WEAK AMENABILITY OF $\mathcal{A}(E)$ AND THE GEOMETRY OF E

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ABSTRACT. We introduce a geometric property, and prove that Banach spaces with this property must carry weakly amenable algebras of approximable operators. We also apply these results to particular examples of Banach spaces, including the James spaces and the Tsirelson space.

1. INTRODUCTION

A Banach algebra $\mathfrak{A}$ is said to be *weakly amenable* if every continuous derivation from $\mathfrak{A}$ into the $\mathfrak{A}$-bimodule $\mathfrak{A}'$ (that is, the topological dual of $\mathfrak{A}$ equipped with the canonical module structure) is inner. Equivalently, $\mathfrak{A}$ is weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}') = \{0\}$, where $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}')$ denotes the first (topological) cohomology group of $\mathfrak{A}$ with coefficients in $\mathfrak{A}'$.

The notion of weak amenability was introduced in [BCD] for commutative Banach algebras, and was extended to the general case in [J1]. The list of examples of weakly amenable Banach algebras includes all group algebras for locally compact groups ([J2]), all C^*-algebras ([Ha]), and all tensor algebras ([DGG]). For a discussion and further examples see [Da].

In this paper we are concerned with the weak amenability of the algebra $\mathcal{A}(E)$ of *approximable operators* on a Banach space E, that is, the closure in $\mathcal{B}(E)$ of the ideal $\mathcal{F}(E)$ of *continuous finite-rank operators* on E, where $\mathcal{B}(E)$ denotes, as usual, the algebra of all *bounded linear operators* on E. Note that $\mathcal{A}(E)$ is the minimum closed (non-zero) ideal in $\mathcal{B}(E)$.

There is much interest in describing weak amenability of this algebra in terms of the geometry of the underlying Banach space. For instance, it is shown in [DGG] that $\mathcal{A}(E)$ is weakly amenable whenever E has one of the following forms : (i) $E = X \oplus C_p$, where C_p denotes any of the universal spaces introduced by W. B. Johnson in [Joh] and X is any Banach space with the bounded approximation property (B.A.P.); and (ii) $E = l_p(Y)$, $1 < p < \infty$, where Y is any reflexive Banach space with the approximation property (A.P.).

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The results we present here improve those from [DGG] mentioned in the previous paragraph. It is known (see [Bl]), that (A.P.) B.A.P. is neither necessary nor sufficient for the weak amenability of $\mathcal{A}(E)$. Here we shall obtain the weak amenability of $\mathcal{A}(E)$ as a consequence of a new property closely related to the local structure of the Banach space E.

In what follows, $\|\,.\,\|_\wedge$ denotes the projective norm on $E'\otimes E$; given a finite-rank operator W, $\operatorname{tr} W$ denotes its trace; and the adjoint of a bounded operator U is denoted by U'. All our Banach spaces are assumed to be over the complex field.

Our starting point will be the following characterization of weak amenability for algebras of approximable operators, given in [Bl].

Theorem 1.1. *Let E be a Banach space, and let $\mathfrak{A}$ be a dense subalgebra of $(E'\otimes E, \|\,.\,\|_\wedge)$ (and hence of $E'\hat{\otimes}E$). The algebra $\mathcal{A}(E)$ is weakly amenable if and only if, whenever $T\in\mathcal{B}(E')$ satisfies*

$$\left|\operatorname{tr}(T(RS-SR)')\right| \le K\|R\|\|S\| \qquad (R,S\in\mathfrak{A})$$

for some constant K, the following holds :

(A^*) *there exists a constant $\widetilde{K}$ such that*

$$|\operatorname{tr}(TW')| \le \widetilde{K}\|W\|$$

for all $W\in\mathfrak{A}$ such that $\operatorname{tr} W = 0$.

We shall give conditions, based on the geometry of the Banach space E, under which condition (A^*) of the above proposition is satisfied. Our main result is Theorem 2.8 at the end of the next section. We then use this result in the subsequent sections to prove the weak amenability of $\mathcal{A}(E)$ for some concrete examples of Banach spaces, including the James spaces, certain spaces of Bochner p-integrable functions, and the Tsirelson space.

Throughout we shall write X^- and X' for the completion and topological dual respectively of a normed space X. If $x_1, x_2, \dots, x_r$ are vectors of some linear space X, $\operatorname{sp}\{x_1, x_2, \dots, x_r\}$ will denote their linear span.

Also, if X and Y are isomorphic (respectively, isometric) normed spaces, we shall write this as $X\simeq Y$ (respectively, $X\cong Y$), and denote by $d(X,Y)$ the Banach–Mazur distance between them, that is, the infimum of numbers $\|T\|\|T^{-1}\|$, where T is an isomorphism between X and Y.

2. Weak amenability as a consequence of a geometric property.

We start with a definition. Note that, by a *sequence of mutually orthogonal projections*, we mean a sequence of projections (P_n) satisfying $P_n P_m = 0$ whenever $n\neq m$ $(n,m\in\mathbb{N})$.

Definition 2.1. *Let E be a Banach space. A pair $(X_0,(P_n))$, where X_0 is a subspace of E and $(P_n)\subset\mathcal{B}(E)$ is a sequence of mutually orthogonal projections, is* linked *to $W\in\mathcal{F}(E)$, or W-linked, if $\operatorname{rg} W\subseteq X_0$ and $\operatorname{rg} P_n\simeq X_0$ $(n\in\mathbb{N})$. Let $X_i := \operatorname{rg} P_i$ $(i\in\mathbb{N})$. A family of linear isomorphisms $T^i_j : X_j\to X_i$ $(i,j\in\mathbb{N}_0 := \mathbb{N}\cup\{0\})$ that* *satisfies* $T^i_k T^k_j = T^i_j$ $(i, j, k \in \mathbb{N}_0)$, *is a* connecting family *for* $(X_0, (P_n))$ (*note that we must have* $T^i_i = I_{X_i}$ $(i \in \mathbb{N}_0)$).

To avoid unnecessary repetition throughout the rest of this section E denotes a Banach space.

Take an operator $W \in \mathcal{F}(E)$, a W-linked pair $(X_0, (P_n))$, and a connecting family for this pair, say (T^i_j). In terms of these we define the sequence $(W_i) \subset \mathcal{F}(E)$ by

$$W_i := T^i_0 W T^0_i P_i \qquad (i \in \mathbb{N}). \tag{1}$$

We define

$$\mathcal{D}_E := \{(R, S) \in \mathcal{B}(E) \times \mathcal{B}(E) : R \in \mathcal{F}(E) \text{ or } S \in \mathcal{F}(E)\}, \tag{2}$$

and, for each $T \in \mathcal{B}(E')$, we set

$$b_T : (R, S) \mapsto \operatorname{tr}(T(RS - SR)') \qquad ((R, S) \in \mathcal{D}_E), \tag{3}$$

and

$$l_T : W \mapsto \operatorname{tr}(TW') \qquad (W \in \mathcal{F}(E)). \tag{4}$$

For future reference also define

$$\Delta_E := \{T \in \mathcal{B}(E') : b_T|_{\mathcal{F}(E)\times\mathcal{F}(E)} \text{ is bounded}\} \tag{5}$$

($\mathcal{F}(E)$ with the operator norm). Given $T \in \Delta_E$, we shall denote by $\|b_T\|$ the norm of $b_T|_{\mathcal{F}(E)\times\mathcal{F}(E)}$.

Proposition 2.2. *Let* $T \in \mathcal{B}(E')$, *and let* $W \in \mathcal{F}(E)$. *Let* $(X_0, (P_n))$ *be linked to* W, *let* (T^i_j) *be a connecting family for* $(X_0, (P_n))$, *and let* $(W_i) \subset \mathcal{F}(E)$ *be as in (1) above. Then, for every positive integer* n *:*

$$\begin{aligned} l_T(W) &= \sum_{k=1}^{n} \frac{1}{2^k} b_T\big(T^0_{2^{k-1}} P_{2^{k-1}}, T^{2^{k-1}}_0 W\big) \\ &\quad + \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} b_T\Big(\sum_{i=1}^{2^k-1} T^i_{2^k+i} P_{2^k+i}, \sum_{i=1}^{2^k-1} T^{2^k+i}_i W_i\Big) \\ &\quad + \frac{1}{2^n} l_T\Big(W + \sum_{i=1}^{2^n-1} W_i\Big). \end{aligned} \tag{6}$$

Proof. We prove this equality by induction on $n \in \mathbb{N}$.

For $n = 1$, the identity is easily verified.

Suppose that (6) holds for some positive integer n. We have

$$\begin{aligned} l_T\Big(W + \sum_{i=1}^{2^n-1} W_i\Big) &= \frac{1}{2} l_T(W - W_{2^n}) + \frac{1}{2} l_T\Big(\sum_{i=1}^{2^n-1} W_i - \sum_{i=2^n+1}^{2^{n+1}-1} W_i\Big) \\ &\quad + \frac{1}{2} l_T\Big(W + \sum_{i=1}^{2^n-1} W_i + \sum_{i=2^n}^{2^{n+1}-1} W_i\Big). \end{aligned} \tag{7}$$

Then since $(X_0,(P_n))$ is a W-linked pair and (T_j^i) is a connecting family for this pair, it follows from (7) and our definition of W_i $(i\in\mathbb{N})$ that

$$\begin{aligned} l_T\Big(W+\sum_{i=1}^{2^n-1}W_i\Big) &= \frac{1}{2}b_T\big(T_{2^n}^0P_{2^n},T_0^{2^n}W\big)+\\ &+\frac{1}{2}b_T\Big(\sum_{i=1}^{2^n-1}T_{2^n+i}^{\,i}P_{2^n+i}\,,\sum_{i=1}^{2^n-1}T_i^{2^n+i}W_i\Big)+\\ &+\frac{1}{2}l_T\Big(W+\sum_{i=1}^{2^{n+1}-1}W_i\Big). \end{aligned}$$

By combining the last identity with the induction hypothesis we see that (6) holds for $n+1$. □

Given n-tuples (respectively, sequences) $(e_i)_{i=1}^n$ (respectively, (e_i)) in E and $(e_i^*)_{i=1}^n$ (respectively, (e_i^*)) in E' such that $e_i^*(e_j)=\delta_{ij}$, $1\le i,j\le n$ (respectively, $i,j\in\mathbb{N}$), we denote by $\{e_i;e_i^*\}_{i=1}^n$ (respectively, $\{e_i;e_i^*\}_{i=1}^\infty$) the biorthogonal system formed by them. Given a sequence of vectors (e_n) in E and an infinite subset I of $\mathbb{N}$, we denote by $[e_n]_{n\in I}$ the closure in E of $\mathrm{sp}\{e_n : n\in I\}$. If I has the form $\{n\in\mathbb{N}:k\le n\}$ for some $k\in\mathbb{N}$, then we also use the notation $[e_n]_{n=k}^\infty$ (or just $[e_n]$ if $I=\mathbb{N}$).

Definition 2.3. *A biorthogonal system $\{e_n;e_n^*\}_{n=1}^\infty$ in a Banach space E such that $\sup_n\{\|e_n\|\|e_n^*\|\}<\infty$ is said to be* trace-unbounded *(t.u. in short) if, for every pair of infinite, disjoint subsets $N_1\subset\mathbb{N}$ and $N_2\subset\mathbb{N}$ the following holds : for every positive integer n, there are subsets*

$$\{i_{n,1},i_{n,2},\dots,i_{n,n}\}\subset N_1 \quad \textit{and} \quad \{j_{n,1},j_{n,2},\dots,j_{n,n}\}\subset N_2\,,$$

and a linear isomorphism

$$T_n:\mathrm{sp}\{e_{i_{n,k}}:1\le k\le n\}\to\mathrm{sp}\{e_{j_{n,k}}:1\le k\le n\}$$

such that

$$\big\|T_nP_{n,1}\big\|\big\|T_n^{-1}P_{n,2}\big\|=o(n)$$

(*where $P_{n,1}=\sum_k e_{i_{n,k}}^*\otimes e_{i_{n,k}}$ and $P_{n,2}=\sum_k e_{j_{n,k}}^*\otimes e_{j_{n,k}}$*).

Note that, if $\{e_n;e_n^*\}_{n=1}^\infty$ is a t.u. biorthogonal system, and N_1, N_2 are arbitrary, infinite, disjoint subsets of $\mathbb{N}$, then the bilinear map

$$\mathcal{F}([e_n]_{n\in N_1},\,[e_n]_{n\in N_2})\times\mathcal{F}([e_n]_{n\in N_2},\,[e_n]_{n\in N_1})\to\mathbb{C}\,,\ (R,S)\mapsto\mathrm{tr}(RS)$$

is indeed unbounded. For, if T_n, $P_{n,1}$ and $P_{n,2}$ $(n\in\mathbb{N})$ are as in the definition above, then we have $\mathrm{tr}(T_nP_{n,1}T_n^{-1}P_{n,2})=\mathrm{tr}(P_{n,2})=n$, while, on the other hand, $\|T_nP_{n,1}\|\|T_n^{-1}P_{n,2}\|=o(n)$. This justifies the name of trace-unbounded.

Example. Let (e_n) be an unconditional basic sequence in a Banach space E such that $[e_n]$ is complemented by a bounded projection P, and let (e_n^*) be the associated sequence of biorthogonal functionals on $[e_n]$. Then the biorthogonal system

$\{e_n; e_n^* \circ P\}_{n=1}^{\infty}$ is t.u. To see this we need the following result of Lindenstrauss and Szankowski (see [T-J, Theorem 48.3]) :

For every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that, if E and F are n-dimensional Banach spaces with 1-unconditional bases, then $d(E, F) \leq c(\varepsilon)n^{2/3+\varepsilon}$.

Without loss of generality, we can assume that (e_n) is 1-unconditional (otherwise, since $[e_n]$ is complemented in E, we can equivalently renorm E so as to achieve this). Let $N_1 = \{i_1, i_2, \dots\}$ and $N_2 = \{j_1, j_2, \dots\}$ be infinite, disjoint subsets of $\mathbb{N}$ written in increasing order. For each n define

$$E_n := \mathrm{sp}\{e_{i_k} : 1 \leq k \leq n\} \text{ and } F_n := \mathrm{sp}\{e_{j_k} : 1 \leq k \leq n\}.$$

Since (e_n) is 1-unconditional, so are the bases $\{e_{i_k} : 1 \leq k \leq n\}$ and $\{e_{j_k} : 1 \leq k \leq n\}$ of E_n and F_n, respectively. Thus, by the preceding result, there is a linear isomorphism $T_n : E_n \to F_n$ such that $\|T_n\|\|T_n^{-1}\| \leq c(\varepsilon)n^{2/3+\varepsilon}$. The 1-unconditionality of (e_n) also implies that $\|\sum_{k=1}^n e_{i_k}^* \otimes e_{i_k}\| = 1$ and $\|\sum_{k=1}^n e_{j_k}^* \otimes e_{j_k}\| = 1$. Let $P_{n,1} = \sum_{k=1}^n (e_{i_k}^* \circ P) \otimes e_{i_k}$ and $P_{n,2} = \sum_{k=1}^n (e_{j_k}^* \circ P) \otimes e_{j_k}$. Then we have

$$\|T_n P_{n,1}\|\|T_n^{-1} P_{n,2}\| \leq c(\varepsilon)\|P\|^2 n^{2/3+\varepsilon}.$$

To achieve our goal, it is clearly enough to take $0 < \varepsilon < 1/3$ (recall that, for a Schauder basis (e_n), it is always true that $\sup_n\{\|e_n\|\|e_n^*\|\} < \infty$). This shows that $\{e_n; e_n^* \circ P\}_{n=1}^{\infty}$ is a t.u. biorthogonal system, as we claimed.

Thus, every Banach space E with an unconditional basic sequence (e_n) for which $[e_n]$ is complemented in E by a bounded projection has t.u. biorthogonal systems. Note that, if E has a t.u. biorthogonal system $\{e_i; e_i^*\}_{i=1}^{\infty}$, then trivially it has infinitely many, as every subsystem $\{e_{i_k}; e_{i_k}^*\}_{k=1}^{\infty}$ of the former is again t.u.

We need the following lemma.

Lemma 2.4. *Let $W \in \mathcal{F}(E)$, and let (E_n) be a sequence of subspaces of E such that $\mathrm{rg}\, W \subseteq E_1$, $E_n \subseteq E_{n+1}$ $(n \in \mathbb{N})$ and $\left(\bigcup_n E_n\right)^- = E$. Then there exists $k \in \mathbb{N}$ and a biorthogonal system $\{e_i; e_i^*\}_{i=1}^m$ on E such that $e_i \in E_k$ $(1 \leq i \leq m)$ and*

$$W = \sum_{1 \leq i,j \leq m} a_{ij} e_j^* \otimes e_i$$

for some scalars $a_{ij} \in \mathbb{C}$ $(1 \leq i, j \leq m)$.

Proof. Let $W = \sum_{i=1}^r \lambda_i \otimes x_i$ with $\lambda_1, \lambda_2, \dots, \lambda_r$ and $x_1, x_2, \dots, x_r$ linearly independent vectors in E' and E, respectively. Since $\mathrm{sp}\{x_1, x_2, \dots, x_r\} = \mathrm{rg}\, W \subseteq E_1$ and $\bigcup_n E_n$ is dense in E, there exist $e_1, e_2, \dots, e_m \in \bigcup_n E_n$ such that $\{e_1, e_2, \dots, e_m\}$ is linearly independent, such that $\mathrm{sp}\{x_1, x_2, \dots, x_r\} \subseteq \mathrm{sp}\{e_1, e_2, \dots, e_m\}$, and such that

$$\det \begin{pmatrix} \lambda_1(e_1) & \lambda_1(e_2) & \cdots & \lambda_1(e_r) \\ \lambda_2(e_1) & \lambda_2(e_2) & \cdots & \lambda_2(e_r) \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_r(e_1) & \lambda_r(e_2) & \cdots & \lambda_r(e_r) \end{pmatrix} \neq 0\,.$$

For $i = r+1, \dots, m$, let e_i^* be any continuous linear functional on E satisfying $e_i^*(e_j) = \delta_{ij}$ $(1 \le j \le m)$, where δ_{ij} is Kronecker's delta symbol. Then define $e_1^*, e_2^*, \dots, e_r^*$ as the unique solution of the system of equations

$$\lambda_i = \sum_{j=1}^{m} \lambda_i(e_j) e_j^* \qquad (1 \le i \le r).$$

The system $\{e_i; e_i^*\}_{i=1}^m$ is biorthogonal, and obviously

$$W = \sum_{1 \le i,j \le m} a_{ij} e_j^* \otimes e_i$$

for some scalars $a_{ij} \in \mathbb{C}$ $(1 \le i, j \le m)$. Moreover, since (E_n) is an increasing sequence, it follows from our choice of the e_i's that there exists $k \in \mathbb{N}$ such that $e_i \in E_k$ $(1 \le i \le m)$. $\square$

Definition 2.5. *Let $W \in \mathcal{F}(E)$, and let $\{x_i; x_i^*\}_{i=1}^m$ be a biorthogonal system on E such that $W = \sum_{1 \le i,j \le m} a_{ij} x_j^* \otimes x_i$ for some scalars $a_{ij} \in \mathbb{C}$, $(1 \le i, j \le m)$ (by Lemma 2.4, such a system always exists). Let $(X_0, (P_n))$ be a W-linked pair such that $x_i \in X_0$ $(1 \le i \le m)$, and let (T_j^i) be a connecting family for $(X_0, (P_n))$. Let the sequences $(e_i) \subset E$ and $(e_i^*) \subset E'$ be defined by*

$$e_{km+i} := T_0^{k+1} x_i \quad \text{and} \quad e_{km+i}^* := x_i^* \circ T_{k+1}^0 \circ P_{k+1} \quad (1 \le i \le m,\, k \in \mathbb{N}_0),$$

*respectively. The triple $\left(\{x_i; x_i^*\}_{i=1}^m, (X_0, (P_n)), (T_j^i)\right)$ is W-*trace-unbounded *(W-t.u. in short) if the biorthogonal system $\{e_n; e_n^*\}_{n=1}^\infty$ is trace-unbounded and the shift operator*

$$S : [e_n] \to [e_n]_{n=2}^\infty, \ \sum_n \alpha_n e_n \mapsto \sum_n \alpha_n e_{n+1}$$

is continuous.

Proposition 2.6. *Let $W \in \mathcal{F}(E)$, and let $\left(\{x_i; x_i^*\}_{i=1}^m, (X_0, (P_n)), (T_j^i)\right)$ be a W-t.u. triple. If $\operatorname{tr} W = 0$, then*

$$\lim_{n\to\infty} \frac{1}{n} l_T\left(W + \sum_{i=1}^{n-1} W_i\right) = 0$$

for all $T \in \Delta_E$ (see (5)).

The proof of Proposition 2.6 is based on the following result.

Lemma 2.7. *Let $T \in \Delta_E$, and let $\{e_i; e_i^*\}_{i=1}^\infty$ be a t.u. biorthogonal system in E. Let $t_{i,j} = (Te_i^*)(e_j)$ $(i, j \in \mathbb{N})$. Then $\lim_i t_{i,i+k}$ exists for every $k \in \mathbb{Z}$, and is equal to zero whenever $k \ne 0$.*

Proof. We show first that $\lim_i t_{i,i}$ exists. Clearly, $\sup_i\{\|e_i\|\|e_i^*\|\} < \infty$ implies that $(t_{i,i})$ is bounded. Assume towards a contradiction that $(t_{i,i})$ has at least two limit points λ_1 and λ_2. Clearly we may suppose that $\lambda_1 = \delta$ and $\lambda_2 = -\delta$ for some $\delta > 0$.

Let $N_1 := \{i : \operatorname{Re} t_{i,i} \ge \delta/2\}$ and $N_2 := \{i : \operatorname{Re} t_{i,i} \le -\delta/2\}$. Since $\{e_i; e_i^*\}_{i=1}^\infty$ is t.u., for every positive integer n, there are subsets $\{i_{n,1}, i_{n,2}, \dots, i_{n,n}\} \subset N_1$ and

$\{j_{n,1}, j_{n,2}, \dots, j_{n,n}\} \subset N_2$, and a linear isomorphism $T_n : \mathrm{sp}\{e_{i_{n,k}}\} \to \mathrm{sp}\{e_{j_{n,k}}\}$ such that $\|T_nP_{n,1}\|\|T_n^{-1}P_{n,2}\| = o(n)$ where $P_{n,1} = \sum_k e^*_{i_{n,k}} \otimes e_{i_{n,k}}$ and $P_{n,2} = \sum_k e^*_{j_{n,k}} \otimes e_{j_{n,k}}$ (see Definition 2.3).

Define $R_n := T_nP_{n,1}$ and $S_n := T_n^{-1}P_{n,2}$ $(n \in \mathbb{N})$. Since $T \in \Delta_E$, we have

$$\left|\mathrm{tr}\big(T(R_nS_n - S_nR_n)'\big)\right| \le \|b_T\|\|R_n\|\|S_n\| = o(n) .$$

On the other hand, taking into account the definition of N_1 and N_2 and the definition of the $t_{i,i}$'s, we find that

$$\begin{aligned} \left|\mathrm{tr}\big(T(R_nS_n - S_nR_n)'\big)\right| &= \left|\mathrm{tr}\big(T(P_{n,1} - P_{n,2})'\big)\right| \\ &\ge \mathrm{Re}\Big(\sum_{k=1}^{n} t_{i_{n,k},i_{n,k}} - \sum_{k=1}^{n} t_{j_{n,k},j_{n,k}}\Big) \ \ge\ n\delta. \quad (n \in \mathbb{N}) \end{aligned}$$

Combining the last inequalities we see that $n\delta \le o(n)$, which is clearly impossible. Thus, $\lim_i t_{i,i}$ exists.

Let us show now that $\lim_i t_{i,i+\varkappa} = 0$ $(\varkappa \in \mathbb{Z} \setminus \{0\})$. Fix $\varkappa$ in $\mathbb{Z} \setminus \{0\}$. Once again assume towards a contradiction that our claim is not true. Then there is $\delta > 0$ such that at least one of the sets $\{i : \mathrm{Re}\, t_{i,i+\varkappa} \ge \delta\}$, $\{i : \mathrm{Re}\, t_{i,i+\varkappa} \le -\delta\}$, $\{i : \mathrm{Im}\, t_{i,i+\varkappa} \ge \delta\}$ or $\{i : \mathrm{Im}\, t_{i,i+\varkappa} \le -\delta\}$ is infinite. Without loss of generality let us suppose that $\{i : \mathrm{Re}\, t_{i,i+\varkappa} \ge \delta\}$ is infinite. Let $\mathcal{C}_l := \big\{n \in \mathbb{N} : n \equiv l \bmod (\varkappa + 1)\big\}$, $0 \le l \le \varkappa$. Since $\{i : \mathrm{Re}\, t_{i,i+\varkappa} \ge \delta\}$ is infinite and $\bigcup_l \mathcal{C}_l = \mathbb{N}$, we see that $\{i : \mathrm{Re}\, t_{i,i+\varkappa} \ge \delta\} \bigcap \mathcal{C}_{l_o}$ must be infinite for some $l_o \in \{0, 1, \dots, \varkappa\}$. Define $N_1 := \{i : \mathrm{Re}\, t_{i,i+\varkappa} \ge \delta\} \bigcap \mathcal{C}_{l_o}$, and $N_2 = \mathbb{N} \setminus N_1$. Note that $\{i + \varkappa : i \in N_1\} \cap N_1 = \emptyset$.

Again, let $\{i_{n,1}, i_{n,2}, \dots, i_{n,n}\} \subset N_1$, $\{j_{n,1}, j_{n,2}, \dots, j_{n,n}\} \subset N_2$, and T_n $(n \in \mathbb{N})$ be as in Definition 2.3. Define $R_n := T_nP_{n,1}$ and $S_n := S^{\varkappa}T_n^{-1}P_{n,2}$ $(n \in \mathbb{N})$ (where S denotes the right-shift operator). Then $R_nS_n = 0$ and $S_nR_n = S^{\varkappa}P_{n,1}$ $(n \in \mathbb{N})$, and, by our definition of N_1, we have that

$$\begin{aligned} \left|\mathrm{tr}\big(T(R_nS_n - S_nR_n)'\big)\right| &= \left|\mathrm{tr}\big(T(S^{\varkappa}P_{n,1})'\big)\right| = \left|\sum_{k=1}^{n} t_{i_{n,k},i_{n,k}+\varkappa}\right| \\ &\ge \sum_{k=1}^{n} \mathrm{Re}\, t_{i_{n,k},i_{n,k}+\varkappa} \ \ge\ n\delta. \end{aligned}$$

However, by our assumption about T, and taking into account that the shift operator S is bounded, we have

$$\left|\mathrm{tr}\big(T(R_nS_n - S_nR_n)'\big)\right| \le \|b_T\|\|R_n\|\|S_n\| = o(n).$$

So, once more we reach the contradiction that $n \le o(n)$. The rest is clear. $\square$

Proof of Proposition 2.6. It is clearly enough to show that $\lim_k l_T(W_k) = 0$. Let $W = \sum_{1\le i,j\le m} a_{ij}x_j^* \otimes x_i$, and let the sequences $(e_n) \subset E$ and $(e_n^*) \subset E'$ be as in Definition 2.5. It follows from the definition of the W_k's that

$$W_{k+1} = \sum_{1\le i,j\le m} a_{ij}e^*_{km+j} \otimes e_{km+i} \qquad (k \in \mathbb{N}_0).$$

Moreover, since $\operatorname{tr} W = 0$, we have $\operatorname{tr} W_k = 0$ $(k \in \mathbb{N})$. Let $\lim_i (Te_i^*)(e_i) = \lambda$, the limit existing by Lemma 2.7. Then we have

$$\begin{aligned}
\lim_k l_T(W_k) &= \lim_k l_{T-\lambda}(W_k) \\
&= \lim_k \sum_{1 \le i,j \le m} a_{ij}\big((Te_{km+j}^*)(e_{km+i}) - \delta_{ij}\lambda\big) \\
&= \sum_{1 \le i,j \le m} a_{ij} \lim_k \big((Te_{km+j}^*)(e_{km+i}) - \delta_{ij}\lambda\big)
\end{aligned}$$

(where δ_{ij} $(1 \le i,j \le m)$ is the Kronecker delta). Clearly, the last expression converges to zero (by Lemma 2.7). □

Theorem 2.8. *Let $\mathfrak{A}$ be a dense subalgebra of $(\mathcal{F}(E), \|\,.\,\|_\wedge)$. Suppose that there exist sequences of positive numbers $(\delta_n^{(i)})$, $(\sigma_n^{(i)})$ $(i = 1,2)$ and a function $h : \Delta_E \to \mathbb{R}_+$ such that for $i = 1,2$:*

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\delta_n^{(i)}\sigma_n^{(i)}}{2^n} < \infty$ *; and*
2. *for every pair W, T, with $W \in \mathfrak{A}$ such that $\operatorname{tr} W = 0$ and $T \in \Delta_E$, there is a W-trace-unbounded triple $\big(\{x_j; x_j^*\}_{j=1}^m, (X_0, (P_n)), (T_k^j)\big)$ such that the sequences*

$$\big(R_n^{(1)}, S_n^{(1)}\big) := \big(T_{2^{n-1}}^0 P_{2^{n-1}}, T_0^{2^{n-1}} W\big)$$

and

$$\big(R_n^{(2)}, S_n^{(2)}\big) := \left(\sum_{j=1}^{2^n-1} T_{2^n+j}^j P_{2^n+j}\,, \sum_{j=1}^{2^n-1} T_j^{2^n+j} W_j\right) \qquad (n \in \mathbb{N})$$

in $\mathcal{D}_E$ (see (2)), satisfy the inequalities

$$\big\|R_n^{(i)}\big\| \le \delta_n^{(i)}\,,$$

$$\big\|S_n^{(i)}\big\| \le \sigma_n^{(i)}\|W\|\,,$$

and

$$\Big|b_T\big(R_n^{(i)}, S_n^{(i)}\big)\Big| \le h(T)\big\|R_n^{(i)}\big\|\big\|S_n^{(i)}\big\| \qquad (n \in \mathbb{N}).$$

Then the algebra $\mathcal{A}(E)$ is weakly amenable.

Proof. We show that (A^*) (of Theorem 1.1) is satisfied.

Let $(\delta_n^{(i)})$, $(\sigma_n^{(i)})$ $(i = 1,2)$, and h be as in the hypotheses of the theorem. Let $T \in \mathcal{B}(E')$ be such that $b_T|_{\mathfrak{A}\times\mathfrak{A}}$ ($\mathfrak{A}$ with the operator norm) is continuous. Since $\mathfrak{A}$ is dense in $(\mathcal{F}(E), \|\,.\,\|_\wedge)$ and $\|\,.\,\| \le \|\,.\,\|_\wedge$, we have $T \in \Delta_E$. Let $W \in \mathfrak{A}$ be such that $\operatorname{tr} W = 0$, and let $\big(\{x_j; x_j^*\}_{j=1}^m, (X_0, (P_n)), (T_k^j)\big)$ be a W-trace-unbounded triple as in 2. By Proposition 2.2, we have

$$\begin{aligned}\big|l_T(W)\big| \;&\leq\; h(T)\bigg(\sum_{k=1}^{n}\frac{\delta_k^{(1)}\sigma_k^{(1)}}{2^k}+\sum_{k=1}^{n-1}\frac{\delta_k^{(2)}\sigma_k^{(2)}}{2^{k+1}}\bigg)\|W\|+\\ &\qquad+\frac{1}{2^n}\bigg|l_T\bigg(W+\sum_{j=1}^{2^n-1}W_j\bigg)\bigg| \qquad (n\in\mathbb{N})\,.\end{aligned}$$

Passing to the limit when n tends to infinity we see from Proposition 2.6 that

$$|l_T(W)| \leq \widetilde{K}_T\,\|W\|,$$

where $\widetilde{K}_T = h(T)\Big(\sum_{k=1}^{\infty}\delta_k^{(1)}\sigma_k^{(1)}/2^k+\sum_{k=1}^{\infty}\delta_k^{(2)}\sigma_k^{(2)}/2^{k+1}\Big)$. Thus (A^*) is satisfied, and, by Theorem 1.1, $\mathcal{A}(E)$ is weakly amenable. ☐

Remark. We point out that the idea of a trace-unbounded triple introduced in this section is, roughly speaking, a revised version of the following property shared by all Banach spaces E of the form $l_p(Y)$ (where Y is another Banach space) : for any finite-dimensional subspace F of E, there is a complemented subspace of E which is the l_p-sum of infinitely many isometric copies of F.

We will see in the remainder of this paper that in many important cases the hypotheses of Theorem 2.8 are relatively easy to verify.

3. The Johnson spaces.

Our next definition is a natural generalization of the one given by W. B. Johnson in [Joh].

Definition 3.1. *A Banach space J is said to be a* Johnson space *if it has the form $(\sum\oplus_n G_n)_p$ ($p=0$ or $1\leq p\leq\infty$), where (G_n) is a sequence of finite-dimensional Banach spaces such that, for each $i\in\mathbb{N}$, the set $\{n\in\mathbb{N}: G_n\cong G_i\}$ is infinite.*

Example 3.1.a. If the sequence (G_n) in the above definition is dense (in the Banach–Mazur sense) in the class of all finite-dimensional Banach spaces, then our definition coincides with the one given in [Joh]. As pointed out there, in this case for p fixed, all the Johnson spaces are pairwise isomorphic and the Banach–Mazur distance between any two of them is 1. Thus, in this situation, for each p there is essentially a unique Johnson space, which is denoted by C_p.

Example 3.1.b. Trivially, l_p $(1\leq p\leq\infty)$ and c_0 are Johnson spaces in the sense of Definition 3.1.

Definition 3.2. *Let $J=(\sum\oplus_n G_n)_p$ be a Johnson space. A Banach space X is a* J-space *if there exists $\lambda\geq 1$ such that, for every finite-dimensional subspace E of X, there is a subspace F of X containing E and such that $d(F,G_i)\leq\lambda$ for some i.*

Example 3.2.a. Any Banach space is a C_p-space for every $1\leq p\leq\infty$ and $p=0$.

Example 3.2.b. Recall from [LP] that a Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-space $(1\leq\lambda<\infty,\ 1\leq p\leq\infty)$ if for every finite-dimensional subspace E of X, there is a finite-dimensional subspace F of X containing E and such that $d(F, l_p^n) \leq \lambda$, where $n = \dim F$. Then a Banach space X is an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some λ. Clearly every $\mathcal{L}_p$-space ($1 \leq p < \infty$) ($\mathcal{L}_\infty$-space) is an l_p-space (c_0-space) for some $\lambda \geq 1$.

Proposition 3.3. *Let $J = \left(\sum \oplus_n G_n\right)_p$ ($p = 0$ or $1 \leq p < \infty$) be a Johnson space, and let X be a J-space. Then the algebra $\mathcal{A}(X \oplus J)$ is weakly amenable.*

Proof. We shall show that all the hypotheses of Theorem 2.8 are satisfied. We give the proof only for $1 \leq p < \infty$, the case $p = 0$ being completely analogous.

Let us start by defining a subalgebra $\mathfrak{A}$ of $\mathcal{F}(X \oplus J)$ as in the hypotheses of Theorem 2.8. Without loss of generality, we may suppose that the norm in $X \oplus J$ is defined by

$$\|(x, y)\| := \left(\|x\|^p + \|y\|^p\right)^{\frac{1}{p}} \qquad (x \in X,\ y \in J).$$

Let $\gamma_0 : X \oplus J \to X$ ($\imath_0 : X \to X \oplus J$) and $\gamma_k : X \oplus J \to G_k$ ($\imath_k : G_k \to X \oplus J$), $k \in \mathbb{N}$, denote the canonical projections (canonical embeddings). Let $\Gamma_n := \sum_{k=0}^{n} \imath_k \circ \gamma_k$ ($n \in \mathbb{N}_0$), and let $\mathfrak{A}$ be the algebra of those operators $W \in \mathcal{F}(X \oplus J)$ such that $\Gamma_\varkappa W = W$ for some $\varkappa \in \mathbb{N}$.

That $\mathfrak{A}$ is in fact an algebra is easily checked using the obvious identity $\Gamma_n \Gamma_m = \Gamma_{\min\{m,n\}}$ ($m, n \in \mathbb{N}_0$). Now $\mathfrak{A}$ is dense in $\mathcal{F}(X \oplus J)$ in the projective norm. To see this, take $\xi \otimes x \in \mathcal{F}(X \oplus J)$. Then we have

$$\|\xi \otimes x - \xi \otimes \Gamma_n x\|_\wedge \leq \|\xi\| \|x - \Gamma_n x\|_p \,. \tag{8}$$

Clearly $\|x - \Gamma_n x\|_p \to 0$. Thus, it follows from (8) that $\Gamma_n \circ (\xi \otimes x) \to \xi \otimes x$ in the projective norm. Since the last holds for every rank-one operator, and each finite-rank operator is in turn a finite sum of operators of rank one, we see that $\Gamma_n W \to W$ in the projective norm for all $W \in \mathcal{F}(X \oplus J)$. Since $\Gamma_n W \in \mathfrak{A}$ ($n \in \mathbb{N}$, $W \in \mathcal{F}(X \oplus J)$) it follows that $\mathcal{F}(X \oplus J) \subset (\mathfrak{A}, \|\,.\,\|_\wedge)^-$, as claimed.

Now we need to find positive sequences $\left(\delta_n^{(i)}\right)$, $\left(\sigma_n^{(i)}\right)$ ($i = 1, 2$) and a map $h : \Delta_{X \oplus J} \to \mathbb{R}_+$ such that conditions (1) and (2) of Theorem 2.8 are satisfied.

Fix $W \in \mathfrak{A}$, and let $\varkappa_o \in \mathbb{N}$ be such that $\Gamma_{\varkappa_o} W = W$. Then, taking advantage of the structure of our Banach space and our definition of $\mathfrak{A}$, we define a W-trace-unbounded triple $\mathcal{U}_W$ as follows.

The definition is given in four steps.

1. We first choose the biorthogonal system. Let $E_n := \operatorname{rg} \Gamma_{\varkappa_o + n - 1}$ ($n \in \mathbb{N}$). It is clear that the sequence (E_n) satisfies all requirements of Lemma 2.4. Thus, there exists $\varkappa \in \mathbb{N}$ and a biorthogonal system $\{x_i; x_i^*\}_{i=1}^m$ on $X \oplus J$ such that

$$W = \sum_{1 \leq i,j \leq m} a_{ij}\, x_j^* \otimes x_i$$

for some scalars $a_{ij} \in \mathbb{C}$ ($1 \leq i, j \leq m$), and $x_i = \Gamma_\varkappa x_i$ ($1 \leq i \leq m$).

2. Let us turn to the second component of $\mathcal{U}_W$, that is, to the W-linked pair. We first define the subspace X_0 of $X \oplus J$. We use the fact that X is a J-space. Let $\lambda \geq 1$ as in Definition 3.2, and let G_0 be a subspace of X such that $\gamma_0(x_i) \in G_0$ $(1 \leq i \leq m)$ and $d(G_0, G_{n_0}) \leq \lambda$ for some $n_0 \in \mathbb{N}$. Then we define $X_0 := \sum_{n=0}^{\varkappa} \imath_n(G_n)$. Note that, since $x_i = \Gamma_\varkappa x_i$ and $\gamma_0(x_i) \in G_0$ $(1 \leq i \leq m)$, we have that $x_i \in X_0$ $(1 \leq i \leq m)$.

Now let us define the sequence (P_n) of mutually orthogonal projections. By our definition of a Johnson space, there is a subsequence $\big(G_{n_k}\big)$ of (G_n) such that $n_1 > n_0$, $G_k \cong G_{n_k}$ $(1 \leq k \leq \varkappa)$, and $G_{n_k} \cong G_{n_l}$ whenever $k \equiv l \bmod \bar{\varkappa}$ (where $\bar{\varkappa} = \varkappa + 1$ and $k, l \in \mathbb{N}_0$). Define $P_n := \sum_{(n-1)\bar{\varkappa} \leq k < n\bar{\varkappa}} \imath_{n_k} \circ \gamma_{n_k}$ $(n \in \mathbb{N})$.

It is not difficult to see that $P_i P_j = 0$ whenever $i \neq j$, and that $\operatorname{rg} P_n \simeq X_0$ $(n \in \mathbb{N})$. Thus $(X_0, (P_n))$ is W-linked.

3. We now define a connecting family for the above pair. For each positive integer n, let $X_n := \operatorname{rg} P_n$, and let $\widetilde{T}_n^1 : X_n \to X_1$ be a linear isometry (it is clear from our definition of P_n that $\operatorname{rg} P_n \cong \operatorname{rg} P_1$ $(n \in \mathbb{N})$). Let $\widetilde{T}_0^1 : X_0 \to X_1$ be a linear isomorphism such that $\|\widetilde{T}_0^1\| \leq 1$ and $\|(\widetilde{T}_0^1)^{-1}\| \leq \lambda + 1$ (it is easily seen that

$$d\Big(\sum_{k=0}^{\varkappa} \imath_k(G_k)\,, \sum_{k=0}^{\varkappa} \imath_{n_k}(G_{n_k})\Big) \leq \lambda,$$

thus $\widetilde{T}_0^1$ exists). Then for every pair $i, j \in \mathbb{N}_0$ we define $T_j^i := (\widetilde{T}_i^1)^{-1} \widetilde{T}_j^1$. It is not difficult to verify that (T_j^i) is a connecting family for $(X_0, (P_n))$.

4. Lastly, it only remains to verify that the sequences $(e_i) \subset X \oplus J$ and $(e_i^*) \subset (X \oplus J)'$ defined by $e_{km+j} = T_0^{k+1} x_j$ and $e_{km+j}^* = x_j^* \circ T_{k+1}^0 \circ P_{k+1}$, respectively, $(1 \leq j \leq m,\ k \in \mathbb{N}_0)$, form a t.u. biorthogonal system, and that the right-shift operator

$$S : [e_i] \to [e_i]_{i=2}^{\infty},\ \sum_i \alpha_i e_i \mapsto \sum_i \alpha_i e_{i+1}$$

is bounded. It is not difficult to see that (e_i) is equivalent to the unit vector basis of l_p, and, since the latter is subsymmetric, so is (e_i). Thus, S is bounded.

Define $P : X \oplus J \to X \oplus J$ by $P := \sum_{i=1}^{\infty} e_i^* \otimes e_i = \sum_{k=1}^{\infty} T_0^k Q T_k^0 P_k$ (where $Q = \sum_{i=1}^{m} x_i^* \otimes x_i$). It is easily seen that P is a continuous projection onto $[e_i]$, and that P satisfies $e_i^* \circ P = e_i^*$ $(i \in \mathbb{N})$. Since (e_i) and (e_i^*) clearly form a biorthogonal system, and (e_i) is subsymmetric (and hence unconditional), it follows (see the example after Definition 2.3) that $\{e_i; e_i^*\}_{i=1}^{\infty}$ is a t.u. biorthogonal system as we need.

With this we conclude the definition of $\mathcal{U}_W$.

Let the sequences $\big((R_n^{(i)}, S_n^{(i)})\big) \subset \mathcal{D}_{X \oplus J}$, $i = 1, 2$, be as in Theorem 2.8. It follows immediately from our definition of $\mathcal{U}_W$ that $\big\|R_n^{(1)}\big\| \leq \lambda + 1$ and $\big\|S_n^{(1)}\big\| \leq \|W\|$ $(n \in \mathbb{N})$.

Let $x \in X \oplus J$. Then we see that

$$\begin{aligned} (9) \qquad \|R_n^{(2)} x\|_{X\oplus J}^p &= \sum_{i=1}^{2^n-1} \|T_{2^n+i}^i P_{2^n+i}\, x\|_{X\oplus J}^p = \sum_{i=1}^{2^n-1} \|P_{2^n+i}\, x\|_{X\oplus J}^p \\ &\le \|x\|_{X\oplus J}^p \end{aligned}$$

and

$$\begin{aligned} (10) \qquad \|S_n^{(2)} x\|_{X\oplus J}^p &= \sum_{i=1}^{2^n-1} \|T_i^{2^n+i} W_i\, x\|_{X\oplus J}^p = \sum_{i=1}^{2^n-1} \|T_0^{2^n+i} W T_i^0 P_i\, x\|_{X\oplus J}^p \\ &\le (\lambda+1)^p \|W\|^p \|x\|_{X\oplus J}^p. \end{aligned}$$

Thus $\|R_n^{(2)}\| \le 1$ and $\|S_n^{(2)}\| \le (\lambda+1)\|W\|$ $(n \in \mathbb{N})$.

Moreover, since $\big(R_n^{(i)}, S_n^{(i)}\big) \in \mathfrak{A} \times \mathfrak{A}$ $(n \in \mathbb{N},\ i = 1, 2)$, we also have

$$\big|b_T\big(R_n^{(i)}, S_n^{(i)}\big)\big| \le \|b_T\| \|R_n^{(i)}\| \|S_n^{(i)}\| \qquad (T \in \Delta_{X\oplus J}).$$

Since $W \in \mathfrak{A}$ is arbitrary, it is clear now that, if we define $\delta_n^{(1)} := \sigma_n^{(2)} := \lambda + 1$, $\sigma_n^{(1)} := \delta_n^{(2)} := 1$ $(n \in \mathbb{N})$, and $h : \Delta_{X\oplus J} \to \mathbb{R}_+,\ T \mapsto \|b_T\|$, then conditions 1 and 2 of Theorem 2.8 are satisfied, and so $\mathcal{A}(X \oplus J)$ is weakly amenable. $\square$

3.1. **The James spaces.** We now consider the family of James spaces. Let us recall briefly its definition. Let $1 < p < \infty$ be fixed. On the linear space c_{00} of sequences of scalars with finite support we define a norm $\|\,.\,\|_{\mathfrak{J}_p}$ by

$$\|(\alpha_n)\|_{\mathfrak{J}_p} := \sup\left\{ \Big(\sum_{n=1}^{m-1} |\alpha_{i_n} - \alpha_{i_{n+1}}|^p \Big)^{\frac{1}{p}} \ :\ m, i_1, i_2, \dots, i_m \in \mathbb{N}, \right. \\ \left. m \ge 2 \text{ and } i_1 < i_2 < \cdots < i_m \right\}$$

(for $(\alpha_n) \in c_{00}$). The p-th James space $\mathfrak{J}_p$ is defined as the completion of c_{00} in the above norm. Equivalently,

$$\mathfrak{J}_p := \left\{ (\alpha_n) : \alpha_n \in \mathbb{C}\ (n \in \mathbb{N}),\ \|(\alpha_n)\|_{\mathfrak{J}_p} < \infty \text{ and } \lim_n \alpha_n = 0 \right\}.$$

We show next that every James space has the form of the spaces considered in Proposition 3.3.

We need the following technical lemma.

Lemma 3.4. *Let X be a Banach space. Let F_0 be a finite-dimensional subspace of X, and let $\{f_1, f_2, \dots, f_n\}$ be an Auerbach basis of F_0. Let $0 < \varepsilon < 1$, and let $\widetilde{f}_1, \widetilde{f}_2, \dots, \widetilde{f}_n \in X$ be such that*

$$\|f_i - \widetilde{f}_i\| \le \frac{\varepsilon}{2n^2} \qquad (1 \le i \le n).$$

If $\widetilde{F}$ is a subspace of X that contains the $\widetilde{f}_i$'s, then there is a subspace F of X containing F_0 and such that $d(F, \widetilde{F}) \le (1+\varepsilon)/(1-\varepsilon)$.

Proof. The proof of this lemma is essentially that of Lemma 0.20 of [Ja]. $\square$

Lemma 3.5. *For every $1 < p < \infty$, there exists a Johnson space J_p such that the James space $\mathfrak{J}_p$ is a J_p-space and $\mathfrak{J}_p \simeq \mathfrak{J}_p \oplus J_p$.*

Proof. For each $n \in \mathbb{N}$, let $\mathfrak{J}_{p,n}$ be the n-dimensional subspace of $\mathfrak{J}_p$ generated by the first n elements of the unit vector basis of $\mathfrak{J}_p$. Let (G_i) be a sequence of finite-dimensional Banach spaces such that :

i) for each $i \in \mathbb{N}$ ($n \in \mathbb{N}$), there exists $n \in \mathbb{N}$ ($i \in \mathbb{N}$) such that $G_i = \mathfrak{J}_{p,n}$; and
ii) for each $i \in \mathbb{N}$, the set $\{j \in \mathbb{N} : G_j = G_i\}$ is infinite.

We define $J_p := \big(\sum \oplus_{i=1}^{\infty} G_i\big)_p$. It is clear that J_p is a Johnson space.

To see that $\mathfrak{J}_p$ is a J_p-space, let E be a finite-dimensional subspace of $\mathfrak{J}_p$, and let $\{x_1, x_2, \dots, x_n\}$ be an Auerbach basis of E. Denote by P_k the k-th natural projection onto the linear span of the first k elements of the unit vector basis of $\mathfrak{J}_p$. Let $0 < \varepsilon < 1$. Let $\varkappa \in \mathbb{N}$ such that

$$\|x_i - P_\varkappa x_i\| \le \frac{\varepsilon}{2n^2} \qquad (1 \le i \le n),$$

and define $\widetilde{x}_i := P_\varkappa x_i$ $(1 \le i \le n)$. Clearly $\widetilde{x}_i \in \mathfrak{J}_{p,\varkappa}$ $(1 \le i \le n)$. Thus, by Lemma 3.4, there exists a finite dimensional subspace F of $\mathfrak{J}_p$ such that $E \subset F$ and $d(F, \mathfrak{J}_{p,\varkappa}) \le (1+\varepsilon)/(1-\varepsilon)$ $(=: \lambda)$. It follows that $\mathfrak{J}_p$ is a J_p-space.

Let us show that $\mathfrak{J}_p \oplus J_p \simeq \mathfrak{J}_p$. In an obvious way, we can build a Schauder basis for J_p out of the canonical bases of the G_i's. Thus, each element of J_p can be naturally identified with a complex sequence. Having this identification in mind, we define a linear map $\Phi : \mathfrak{J}_p \oplus J_p \to c_0$ as follows. Let $m_i := \dim G_i$, $n_0 := 0$ and $n_i := m_1 + m_2 + \dots + m_i$ $(i \in \mathbb{N})$. Then let Φ be the map that takes each pair of sequences $(y_j) \in \mathfrak{J}_p$, $(z_j) \in J_p$ to the sequence (x_j) defined by

$$x_j = \begin{cases} y_{n_i+1} + z_{j-n_i} & (2n_i < j \le 2n_i + m_{i+1}), \\ y_{j-n_{i+1}} & (2n_i + m_{i+1} < j \le 2n_{i+1}) \end{cases} \qquad (j \in \mathbb{N}, i \in \mathbb{N}_0),$$

that is, (x_j) is the sum of the sequences

$$\underbrace{y_1, \dots, y_1}_{m_1}, y_1, y_2, \dots, y_{m_1}, \underbrace{y_{m_1+1}, \dots, y_{m_1+1}}_{m_2}, y_{m_1+1}, \dots, y_{m_2}, \dots$$

and

$$z_1, z_2, \dots, z_{m_1}, \underbrace{0, 0, \dots, 0}_{m_1}, z_{m_1+1}, \dots, z_{m_2}, \underbrace{0, 0, \dots, 0}_{m_2}, z_{m_2+1}, \dots$$

The first of these sequences belongs to $\mathfrak{J}_p$ since obviously its norm as an element of $\mathfrak{J}_p$ equals the norm of (y_j), or what is the same,

$$\big\|\Phi\big((y_j), \mathrm{O}_J\big)\big\|_{\mathfrak{J}_p} = \big\|(y_j)\big\|_{\mathfrak{J}_p}$$

(where O_J denotes the null sequence of J_p).

It turns out that the second sequence belongs to $\mathfrak{J}_p$ as well. To see this let m, $k_1, k_2, \dots, k_m \in \mathbb{N}$ be such that $m > 1$ and $k_1 < k_2 < \dots < k_m$. Let $I_0 := \big\{1 \le i \le m-1 : k_i \le n_j < k_{i+1}$ for some $j \in \mathbb{N}\big\}$ and $I := \{1, 2, \dots, m-1\} \setminus I_0$. It is not difficult to realize that in estimating the $\|\,.\,\|_{\mathfrak{J}_p}$ norm of the second sequence we just need to take into account sums of the form

$$\sum_{i\in I}\big|z_{k_i}-z_{k_{i+1}}\big|^p+\sum_{i\in I_0}\max\Big\{\big|z_{k_i}-z_{k_{i+1}}\big|^p,\big|z_{k_i}\big|^p+\big|z_{k_{i+1}}\big|^p\Big\}.$$

On the other hand,

$$\begin{aligned}\big\|(z_j)\big\|^p_{J_p} &= \sum_{i=0}^{\infty}\big\|\big(z_{n_i+1},z_{n_i+2},\dots,z_{n_{i+1}},0,0,\dots,0,\dots\big)\big\|^p_{\mathfrak{J}_p}\\ &= \sum_{i=0}^{\infty}\max_{\substack{n,l_1,l_2,\dots,l_n\in\mathbb{N}\\ n_i<l_1<\cdots<l_n\le n_{i+1}}}\Big\{\sum_{j=1}^{n-1}\big|z_{l_j}-z_{l_{j+1}}\big|^p+\big|z_{l_n}\big|^p\Big\}.\end{aligned}$$

Thus we have

$$\sum_{i\in I}\big|z_{k_i}-z_{k_{i+1}}\big|^p\le\big\|(z_j)\big\|^p_{J_p}$$

and

$$\begin{aligned}&\sum_{i\in I_0}\max\Big\{\big|z_{k_i}-z_{k_{i+1}}\big|^p,\big|z_{k_i}\big|^p+\big|z_{k_{i+1}}\big|^p\Big\}\\ &\qquad\le 2^{p-1}\sum_{i\in I_0}\big|z_{k_i}\big|^p+2^{p-1}\sum_{i\in I_0}\big|z_{k_{i+1}}\big|^p\\ &\qquad\le 2^p\big\|(z_j)\big\|^p_{J_p}.\end{aligned}$$

Combining the last two inequalities, we see that the $\|\,.\,\|_{\mathfrak{J}_p}$ norm of the second sequence is not greater than $(2^p+1)^{\frac{1}{p}}\|(z_j)\|_{J_p}$, or, equivalently,

$$\big\|\Phi\big(\mathrm{O}_{\mathfrak{J}},(z_j)\big)\big\|^p_{\mathfrak{J}_p}\le(2^p+1)\big\|(z_j)\big\|^p_{J_p}$$

(where $\mathrm{O}_{\mathfrak{J}}$ denotes the null sequence of $\mathfrak{J}_p$).

Since the above argument applies to any pair of sequences $(y_j)\in\mathfrak{J}_p$ and $(z_i)\in J_p$, we have proved that Φ is a continuous linear map from $\mathfrak{J}_p\oplus J_p$ into $\mathfrak{J}_p$.

The injectivity of Φ follows easily from its definition. Thus to finish our proof it is enough, by the open mapping theorem, to show that the image of Φ coincides with $\mathfrak{J}_p$.

Let $(x_j)\in\mathfrak{J}_p$ arbitrary. We define two complex sequences (y_j) and (z_j) as follows : $y_j=x_{j+n_i}$ and $z_j=x_{j+n_{i-1}}-x_{n_i+n_{i-1}+1}$ for $n_{i-1}<j\le n_i$ $(i,j\in\mathbb{N})$. Then

$$\begin{aligned}\big\|(z_j)\big\|_{J_p} &= \big\|\big(x_1-x_{n_1+1},x_2-x_{n_1+1},\\ &\qquad\dots,x_{n_1}-x_{n_1+1},x_{2n_1+1}-x_{n_2+n_1+1},\dots\big)\big\|_{J_p}\le\|(x_j)\|_{\mathfrak{J}_p}\end{aligned}$$

so that $(z_j)\in J_p$.

Obviously $(y_j)\in\mathfrak{J}_p$ and $\Phi\big((y_j),(z_j)\big)=(x_j)$. Thus, the image of Φ is the whole of $\mathfrak{J}_p$ as needed and this concludes our proof. □

Remark. A similar decomposition to the above has been independently obtained by N. J. Laustsen in his investigations of the ideal structure of $\mathfrak{J}_p$.

Theorem 3.6. *For every $1 < p < \infty$, $\mathcal{A}(\mathfrak{J}_p)$ is weakly amenable.*

Proof. This follows immediately from Proposition 3.3 and the previous lemma. $\square$

4. Spaces of the form $L_p(\mu, E)$.

Our next class of examples is based on Banach spaces of the form $L_p(\mu, E)$, that is, spaces of (equivalence classes of) Bochner p-integrable functions with values in a Banach space E. Precisely, we shall prove the following.

Theorem 4.1. *Let E be a Banach space, and let (Ω, Σ, μ) be a measure space. If E' has the B.A.P. and there exists a sequence (Ω_n) of pairwise disjoint sets in Σ with $0 < \mu(\Omega_n) < \infty$ $(n \in \mathbb{N})$, then $\mathcal{A}(L_p(\mu, E))$ $(1 \le p < \infty)$ is weakly amenable.*

Proof. Let the sequence (Ω_n) be as in the hypotheses, and let $P : L_p(\mu, E) \to L_p(\mu, E)$ be defined by

$$P(f) = \sum_n \frac{1}{\mu(\Omega_n)} \left(\int_{\Omega_n} f \, d\mu \right) \chi_{\Omega_n} \qquad (f \in L_p(\mu, E)).$$

It is easily verified that P is a continuous projection with $\operatorname{rg} P \cong l_p(E)$. Thus

$$L_p(\mu, E) \simeq \ker P \oplus l_p(E) \simeq \ker P \oplus l_p(E) \oplus l_p(E) \simeq L_p(\mu, E) \oplus l_p(E)$$

and, consequently, the weak amenability of $\mathcal{A}(L_p(\mu, E))$ is equivalent to the weak amenability of $\mathcal{A}(L_p(\mu, E) \oplus l_p(E))$. We shall prove that the algebra $\mathcal{A}(L_p(\mu, E) \oplus l_p(E))$ is weakly amenable.

Let (n_i) be a sequence of positive integers such that the map $i \mapsto n_i$, $\mathbb{N} \to \mathbb{N}$, is surjective and, for every i, the set $\{k \in \mathbb{N} : n_k = n_i\}$ is infinite. Define $G_i := l_p^{n_i}(E)$ $(i \in \mathbb{N})$ and $X := L_p(\mu, E)$. Clearly $l_p(E) \cong \left(\sum \oplus_i G_i \right)_p$.

Let F_0 be a finite-dimensional subspace of $L_p(\mu, E)$, and let $\{f_1, f_2, \dots, f_n\}$ be an Auerbach basis of F_0. Let $\mathrm{St}_p(\mu, E)$ denote the linear subspace of (equivalence classes of) μ-step functions. Since $\mathrm{St}_p(\mu, E)$ is dense in $L_p(\mu, E)$, given $0 < \varepsilon < 1$, there exist $\widetilde{f}_1, \widetilde{f}_2, \dots, \widetilde{f}_n \in \mathrm{St}_p(\mu, E)$ such that

$$\left\| f_i - \widetilde{f}_i \right\| \le \frac{\varepsilon}{2n^2} \qquad (1 \le i \le n).$$

It is easy to see that there is a subspace $\widetilde{F}$ of $\mathrm{St}_p(\mu, E)$ that contains the $\widetilde{f}_i$'s and is isometric to $l_p^m(E)$ for some positive integer m. Thus, by Lemma 3.4, there exists a subspace F of $L_p(\mu, E)$ such that $F_0 \subset F$ and

$$d(F, l_p^m(E)) \le \frac{1+\varepsilon}{1-\varepsilon}.$$

With this we have shown that for any $\lambda > 1$, if F_0 is a finite-dimensional subspace of X, then there exists a subspace F of X such that $F_0 \subset F$ and $d(F, G_n) \le \lambda$ for some $n \in \mathbb{N}$. Thus the only difference between the present situation and the one of Proposition 3.3 is that the spaces G_i are not finite-dimensional unless $\dim E < \infty$. If $\dim E < \infty$, then it is clear from the preceding argument that $l_p(E)$ is a Johnson space and $L_p(\mu, E)$ is an $l_p(E)$-space, so that we can apply Proposition 3.3 to obtain the desired result.

If $\dim E = \infty$, it turns out that most of the proof of Proposition 3.3 remains valid. In fact, we can define the algebra $\mathfrak{A}$ and then a trace-unbounded triple, $\mathcal{U}_W$, for each $W \in \mathfrak{A}$, exactly as we did in the proof of Proposition 3.3. It is clear that we may take as λ any number in the interval $(1, \infty)$.

Also as in the proof of Proposition 3.3 we see that for each t.u. triple $\mathcal{U}_W$, the corresponding sequences $\big((R_n^{(1)}, S_n^{(1)})\big)$ and $\big((R_n^{(2)}, S_n^{(2)})\big)$ satisfy the inequalities : $\|R_n^{(1)}\| \le \lambda + 1$, $\|S_n^{(1)}\| \le \|W\|$, $\|R_n^{(2)}\| \le 1$, and $\|S_n^{(2)}\| \le (\lambda+1)\|W\|$ $(n \in \mathbb{N})$.

The next step in the proof of Proposition 3.3 was to show that

$$\big|b_T\big(R_n^{(i)}, S_n^{(i)}\big)\big| \le h(T)\|R_n^{(i)}\|\|S_n^{(i)}\|$$

for some constant $h(T)$ depending only on T (for $T \in \Delta_{X\oplus J}$). In our present situation, the $R_n^{(i)}$'s are not even finite-rank operators, so we no longer have $\big(R_n^{(i)}, S_n^{(i)}\big) \in \mathfrak{A} \times \mathfrak{A}$ $(n \in \mathbb{N}$, $i = 1, 2)$. We shall show, however, that, if E' has the δ-A.P., then we still have

$$\big|b_T\big(R_n^{(i)}, S_n^{(i)}\big)\big| \le \delta(\lambda+1)^2\|b_T\|\|R_n^{(i)}\|\|S_n^{(i)}\| \qquad (T \in \Delta_{X\oplus l_p(E)}).$$

We give the details of the proof only for $i = 1$, the details for the case $i = 2$ are very similar.

Let us start by noting that, if E' has the δ-A.P., then, for every positive integer n, $l^n_{p'}(E')$ $(p' = \frac{p}{p-1})$ has the δ-A.P. and by [GW, Theorem 3.3], $\mathcal{A}(l^n_p(E))$ has a bounded approximate identity (b.a.i.) with bound δ. It is clear from the definition of X_0 of Proposition 3.3 (see part 2 of the definition of $\mathcal{U}_W$) that $d(X_0, l^n_p(E)) \le \lambda$ for some n. Thus, by the previous argument, the algebra $\mathcal{A}(X_0)$ has a b.a.i., $\{F_\alpha\}_{\alpha\in I}$, with bound $\delta\lambda$, that is, such that $\|F_\alpha\| \le \delta\lambda$ $(\alpha \in I)$.

Let us fix $n \in \mathbb{N}$, and, for each $\alpha \in I$, define

$$\widetilde{F}_\alpha := F_\alpha\Gamma_\varkappa + T_0^{2^{n-1}} F_\alpha T^0_{2^{n-1}} P_{2^{n-1}}$$

(see part 1 of the definition of $\mathcal{U}_W$ for the meaning of $\Gamma_\varkappa$). Without loss of generality, we shall suppose that $n_0 > \varkappa$ (see part 2 of the definition of $\mathcal{U}_W$).

Let $x \in X \oplus l_p(E)$. Then we see that

$$\begin{aligned}\big\|\widetilde{F}_\alpha x\big\|^p &= \big\|F_\alpha\Gamma_\varkappa x\big\|^p + \big\|T_0^{2^{n-1}} F_\alpha T^0_{2^{n-1}} P_{2^{n-1}} x\big\|^p \\ &\le \delta^p\lambda^p\big\|\Gamma_\varkappa x\big\|^p + \delta^p\lambda^p(\lambda+1)^p\big\|P_{2^{n-1}}x\big\|^p \\ &\le \delta^p(\lambda+1)^{2p}\|x\|^p.\end{aligned}$$

That is, $\big\|\widetilde{F}_\alpha\big\| \le \delta(\lambda+1)^2$ $(\alpha \in I)$.

It is easily checked that $R_n^{(1)}\widetilde{F}_\alpha = \widetilde{F}_\alpha R_n^{(1)}$ $(\alpha \in I)$ since $\Gamma_\varkappa$ and $P_{2^{n-1}}$ are orthogonal projections for every $n \in \mathbb{N}$ whenever $n_0 > \varkappa$. Taking this into account we find that (we write R and S instead of $R_n^{(1)}$ and $S_n^{(1)}$, respectively)

$$\begin{aligned}\left|b_T(R,S)\right| &= \Big|\operatorname{tr}\Big(T\big(R(\widetilde{F}_\alpha S)-(S\widetilde{F}_\alpha)R\big)'\Big)+\\ &\qquad +\operatorname{tr}\Big(T\big(R(S-\widetilde{F}_\alpha S)-(S-S\widetilde{F}_\alpha)R\big)'\Big)\Big|\\ &\le \|b_T\|\|R\widetilde{F}_\alpha\|\|S\| + \|R'T\|_\wedge\|S-\widetilde{F}_\alpha S\| +\\ &\qquad +\|TR'\|_\wedge\|S-S\widetilde{F}_\alpha\|.\end{aligned}$$

Then, passing to the limit with respect to α and using the fact that $\|\widetilde{F}_\alpha\|\le\delta(\lambda+1)^2$ ($\alpha\in I$), we obtain the desired inequality.

If $i=2$, define

$$\widetilde{F}_\alpha := \sum_{i=1}^{2^n-1} T_0^i F_\alpha T_i^0 P_i + \sum_{i=1}^{2^n-1} T_0^{2^n+i} F_\alpha T_{2^n+i}^0 P_{2^n+i} \qquad (\alpha\in I).$$

As in the case where $i=1$, it is easily verified that $R_n^{(2)}\widetilde{F}_\alpha = \widetilde{F}_\alpha R_n^{(2)}$ and that $\|\widetilde{F}_\alpha\|\le\delta(\lambda+1)^2$ ($\alpha\in I$). The rest remains exactly the same.

Now it is clear that all hypotheses of Theorem 2.8 are satisfied, for, if we take $\big(\delta_n^{(i)}\big)$ and $\big(\sigma_n^{(i)}\big)$ ($i=1,2$) as in the proof of Proposition 3.3 and define $h:\Delta_{X\oplus l_p(E)}\to\mathbb{R}_+$ by $h(T):=\delta(\lambda+1)^2\|b_T\|$ ($T\in\Delta_{X\oplus l_p(E)}$), then conditions 1 and 2 of Theorem 2.8 are automatically satisfied. Thus $\mathcal{A}(L_p(\mu,E)\oplus l_p(E))$ is weakly amenable, and hence so is $\mathcal{A}(L_p(\mu,E))$. □

Remark. Note that the hypothesis about the measure space (Ω,Σ,μ) on Proposition 4.1 is equivalent to $\dim L_p(\Omega,\mu)=\infty$ for some (and hence for all) $1\le p\le\infty$. One implication is trivial. To see the other, suppose that $\dim L_p(\Omega,\mu)=\infty$. We can clearly forget about atoms of infinite measure as L_p functions must vanish on such. If (Ω,Σ,μ) contains infinitely many atoms (of finite measure), then our claim is obvious. Otherwise, since $\dim L_p(\Omega,\mu)=\infty$, there is a σ-algebra, $\Sigma_1\subset\Sigma$, such that (Ω,Σ_1,μ_1) (where μ_1 denotes the restriction of μ to Σ_1) is isomorphic to the interval $[0,1]$ with Lebesgue measure (see [W, I.B.1, III.A.1 & III.A.2]). It is clear then that Σ_1 (and hence Σ) contains a sequence (Ω_n) as in Proposition 4.1.

Remark. Note that, if Σ is finite, then $L_p(\mu,E)\simeq l_p^m(E)$ for some m. In this case $\mathcal{A}(E)$ is weakly amenable if $\mathcal{A}(l_p^m(E))$ is weakly amenable, and the converse is also true whenever E has the B.A.P. (see [Bl, Corollary 3.9] and the remark after it).

Since the dual of a reflexive Banach space E with the A.P. has the B.A.P. (see [DF, § 16.4, Corollary 4]), Proposition 4.1 generalizes Corollary 5.8 of [DGG]. However, Proposition 4.1 is still far from being the best possible. In fact, as shown in [Bl, Proposition 4.6], there are Banach spaces E without the A.P. and such that $\mathcal{A}(l_2(E))$ is weakly amenable.

5. The Tsirelson space.

In proving that the hypotheses of Theorem 2.8 are satisfied for the examples of Banach spaces considered in Sections 3 and 4, we have taken advantage of the abundance in those spaces of complemented subspaces of the form $l_p(X)$, $1 \le p < \infty$, (or $c_0(X)$), where X denotes another subspace, and in turn of the properties of the unit vector basis of l_p, $1 \le p < \infty$, (c_0) (see (9) and (10)). In order to clarify the relevance of this situation in the solution to our problem, it is thus necessary to study the weak amenability of $\mathcal{A}(X)$ for Banach spaces X without basic sequences equivalent to the unit vector basis of c_0 or l_p, $1 \le p < \infty$. The example considered in this section is of this kind. In fact, one of the most relevant features of the Tsirelson space and its dual is the total absence of subsymmetric basic sequences, yet both of them have a 1-unconditional basis.

We shall work with the dual T' of the original Tsirelson space T, and use the analytic definition of T' as given by Figiel and Johnson in [FJ]. This will make little difference for our purposes since T is reflexive and so by [Bl, Proposition 3.11], $\mathcal{A}(T)$ is weakly amenable if and only if $\mathcal{A}(T')$ is weakly amenable. On the other hand, as we have already pointed out, neither T nor T' have subsymmetric basic sequences.

Let us pass to the definition of T'. Let (t_n) denote the unit vector basis of c_{00} (the space of scalar sequences with finite support). If E, F are finite, non-empty subsets of $\mathbb{N}$, we write $E < F$ to mean that $\max E < \min F$. For any $E \subset \mathbb{N}$ and any $x = \sum_n a_n t_n \in c_{00}$, define $Ex := \sum_{n \in E} a_n t_n$. Next, let $\|\,.\,\|_0 := \|\,.\,\|_{c_0}$ and, for $m \ge 0$, define

$$\|x\|_{m+1} := \max\left\{ \|x\|_m \,, \frac{1}{2} \max\left[\sum_{j=1}^{k} \left\|E_j x\right\|_m \right] \right\} \qquad (x \in c_{00}),$$

where the inner maximum is taken over all possible choices of finite subsets $E_1, E_2, \dots, E_k$ of $\mathbb{N}$, such that : $\{k\} \le E_1 < E_2 < \cdots < E_k$. It is easily verified that $\|\,.\,\|_m$ is a norm on c_{00} for every m, and that, for each $x \in c_{00}$ the sequence $\big(\|x\|_m\big)$ is non-decreasing and majorized by $\|x\|_{l_1}$. Thus we can define

$$\|x\| := \lim_{m \to \infty} \|x\|_m \qquad (x \in c_{00}).$$

The latter is clearly a norm on c_{00}. The dual of the Tsirelson space, T', is defined as the completion of c_{00} in the last norm.

As immediate consequences of the above definition we have that (t_n) is a normalized 1-unconditional basis for T' and that, whenever (k_n) and (j_n) are increasing sequences of positive integers such that $k_n \le j_n$ for all n, then

$$\left\| \sum_{n=1}^{\infty} a_n t_{k_n} \right\| \le \left\| \sum_{n=1}^{\infty} a_n t_{j_n} \right\| \tag{11}$$

for any sequence $(a_n) \subset \mathbb{C}$ such that $\sum_n a_n t_{j_n} \in T'$. Moreover, it can be shown (see [CJT, Theorem 10]) that every subsequence of (t_n) in turn contains an equivalent, 'fast growing' subsequence. In particular, we shall use later on the fact that (t_n) is equivalent to (t_{2^n+1}).

As we have already mentioned, the goal of this section is to prove the following result.

Theorem 5.1. *The Banach algebra $\mathcal{A}(T)$ ($\mathcal{A}(T')$) is weakly amenable.*

We shall need some basic results about T'. Let $S : T' \to T'$ be the right-shift operator relative to the basis (t_n) of T'. The following lemma is [CS, Proposition III.10].

Lemma 5.2. *If $x \in c_{00}$ and $\min\{\mathrm{supp}(Sx)\} =: n > 3$ then*

$$\|Sx\| \leq \left(1 - \frac{3}{n}\right)^{-1} \|x\|.$$

□

Lemma 5.3. *For every positive integer n, we have*

$$\|S^n x\| \leq c\, n^3 \, \|x\| \qquad (x \in T') \tag{12}$$

where $c = \max\{\|S\|, \|S^2\|\}$.

Proof. Clearly, it is enough to prove this for $x \in c_{00}$. So let $x \in c_{00}$. Since (t_i) is a 1-unconditional basis, we may suppose without loss of generality that $t_i^*(x) \geq 0$ $(i \in \mathbb{N})$, where t_i^* denotes the i-th biorthogonal functional associated with the basis (t_i). Then, by Lemma 5.2, we have for $n \geq 2$ that

$$\begin{aligned} \|S^n x\| \;&=\; \|S(S^{n-1}x)\| \;\leq\; \left(1 - \frac{3}{n+1}\right)^{-1} \|S^{n-1}x\| \;\leq\; \cdots \\ &\leq\; \prod_{i=4}^{n+1} \left(1 - \frac{3}{i}\right)^{-1} \|S^2 x\| \;\leq\; n^3 \|S^2\| \|x\|. \end{aligned}$$

The desired result follows. □

Lemma 5.4. *For any positive integer n, let $\{I_n, I_{n+1}, \ldots, I_{n2^n}\}$ be a partition of $\mathbb{N} \cap [n, \infty[$. Set $X_j = \mathrm{sp}\{t_k : k \in I_j\}$ $(n \leq j \leq n2^n)$. Then there exists a constant $M > 0$, independent of n and the partition chosen, such that $\|I\|\,\|I^{-1}\| \leq M$, where I denotes the formal identity map from $[t_k]_{k=n}^{\infty}$ to $\left(\sum \oplus_{n \leq j \leq n2^n} X_j\right)_1$.*

Proof. This lemma is a special case of [CS, Proposition V.12]. □

Proof of Theorem 5.1. Once again we shall show that all hypotheses of Theorem 2.8 are satisfied. Let $\mathfrak{A}$ be the algebra of those operators of the form :

$$W = \sum_{1 \leq i,j \leq n} c_{ij} t_j^* \otimes t_i \qquad (n \in \mathbb{N}, c_{ij} \in \mathbb{C}, 1 \leq i,j \leq n)$$

where (t_i^*) is the sequence of biorthogonal functionals associated with the unit vector basis of T'. Since T' is reflexive, (t_i^*) is a Schauder basis for $T''(\cong T)$. Thus, it is very easy to verify that $\mathcal{F}(T') \subset (\mathfrak{A}, \|\,.\,\|_\wedge)^-$.

Let $W = \sum_{1\le i,j\le m} a_{ij} t_j^* \otimes t_i$. We define a sequence $E_0, E_1, E_2, \dots$ of finite subsets of $\mathbb{N}$ as follows : $E_0 := \{1, 2, \dots, m\}$, $E_n := \{2^i + n : \rho(n) \le i < \rho(n) + m\}$ for $1 \le n \le m$ (where $\rho(n) = [\log_2 n] + 1$), and

$$E_n := \{\varkappa_0 + (n-m-1)m + 1, \varkappa_0 + (n-m-1)m + 2, \dots, \varkappa_0 + (n-m)m\}$$

for $n > m$ (where $\varkappa_0 = \max E_m$). Note that $E_1, E_2, \dots, E_n, \dots$ are pairwise disjoint.

Next define $P_n \in \mathcal{F}(T')$ by $P_n := \sum_{i\in E_n} t_i^* \otimes t_i$ $(n \in \mathbb{N}_0)$ and $T_k^l : P_k T' \to P_l T'$ by

$$T_k^l := \sum_{i\in E_k} t_i^* \otimes t_{\phi_{k,l}(i)} \Big|_{P_k T'} \qquad (k, l \in \mathbb{N}_0),$$

where $\phi_{k,l}$ $(k, l \in \mathbb{N}_0)$ denotes the unique, monotone, increasing, bijective map from E_k onto E_l.

It is not difficult to see that $\big(\{t_i, t_i^*\}_{i=1}^m, (\operatorname{rg} P_0, (P_n)), (T_k^l)\big)$ is a W- t.u. triple. We only need to verify that the sequences $e_{km+j} = T_0^{k+1} t_j$ and $e_{km+j}^* = t_j^* \circ T_{k+1}^0 \circ P_{k+1}$ $(1 \le j \le m,\ k \in \mathbb{N}_0)$, form a t.u. biorthogonal system, and that the associated right-shift operator is continuous. But these follow easily from our previous definitions. In fact, we know that (t_n) is unconditional, and that $S : T' \to T'$, $\sum_n \alpha_n t_n \mapsto \sum_n \alpha_n t_{n+1}$ is bounded. On the other hand, it is easily seen that, for some $\kappa \in \mathbb{N}$ and every n big enough, we have $e_n = t_{n+\kappa}$. It follows that the right-shift operator on $[e_n]$ relative to the basis (e_n) must be bounded. Moreover, since (t_n) is unconditional, $\{t_n; t_n^*\}_{n=1}^\infty$ is t.u. and, consequently, so is $\{e_n; e_n^*\}_{n=1}^\infty$.

Now let us turn our attention to the connecting family (T_k^l). We need to find upper estimates for the norm of these operators. It is clear from the definition of the norm in T' and our definition of E_n $(n > m)$, that, whenever $n > m$, we have

$$\Big\| \sum_{i\in E_n} a_i t_i \Big\| = \frac{1}{2} \sum_{i\in E_n} |a_i| \qquad (a_i \in \mathbb{C},\ i \in E_n).$$

Thus, if $l, k \ge m$, then T_k^l is clearly an isometry. On the other hand, it follows easily from (11) that $\|T_k^l\| = 1$ whenever $l \le k$ $(k, l \in \mathbb{N}_0)$. Taking into account the fact that $T_k^l = T_m^l T_k^m$ $(k, l \in \mathbb{N}_0)$, it is clear that we just need to consider the case where $0 \le k < l \le m$.

Suppose first that $k > 0$. Since (t_i) is equivalent to (t_{2^i+1}) (see the comments following the definition of T'), there exists $K > 0$ such that

$$\Big\| \sum_{i=1}^\infty \alpha_i t_{2^i} \Big\| \le \Big\| \sum_{i=1}^\infty \alpha_i t_{2^i+1} \Big\| \le K \Big\| \sum_{i=1}^\infty \alpha_i t_i \Big\| \tag{13}$$

for any sequence $(\alpha_i) \subset \mathbb{C}$ such that $\sum_{i=1}^{\infty} \alpha_i t_i$ converges. Then, using Lemma 5.3, we see that

$$\begin{aligned}
\left\| \sum_{i=1}^{m} \alpha_i t_{2^{\rho(l)+i-1}+l} \right\| &\leq \left\| \sum_{i=1}^{m} \alpha_i t_{2^{\rho(l)+i}} \right\| \\
&\leq K \left\| \sum_{i=1}^{m} \alpha_i t_{\rho(l)+i} \right\| \leq K c\, \rho(l)^3 \left\| \sum_{i=1}^{m} \alpha_i t_i \right\| \\
&\leq K c\, \rho(l)^3 \left\| \sum_{i=1}^{m} \alpha_i t_{2^{\rho(k)+i-1}+k} \right\| \qquad (\alpha_i \in \mathbb{C}, 1 \leq i \leq n),
\end{aligned}$$

that is, $\|T_k^l\| \leq K c\, \rho(l)^3$ $(0 < k < l \leq m)$. If $k = 0$, then, using the upper bound found and (13), we see that $\|T_0^l\| \leq \|T_1^l\| \|T_0^1\| \leq c\, K^2 \rho(l)^3$ $(l \in \mathbb{N})$.

We can now estimate $\|R_n^{(i)}\|$ and $\|S_n^{(i)}\|$ $(n \in \mathbb{N},\ i = 1, 2)$ (see Theorem 2.8). By the definition of the W_i's (see (1)), the preceding result, and Propositions 5.3 and 5.4, we have

$$\begin{aligned}
\|S_n^{(2)} x\| &= \left\| \sum_{i=1}^{2^n-1} T_i^{2^n+i} W_i x \right\| \leq \|W\| \left(\sum_{i=1}^{2^n-1} \|T_0^{2^n+i}\| \|P_i x\| \right) \\
&\leq cK^2 \|W\| \left(\sum_{i=1}^{2^n-1} \rho(2^n+i)^3 \|P_i x\| \right) \\
&\leq cK^2 (n+1)^3 \|W\| \left(\sum_{i=1}^{2^n-1} \|S^n P_i x\| \right) \\
&\leq McK^2 (n+1)^3 \|W\| \left\| \sum_{i=1}^{2^n-1} S^n P_i x \right\| \\
&\leq McK^2 (n+1)^3 \|W\| \|S^n\| \|x\| \\
&\leq Mc^2 K^2 n^3 (n+1)^3 \|W\| \|x\| \qquad (n \in \mathbb{N},\ x \in T'),
\end{aligned}$$

that is, $\|S_n^{(2)}\| \leq Mc^2 K^2 n^3 (n+1)^3 \|W\|$ $(n \in \mathbb{N})$.

Also by Propositions 5.3 and 5.4, and taking into account the fact that $\|T_k^l\| = 1$ for $l \leq k$, we have

$$\begin{aligned}
\|R_n^{(2)} x\| &= \left\| \sum_{i=1}^{2^n-1} T_{2^n+i}^{i} P_{2^n+i} x \right\| \\
&\leq \sum_{i=1}^{2^n-1} \|P_{2^n+i} x\| \leq \sum_{i=1}^{2^n-1} \|S^n P_{2^n+i} x\| \\
&\leq M \left\| \sum_{i=1}^{2^n-1} S^n P_{2^n+i} x \right\| \leq M \|S^n\| \|x\| \\
&\leq M c\, n^3 \|x\| \qquad (n \in \mathbb{N}, x \in T'),
\end{aligned}$$

that is, $\|R_n^{(2)}\| \leq M c\, n^3$ $(n \in \mathbb{N})$.

Obviously, $\|R_n^{(1)}\| \leq \|T^0_{2^{n-1}}P_{2^{n-1}}\| \leq 1$ and $\|S_n^{(1)}\| \leq \|T_0^{2^{n-1}}W\| \leq cK^2n^3\|W\|$ ($n \in \mathbb{N}$). Moreover, since $\big(R_n^{(i)}, S_n^{(i)}\big) \in \mathcal{F}(T') \times \mathcal{F}(T')$ ($n \in \mathbb{N}$, $i = 1, 2$), it is clear that, for every $L \in \Delta_{T'}$,

$$\big|b_L\big(R_n^{(i)}, S_n^{(i)}\big)\big| \leq \|b_L\|\|R_n^{(i)}\|\|S_n^{(i)}\| \qquad (n \in \mathbb{N},\, i = 1, 2).$$

Thus we can define $\delta_n^{(1)} := \sigma_n^{(1)} := \delta_n^{(2)} := \sigma_n^{(2)} := C(n+1)^6$ ($n \in \mathbb{N}$) (where $C = Mc^2K^2$) and $h(L) := \|b_L\|$ ($L \in \Delta_{T'}$). The rest is clear. $\square$

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