$. Therefore, F is a CR-automorphism of S^{2n-1}, and the CR-structure of $\tilde{S}^{2n-1}$ is the standard structure of S^{2n-1}. Hence $\tilde{\pi}$ is a CR-map, and so is f. In this case we set $\mathcal{F} := f$, and $\mathcal{F}$ satisfies (2.1).

Assume now that γ has the form (2.4). Then (3.9) shows that

$$h\overline{J_q}h^{-1} = J_{F_s(q)} \tag{3.11}$$

for every $q \in S^{2n-1}$. Further, for every $q \in S^{2n-1}$ there exists $h' \in SU_n$ such that $q = h'q_0$. Then $J_q = h'J_{q_0}h'^{-1} = h'SU_{n-1}h'^{-1}$, where SU_{n-1} is embedded in SU_n in the standard way. Therefore,

$$\overline{J_q} = \overline{h'}J_{q_0}\overline{h}'^{-1} = J_{\overline{h'}q_0} = J_{\overline{h'q_0}} = J_{\overline{q}}.$$

We choose $q = q' := \overline{h^{-1}q_0}$. Then from (3.11) we obtain the equality $F_s(q') = e^{i\alpha}h\overline{q'}$ for some $\alpha \in \mathbb{R}$, and (3.8) shows that $F_s(gq') = e^{i\alpha}h\overline{gq'}$, for all $g \in SU_n$, i.e., $F_s(p) = A\overline{p}$, with $A \in U_n$, for all $p \in S^{2n-1}$. Therefore, the CR-structure of $\tilde{S}^{2n-1}$ is obtained in this case from the CR-structure of S^{2n-1} by complex conjugation, and the map $\mathcal{F}(t) := f(\overline{t})$, $t \in \mathcal{L}_m^{2n-1}$, is a CR-diffeomorphism. Clearly, $\mathcal{F}$ satisfies in this case (2.2).

Now let $n = 2$. Since every automorphism of SU_2 has the form (2.3), identity (3.8) shows that the map $\hat{F} := \hat{h}^{-1} \circ F_s$, where $\hat{h}$ is the element of $\mathrm{Aut}(S^3)$ corresponding to h, is SU_2-equivariant.

We now find the general form of SU_2-equivariant diffeomorphisms of S^3. Let G be one such diffeomorphism. Clearly, it is entirely defined by the image of one point, for instance, $q_0 := (1, 0)$. Indeed, for every $q \in S^3$ there exists a unique $g \in SU_2$ such that $q = gq_0$. Then $G(q) = G(gq_0) = gG(q_0)$. Let

$$Gq_0 = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix},$$

where $|a_0|^2 + |b_0|^2 = 1$. Then we have

$$G\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_0a - b_0\overline{b} \\ a_0b + b_0\overline{a} \end{pmatrix}, \tag{3.12}$$

for all (a, b) such that $|a|^2 + |b|^2 = 1$.

We now regard G as a map between two copies of S^3, $G : S^3_1 \to S^3_2$, and find the push-forward of the standard CR-structure on S^3_1 to S^3_2 by means of G. Let $T^c_p(S^3_j)$, $j = 1, 2$, be the complex tangent subspaces at p for the two CR-structures respectively. Let z, w be complex variables in $\mathbb{C}^2$. We find equations for the following subspaces:

$$
\begin{array}{ll}
T^c_{q_0}(S^3_1): & z = 0,\\
T^c_{(a_0,b_0)}(S^3_2): & a_0\overline{z} + \overline{b_0}w = 0,\\
T^c_{e^{\frac{2\pi i}{m}}(a_0,b_0)}(S^3_2): & -2i\overline{a_0}|b_0|^2\Big(\sin\frac{2\pi}{m}\Big)z + \overline{b_0}\Big(\cos\frac{2\pi}{m} + i(|a_0|^2 - |b_0|^2)\sin\frac{2\pi}{m}\Big)w+\\
\multicolumn{2}{l}{a_0\Big(\cos\frac{2\pi}{m} + i(|a_0|^2 - |b_0|^2)\sin\frac{2\pi}{m}\Big)\overline{z} + 2ib_0|a_0|^2\Big(\sin\frac{2\pi}{m}\Big)\overline{w} = 0,}\\
e^{\frac{2\pi i}{m}}T^c_{(a_0,b_0)}(S^3_2): & a_0e^{\frac{2\pi i}{m}}\overline{z} + b_0e^{-\frac{2\pi i}{m}}w = 0.
\end{array}
$$

Since $m > 1$ is odd, it is clear that $T^c_{e^{\frac{2\pi i}{m}}(a_0,b_0)}(S^3_2) = e^{\frac{2\pi i}{m}}T^c_{(a_0,b_0)}(S^3_2)$ if and only if $a_0 = 0$ or $b_0 = 0$.

We now apply the above to $G = \hat{F}^{-1}$. Since the CR-structure on $\tilde{S}^3$ is the lift of a CR-structure on $\mathcal{L}^3_m$, it is $\mathbb{Z}_m$-invariant, where we set $\mathbb{Z}_m = \{\alpha \cdot \mathrm{id} : \alpha^m = 1\} \subset U_n$. Therefore, we have in (3.12) that either $a_0 = 0$, or $b_0 = 0$.

Assume first that $b_0 = 0$. In this case $\hat{F}$ is a CR-automorphism of S^3, and so is F_s. Thus, the CR-structure of $\tilde{S}^3$ is the standard CR-structure of S^3. Hence $\tilde{\pi}$ is a CR-map, and so is f. In this case we set $\mathcal{F} := f$, and this is clearly a SU_2-equivariant map.

Now let $a_0 = 0$. In this case $\hat{F}$ is an anti-CR-automorphism of S^3, and so is F_s. Therefore, the CR-structure of $\tilde{S}^3$ is obtained from the CR-structure of S^3 by complex conjugation. Thus, $\tilde{f} := f(\overline{t})$, $t \in \mathcal{L}^3_m$, is a CR-diffeomorphism. Clearly, $\tilde{f}$ satisfies (2.2). We note that for $n = 2$ identity (2.2) can be written as follows:

$$\tilde{f}(gq) = h_0gh_0^{-1}\tilde{f}(q)$$

for all $g \in SU_2$ and $q \in \mathcal{L}^3_m$, where

$$h_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.13}$$

Thus, we can set $\mathcal{F} := \Psi(h_0^{-1}) \circ \tilde{f}$, where $\Psi : SU_2 \to M$ is introduced in the proof of Proposition 1.1. Clearly, $\mathcal{F}$ is an equivariant CR-diffeomorphism.

The proof of the proposition is complete. □

Remark 3.5 For $n \geq 4$ Proposition 3.4 can be proved without referring to the sphericity of $O(p)$. Indeed, one can show that for $n \geq 4$ there exist only two CR-structures on S^{2n-1} invariant under the standard action of SU_n on S^{2n-1}: the standard CR-structure and the CR-structure obtained by its complex conjugation.

For $n = 3$, the contact structure on S^7 invariant under the standard action of SU_3 is unique, but it admits many non-spherical CR-structures. Hence it is essential to use the sphericity of $O(p)$ in the proof of Proposition 3.4 for $n = 3$.

It is clear from the proof that the sphericity is also essential for $n = 2$. In fact, in addition to sphericity we had to use the $\mathbb{Z}_m$-invariance of the CR-structure in question. One can see from the proof that, besides the standard CR-structure and its conjugate, there exist other spherical CR-structures on S^3 that are invariant under the standard action of SU_2. These CR-structures are equivalent to the standard one by means of an equivariant CR-diffeomorphism (see (3.12)).

We now introduce additional notation.

Definition 3.6 *Let $d \in \mathbb{C} \setminus \{0\}$, $|d| \neq 1$, let M_d^n be the Hopf manifold constructed by means of the identification of $z \in \mathbb{C}^n \setminus \{0\}$ with $d \cdot z$, and let $[z] \in M_d^n$ be the equivalence class of z. Then we denote by $M_d^n/\mathbb{Z}_m$, with $m \in \mathbb{N}$, the complex manifold obtained from M_d^n by means of the identification of $[z]$ and $[e^{\frac{2\pi i}{m}} z]$.*

We also denote by $S_{r,R}^n := \{z \in \mathbb{C}^n : r < |z| < R\}$, $0 \leq r < R \leq \infty$, a spherical layer in $\mathbb{C}^n$.

We are now ready to prove the following theorem for $n \geq 3$.

THEOREM 3.7 *Let M be a connected complex manifold of dimension $n \geq 3$ endowed with an effective action of SU_n by biholomorphic transformations. Assume that all orbits of this action are real hypersurfaces. Then there exists $m \in \mathbb{N}$, $(m, n) = 1$, such that, M is biholomorphically equivalent to either*

(i) $S_{r,R}^n/\mathbb{Z}_m$, or

(ii) $M_d^n/\mathbb{Z}_m$.
The biholomorphic equivalence f *can be chosen to satisfy either (2.1) or (2.2) for all* $g \in SU_n$ *and* $q \in M$ *(here* $S_{r,R}^n/\mathbb{Z}_m$ *and* $M_d^n/\mathbb{Z}_m$ *are equipped with the standard actions of* SU_n*).*

Proof: Assume first that M is non-compact. Let $p \in M$. By Propositions 3.3 and 3.4, for some $m \in \mathbb{N}$ there exists a CR-diffeomorphism $f : O(p) \to \mathcal{L}_m^{2n-1}$ such that either (2.1) or (2.2) holds for all $q \in O(p)$. Assume that (2.1) holds. Then the map f extends to a biholomorphic map of a neighborhood U of $O(p)$ onto a neighborhood of $\mathcal{L}_m^{2n-1}$ in $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_m$. We can take U to be a connected union of orbits. Then the extended map satisfies (2.1) on U and maps therefore U biholomorphically onto the quotient of a spherical layer by the action of $\mathbb{Z}_m$.

Let D be a maximal domain in M such that there exists a biholomorphic map f from D onto the quotient of a spherical layer by the action of $\mathbb{Z}_m$ that satisfies a relation of the form (2.1) for all $g \in U_n$ and $q \in D$. As shown above, there exists such a domain D. Assume that $D \neq M$ and let x be a boundary point of D. Consider the orbit $O(x)$. Extending as before a map from $O(x)$ into a lense manifold to a neighborhood of $O(x)$ we see that the orbits of all points close to x have the same type as $O(x)$. Hence $O(x)$ is also equivalent to $\mathcal{L}_m^{2n-1}$. Let $h : O(x) \to \mathcal{L}_m^{2n-1}$ be a CR-isomorphism. It satisfies either relation (2.1) or relation (2.2) for all $g \in U_n$ and $q \in O(x)$.

Assume first that (2.1) holds for h. The map h extends to some neighborhood V of $O(x)$ that we can assume to be a connected union of orbits. The extended map satisfies (2.1) on V. For $s \in V \cap D$ we consider the orbit $O(s)$. The maps f and h take $O(s)$ into some surfaces $r_1 S^{2n-1}/\mathbb{Z}_m$ and $r_2 S^{2n-1}/\mathbb{Z}_m$, respectively, where $r_1, r_2 > 0$. Hence $F := h \circ f^{-1}$ maps $r_1 S^{2n-1}/\mathbb{Z}_m$ onto $r_2 S^{2n-1}/\mathbb{Z}_m$ and satisfies the relation

$$F(ut) = uF(t), \tag{3.14}$$

for all $u \in SU_n$ and $t \in r_1 S^{2n-1}/\mathbb{Z}_m$. Let $\pi_1 : r_1 S^{2n-1} \to r_1 S^{2n-1}/\mathbb{Z}_m$ and $\pi_2 : r_2 S^{2n-1} \to r_2 S^{2n-1}/\mathbb{Z}_m$ be the standard projections. Clearly, F can be lifted to a map between $r_1 S^{2n-1}$ and $r_2 S^{2n-1}$, i.e., there exists a CR-isomorphism $G : r_1 S^{2n-1} \to r_2 S^{2n-1}$ such that

$$F \circ \pi_1 = \pi_2 \circ G. \tag{3.15}$$

We see from (3.14) and (3.15) that, for all $g \in SU_n$ and $y \in r_1 S^{2n-1}$,

$$\pi_2(G(gy)) = F(\pi_1(gy)) = F(g\pi_1(y)) = \\ gF(\pi_1(y)) = g\pi_2(G(y)) = \pi_2(gG(y)).$$

Since the fibers of π_2 are discrete, this yields the relation

$$G(gy) = gG(y), \tag{3.16}$$

for all $g \in SU_n$ and $y \in r_1 S^{2n-1}$.

The map G extends to a biholomorphic map of the corresponding balls $r_1 B^n$, $r_2 B^n$, and the extended map satisfies (3.16) on $r_1 B^n$. Setting $y = 0$ in (3.16) we see that $G(0)$ is a fixed point of the standard action of SU_n on $r_2 B^n$, and therefore $G(0) = 0$. Combined with (3.16) this shows that $G = d \cdot \mathrm{id}$, where $d \in \mathbb{C} \setminus \{0\}$. This means, in particular, that F is biholomorphic on $(\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_m$. Now,

$$\mathcal{F} := \begin{cases} F \circ f & \text{on } D \\ h & \text{on } V \end{cases}$$

is a holomorphic map on $D \cup V$, provided that $D \cap V$ is connected.

We now claim that we can choose V such that $D \cap V$ is connected. We assume that V is small enough, so that the strictly pseudoconvex orbit $O(x)$ partitions V into two pieces: $V = V_1 \cup V_2 \cup O(x)$, where $V_1 \cap V_2 = \emptyset$ and each intersection $V_j \cap D$ is connected. Indeed, there exist holomorphic coordinates on D in which $V_j \cap D$ is a union of the quotients of spherical layers by the action of $\mathbb{Z}_m$. If there are several such 'factorized' layers, then there exists a layer with closure disjoint from $O(x)$, and D is disconnected, which is impossible. Therefore, $V_j \cap D$ is connected, and if V is sufficiently small, then each V_j is either a subset of D or is disjoint from D. If $V_j \subset D$ for $j = 1, 2$, then $M = D \cup V$ is compact, which contradicts our assumption. Thus, only one of the two sets V_1 and V_2 lies in D, and therefore $D \cap V$ is connected. Hence the map $\mathcal{F}$ is well-defined. Clearly, it satisfies (2.1) for all $g \in SU_n$ and $q \in D \cup V$.

We now claim that $\mathcal{F}$ is one-to-one on $D \cup V$. Obviously, $\mathcal{F}$ is one-to-one on each of V and D. Assume that there exist points $p_1 \in D$ and $p_2 \in V$ such that $\mathcal{F}(p_1) = \mathcal{F}(p_2)$. Since $\mathcal{F}$ satisfies (2.1) for all $g \in SU_n$ and $q \in D \cup V$, it follows that $\mathcal{F}(O(p_1)) = \mathcal{F}(O(p_2))$. Let $\Gamma(\tau)$, $0 \le \tau \le 1$ be a continuous path in $D \cup V$ joining p_1 to p_2. For each $0 \le \tau \le 1$ we set $\rho(\tau)$ to be the radius of the sphere corresponding to the lense manifold $\mathcal{F}(O(\Gamma(\tau)))$. Since ρ is continuous and $\rho(0) = \rho(1)$, there exists a point $0 < \tau_0 < 1$ at which ρ attains either its maximum or its minimum on $[0, 1]$. Then $\mathcal{F}$ is not one-to-one in a neighborhood of $O(\Gamma(\tau_0))$, which is a contradiction.

We have thus constructed a domain containing D as a proper subset that can be mapped onto the quotient of a spherical layer by the action of $\mathbb{Z}_m$ by means of a map satisfying (2.1). This is a contradiction showing that in fact $D = M$.

Assume now that h satisfies (2.2) (rather than (2.1)) for all $g \in SU_n$ and $q \in O(x)$. Then h extends to a neighborhood V of $O(x)$ and satisfies (2.2) there. For a point $s \in V \cap D$ we consider its orbit $O(s)$. The maps f and h take $O(s)$ into some lense manifolds $r_1 S^{2n-1}/\mathbb{Z}_m$ and $r_2 S^{2n-1}/\mathbb{Z}_m$, respectively, where $r_1, r_2 > 0$. Hence $F := h \circ f^{-1}$ maps $r_1 S^{2n-1}/\mathbb{Z}_m$ onto $r_2 S^{2n-1}/\mathbb{Z}_m$ and satisfies the relation

$$F(ut) = \overline{u} F(t), \tag{3.17}$$

for all $u \in SU_n$ and $t \in r_1 S^{2n-1}/\mathbb{Z}_m$. As above, F can be lifted to a map G from $r_1 S^{2n-1}$ into $r_2 S^{2n-1}$. By (3.17) and (3.15), for all $g \in SU_n$ and $y \in r_1 S^{2n-1}$ we obtain

$$\begin{array}{l} \pi_2(G(gy)) = F(\pi_1(gy)) = F(g\pi_1(y)) = \\ \overline{g}F(\pi_1(y)) = \overline{g}\pi_2(G(y)) = \pi_2(\overline{g}G(y)). \end{array}$$

As before, this shows that

$$G(gy) = \overline{g} G(y), \tag{3.18}$$

for all $g \in SU_n$ and $y \in r_1 S^{2n-1}$.

The map G extends to a biholomorphic map between the corresponding balls $r_1 B^n$ and $r_2 B^n$, and the extended map satisfies (3.18) on $r_1 B^n$. By setting $y = 0$ in (3.18) we see similarly to the above that $G(0)$ is a fixed point of the standard action of U_n on $r_1 B^n$, and thus $G(0) = 0$. Hence $G = d \cdot U$, where $d \in \mathbb{C} \setminus \{0\}$ and U is a unitary matrix. However, this contradicts (3.18), and therefore h cannot satisfy (2.2) on $O(x)$.

The proof in the case when f satisfies (2.2) on $O(p)$ is similar to the above. In that case we obtain an extension to the whole of M satisfying (2.2). This completes the proof in the case of non-compact M.

Assume now that M is compact. We consider a domain D as above and assume first that the corresponding map f satisfies (2.1). Since M is compact, $D \neq M$. Let x be a boundary point of D, and consider the orbit $O(x)$. We choose a connected neighborhood V of $O(x)$ as above, and let $V = V_1 \cup V_2 \cup O(x)$, where $V_1 \cap V_2 = \emptyset$ and each V_j is either a subset of D or is disjoint from D. If one domain of V_1, V_2 is disjoint from D, then arguing as above we arrive at a contradiction with the maximality of D. Hence $V_j \subset D$, $j = 1, 2$, and $M = D \cup O(x)$.

We can now extend $f|_{V_1}$ and $f|_{V_2}$ to biholomorphic maps f_1 and f_2, respectively, that are defined on V, map it onto spherical layers factorized by the action of $\mathbb{Z}_m$, and satisfy (2.1) on V. Then f_1 and f_2 map $O(x)$ onto $r_1 S^{2n-1}/\mathbb{Z}_m$ and $r_2 S^{2n-1}/\mathbb{Z}_m$, respectively, for some $r_1, r_2 > 0$. Clearly, $r_1 \neq r_2$. Hence $F := f_2 \circ f_1^{-1}$ maps $r_1 S^{2n-1}/\mathbb{Z}_m$ onto $r_2 S^{2n-1}/\mathbb{Z}_m$ and satisfies (3.14). This shows, similarly to the above, that $F(< t >_1) =< d \cdot t >_2$ for all $< t >_1 \in r_1 S^{2n-1}/\mathbb{Z}_m$, where $d \in \mathbb{C} \setminus \{0\}$ and $< t >_j \in r_j S^{2n-1}/\mathbb{Z}_m$ is the equivalence class of $t \in r_j S^{2n-1}$, $j = 1, 2$. Since $r_1 \neq r_2$, it follows that $|d| \neq 1$. Now, the map

$$\mathcal{F} := \begin{cases} f & \text{on } D \\ f_1 & \text{on } O(x) \end{cases}$$

establishes a biholomorphic equivalence between M and $M_d^n/\mathbb{Z}_m$ and satisfies (2.1).

The proof in the case when f satisfies (2.2) on D is similar. In that case we obtain an extension $\mathcal{F}$ satisfying (2.2).

The proof of the theorem is complete. □

We now discuss the case $n = 2$. First, we recall the example of a non-standard complex structure on $\mathbb{CP}^2 \setminus \{0\}$ given by Rossi in [R1]. Let $(w_0 : w_1 : w_2 : w_3)$ be the homogeneous coordinates in $\mathbb{CP}^3$. Consider in $\mathbb{CP}^3$ the variety V given by the equation

$$w_1 w_2 = w_3(w_3 + w_0).$$

Let $(z_0 : z_1 : z_2)$ be the homogeneous coordinates in $\mathbb{CP}^2$. We consider the map $\Phi : \mathbb{CP}^2 \setminus \{0\} \to V$ defined by the formulas

$$\begin{aligned} w_0 &= z_0^2, \\ w_1 &= z_1^2 - \frac{z_1\overline{z_2}}{|z_1|^2 + |z_2|^2} z_0^2, \\ w_2 &= z_2^2 + \frac{\overline{z_1}z_2}{|z_1|^2 + |z_2|^2} z_0^2, \\ w_3 &= z_1 z_2 - \frac{|z_2|^2}{|z_1|^2 + |z_2|^2} z_0^2. \end{aligned}$$

The map Φ is everywhere 2-to-1. Consider the unique complex structure on $\mathbb{CP}^2 \setminus \{0\}$ making Φ locally biholomorphic. We denote $\mathbb{CP}^2 \setminus \{0\}$ with this new complex structure by X.

Clearly, SU_2 acts on X by diffeomorphisms in the usual way: for $(z_0 : z_1 : z_2) \in X$ and $g \in SU_2$ we have

$$g(z_0 : z_1 : z_2) := (z_0 : u_1 : u_2), \tag{3.19}$$

where $(u_1, u_2) := g(z_1, z_2)$. It can be verified directly that this is an action by biholomorphic automorphisms of X. Let $\mathfrak{S}_R^3$ be the sphere of radius R in X. It is an SU_2-orbit in X and therefore the CR-structure it has as a real hypersurface in X is invariant under the standard action of SU_2. It follows from the results of [R1] (see also [R2]) that none of the $\mathfrak{S}_R^3$ is CR-equivalent to the ordinary sphere S^3 and hence none of the $\mathfrak{S}_R^3$ is spherical (cf. Proposition 3.4 and Remark 3.5).

Further, it can be shown (directly or using an approach based on classifying algebras, as in [Kr]) that a CR-structure on S^3 invariant under the standard action of SU_2 is equivalent to either the standard CR-structure or to the CR-structure of one of $\mathfrak{S}_R^3$ by means of an SU_2-equivariant CR-diffeomorphism, and the manifolds $\mathfrak{S}_R^3$, $0 < R < \infty$, are pairwise non-CR-equivalent.

We now give the following definition.

Definition 3.8 *We denote by $\mathfrak{S}_{r,R}^2$, $0 \le r < R \le \infty$, the spherical layer $S_{r,R}^2$ equipped with the non-standard complex structure induced by the complex structure of X.*

We are now ready to prove the following classification theorem for $n = 2$.

THEOREM 3.9 *Let M be a connected complex manifold of dimension 2 endowed with an effective action of SU_2 by biholomorphic transformations. Suppose that all orbits of this action are real hypersurfaces. Then M is biholomorphically equivalent to either*
(i) $S^2_{r,R}/\mathbb{Z}_m$, where m is odd, or
(ii) $M^2_d/\mathbb{Z}_m$, where m is odd, or
(iii) $\mathfrak{S}^2_{r,R}$.
The biholomorphic equivalence can be chosen to be SU_2-equivariant (here $S^2_{r,R}/\mathbb{Z}_m$, $M^2_d/\mathbb{Z}_m$ and $\mathfrak{S}^2_{r,R}$ are equipped with the standard actions of SU_2).

Proof: Assume first that there exists $p \in M$ such that $O(p)$ is equivariantly diffeomorphic to $\mathcal{L}^3_m$, where $m > 1$ is odd. It then follows from Proposition 3.3 that $O(p)$ is spherical and therefore, by Proposition 3.4, $O(p)$ is equivalent to $\mathcal{L}^3_m$ by means of an equivariant CR-diffeomorphism. We can now proceed as in the proof of Theorem 3.7. For the proof to go through in the present case as well, we must only show that (using the notation of the proof of Theorem 3.7) $O(x)$ has the same type as $O(p)$. Indeed, we know that $O(x)$ must be equivalent to $\mathcal{L}^3_k$ for some $k \geq 1$, or to $\mathfrak{S}^3_\rho$ for some ρ, by means of an equivariant CR-map. This map extend to an equivariant biholomorphic map between a neighborhood of $O(x)$ and a neighborhood of one of these manifolds in $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}_k$ or X, respectively. Since we know that each neighborhood of x contains points with orbits of the same type as $O(p)$, it follows that $O(x)$ has the same type as $O(p)$. Hence M is equivalent to $S^2_{r,R}/\mathbb{Z}_m$ or to $M^2_d/\mathbb{Z}_m$, with odd $m > 1$, by means of an equivariant biholomorphic map.

Assume now that the orbit of every point in M is equivariantly diffeomorphic to S^3 and let $p \in M$ be a point such that $O(p)$ is a spherical hypersurface. It then follows from the explicit classification of SU_2-invariant CR-structures on S^3 discussed above that $O(p)$ is equivalent to S^3 by means of an equivariant CR-map. We can now proceed as in the proof of Theorem 3.7. For the proof to go through in the present case, we must only show that $O(x)$ is also spherical. We know that $O(x)$ is equivalent to S^3 or to $\mathfrak{S}^3_\rho$ for some ρ by means of an equivariant CR-map. This map extends to an equivariant biholomorphic map between a neighborhood of $O(x)$ and a neighborhood of one of these manifolds in $\mathbb{C}^2$ or in X, respectively. Since we know that in each neighborhood of x there exist points with spherical orbits and since none of the $\mathfrak{S}^3_t$ is spherical, $O(x)$ must be spherical. Hence M is equivalent to $S^2_{r,R}$ or to M^2_d by means of an equivariant biholomorphic map.

Assume now that the orbits of points in M are non-spherical. Suppose first that M is non-compact. Let $p \in M$. Then there exists $0 < \rho < \infty$ such that $O(p)$ is equivalent to $\mathfrak{S}^3_\rho$ by means of an equivariant CR-diffeomorphism f. The map f extends to an equivariant biholomorphic map between a neighborhood U of $O(p)$ (here U can be taken to be a connected union of orbits) and $\mathfrak{S}^2_{\rho_1,\rho_2} \subset X$ with $\rho_1 < \rho < \rho_2$.

Let D be a maximal domain in M such that there exists an equivariant biholomorphic map f from D onto $\mathfrak{S}^2_{\rho',\rho''}$ for some $\rho' < \rho''$. As shown above, there exists such a domain D. Assume that $D \neq M$ and let x be a boundary point of D. Consider the orbit $O(x)$. Since $O(x)$ is non-spherical, there exists an equivariant CR-diffeomorphism from $O(x)$ onto $\mathfrak{S}^3_{\tilde\rho}$ for some $\tilde\rho$, $0 < \tilde\rho < \infty$. This diffeomorphism extends to an equivariant biholomorphic mapping between a neighborhood V of $O(x)$ (that can be taken to be a union of orbits) and $\mathfrak{S}^2_{\tilde\rho_1,\tilde\rho_2}$ for some $\tilde\rho_1 < \tilde\rho < \tilde\rho_2$. For $s \in V \cap D$ we consider the orbit $O(s)$. The maps f and h take $O(s)$ into some surfaces $\mathfrak{S}^3_{r_1}$ and $\mathfrak{S}^3_{r_2}$ Hence $F := h \circ f^{-1}$ maps $\mathfrak{S}^3_{r_1}$ equivariantly onto $\mathfrak{S}^3_{r_2}$. Therefore, $r_1 = r_2 = t$, and F is an equivariant holomorphic automorphism of $\mathfrak{S}^3_t$.

The points in $\mathfrak{S}^3_t$ are non-umbilic, therefore it is clear from what we wrote on the Chern–Moser normal forms in the proof of Proposition 3.3 that the group of CR-automorphisms of $\mathfrak{S}^3_t$ is a two-component Lie group with identity component SU_2. This shows that every equivariant CR-automorphism of $\mathfrak{S}^3_t$ is an element of the center of SU_2 and thus extends to an automorphism of X.

Hence

$$\mathcal{F} := \begin{cases} F \circ f & \text{on } D \\ h & \text{on } V \end{cases}$$

is a holomorphic map on $D \cup V$, provided that $D \cap V$ is connected.

As in the proof of Theorem 3.7, we can now show that $D \cap V$ is indeed connected and $\mathcal{F}$ is one-to-one on $D \cup V$. This contradiction shows that in fact $D = M$.

Assume now that M is compact. We consider a domain D defined as above. Since M is compact, $D \neq M$. For a boundary point x of D we consider the orbit $O(x)$. We choose a connected neighborhood V of $O(x)$ as above, and let $V = V_1 \cup V_2 \cup O(x)$. As in the proof of Theorem 3.7, it turns out that $V_j \subset D$, $j = 1, 2$, and $M = D \cup O(x)$.

We can now extend $f|_{V_1}$ and $f|_{V_2}$ to equivariant biholomorphic maps f_1 and f_2, respectively, that are defined on V and map it onto spherical layers in X. Then f_1 and f_2 map $O(x)$ onto $\mathfrak{S}^3_{r_1}$ and $\mathfrak{S}^3_{r_2}$, respectively, for some $r_1, r_2 > 0$. Clearly, $r_1 \neq r_2$. However, the surfaces $\mathfrak{S}^3_R$, $0 < R < \infty$ are not CR-equivalent. This contradiction shows that M cannot be compact.

The proof is now complete. □

4 The Case of Complex Hypersurface Orbits

We now discuss orbits that are complex hypersurfaces. We start with several examples.

Example 4.1 Let B^n_R be the ball of radius $0 < R \leq \infty$ in $\mathbb{C}^n$ and $\widehat{B^n_R}$ its blow-up at the origin, i.e.,

$$\widehat{B^n_R} := \left\{(z,w) \in B^n_R \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i, \text{ for all } i,j\right\},$$

where $z = (z_1, \dots, z_n)$ are the standard coordinates in $\mathbb{C}^n$ and $w = (w_1 : \dots : w_n)$ are the homogeneous coordinates in $\mathbb{CP}^{n-1}$. We define an action of U_n on $\widehat{B^n_R}$ as follows. For $(z,w) \in \widehat{B^n_R}$ and $g \in U_n$ we set

$$g(z,w) := (gz, gw),$$

where in the right-hand side we use the standard actions of U_n on $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$. Consider now the induced action of SU_n. The points $(0,w) \in \widehat{B^n_R}$ form an orbit O, which is a complex hypersurface biholomorphically equivalent to $\mathbb{CP}^{n-1}$. All other orbits are real hypersurfaces that are the boundaries of strongly pseudoconvex neighborhoods of O.

We fix $m \in \mathbb{N}$ and denote by $\widehat{B^n_R}/\mathbb{Z}_m$ the quotient of $\widehat{B^n_R}$ by the equivalence relation $(z,w) \sim e^{\frac{2\pi i}{m}}(z,w)$. Let $\{(z,w)\} \in \widehat{B^n_R}/\mathbb{Z}_m$ be the equivalence class of $(z,w) \in \widehat{B^n_R}$. We now define in a natural way an action of SU_n on $\widehat{B^n_R}/\mathbb{Z}_m$: for $\{(z,w)\} \in \widehat{B^n_R}/\mathbb{Z}_m$ and $g \in SU_n$ we set

$$g\{(z,w)\} := \{g(z,w)\}.$$

The points $\{(0,w)\}$ form the unique complex hypersurface orbit O, which is biholomorphically equivalent to $\mathbb{CP}^{n-1}$, and each real hypersurface orbit is the boundary of a strongly pseudoconvex neighborhood of O.

Now let $S^n_{r,\infty} = \{z \in \mathbb{C}^n : |z| > r\}$, $r > 0$, be a spherical layer with infinite outer radius and let $\widetilde{S^n_{r,\infty}}$ be the union of $S^n_{r,\infty}$ and the hypersurface at infinity in $\mathbb{CP}^n$, namely,

$$\widetilde{S^n_{r,\infty}} := \{(z_0 : z_1 : \ldots : z_n) \in \mathbb{CP}^n : (z_1, \ldots, z_n) \in S^n_{r,\infty},\ z_0 = 0, 1\}.$$

We shall equip $\widetilde{S^n_{r,\infty}}$ with the standard action of U_n. For $(z_0 : z_1 : \ldots : z_n) \in \widetilde{S^n_{r,\infty}}$ and $g \in U_n$ we set

$$g(z_0 : z_1 : \ldots : z_n) := (z_0 : u_1 : \ldots : u_n),$$

where $(u_1, \ldots, u_n) := g(z_1, \ldots, z_n)$. Consider now the induced action of SU_n. The points $(0 : z_1 : \ldots : z_n)$ at infinity form an orbit O, which is a complex hypersurface biholomorphically equivalent to $\mathbb{CP}^{n-1}$. All other orbits are real hypersurfaces that are the boundaries of strongly pseudoconcave neighborhoods of O.

We fix $m \in \mathbb{N}$ and denote by $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ the quotient of $\widetilde{S^n_{r,\infty}}$ by the equivalence relation $(z_0 : z_1 : \ldots : z_n) \sim e^{\frac{2\pi i}{m}}(z_0 : z_1 : \ldots : z_n)$. Let $\{(z_0 : z_1 : \ldots : z_n)\} \in \widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ be the equivalence class of $(z_0 : z_1 : \ldots : z_n) \in \widetilde{S^n_{r,\infty}}$. We endow $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ with the standard action of SU_n; namely, for $\{(z_0 : z_1 : \ldots : z_n)\} \in \widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$ and $g \in SU_n$ we set

$$g\{(z_0 : z_1 : \ldots : z_n)\} := \{g(z_0 : z_1 : \ldots : z_n)\}.$$

The points $\{(0 : z_1 : \ldots : z_n)\}$ form a unique complex hypersurface orbit O which is biholomorphically equivalent to $\mathbb{CP}^{n-1}$, and each real hypersurface orbit is the boundary of a strongly pseudoconcave neighborhood of O.

Finally, let $\widehat{\mathbb{CP}^n}$ be the blow-up of $\mathbb{CP}^n$ at the point $(1 : 0 : \ldots : 0) \in \mathbb{CP}^n$:

$$\widehat{\mathbb{CP}^n} := \Big\{\Big((z_0 : z_1 : \ldots : z_n), w\Big) \in \mathbb{CP}^n \times \mathbb{CP}^{n-1} : z_i w_j = z_j w_i \\ \text{for all } i, j \neq 0,\ z_0 = 0, 1\Big\},$$

where $w = (w_1 : \ldots : w_n)$ are the homogeneous coordinates in $\mathbb{CP}^{n-1}$. We define an action of U_n in $\widehat{\mathbb{CP}^n}$ as follows. For $\Big((z_0 : z_1 : \ldots : z_n), w\Big) \in \widehat{\mathbb{CP}^n}$ and $g \in U_n$ we set

$$g\Big((z_0 : z_1 : \ldots : z_n), w\Big) := \Big((z_0 : u_1 : \ldots : u_n), gw\Big),$$

where $(u_1, \ldots, u_n) := g(z_1, \ldots, z_n)$. Consider now the induced action of SU_n. This action has exactly two orbits that are complex hypersurfaces: the orbit O_1 consisting of the points $\big((1 : 0 : \ldots : 0), w\big)$ and the orbit O_2 consisting of the points $\big((0 : z_1 : \ldots : z_n), w\big)$. Both O_1 and O_2 are biholomorphically equivalent to $\mathbb{CP}^{n-1}$. The real hypersurface orbits are the boundaries of strongly pseudoconvex neighborhoods of O_1 and strongly pseudoconcave neighborhoods of O_2.

Fixing $m \in \mathbb{N}$ we denote by $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ the quotient of $\widehat{\mathbb{CP}^n}$ by the equivalence relation $\big((z_0 : z_1 : \ldots : z_n), w\big) \sim e^{\frac{2\pi i}{m}}\big((z_0 : z_1 : \ldots : z_n), w\big)$. Let $\Big\{\big((z_0 : z_1 : \ldots : z_n), w\big)\Big\} \in \widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ be the equivalence class of $\big((z_0 : z_1 : \ldots : z_n), w\big) \in \widehat{\mathbb{CP}^n}$. We endow $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ with an action of SU_n: for $\Big\{\big((z_0 : z_1 : \ldots : z_n), w\big)\Big\} \in \widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ and $g \in SU_n$ we set

$$g\Big\{\big((z_0 : z_1 : \ldots : z_n), w\big)\Big\} := \Big\{g\big((z_0 : z_1 : \ldots : z_n), w\big)\Big\}.$$

As above, there exist exactly two orbits that are complex hypersurfaces: the orbit O_1 consisting of the points $\Big\{\big((1 : 0 : \ldots : 0), w\big)\Big\}$ and the orbit O_2 consisting of the points $\Big\{\big((0 : z_1 : \ldots : z_n), w\big)\Big\}$. Both O_1 and O_2 are biholomorphically equivalent to $\mathbb{CP}^{n-1}$. The real hypersurface orbits are the boundaries of strongly pseudoconvex neighborhoods of O_1 and strongly pseudoconcave neighborhoods of O_2.

We show below that the complex hypersurface orbits in Example 4.1 are in fact the only ones occurring for $n \geq 3$. For $n = 2$ the above list must be augmented by another family of examples, coming from the manifold X discussed in Section 3.

Example 4.2 Let $\mathfrak{S}^2_{r,\infty} \subset X$, $r \geq 0$, be a spherical layer with infinite outer radius and let $\widetilde{\mathfrak{S}^2_{r,\infty}}$ be the union of $\mathfrak{S}^2_{r,\infty}$ and the hypersurface at infinity:

$$\widetilde{\mathfrak{S}^2_{r,\infty}} := \{(z_0 : z_1 : z_2) \in X : (z_1, z_2) \in \mathfrak{S}^2_{r,\infty}, z_0 = 0, 1\}.$$

The group SU_2 acts on $\widetilde{\mathfrak{S}^2_{r,\infty}}$ by biholomorphic transformations as defined in (3.19). The points $(0 : z_1 : z_2)$ form an orbit O, which is a complex hypersurface biholomorphically equivalent to $\mathbb{CP}^1$. All other orbits are real hypersurfaces, which are the boundaries of strongly pseudoconcave neighborhoods of O.

We establish now the following result.

Proposition 4.3 *Let M be a connected complex manifold of dimension $n \geq 2$ endowed with an effective action of SU_n by biholomorphic transformations. Suppose that each orbit is a real or a complex hypersurface in M. Then there exist at most two complex hypersurface orbits.*

Proof: We fix a smooth SU_n-invariant distance function ρ on M. Let O be an orbit that is a complex hypersurface. Consider the ϵ-neighborhood of $U_\epsilon(O)$ of O in M:

$$U_\epsilon(O) := \left\{ p \in M : \inf_{q \in O} \rho(p, q) < \epsilon \right\}.$$

If ϵ is sufficiently small, then the boundary of $U_\epsilon(O)$,

$$\partial U_\epsilon(O) = \left\{ p \in M : \inf_{q \in O} \rho(p, q) = \epsilon \right\},$$

is a smooth connected real hypersurface in M. Clearly, ∂U_ϵ is SU_n-invariant, and therefore it is a union of orbits. If $\partial U_\epsilon(O)$ contains an orbit that is a real hypersurface, then $\partial U_\epsilon(O)$ obviously coincides with that orbit.

Assume that $\partial U_\epsilon(O)$ contains an orbit that is a complex hypersurface. Then $\partial U_\epsilon(O)$ must be a union of such orbits. It follows from the proof of Proposition 1.1 (see Case 2 there) that if an orbit $O(p)$ is a complex hypersurface, then I_p^c is conjugate in SU_n to the group H^n of matrices (1.2). Therefore, I_p contains the center of SU_n. Since the center acts trivially on $O(p)$, the action of SU_n on $O(p)$, which is an arbitrary complex hypersurface orbit, is not effective. Hence it is not effective on $\partial U_\epsilon(O)$ and therefore on M.

The above contradiction shows that if ϵ is sufficiently small, then $U_\epsilon(O)$ contains no complex hypersurface orbits distinct from O itself, and the boundary of $U_\epsilon(O)$ is a real hypersurface orbit. Let $\tilde{M}$ be the manifold obtained by the removal of all complex hypersurface orbits from M. Since every such orbit has a neighborhood containing no other complex hypersurface orbits, $\tilde{M}$ is connected. It is also clear that $\tilde{M}$ is non-compact. Hence, by Theorems 3.7 and 3.9, $\tilde{M}$ is biholomorphically equivalent to $S^n_{r,R}/\mathbb{Z}_m$ or $\mathfrak{S}^2_{r,R}$, for some r and R, $0 \leq r < R \leq \infty$. Each of these manifolds has two ends at infinity, and therefore the number of removed complex hypersurfaces is at most two, which completes the proof. □

We shall now discuss the case $n \geq 3$. First, we require the following lemma.

Lemma 4.4 *Let G be a closed subgroup of SU_n, $n \geq 2$, such that $G^c = H^n$, where H^n is defined in (1.2). If $n \geq 3$, then G is connected and hence $G = H^n$. If $n = 2$, then either $G = H^2$, or or $G = H^2 \cup h_0 H^2$, where h_0 is defined in (3.13).*

Proof: We proceed as in the proof of Lemma 4.4 in [IKruzh]. Let $C_1, \dots, C_m$ be the connected components of G with $C_1 = H^n$. Clearly, there exist $g_1 = \mathrm{id}, g_2, \dots, g_m$ in SU_n such that $C_j = g_j H^n$, $j = 1, \dots, m$. Moreover, for each pair of indices i, j there exists k such that $g_i H^n \cdot g_j H^n = g_k H^n$, and therefore

$$g_k^{-1} g_i H^n g_j = H^n. \tag{4.1}$$

We now apply (4.1) to the vector $v := (1, 0, \dots, 0)$, which is preserved by elements of H^n up to multiplication. This shows that for every $h \in H^n$ the vector $hg_j v$ is proportional to $g_i^{-1} g_k v$.

Assume first that $n \geq 3$. Then $g_j v = (\alpha_j, 0, \dots, 0)$, $|\alpha_j| = 1$, $j = 1, \dots, m$ and therefore g_j has the form (1.2). This shows that $C_j = C_1$ for all j, and thus G is connected. Hence $G = H^n$.

Assume now that $n = 2$. Then either $g_j v = (\alpha_j, 0)$ or $g_j v = (0, \alpha_j)$, where $|\alpha_j| = 1$, $j = 1, \dots, m$. This shows that either $G = H^2$ or $G = H^2 \cup h_0 H^2$.

The proof is complete. □

We now prove the following classification theorem.

THEOREM 4.5 *Let M be a connected complex manifold of dimension $n \geq 3$ endowed with an effective action of SU_n by biholomorphic transformations. Suppose that each orbit of this action is a real or a complex hypersurface and at least one orbit is a complex hypersurface. Then there exists $m \in \mathbb{N}$, $(m, n) = 1$, such that M is biholomorphically equivalent to either*

(i) $\widehat{B^n_R}/\mathbb{Z}_m$, $0 < R \leq \infty$, *or*

(ii) $\widehat{S^n_{r,\infty}}/\mathbb{Z}_m$, $0 \leq r < \infty$, *or*

(iii) $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$.

The biholomorphic equivalence f can be chosen to satisfy (2.1) or (2.2) for all $g \in SU_n$ and $q \in M$.

Proof: Assume first that only one orbit O is a complex hypersurface. Consider $\tilde{M} := M \setminus O$. Since $\tilde{M}$ is clearly non-compact, by Theorem 3.7 there exists $m \in \mathbb{N}$, $(m,n) = 1$, and $0 \le r < R \le \infty$, such that the manifold $\tilde{M}$ is biholomorphically equivalent to $S^n_{r,R}/\mathbb{Z}_m$ by means of a map f satisfying either (2.1) or (2.2) for all $g \in SU_n$ and $q \in \tilde{M}$. We shall assume that f satisfies (2.1) because the latter case can be dealt with in the same manner.

Fix $p \in O$ and consider I_p. As mentioned in the proof of Proposition 4.3, there exists $g \in SU_n$ such that $I_p^c = g^{-1}H^n g$. Therefore, by Lemma 4.4, I_p is connected and $I_p = g^{-1}H^n g$. For an arbitrary real hypersurface orbit $O(q)$ we set

$$N_{p,q} := \{s \in O(q) : I_s \subset I_p\}\,.$$

Since I_s is conjugate in SU_n to $G^n_m \cdot SU_{n-1}$ (see (3.2)), it follows that

$$N_{p,q} = \Big\{s \in O(q) : I_s = g^{-1}G^n_m \cdot SU_{n-1}g\Big\}\,.$$

Let N_p be the union of the $N_{p,q}$ over all real hypersurface orbits $O(q)$. Also let N'_p be the set of points in $S^n_{r,R}/\mathbb{Z}_m$ whose isotropy subgroups with respect to the standard action of SU_n coincide with $g^{-1}G^n_m \cdot SU_{n-1}g$.

It is easy to verify that N'_p is a complex curve in $S^n_{r,R}/\mathbb{Z}_m$ biholomorphically equivalent to either an annulus of modulus $(R/r)^m$ (if $0 < r < R < \infty$), or a punctured disk (if $r = 0$, $R < \infty$ or $r > 0$, $R = \infty$), or $\mathbb{C} \setminus 0$ (if $r = 0$ and $R = \infty$). Clearly, $f^{-1}(N'_p) = N_p$, and hence N_p is a complex curve in $\tilde{M}$.

Obviously, N_p is invariant under the action of I_p. By Bochner's theorem there exist local holomorphic coordinates in a neighborhood of p such that the action of I_p is linear in these coordinates and coincides with the action of the linear isotropy subgroup L_p introduced in the proof of Proposition 1.1 (upon the natural identification of the coordinate neighborhood in question and a neighborhood of the origin in $T_p(M)$). Recall that L_p has two invariant complex subspaces in $T_p(M)$: $T_p(O)$ and a one-dimensional subspace, which correspond in our coordinates to O and some holomorphic curve. It can be easily seen that $\overline{N_p}$ is precisely this curve. Hence $\overline{N_p}$ near p is an analytic disc with center at p, and therefore N'_p cannot in fact be equivalent to an annulus, and we have either $r = 0$ or $R = \infty$.

Assume first that $r = 0$ and $R < \infty$. We consider the holomorphic embedding $\nu : S^n_{0,R}/\mathbb{Z}_m \to \widehat{B^n_R}/\mathbb{Z}_m$ defined by the formula

$$\nu(< z >) := \{(z, w)\},$$

where $w = (w_1 : \ldots : w_n)$ is uniquely determined by the conditions $z_i w_j = z_j w_i$ for all i, j, and $< z > \in (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_m$ is the equivalence class of the point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n \setminus \{0\}$. Clearly, ν is SU_n-equivariant. Now let $f_\nu := \nu \circ f$. We claim that f_ν extends to O as a biholomorphic map of M onto $\widehat{B^n_R}/\mathbb{Z}_m$.

Let $\hat{O}$ be the orbit in $\widehat{B^n_R}/\mathbb{Z}_m$ that is a complex hypersurface and let $\hat{p} \in \hat{O}$ be the (unique) point such that the isotropy subgroup $I_{\hat{p}}$ at this point (with respect to the action of SU_n on $\widehat{B^n_R}/\mathbb{Z}_m$ described in Example 4.1) coincides with I_p. Then $\{\hat{p}\} \cup \nu(N'_p)$ is a smooth complex curve. We define the extension F_ν of f_ν by setting $F_\nu(p) := \hat{p}$ for each $p \in O$.

We must show that F_ν is continuous at each point $p \in O$. Let $\{q_j\}$ be a sequence of points in M accumulating to p. Since all accumulation points of the sequence $\{F_\nu(q_j)\}$ lie in $\hat{O}$ and $\hat{O}$ is compact, it suffices to show that each convergent subsequence $\{F_\nu(q_{j_k})\}$ of $\{F_\nu(q_j)\}$ converges to $\hat{p}$. For every q_{j_k} there exists $g_{j_k} \in SU_n$ such that $g^{-1}_{j_k} I_{q_{j_k}} g_{j_k} \subset I_p$, i.e., $g^{-1}_{j_k} q_{j_k} \in \overline{N_p}$. We select a convergent subsequence $\{g_{j_{k_l}}\}$ and denote its limit by g. Then $\{g^{-1}_{j_{k_l}} q_{j_{k_l}}\}$ converges to $g^{-1}p$. Since $g^{-1}p \in O$ and $g^{-1}_{j_{k_l}} q_{j_{k_l}} \in \overline{N_p}$, it follows that $g^{-1}p = p$, i.e, $g \in I_p = I_{\hat{p}}$. The map F_ν satisfies (2.1) for all $g \in SU_n$ and $q \in M$, hence $F_\nu(q_{j_{k_l}}) \in \overline{N_{g_{j_{k_l}}\hat{p}}}$, where $N_{g_{j_{k_l}}\hat{p}} \subset \widehat{B^n_R}/\mathbb{Z}_m$ is constructed similarly to $N_p \subset \tilde{M}$. Therefore, the limit of $\{F_\nu(q_{j_{k_l}})\}$ (which is equal to the limit of $\{F_\nu(q_{j_k})\}$) is $\hat{p}$. Hence F_ν is continuous, and therefore holomorphic on M. It obviously maps M biholomorphically onto $\widehat{B^n_R}/\mathbb{Z}_m$.

The case when $r > 0$ and $R = \infty$ can be treated along the same lines, but one must consider the holomorphic embedding $\sigma : S^n_{r,\infty}/\mathbb{Z}_m \to \widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$

$$\sigma(< z >) := \{(1 : z_1 : \ldots : z_n)\},$$

the map $f_\sigma := \sigma \circ f$, and prove that f_σ extends to O as a biholomorphic map of M onto $\widetilde{S^n_{r,\infty}}/\mathbb{Z}_m$.

If $r = 0$ and $R = \infty$, then precisely one of the maps f_ν and f_σ extends to O, and the extension defines a biholomorphic map from M to either $\widehat{\mathbb{C}^n}/\mathbb{Z}_m$, or $\widetilde{S^n_{0,\infty}}/\mathbb{Z}_m$.

Assume now that two orbits O_1 and O_2 in M are complex hypersurfaces. As above, we consider the manifold $\tilde{M}$ obtained from M by removing O_1 and O_2. For some $m \in \mathbb{N}$, $(m,n)=1$, and $0 \le r < R \le \infty$, it is biholomorphically equivalent to $S^n_{r,R}/\mathbb{Z}_m$ by means of a map f satisfying either (2.1) or (2.2). Arguments very similar to the ones used above show that in this case we have $r=0$, $R=\infty$, and the map $f_\tau := \tau \circ f$ extends to a biholomorphic map $M \to \widehat{\mathbb{CP}^n}/\mathbb{Z}_m$. Here $\tau : (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_m \to \widehat{\mathbb{CP}}^n/\mathbb{Z}_m$ is the SU_n-equivariant map defined as follows:

$$\tau(< z >) := \Big\{\Big((1 : z_1 : \ldots : z_n), w\Big)\Big\},$$

where $w = (w_1 : \ldots : w_n)$ is uniquely determined from the conditions $z_i w_j = z_j w_i$ for all i, j.

The proof is complete. □

We consider now the case $n = 2$ and obtain the following result.

THEOREM 4.6 *Let M be a connected complex manifold of dimension 2 endowed with an effective action of SU_2 by biholomorphic transformations. Suppose that each orbit of this action is either a real or a complex hypersurface and at least one orbit is a complex hypersurface. Then M is biholomorphically equivalent to either*
(i) $\widehat{B^2_R}/\mathbb{Z}_m$, where $0 < R \le \infty$ and m is odd, or
(ii) $\widetilde{S^2_{r,\infty}}/\mathbb{Z}_m$, where $0 \le r < \infty$ and m is odd, or
(iii) $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$, where m is odd, or
(iv) $\widetilde{\mathfrak{S}^2_{r,\infty}}$, where $0 \le r < \infty$.
The biholomorphic equivalence can be chosen to be SU_2-equivariant.

Proof: Assume that there exists a real hypersurface orbit in M that is equivariantly diffeomorphic to $\mathcal{L}^3_m$ with odd $m > 1$. Then we proceed similarly to the proof of Theorem 4.5. Assume first that only one orbit O is a complex hypersurface. Let $\tilde{M} := M \setminus O$. Since $\tilde{M}$ is clearly non-compact, by Theorem 3.9 it is equivalent to $S^2_{r,R}/\mathbb{Z}_m$ by means of a biholomorphic equivariant map f.

We fix $p \in O$ and consider I_p. By Lemma 4.4, I_p is conjugate in SU_2 to H^2 or $H^2 \cup h_0 H^2$. It follows from the proof of Proposition 1.1 (see Case 2 there) that I_p is isomorphic to a subgroup of $U_1 \times U_1$ and hence is Abelian.

Since $H^2 \cup h_0 H^2$ is not Abelian, this shows that I_p is actually conjugate to H^2. Thus, there exists $g \in SU_2$ such that $I_p = g^{-1}H^2 g$. For an arbitrary real hypersurface orbit $O(q)$ we set

$$N_{p,q} := \{s \in O(q) : I_s \subset I_p\}.$$

Since I_s is conjugate in SU_2 to G^2_m (see (3.3)), it follows that

$$N_{p,q} = \left\{s \in O(q) : I_s = g^{-1}G^2_m g\right\}.$$

The set $N_{p,q}$ has two connected components; namely, if we fix $t \in N_{p,q}$, then $N_{p,q} = N^1_{p,q} \cup N^2_{p,q}$, where

$$N^1_{p,q} := \{ht\} \qquad \text{and} \qquad N^2_{p,q} := \{g^{-1}h_0 g h t\},$$

with $h \in g^{-1}H^2 g$ and h_0 defined in (3.13).

We now consider the corresponding sets N^1_p and N^2_p. The point p is the accumulation point in O for exactly one of these sets. As in the proof of Theorem 4.5, we obtain that either $r = 0$, or $R = \infty$. For example, assume that $r = 0$ and $R < \infty$. Let $\hat{O}$ be the orbit in $\widehat{B^2_R}/\mathbb{Z}_m$ that is a complex hypersurface. There exist precisely two points in $\hat{O}$ whose isotropy subgroups in SU_2 coincide with I_p. These points $\hat{p}_1$ and $\hat{p}_2$ are the accumulation points in $\hat{O}$ of $\nu(N'^1_p)$ and $\nu(N'^2_p)$ respectively, where $N'^1_p, N'^2_p \subset S^2_{0,R}/\mathbb{Z}_m$ are the two connected components of the set of points with isotropy subgroup equal to $g^{-1}G^2_m g$. Clearly we have either $f(N^1_p) = N'^1_p, f(N^2_p) = N'^2_p$, or $f(N^1_p) = N'^2_p, f(N^2_p) = N'^1_p$.

We now define the extension F_ν of f_ν by setting $F_\nu(p) = \hat{p}_1$ if N^1_p accumulates to p and $f(N^1_p) = N'^1_p$, or if N^2_p accumulates to p and $f(N^2_p) = N'^1_p$. In a similar way we set $F_\nu(p) = \hat{p}_2$ if N^2_p accumulates to p and $f(N^2_p) = N'^2_p$, or if N^1_p accumulates to p and $f(N^1_p) = N'^2_p$ The proof of the continuity of F_ν proceeds as for $n \geq 3$. The arguments in the cases $r > 0$, $R = \infty$ and $r = 0$, $R = \infty$ are analogous to the above.

The case of two complex hypersurface orbits can be dealt with in a similar manner.

Hence if there exists a real hypersurface orbit in M that is equivariantly diffeomorphic to $\mathcal{L}^3_m$, where $m > 1$ is odd, then we see that M is equivalent to either $\widehat{B^2_R}/\mathbb{Z}_m$, or $\widetilde{S^2_{r,\infty}}/\mathbb{Z}_m$, or $\widehat{\mathbb{CP}^n}/\mathbb{Z}_m$ by means of a biholomorphic SU_2-equivariant map.

Assume now that every real hypersurface orbit in M is equivariantly diffeomorphic to S^3 and suppose first of all that there exists a spherical real hypersurface orbit. First, let M have only one complex hypersurface orbit O. Consider $\tilde{M} := M \setminus O$. By Theorem 3.9 it is equivalent to $S^2_{r,R}$ by means of a biholomorphic equivariant map f.

Let L be a complex line and S^3_R the sphere of radius R with center at the origin in $\mathbb{C}^2$. It is not difficult to show that there exists $g \in SU_2$ such that for all $0 < R < \infty$, $L \cap S^3_R$ is an orbit of $g^{-1}H^2 g$. Furthermore, for fixed $g \in SU_2$ there are exactly two complex lines, say L^1_g and L^2_g, that possess the above property, and $L^2_g = g^{-1}h_0 g L^1_g$. Let $N^j_g := f^{-1}(L^j_g)$, $j = 1, 2$. Clearly, the N^j_g are holomorphic curves in $\tilde{M}$. We will use them instead of the N^j_p in the above proof to extend the map f to O.

Fix $p \in O$ such that $I_p = g^{-1}H^2 g$ (there are precisely two such points in O). The point p is the accumulation point in O for exactly one of N^j_g, say N^1_g. Obviously, N^1_g is invariant under the action of I_p. As in the proof of Theorem 4.5, we use Bochner's theorem to show that $\overline{N^1_g}$ near p is an analytic disc with center at p. Further, N^1_g is biholomorphically equivalent to $L^1_g \cap S^2_{r,R}$, that is, to the annulus with inner radius r and outer radius R by means of the map f. Therefore, we have either $r = 0$ or $R = \infty$.

Assume first that $r = 0$ and $R < \infty$. We consider a holomorphic embedding $\nu : S^2_{0,R} \to \widehat{B^2_R}$ defined by the formula

$$\nu(z) := \{(z, w)\},$$

where $w = (w_1 : w_2)$ is uniquely determined by the condition $z_1 w_2 = z_2 w_1$. Clearly, ν is SU_2-equivariant. Now let $f_\nu := \nu \circ f$. We claim that f_ν extends to O as a biholomorphic map of M onto $\widehat{B^2_R}$.

Let $\hat{O}$ be the orbit in $\widehat{B^2_R}$ that is a complex hypersurface and let $\hat{p} \in \hat{O}$ be the point to which L^1_g accumulates. Clearly, $I_{\hat{p}} = g^{-1}H^2 g = I_p$. We define the extension F_ν of f_ν by setting $F_\nu(p) := \hat{p}$.

We must show that F_ν is continuous at each point $p \in O$. Let $\{q_j\}$ be a sequence of points in M accumulating to p. Since all accumulation points of the sequence $\{F_\nu(q_j)\}$ lie in $\hat{O}$ and $\hat{O}$ is compact, it suffices to show that each convergent subsequence $\{F_\nu(q_{j_k})\}$ of $\{F_\nu(q_j)\}$ converges to $\hat{p}$. For every q_{j_k} there exists $g_{j_k} \in SU_2$ such that $g_{j_k} q_{j_k} \in \overline{N^1_g}$. We select a convergent subsequence $\{g_{j_{k_l}}\}$ and denote its limit by g. Then $\{g_{j_{k_l}} q_{j_{k_l}}\}$ converges to gp. Since $gp \in O$ and $g_{j_{k_l}} q_{j_{k_l}} \in \overline{N^1_g}$, it follows that $gp = p$, i.e, $g \in I_p = I_{\hat{p}}$.

The map F_ν is SU_2-equivariant, hence $g_{j_{k_l}} F_\nu(q_{j_{k_l}}) \in \overline{L_g^1}$. Let $\tilde{p} \in O$ be the limit of $\{F_\nu(q_j)\}$. Then $g\tilde{p} = \hat{p}$. Since $g \in I_{\hat{p}}$, it follows that $\tilde{p} = \hat{p}$. Hence F_ν is continuous, and therefore holomorphic on M. It obviously maps M biholomorphically onto $\widehat{B_R^2}$.

The case when $r > 0$ and $R = \infty$ can be treated along the same lines, but one must consider the holomorphic embedding $\sigma : S^2_{r,\infty} \to \widetilde{S^2_{r,\infty}}$

$$\sigma(z) := \{(1 : z_1 : z_2)\}, \tag{4.2}$$

the map $f_\sigma := \sigma \circ f$, and prove that f_σ extends to O as a biholomorphic map of M onto $\widetilde{S^2_{r,\infty}}$.

If $r = 0$ and $R = \infty$, then precisely one of f_ν and f_σ extends to O, and the extension defines a biholomorphic map from M to either $\widehat{\mathbb{C}^2}$, or $\widetilde{S^2_{0,\infty}}$.

Assume now that two orbits O_1 and O_2 in M are complex hypersurfaces. As above, we consider the manifold $\tilde{M}$ obtained from M by the removal of O_1 and O_2. It is biholomorphically equivalent to $S^2_{r,R}$ by means of an equivariant map f. Arguments very similar to the ones used above show that in this case $r = 0$, $R = \infty$, and $f_\tau := \tau \circ f$ extends to a biholomorphic map $M \to \widehat{\mathbb{CP}^2}$. Here $\tau : (\mathbb{C}^2 \setminus \{0\}) \to \widehat{\mathbb{CP}}^2$ is an SU_2-equivariant map defined as

$$\tau(z) := \Big\{\Big((1 : z_1 : z_2), w\Big)\Big\},$$

where $w = (w_1 : w_2)$ is uniquely determined from the condition $z_1 w_2 = z_2 w_1$.

Assume now that all real hypersurface orbits in M are non-spherical. Then every such orbit is equivalent to some $\mathfrak{S}^3_R$ by means of an equivariant CR-diffeomorphism. First, we note that in this case there is exactly one complex hypersurface orbit. Let O be a complex hypersurface orbit in M. Consider the ϵ-neighborhood $U_\epsilon(O)$ of O as in the proof of Proposition 4.3. As shown there, if ϵ is sufficiently small, then $\partial U_\epsilon(O)$ is a real hypersurface orbit. Hence $U_\epsilon(O)$ is either strongly pseudoconvex or strongly pseudoconcave if ϵ is sufficiently small.

Assume that $U_\epsilon(O)$ is strongly pseudoconvex. Then blowing down O in $U_\epsilon(O)$ we obtain a Stein analytic space with boundary $\partial U_\epsilon(O)$ (see e.g., [GR]). But this is impossible since it is shown in [R1] (see also [R2]) that none of $\mathfrak{S}^3_R$ can bound a Stein analytic space. Hence $U_\epsilon(O)$ can only be strongly pseudoconcave. Therefore, there exists exactly one complex hypersurface orbit in M.

Let O be the unique complex hypersurface orbit. We consider $\tilde{M} := M \setminus O$. By Theorem 3.9 it is equivalent to $\mathfrak{S}^2_{r,R}$ by means of a biholomorphic equivariant map f. It follows from the explicit description of the complex structure on X that for every complex line L in $\mathbb{C}^2$ the punctured line $L \setminus \{0\}$ is a complex curve in X'. Hence we can construct holomorphic curves N^j_g as above. By the same argument we have either $r = 0$ or $R = \infty$. Assume first that $r = 0$. Let $\{p_j\} \subset \tilde{M}$ be a sequence convergent to a point $p \in O$. Denote by r_j the radius of the sphere $f(O(p_j))$ and assume that $r_j \to 0$ as $j \to \infty$. Then U_ϵ is strongly pseudoconvex for some ϵ, which is impossible, as shown in the preceding paragraph. This contradiction shows that we have $R = \infty$ and $r_j \to \infty$.

We now regard the map σ defined in (4.2) as a map from $\mathfrak{S}^2_{r,\infty}$ into $\widetilde{\mathfrak{S}^2_{r,\infty}}$. This map remains holomorphic and SU_2-equivariant. We now define the map $f_\sigma := \sigma \circ f$, and prove that f_σ extends to O as a biholomorphic map of $F_\sigma : M \to \widetilde{\mathfrak{S}^2_{r,\infty}}$. We construct F_σ as above, by extending f_σ along the curves N^j_g and prove that it is continuous on O.

The proof is complete. □

5 The Homogeneous Case

We consider now the case when the action of SU_n on M is transitive. We claim that there are in fact no manifolds admitting such actions. We start with the following result.

Lemma 5.1 *Let G be a connected closed subgroup of SU_n of dimension $n^2 - 2n - 1$, $n \geq 3$. Then either*
(i) $n = 3$ and G is conjugate in SU_3 to $(U_1 \times U_1 \times U_1) \cap SU_3$ embedded in SU_3 in the standard way, or
(ii) $n = 4$ and G is conjugate in SU_4 to $(U_2 \times U_2) \cap SU_4$ embedded in SU_4 in the standard way.

Proof: Since G is compact, it is completely reducible, i.e., $\mathbb{C}^n$ decomposes into a sum of G-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \ldots \oplus V_m$, such that the restriction G_j of G to every V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence $n_1 + \ldots + n_m = n$) and let U_{n_j} be the unitary transformation group of V_j. Clearly, $G_j \subset U_{n_j}$, and therefore $\dim G \leq n_1^2 + \ldots + n_m^2$. On the other hand $\dim G = n^2 - 2n - 1$, which shows that $m \leq 2$ for $n \neq 3$. If $n = 3$, then it is also possible that $m = 3$, which means that G is conjugate in SU_3 to $(U_1 \times U_1 \times U_1) \cap SU_3$ embedded in U_3 in the standard way.

Now let $m = 2$. Then either $n = 4$ and G is conjugate in SU_4 to $(U_2 \times U_2) \cap SU_4$ embedded in SU_4 in the standard way, or G is conjugate in SU_n to a subgroup $\hat{G}$ of the group H^n (see (1.2)). The group H^n has dimension $(n-1)^2$ and is isomorphic to U_{n-1} in the obvious way. Hence $\hat{G}$ is isomorphic to a subgroup of U_{n-1} of codimension 2. It was shown in Lemma 2.1 in [IKran] that U_{n-1} does not have subgroups of codimension 2 unless $n = 3$, in which case $\hat{G}$ is conjugate to the group $(U_1 \times U_1 \times U_1) \cap SU_3$. But this is impossible since for this group $m = 3$.

Let $m = 1$. We proceed as at the beginning of the proof of Proposition 3.1. Let $\mathfrak{g} \subset \mathfrak{su}_n \subset \mathfrak{sl}_n$ be the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} + i\mathfrak{g} \subset \mathfrak{sl}_n$ its complexification. Then $\mathfrak{g}^{\mathbb{C}}$ acts irreducibly on $\mathbb{C}^n$ and by a theorem of É. Cartan is semisimple.

Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into the direct sum of simple ideals. Then the irreducible n-dimensional representation of $\mathfrak{g}^{\mathbb{C}}$ given by the embedding of $\mathfrak{g}^{\mathbb{C}}$ in $\mathfrak{gl}_n$ is the tensor product of some irreducible faithful representations of the $\mathfrak{g}_j$. Let n_j be the dimension of the corresponding representation of $\mathfrak{g}_j$, $j = 1, \ldots, k$. Then $n_j \geq 2$, $\dim_{\mathbb{C}} \mathfrak{g}_j \leq n_j^2 - 1$, and $n = n_1 \cdot \ldots \cdot n_k$.

Since $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} = n^2 - 2n - 1$, it follows from the claim in the proof of Proposition 3.1 that $k = 1$, i.e., $\mathfrak{g}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras are listed in table (3.1). It follows from the table that a simple complex Lie algebra of dimension $n^2 - 2n - 1$ cannot have an n-dimensional irreducible representation. Hence, in fact, $m \neq 1$.

The lemma is proved. □

The following theorem is an easy consequence of Lemma 5.1.

THEOREM 5.2 *There exists no real manifold of dimension $2n \geq 4$ admitting an effective transitive action of SU_n.*

Proof: Let M be the manifold, $p \in M$ and I_p be as before the isotropy subgroup of p. Obviously, $\dim I_p = n^2 - 2n - 1$ (clearly, we have $n \geq 3$).

Therefore, from Lemma 5.1 we see that either $n = 3$ and I_p^c is conjugate in SU_3 to $(U_1 \times U_1 \times U_1) \cap SU_3$ embedded in SU_3 in the standard way, or $n = 4$ and I_p^c is conjugate in SU_4 to $(U_2 \times U_2) \cap SU_4$ embedded in SU_4 in the standard way. In these cases, however, I_p^c clearly contains the center of SU_n for $n = 3, 4$, and hence the action of SU_n on M is not effective.

This contradiction proves the theorem. □

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