A family of simple Lie algebras in characteristic two

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In this paper we describe a family of simple Lie algebras defined over a field of characteristic two depending on two integer parameters. We also describe their loop algebras.

Key Words: Simple Lie algebras, Zassenhaus algebras.

1. INTRODUCTION

As reported by A.I. Kostrikin in his paper [21], the very first algebra which distinguished modular (i.e. over fields of positive characteristic $p$) Lie algebra theory from the classical one was the Witt algebra $W(1 : k)$, where $k$ is a power of $p$. This algebra is a generalization due to H. Zassenhaus [31] in the thirties of an analogous structure defined by E. Witt over the integers. This algebra is graded over the elementary abelian additive group of the field $F_k$ and, for $p > 2$, it is simple. When the characteristic is two $W(1 : k)$ has exactly one non-trivial ideal, namely its derived subalgebra $Z_k$. The simple object is called a Zassenhaus algebra. In characteristic two, a Zassenhaus algebra admits a non-singular outer derivation, which is quite an important feature (see [3]).

In the following years, further generalizations of the above structure led to the construction of other families of simple Lie algebras, called algebras of Cartan type (see [29]), namely the generalized Jacobson-Witt algebras, the special algebras, the hamiltonian algebras and the contact algebras. Kostrikin and I.R. Shafarevich during the sixties conjectured that, apart from the small characteristic case, every finite-dimensional simple Lie algebra over an algebraically closed field is either classical or of Cartan type. Recently this was proved true for $p > 7$ by H. Strade.
and R.L. Wilson in [30]. For small characteristic, the corresponding result does not hold: in fact, several families of algebras not included in the above list have been found, and the classification problem in the small characteristic case still remains an open problem. Kostrikin has said that the classification of simple Lie algebras in characteristic two is a very hard problem, due to the existence of many simple objects. For some examples, see for instance [4], [5], [6], [14], [19], [22].

In this paper, we show a new family of simple Lie algebras in characteristic two depending on two integer parameters, built by means of a “doubling process” starting from Zassenhaus algebras: we will call them Bi-Zassenhaus algebras and we denote them by \( B(g, h) \). After a background section on Zassenhaus algebras, we give a construction for Bi-Zassenhaus algebras and we prove they are simple. Then we investigate their second cohomology groups and central extensions: an important topic in its own, this will be needed when dealing with the almost finite presentation of the Bi-Zassenhaus loop algebras, described in [17]. We also show that the family of Bi-Zassenhaus algebras does not coincide with any known family of algebras, although occasional coincidences may occur. The following two sections are dedicated to the construction of two further gradings of these simple algebras: the former is used to define the algebras over the prime field, while the latter is used to build the twisted loop algebras, which is the topic of the last section. We then discuss some invariants of Bi-Zassenhaus loop algebras which show that they are different from already known graded Lie algebras of maximal class; we conclude by describing their cohomology and their central extensions. The family of Bi-Zassenhaus loop algebras plays an important rôle in the classification [18] of infinite-dimensional modular \( N \)-graded Lie algebras of maximal class. In the odd characteristic case, every infinite-dimensional modular \( N \)-graded Lie algebra of maximal class (generated by its first homogeneous component) can be built starting from an Albert-Frank-Shalev algebra via some suitable constructions, (see A.Caranti and M.F. Newman [9]). The family of Albert-Frank-Shalev algebras (AFS, for short) was built by A. Shalev [28] by looping simple algebras constructed in the fifties by A.A. Albert and M.S. Frank in [1]. It may be interesting to observe that the Albert-Frank and the Zassenhaus algebras are isomorphic when the characteristic is two. In characteristic two, a similar classification result holds if we are allowed to start from either an AFS-algebra or a Bi-Zassenhaus loop algebra. In the latter half of this paper we show the construction of the Bi-Zassenhaus loop algebras (\( B_l \), for short) by using a method already employed by Caranti, S. Mattarei and Newman in [8].

A significant amount of computational work was required throughout the whole study, although results have been proved in full generality and none of them relies on such computations. Use of mathematical software has been indispensable: the \( p \)-Quotient program [15] of the Australian National University and GAP [26] have provided the examples for the construction of the algebras involved. It is perhaps worth noting that the first suggestion of the existence of some loop algebras different from the AFS ones came from machine computation: the following
2. THE ZASSENHAUS ALGEBRAS

We start by recalling some facts about Zassenhaus algebras. If $k = p^r$ for some $p > 0$, the $F_k$-vector space

$$W(1 : k) = \langle e_\alpha; \alpha \in F_k \rangle$$

becomes a Lie algebra by extending by bilinearity the following bracket product among basis elements:

$$[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta}.$$ 

In characteristic $p > 2$, the resulting algebra is simple, while in the remaining case there exists a unique non-trivial ideal

$$Z_k = \langle e_\alpha; \alpha \in F_k^* \rangle ,$$

where by $F^*$ we denote the non-zero elements of the field. It is immediate to check that the algebra $Z_k$ is simple. The adjoint map of the element $e_0$ in $W(1 : k)$ is an outer derivation $D$ for $Z_k$.

In both cases, we will call Zassenhaus algebra the simple object.

The algebra $W(1 : k)$ can be differently graded by applying the basis transformation

$$\begin{cases}
y_{-1} = e_0 + \sum_{\alpha \in F_k^*} e_{\alpha} \\
y_j = -\sum_{\alpha \in F_k^*} \alpha^{k-2-j} e_{\alpha} \\
y_{k-2} = -\sum_{\alpha \in F_k^*} e_{\alpha} ,
\end{cases} \quad (1)$$

whose inverse is

$$\begin{cases}
e_0 = y_{-1} + y_{k-2} \\
e_\beta = \sum_{i=1}^{\beta+1} \beta_i e_i + y_{k-2} ,
\end{cases}$$

and by using the identity

$$\sum_{\alpha \in F_k} \alpha z = -\delta_{z/(k-1)}z(z,0) , \quad (2)$$
where the map delta is a generalization of the Kronecker map, and for a ring $A$ it is defined as

$$
\delta_A(x, y) = \begin{cases} 
1 & \text{if } x = y \text{ in } A \\
0 & \text{otherwise} 
\end{cases}
$$

when $A$ is not specified, we understand it as $\mathbb{Z}$.

Under the map (1), the space $W(1 : k)$ gets a new basis

$$
W(1 : k) = \langle y_i; -1 \leq i \leq k - 2 \rangle
$$

and the Lie product now reads as

$$
[y_i, y_j] = a_{i,j} y_{i+j},
$$

where the coefficients $a_{i,j}$ are defined as

$$
a_{i,j} = \binom{i + j + 1}{j} - \binom{i + j + 1}{i} = \binom{i + j}{i + 1} - \binom{i + j}{j + 1},
$$

and they satisfy the properties stated in the following lemma, which is Lemma 1.5 of [2]:

**Lemma 2.1.** Let $i, j, l$ be arbitrary integers, $a_{i,j}$ defined as above. Then the following identities hold:

1. $a_{i,i} = 0$ and $a_{i,j} = -a_{i,j}$; and $a_{-1,j} = 1$ for all $j \geq 0$.
2. If $\{i, j\} \neq \{-1, 0\}$, then $a_{i,j} = a_{i-1,j} + a_{i,j-1}$.
3. $a_{i,j} a_{i+j,l} + a_{j,l} a_{i+j,j+i} + a_{i,l} a_{i+j,l} = 0$.
4. $a_{j,l} a_{i,j+l} = a_{i,j} a_{j+l,j} + a_{i,j} a_{i,j+l}$.

In this new basis, the algebra $W(1 : k)$ is defined over the prime field and it is graded over the integers by giving to every basis element $y_i$ its index as weight. Furthermore, it is possible to get a grading on the additive cyclic group $C_{k-1}$, too by declaring $y_{-1}$ and $y_{k-2}$ both of weight $k - 1$. With respect to this new basis, the simple ideal in characteristic two becomes

$$
Z_k = \langle y_i; -1 \leq i \leq k - 2 \rangle,
$$

while the derivation reads as

$$
D = a_{-1} (y_{-1} + y_{k-2}).
$$

In order to investigate the behaviour of $D$, we recall Lucas’ Theorem (see [23], [20]) to evaluate binomial coefficients over fields of positive characteristic: if
\[ a = \sum_{i=0}^{n} a_i \cdot p^i \quad \text{and} \quad b = \sum_{i=0}^{n} b_i \cdot p^i \]
is the \( p \)-adic expansion of the two integers \( a \) and \( b \), then we have the equivalence

\[
\left( \frac{a}{b} \right) \equiv \prod_{i=0}^{n} \left( \frac{a_i}{b_i} \right) \pmod{p}.
\]

Note that we can safely ignore signs here since \( p = 2 \).

For \( k = 2^r \) we can show that

\[
\alpha_{i,j} = 0 \quad \text{for} \quad i + j < -1 \quad \text{or} \quad i + j > k - 3, \tag{3}
\]

where the former case is immediate since it happens only for \( i = j = -1 \), while for the latter one we write \( i + j = k - 2 + \psi \) for \( 0 \leq \psi \leq k - 2 \) and compute

\[
\alpha_{i,j} = \binom{i + j + 1}{i} + \binom{i + j + 1}{j} = \binom{i + j + 1}{i} + \binom{i + j + 1}{i + 1} = \binom{i + j + 2}{i + 1} = \binom{k + \psi}{i + 1} \equiv \binom{2^{q+k}}{0} \binom{\psi}{i + 1} \pmod{2} \equiv 0 \pmod{2},
\]

since \( j \leq k - 2 \) and then \( \psi < i + 1 < k \). Moreover, if \( y_{k} \) is a basis element for \( \mathbb{Z}_k \), we have that

\[
y_k D = a_{i,-1} y_{i-1} + a_{i,k-2} y_{k+i-2};
\]
in fact, we have that \( a_{i,-1} = 1 - \delta(i,-1) \), while by using the properties of binomial coefficients and Lucas’ Theorem we can evaluate the latter coefficient as

\[
a_{i,k-2} = \binom{i + k - 2 + 1}{k - 2} - \binom{i + k - 2 + 1}{i} = \binom{k + i - 1}{k - 2} - \binom{k + i - 1}{k - 1} = \binom{k + i - 1}{k - 2} + \binom{k + i - 1}{k - 1}.
\]
This means that $D$ cycles on the elements of the basis as

$$y_{k-3} \mapsto y_{k-4} \mapsto \ldots \mapsto y_1 \mapsto y_0 \mapsto y_{-1} \mapsto y_{k-3}.$$  

### 3. THE BI-ZASSENHAUS ALGEBRAS

We can now begin the construction of the new family.

Let $g \geq 2$, $h \geq 1$ be positive integers. Define $k = 2^g + h$ and $\eta = 2^g - 1$. Build the vector space $B(g, h)$ over $F_k$ of dimension $2k$ with basis

$$\{ e_{(i, \alpha)} : (i, \alpha) \in F_2 \times F_k \}.$$

Define on the basis elements a bracket operation

$$[e_{(i, \alpha)}, e_{(j, \beta)}] = (\alpha^{1+i(j-1)} + \beta^{1+ij(\eta-1)})e_{(i+j, \alpha+\beta)}, \quad (4)$$

and extend it by bilinearity to the whole of $B(g, h)$. Note that, in expanded form, it reads

$$[e_{(0, \alpha)}, e_{(0, \beta)}] = (\alpha + \beta)e_{(0, \alpha+\beta)}$$

$$[e_{(0, \alpha)}, e_{(1, \beta)}] = (\alpha + \beta)e_{(1, \alpha+\beta)}$$

$$[e_{(1, \alpha)}, e_{(1, \beta)}] = (\alpha^{\eta} + \beta^{\eta})e_{(0, \alpha+\beta)}.$$

Endowed with (4), $B(g, h)$ becomes a Lie algebra over $F_k$. Actually, since the characteristic of the underlying field is two, the condition $[x, x] = 0$ immediately follows, so we have only to show that the Jacobi identity holds. Moreover, if at least two elements out of three have zero first index, then the Lie product coincides with the Zassenhaus algebra one, so the check is superfluous. Hence we are left with only two identities to check:

$$J_1 = [e_{(0, \alpha)}, e_{(1, \beta)}, e_{(1, \gamma)}] + [e_{(1, \beta)}, e_{(1, \gamma)}, e_{(0, \alpha)}] + [e_{(1, \gamma)}, e_{(0, \alpha)}, e_{(1, \beta)}]$$

$$J_2 = [e_{(1, \alpha)}, e_{(1, \beta)}, e_{(1, \gamma)}] + [e_{(1, \beta)}, e_{(1, \gamma)}, e_{(1, \alpha)}] + [e_{(1, \gamma)}, e_{(1, \alpha)}, e_{(1, \beta)}],$$

that expand as

$$J_1 = (\alpha + \beta)[e_{(1, \alpha+\beta)}, e_{(1, \gamma)}] + (\beta^{\eta} + \gamma^{\eta})[e_{(0, \beta+\gamma)}, e_{(0, \alpha)}]$$
generated by the elements of the basis with zero first index. The algebra then choose an index where $A$ is a non-trivial ideal of the algebra $I$.

The subspace $J = \langle e_{(0, \alpha + \beta, \gamma)}(\alpha + \beta + \gamma) \rangle_{\alpha, \beta, \gamma}$ has a grading over the abelian additive group of $F_2 \times F_k$ obtained by weighting every basis element by its index:

$$\overline{B}(g, h) = \bigoplus_{(i, \alpha) \in F_2 \times F_k} \overline{B}(g, h)(i, \alpha) = \bigoplus_{(i, \alpha) \in F_2 \times F_k} F_k \cdot e_{(i, \alpha)}.$$ 

The subspace

$$B(g, h) = \langle e_{(i, \alpha)}: (i, \alpha) \in F_2 \times F_k^* \rangle$$

is a non-trivial ideal of the algebra $\overline{B}(g, h)$: now look at it as a Lie algebra on its own, and let $I$ be a non-zero ideal of $B(g, h)$. We can write any non-zero element in $I$ as

$$\xi = \sum_{\alpha \in \Gamma_{\xi} \subseteq F_k^*} (A_{\alpha} e_{(0, \alpha)} + B_{\alpha} e_{(1, \alpha)}) ,$$

where $A_{\alpha}, B_{\alpha} \in F_k$ are not both zero for every $\alpha \in \Gamma_{\xi} \subseteq F_k^*$. If $\text{Card}(\Gamma_{\xi}) > 1$, then choose an index $\gamma \in \Gamma_{\xi}$ and compute

$$I \ni \nu = [\xi, e_{(0, \gamma)}] = \sum_{\alpha \in \Gamma_{\xi} \subseteq F_k^*} (A_{\alpha} (\alpha + \gamma) e_{(0, \alpha + \gamma)} + B_{\alpha} (\alpha + \gamma) e_{(1, \alpha + \gamma)})$$

$$= \sum_{\delta \in \Delta, \gamma \in F_k^*} (\tilde{A}_{\delta} e_{(0, \delta)} + \tilde{B}_{\delta} e_{(1, \delta)}) ,$$
where
\[ \text{Card}(\Delta_x) = \text{Card}\{ \alpha + \gamma | \alpha \in \Gamma_x - \{ \gamma \} \} = \text{Card}(\Gamma_x) - 1, \]
so we can always assume \( \Gamma_x = \{ \alpha \} \) and then write \( \xi \) as \( Ae_{(0,0)} + Be_{(1,0)}. \) If \( B = 0 \), then the ideal \( I \) includes a basis element \( e_{(0,0)}. \) Otherwise consider the following polynomial in \( \mathbb{F}_k[t]: \)
\[ f(t) = B^2(t + \alpha)^{\eta-1} + A^2. \]
It has degree \( \eta - 1 = 2^g - 2 > 0 \), since \( g > 1 \), so it has at most \( \eta - 1 \) distinct roots in \( \mathbb{F}_k. \) As \( h \geq 1 \), then \( \text{Card}(\mathbb{F}_k - \{ 0, \alpha \}) = k - 2 > \eta - 1 \) and thus there exists \( \beta \in \mathbb{F}_k - \{ 0, \alpha \} \) such that \( f(\beta) \neq 0. \) Hence the linear system
\[
\begin{cases}
A\beta x + B(\alpha^n + (\alpha + \beta)^n)y = 1 \\
B\beta x + A\beta y = 0
\end{cases}
\]
has an unique solution \( (C, D) \), since its discriminant is \( \beta f(\beta) \neq 0. \) Define the element \( \mu = Ce_{(0,0)+\beta} + De_{(1,0)+\beta} \) and compute
\[ I \ni [\xi, \mu] = 1 \cdot e_{(0,0)} + 0 \cdot e_{(1,0)} = e_{(0,0)}. \]
Thus \( I \) always includes an element of the basis with zero first index, say \( e_{(0,0)}. \) But then
\[ I \ni \left[ e_{(0,0)}, \frac{1}{\gamma}e_{(i,\alpha+\gamma)} \right] = e_{(i,\gamma)} \quad \forall (i, \gamma) \in \mathbb{F}_2 \times \mathbb{F}_k^*. \]
So \( I = B(g, h) \) is a simple Lie algebra, of dimension \( d = 2(k-1) = 2^g+h+1 - 2; \) we call it Bi-Zassenhaus algebra.

We now move towards the construction of the central extensions of the Bi-Zassenhaus algebras. A first step is to describe their derivations.

On every Bi-Zassenhaus algebra act the two natural outer derivations \( D = \text{ad} e_{(0,0)} \) and \( E = \text{ad} e_{(1,0)} \)
\[ e_{(i,\alpha)}D = \alpha e_{(i,\alpha)} \quad e_{(i,\alpha)}E = \alpha^{1+\eta-1} e_{(i,\alpha)}, \]
and two more \( F_t \) \((t = 0, 1)\) obtained by linear extension of the maps acting on the basis elements as
\[ e_{(i,\alpha)}F_t = i e_{(i+t,\alpha)}, \]
where \( i + t \) is obviously to be taken modulo two. By using induction and equation (4), the action of the 2-powers of these and of the adjoint maps is the following.
for $s > 0$:

$$e_{(i,\alpha)} F^2_t = i(i + t) e_{(i,\alpha)} ,$$
$$e_{(i,\alpha)} E^{2^r} = \alpha^{2^r+s-1} e_{(i,\alpha)} ,$$
$$e_{(i,\alpha)} D^{2^r} = \alpha^{2^r} e_{(i,\alpha)} ,$$
$$e_{(i,\alpha)} (\text{ad} e_{(t,\gamma)})^{2^r} = (\alpha(\alpha^{1+i(\eta-1)} + \gamma^{1+i(\eta-1)})^{2^r-1} - e_{(i,\alpha)} .$$

This implies that, for $s > 0$,

$$F^2_t = (1-t) F^1_t ,$$
$$E^{2^{s+h+1}} = E^{2^s} ,$$
$$D = E^{2^s} g_{i;} h ,$$
$$\text{(ad} e_{(t,\gamma)})^{2^r} = E^{2^{s+(i+1)(\eta-1)} + \gamma^{2^{r-1}(1+i(\eta-1))} E^{2^r+h} .$$

In particular, among all the 2-powers of $D$ and $E$ only $g + h + 1$ of them represent distinct derivations (for instance, the powers $E^{2^r}$ for $0 \leq s \leq g + h$). Furthermore, $F^1_t$ is nilpotent and $F^0_t$ is unipotent. It is actually possible to describe the whole algebra $\text{Der}(B(g, h))$ of derivation of a Bi-Zassenhaus algebra. By the results in [29, Ch.3, Prop.4.4, Th.4.5], the space $\text{Der}(B(g, h))$ is $(\mathbb{F}_2 \times \mathbb{F}_k)$-graded by

$$\text{Der}(B(g, h))(t,\zeta) = \{ \Phi \in \text{Der}(B(g, h)) : \Phi(B(g, h)(i,\alpha)) \subseteq B(g, h)(i+t,\alpha+\zeta) \} .$$

The latter condition can be expressed by introducing, for every $(t,\zeta) \in \mathbb{F}_2 \times \mathbb{F}_k$, the maps

$$\Lambda_{(t,\zeta)} : \mathbb{F}_2 \times \mathbb{F}_k \rightarrow \mathbb{F}_k$$

such that $\Phi(e_{(i,\alpha)}) = \Lambda_{(t,\zeta)} e_{(i+t,\alpha+\zeta)}$. In this notation, the Leibniz rule for derivation applied to two basis elements $e_{(i,\alpha)}$, $e_{(j,\beta)}$ reads as

$$\left(\alpha^{1+ij(\eta-1)} + \beta^{1+i(j+1)(\eta-1)}\right) \cdot \Lambda_{(t,\zeta)} (i + j,\alpha + \beta) +$$

$$\left(\alpha + \zeta\right)^{1+i(j+1)(\eta-1)} \cdot \Lambda_{(t,\zeta)} (i,\alpha) +$$

$$\left(\alpha^{1+i(j+1)(\eta-1)} + \beta^{1+ij(\eta-1)}\right) \cdot \Lambda_{(t,\zeta)} (j,\beta) = 0 ,$$

where $i + t$ and $j + t$ are taken modulo two. When $\zeta \neq 0$, equation (6) in the four cases $\{i = 1, j = 0, \beta = \alpha\}$, $\{i = j = 0, \beta = \alpha + \zeta\}$, $\{i = 1, j = 0, \alpha = \beta = \zeta\}$, and $\{i = j = 0\}$ gives the relations

$$\Lambda_{(t,\zeta)} (1,\alpha) = \begin{cases} \Lambda_{(t,\zeta)} (0,\alpha) & \text{when } t = 0 \\ \frac{\alpha^0 + \zeta^\eta}{\alpha + \zeta} \Lambda_{(t,\zeta)} (0,\alpha) & \text{when } t = 1 , \end{cases}$$
Since $\Lambda(t,\zeta)(0,\cdot)$ is a function from the finite field $\mathbb{F}_k$ to itself, it can be represented as a polynomial in $\mathbb{F}_k[x]$ of degree at most $k-1$. Then the above conditions imply that $\Lambda(t,\zeta)(i,\alpha)$ is a multiple of $\alpha^{1+i(n-1)} + \zeta^{1+i(n-1)}$. This means that, for $\zeta \neq 0$, $\text{Der}(B(g,h))(t,\zeta) = \langle \text{ad} \, e(t,\zeta) \rangle$. When $(t, \zeta) = (0, 0)$, the relation (6) gives

$$\Lambda_{(0,0)}(i + j, \alpha + \beta) = \Lambda_{(0,0)}(i, \alpha) + \Lambda_{(0,0)}(j, \beta).$$

This implies that $\Lambda_{(0,0)}(0, \cdot)$ is linear over $\mathbb{F}_2$, while $\Lambda_{(0,0)}(1, \alpha) + \Lambda_{(0,0)}(0, \alpha)$ is constant. By [25, Th.9.4.4], every $\mathbb{F}_s$-linear operator over $\mathbb{F}_s$ can be uniquely represented as a polynomial $\sum_{i=0}^{n-1} a_i x^i$ over $\mathbb{F}_s$: thus $\Lambda_{(0,0)}(0, \alpha)$ is a $\mathbb{F}_s$-linear combination of $\{\alpha^{2^j i + h + 1 - 1} \}_{j=0}^{n-1}$. Meanwhile, the above cited constant part can be expressed by a multiple of the derivation $F_0$. Summarizing this, we can write $\text{Der}(B(g,h))(0,0) = \langle D^{t_1}; 0 \leq t_1 \leq g + h - 1 \rangle \oplus \langle F_0 \rangle$. Finally, for $(t, \zeta) = (1, 0)$ the equation (6) again gives linearity for $\Lambda_{(1,0)}(0, \cdot)$, with the additional condition

$$\frac{\alpha^y + \beta^y}{\alpha + \beta} \Lambda_{(1,0)}(0, \alpha + \beta) = \Lambda_{(1,0)}(1, \alpha) + \Lambda_{(1,0)}(1, \beta),$$

which restricts the possible choices of the maps $\Lambda_{(1,0)}(i, \alpha)$ to multiples of $\alpha^{1+i(n-1)}$ or $i$. Thus $\text{Der}(B(g,h))(1,0) = \langle E \rangle \oplus \langle F_1 \rangle$. So, the graded Lie algebra of derivations of a Bi-Zassenhaus algebra has dimension $2^{2^h + h + 1} + g + h + 1$ and it is generated by

$$\text{Der}(B(g,h)) = \langle E^{2^s}; F_i; s = 0, \ldots, g + h, t = 0, 1 \rangle \oplus \text{ad} \, B(g,h).$$

The Lie products among its basis elements which are not immediate can be obtained by the equations (5) and are summarized in the following table, for $s > 0$:

\[
\begin{align*}
[E,F_0] & = E, & [F_0, \text{ad} \, e(t,\gamma)] & = \text{lad} \, e(t,\gamma), \\
[E,F_1] & = D = E^{2^{h+1}}, & [E, \text{ad} \, e(t,\gamma)] & = \gamma^{1+i(n-1)} \text{ad} \, e(1-t,\gamma), \\
[E^{2^s}, F_0] & = 0, & \quad [E^{2^s}, \text{ad} \, e(t,\gamma)] & = \gamma^{2^{s+2h+1}} \text{ad} \, e(t,\gamma), \\
[E^{2^s}, F_1] & = 0, & \quad [F_1, \text{ad} \, e(t,\gamma)] & = \text{lad} \, e(1-t,\gamma), \\
[F_0,F_1] & = F_1.
\end{align*}
\]

Then $\text{Der}(B(g,h))^{(1)} = \text{Der}(B(g,h))^r = \langle E, D, F_1 \rangle \oplus \text{ad} \, B(g,h)$ for $r \geq 1$, while $\text{Der}(B(g,h))^{(2)} = \langle D \rangle \oplus \text{ad} \, B(g,h)$ and $\text{Der}(B(g,h))^{(s)} = \text{ad} \, B(g,h)$ for $s \geq 3$.

By a classical result of Zassenhaus (see [29, Ch.1, Th.7.9]), the existence of an outer derivation implies that the Killing form of the algebra is degenerate. Does
\[ B(g, h) \) admit any nondegenerate associative form? We first recall that, for any Lie algebra \( L \) over a field \( F \), a symmetric bilinear form \( \lambda: L \times L \to F \) is called associative if \( \lambda([x, y], z) = \lambda(x, [y, z]) \) for every \( x, y, z \in L \) and nondegenerate if its radical \( \text{Rad}(\lambda) = \{ x \in L : \lambda(x, y) = 0 \ \forall y \in L \} \) is the zero subspace. By linearity, \( \lambda(0, \cdot) = 0 \), so we can write

\[
0 = \lambda(0, e_{(j, \beta)}) = \lambda(e_{(i, \alpha)} e_{(i, \alpha)} e_{(j, \beta)}) = \lambda(e_{(i, \alpha)} e_{(i, \alpha)} e_{(j, \beta)}) = (\alpha^\gamma + \beta^\gamma) \lambda(e_{(i, \alpha)} e_{(i, \alpha)} e_{(j, \beta)}) ,
\]

which yields

\[
\lambda(B(g, h)_{(i, \alpha)}, B(g, h)_{(j, \beta)}) = 0 \quad \text{for } \alpha \neq \beta .
\]

So we have to look for possible non-zero values of the form on pairs of elements whose second indices are the same. Associativity condition tested on the sets \( \{ e_{(i, \alpha)}, e_{(i, \alpha + \beta)} e_{(i, \beta)} \} \) when either \( i = j = l = 0 \) or exactly one of them is zero gives the equation

\[
\beta \lambda(e_{(0, \beta)} e_{(0, \beta)}) = \alpha \lambda(e_{(0, \alpha)} e_{(0, \alpha)}) = \frac{\alpha + \beta}{\alpha^\gamma + \beta^\gamma} \lambda(e_{(1, \beta)} e_{(1, \beta)}) ,
\]

which has

\[
\lambda(e_{(0, \gamma)} e_{(0, \gamma)}) = \lambda(e_{(1, \gamma)} e_{(1, \gamma)}) = 0 \quad \forall \gamma \in F^*
\]
as its only solution, otherwise the identity

\[
\frac{\lambda(e_{(1, \beta)} e_{(1, \beta)})}{\lambda(e_{(0, \beta)} e_{(0, \beta)})} = \frac{\alpha^\gamma + \beta^\gamma}{\alpha + \beta}
\]

should be satisfied for every values of \( \alpha \in F^* - \{ \beta \} \), which cannot occur. Then the only possible non-zero values can be assumed for \( \lambda(e_{(0, \gamma)} e_{(1, \gamma)}) \): the sole equations coming from associativity and involving these elements come from the sets \( \{ e_{(i, \alpha)} e_{(j, \alpha + \beta)} e_{(i, \beta)} \} \) when either \( i = j = l = 1 \) or exactly one of them is one; all these equations state that the product

\[
\gamma \lambda(e_{(1, \gamma)} e_{(0, \gamma)})
\]

must be constant as a function of \( \gamma \). No more associativity conditions remain to be tested, so we can say that the Bi-Zassenhaus algebra admits an unique (up to scalar multiple) associative form defined by

\[
\lambda(e_{(i, \alpha)} e_{(j, \beta)}) = (i + j) \frac{\delta_{F^*}(\alpha, \beta)}{\alpha} , \tag{7}
\]
and extended by bilinearity and symmetry. It is nondegenerate, since for every basis element $e_{(i, \gamma)}$ the form is non-zero on the couple $(e_{(0, \gamma)}, e_{(1, \gamma)})$ and then its radical is the zero subspace. For a given non-singular associative form $\mu$ on a finitely dimensional Lie algebra $L$, the space $\text{SkDer}_\mu(L)$ of the outer skew derivations is defined as the subspace of $\text{Der}(L)$ generated by the derivations which satisfy $\mu(a \Delta, b) + \mu(a, b \Delta) = 0$ and $\mu(a \Delta, a) = 0$ for all $a, b \in L$ (note that the latter equality follows from the former one when the characteristic is not two). By using equations (7) and linearity, a check of the stated conditions on the basis elements shows that

$$\text{SkDer}_\lambda(B(g, h)) = \langle E^c : 1 \leq c \leq g + h \rangle = \langle D^c : 0 \leq c \leq g + h - 1 \rangle.$$  

Note that $E$ itself is not a skew derivation for the associative form $\lambda$. By a standard result (see [27]), there is a isomorphism $\Phi$ between the space of skew derivations $\text{SkDer}_\mu(L)$ and the second cohomology group $H^2(L, \mathbb{F})$: the image of a derivation $\Delta$ under this map is the 2-cocycle defined by $\Phi(\Delta)(a, b) = \mu(a \Delta, b)$ for every $a, b \in L$. Note that $\Phi$ can be defined over all the skew derivations, and in that case it has the inner ones as kernel. In the case of the Bi-Zassenhaus algebra, by such isomorphism we obtain $g + h$ independent 2-cocycles $\phi_c$ for $0 \leq c \leq g + h - 1$ whose actions on the basis elements is the following:

$$\phi_c(e_{(i, \alpha)}, e_{(j, \beta)}) = \Phi(D^c)(e_{(i, \alpha)}, e_{(j, \beta)}) = \lambda(e_{(i, \alpha)}D^c, e_{(j, \beta)}) = \alpha^{2^c-1}(i + j)\delta_{\alpha, \beta}.$$

The introduction of $g + h$ central elements denoted by $\zeta_c$ for $0 \leq c \leq g + h - 1$ as a basis for the vector space

$$W = \sum_{c=0}^{g+h-1} \mathbb{F} k \zeta_c$$

allows us to construct the central extension

$$\hat{B}(g, h) = B(g, h) \oplus W,$$

where the Lie operation $[\cdot, \cdot]$ is defined as follows:

$$[a, b] = [a, b] + \sum_{c=0}^{g+h-1} \phi_c(a, b) \zeta_c.$$

The sequence

$$0 \to W \to \hat{B}(g, h) \to B(g, h) \to 0$$
is the universal covering of the Bi-Zassenhaus algebra (see [13]).
Furthermore, note that \( E \) and \( F_1 \) satisfy only the first identity for being skew: hence the two symmetric maps

\[
\phi_{E}(e_{(i,a)},e_{(j,b)}) = \Phi(E)(e_{(i,a)},e_{(j,b)}) = \alpha^i \beta^j(1 + i + j)\delta_{F_2}^e(\alpha,\beta)
\]

and

\[
\phi_{F_1}(e_{(i,a)},e_{(j,b)}) = \Phi(F_1)(e_{(i,a)},e_{(j,b)}) = i(1 + i + j)\delta_{F_1}^e(\alpha,\beta),
\]

extended by bilinearity satisfy the 2-cocycle identity, but not the further condition \( \phi(a,a) = 0 \) for every \( a \in B(g,h) \).

Now we are interested in extending the Bi-Zassenhaus algebra by its outer derivation \( E \), so we have to extend consistently the 2-cocycles. By a result of [11], given a perfect Lie algebra \( H \) and an outer derivation \( \Delta \) of \( H \), all the cocycles of \( H \oplus \langle \Delta \rangle \) are obtained by extending the cocycles of \( H \) and setting \( \phi([a,b],\Delta) = \phi(a\Delta,b) + \phi(a,b\Delta) \) for every \( \phi \in H^2(H,F) \) and \( a,b \in H \). This is equivalent to introduce a derivation \( \hat{\Delta} \) on the central extension \( H \oplus W \) defined as

\[
u \hat{\Delta} = \nu \Delta + \sum_{\phi \in H^2(H,F)} \phi(u,\Delta) \cdot \zeta_{\phi}, \tag{8}
\]

where the central elements \( \zeta_{\phi} \) span \( W \). Now, since

\[
\phi_{c}([e_{(i,a)},e_{(j,b)}],E) = \phi_{c}(e_{(i,a)}E,e_{(j,b)}) + \phi_{c}(e_{(i,a)},e_{(j,b)}E) = \alpha^{i+1}(\alpha^{j} - 1) + \alpha^{j+1}(\alpha^{i} - 1) = 0,
\]

we obtain that the extended cocycles satisfy \( \phi_{c}(a,E) = 0 \) for all \( 0 \leq c \leq g + h - 1 \) and \( a \in B(g,h) \). In this way we have obtained the cohomology of the extension \( \hat{B}(g,h) \oplus \langle \hat{E} \rangle \), where the extended derivation \( \hat{E} \) defined as above coincide with \( E \).

Is it even possible to extend the associative form \( \lambda \) to the whole of \( \hat{B}(g,h) \oplus \langle \hat{E} \rangle \)? Carrara proved in [11] that it is a general fact that \( \lambda(W,W) = \lambda(B(g,h),W) = 0 \); moreover, the above equations \( \phi_{c}(a,E) = 0 \) imply \( \lambda(B(g,h),\hat{E}) = 0 \). Finally, the equations

\[
\lambda([e_{(1,a)},e_{(0,a)}],\hat{E}) = \lambda(e_{(1,a)},[e_{(0,a)},\hat{E}])
\]

when \( \alpha \) runs over \( F_{k} \) give the homogeneous linear system

\[
\sum_{c=0}^{g+h-1} \alpha^{2^c-1} \cdot \lambda(z_{c},\hat{E}) = 0,
\]
whose matrix has maximal rank $g + h$, so its only solution is the trivial one, which implies $\lambda(W, \hat{E}) = 0$). Then in this presentation the only possible extension of $\lambda$ is always zero out of $B(g, h)$.

To end with, we take a look at the isomorphism problem for the Bi-Zassenhaus algebras, following a method employed in [6]. For any Lie algebra $L$, define $\mathcal{M}_n(L)$ as the $\mathbb{F}_k$-vector space generated by $\sum_{i=0}^{n} (\text{ad } L)^{2^i}$. Due to the characteristic, a spanning set for $\mathcal{M}_n(B(g, h))$ is given by all the maps $(\text{ad } e_{(t, a)})^{2^i}$ for $i$ between zero and $n$, whose expression is in equation (6). If we define, for $x_1, x_2, n \in \mathbb{N}$, the two functions $f_n(x_1, x_2) = \text{Card}\{i \in \{1, 2\}: x_i > t\}$ and $\xi_n(L) = \dim \mathcal{M}_n(L) - \dim \mathcal{M}_{n-1}(L)$, we have that

$$\xi_n(B(g, h)) = \begin{cases} 3 & \text{for } n = 1, \\ f_{n-1}(g - 1, h) & \text{for } n > 1. \end{cases}$$

By using this result, we can discuss possible isomorphisms in the Bi-Zassenhaus family or with the four families of known algebras which can assume the same dimension, namely the special Cartan type $S$, Brown’s $G_2$, Kaplansky’s $G$ and the non-alternating Hamiltonian $P$.

- **Bi-Zassenhaus algebras $B$.**
  Since $\xi_n(B(a + 1, b)) = \xi_n(B(b + 1, a))$, two Bi-Zassenhaus algebras of parameters $(g_1, h_1), (g_2, h_2)$ are not isomorphic if $g_1 + h_1 \neq g_2 + h_2$ or $h_1 \neq g_2 - 1$ or $h_2 \neq g_1 - 1$.

  - Open problem: Is $B(a + 1, b) \simeq B(b + 1, a)$?

  - **Special algebras $S$** (see [29, Ch.4]).
    $\dim S(3 : (m_1, m_2, m_3)) = 2^{m_1 + m_2 + m_3 + 1} - 2$, for $m_i > 0$. We know that $S(3 : (1, 1, 1)) \not\simeq B(2, 1)$ since for such values of the parameters the special algebra is restrictable, i.e. $(\text{ad } x)^2 \in \text{ad } S(3 : (1, 1, 1))$ for any element $x$.

  - Open problem: Is $B(g, h) \simeq S(3 : (m_1, m_2, g + h - (m_1 + m_2)))$ for $(g, h) \neq (2, 1)$?

- **Brown’s algebras $G_2$** (see [6]).
  $\dim G_2(2 : (m_1, m_2)) = 2^{m_1 + m_2 + 2} - 2$, for $m_1 \geq m_2 > 0$.

  $$\xi_n(G_2(2 : (m_1, m_2))) = \begin{cases} 2 + f_1(m_1, m_2) & \text{for } n = 1, \\ f_n(m_1, m_2) & \text{for } n > 1. \end{cases}$$

Now we discuss the possibility that $\xi_n(B(g, h)) = \xi_n(G_2(2 : (m_1, m_2)))$; by symmetry, we can suppose $h + 1 \geq g \geq 2$. To obtain the same dimension and to satisfy the above identity for $n = 1$, we have $(m_1, m_2) = (g + h - 2, 1)$ with $g + h \geq 4$. If $g + h = 4$, then $\xi_2(G_2(2 : (2, 1))) = 0 \neq 1 = \xi_2(B(2, 2))$, so we can suppose $g + h \geq 5$. Moreover, if $g \geq 3$, then $\xi_2(B(g, h)) = 2 > 1 \geq \xi_2(G_2(2 : (g + h - 2, 1)))$. Thus we must choose $g = 2$, but then $\xi_2(B(2, h)) = \xi_2(G_2(2 : (h - 1, 1))) = 3 \neq 2$. Therefore, it is not possible to find such algebras.
Then Bi-Zassenhaus algebras and \( G_2 \) algebras are never isomorphic.

- Kaplansky's algebras \( G \) (see [19]) and non-alternating Hamiltonian algebras \( P \) (see [22]).

\[
\dim G(n) = \dim P(n, 1) = 2^n - 2 \quad \text{and} \quad G(n) \simeq P(n, 1) \quad \text{for every} \quad n \geq 4 \quad ([22, \text{Th.5.12}]).
\]

Brown proved in [6] that \( G(n) \neq P(n, 1) \), so there is no Bi-Zassenhaus algebra isomorphic to a Kaplansky's or non-alternating Hamiltonian algebra for \( n > 4 \).

Open problem: Is \( P(4, 1) \simeq G(4) \sim B(2, 1) \) ?

### 4. A DIFFERENT PRESENTATION

In order to be able to build a loop algebra, a cyclic grading of the simple algebra is needed: this will be shown in the next section. An intermediate step to take is to find a grading which allows the simple algebra to be defined over the prime field. To do this, we resemble the construction for Zassenhaus algebras.

Define a new basis

\[
\{ y_{(t,j)}; (t, j) \in F_2 \times \{-1, 0, \cdots, k - 2}\}
\]

for \( B(g, h) \), where the \( y_{(a,j)} \) are related to the \( e_{(a,0)} \) by the formulas (1), for \( a = 0, 1 \). By using the identity (2), it can be shown that the Lie product in the new basis is

\[
[ y_{(t,i)}, y_{(t,j)} ] = b_{(s,i), (t,j)} y_{s+i+t+i+j+s(1-\eta)} ,
\]

where the coefficients read now as

\[
\left( i + j + st(2 - \eta + A(i,j)(k - 1)) \right) + \left( i + j + st(2 - \eta + A(i,j)(k - 1)) \right),
\]

and the function \( A(i,j) \) is

\[
A(i,j) = \begin{cases} 0 & \text{for} \quad i + j \geq \eta - 2 \\ 1 & \text{otherwise} \end{cases}.
\]

Note that this is the compact form of the following expanded product, where \( b_{i,j} = b_{(1,i),(1,j)} \):

\[
\begin{align*}
[ y_{(0,i)}, y_{(0,j)} ] &= a_{i,j} y_{(0,i+j)} \\
[ y_{(1,i)}, y_{(0,j)} ] &= a_{i,j} y_{(1,i+j)} \\
[ y_{(1,i)}, y_{(1,j)} ] &= b_{i,j} y_{(0,i+j+1-\eta)} .
\end{align*}
\]

Now we investigate some properties of the coefficients: first of all, they lie in \( \{0, 1\} \) and so the Bi-Zassenhaus algebra is defined over the prime field \( F_2 \).
Moreover, they are obviously symmetric in \((s, i), (t, j)\). They also satisfy the following identity:

\[
b_{(s, i), (t, j)} = 0 \text{ for } i + j + st(1 - \eta) < -1 \text{ or } i + j + st(1 - \eta) > k - 3. \quad (9)
\]

When \(st = 0\) the result directly follows from the analogous one (3) proved above for the \(a_{i,j}\), so we can assume \(s = t = 1\). When \(i + j < \eta - 2\), we have

\[
b_{i,j} = \binom{i + j + 2 + k - \eta - 1}{i + 1} + \binom{i + j + 2 + k - \eta - 1}{j + 1}
\]
\[
= \binom{i + j + 2 + 2^q + h - 2^q}{i + 1} + \binom{i + j + 2 + 2^q + h - 2^q}{j + 1}
\]
\[
\equiv \binom{2^q + h - 2^q}{0} \left( \binom{i + j + 2}{i + 1} + \binom{i + j + 2}{j + 1} \right) \pmod{2}
\]
\[
\equiv 1 \cdot \left( \binom{i + j + 2}{i + 1} + \binom{i + j + 2}{i + 1} \right) \pmod{2}
\]
\[
\equiv 0 \pmod{2}.
\]

In the latter case, when \(i + j + 1 - \eta > k - 3\), two subcases may occur; either \(i + j + 1 - \eta = k - 2\), and then

\[
b_{i,j} = \binom{k - 1}{i + 1} + \binom{k - 1}{j + 1} = 1 + 1 \equiv 0 \pmod{2}
\]

or \(i + j + 1 - \eta \geq k - 1\), so we can write \(i + j + 2 - \eta = k + \psi\) for some \(0 \leq \psi \leq k - 1\), so that \(\psi < i + 1 - \eta < i + 1 < k\) and we get

\[
b_{i,j} = \binom{i + j + 2 - \eta}{i + 1} + \binom{i + j + 2 - \eta}{j + 1}
\]
\[
= \binom{i + j + 2 - \eta}{i + 1} + \binom{i + j + 2 - \eta}{i + 1 - \eta}
\]
\[
= \binom{k + \psi}{i + 1} + \binom{k + \psi}{i + 1 - \eta}
\]
\[
\equiv \binom{k}{0} \left( \binom{\psi}{i + 1} + \binom{\psi}{i + 1 - \eta} \right) \pmod{2}
\]
\[
\equiv 0 \pmod{2}.
\]

Then we take a closer look on \(b_{i,-1}\) and \(b_{i,k-2}\). When \(\eta - 1 \leq i \leq k - 3\) then

\[
b_{i,-1} = \binom{i + 1 - \eta}{i + 1} + \binom{i + 1 - \eta}{0} = 1,
\]
while
\[ b_{i,k-2} = \binom{i + k - \eta}{i + 1} + \binom{i + k - \eta}{k - 1} = \delta(i, \eta - 1) + \delta(i, \eta - 1) \equiv 0 \pmod{2}; \]

the identity above is a direct application of Lucas’ Theorem for the latter binomial coefficient, while for the former one it comes from the fact that, if \( \psi = \eta + \psi \) for some \( 0 \leq \psi \leq k - 3 - \eta \), then
\[
\binom{i + k - \eta}{i + 1} = \binom{i + k - \eta}{k - 1 - \eta} \equiv \left( \frac{2^{g+h}}{0} \right) \left( \frac{\psi}{2^{g+h} - 2^{g}} \right) \equiv 0 \pmod{2}.
\]

When \(-1 \leq i \leq \eta - 2\), we have
\[
b_{i,-1} = \left( i + k - \eta \right) + \binom{i + k - \eta}{0} = \left( \frac{2^{g+h} - 2^{g} + i + 1}{i + 1} \right) + 1 \\
\equiv \left( \frac{2^{g+h} - 2^{g}}{0} \right) \left( \frac{i + 1}{i + 1} \right) + 1 \pmod{2} \\
\equiv 1 + 1 \pmod{2},
\]

and
\[
b_{i,k-2} = \left( i + k - \eta \right) + \binom{i + k - \eta}{k - 1} \equiv \left( \frac{2^{g+h} - 2^{g}}{0} \right) \left( \frac{i + 1}{i + 1} \right) + \left( \frac{2^{g+h} - 2^{g}}{2^{g} - 2^{g}} \right) \left( \frac{i + 1}{2^{g} - 1} \right) \pmod{2} \\
\equiv 1 + 0 \pmod{2};
\]

As a consequence, \( b_{i,k-2} = 1 - b_{i,-1} \) for every index \( i \). By using the properties above, we can see that in the new presentation, the Bi-Zassenhaus algebra is generated by the basis elements whose second index is not \( k - 2 \). In view of property (9), nothing changes if we present \( B(g, h) \) in a slightly different way: let \( G_k \) be the ring isomorphic to \( \mathbb{Z}/(k-1)\mathbb{Z} \) where we choose as representatives the set \( \{ n \colon 0 \leq n \leq k - 3 \} \); then we have
\[
B(g, h) = \langle \gamma_{i,s,[i]} : (s,i) \in \mathbb{F}_2 \times G_k \rangle.
\]

This notation is useful when dealing with the outer derivations; in the new basis, \( D = \text{ad} \left( y_{(0,-1)} + y_{(0,k-2)} \right) \) and \( E = \text{ad} \left( y_{(1,-1)} + y_{(1,k-2)} \right) \). Since
\[
y_{(0,i)}E = a_{i,-1}y_{(1,i-1)} + a_{i,k-2}y_{(1,i+k-2)} = (a_{i,-1} + a_{i,k-2})y_{(1,i-1)} \\
y_{(1,i)}E = b_{i,-1}y_{(1,i-\eta)} + b_{i,k-2}y_{(0,i-\eta+k-1)} = (b_{i,-1} + b_{i,k-2})y_{(0,i-\eta)} ,
\]
and the behaviour of the involved coefficients has been shown above, we can write

\[ y_{(s,[i])}E = y_{(1-s,|i-1+s(1-\eta)|)} \cdot \tag{10} \]

An alternative equivalent construction of \( E \) can be described in terms of \( D \) in the old notation as

\[ y_{(s,[i])}E = y_{(1-s,|i|)}D^{1+s(1-\eta)}. \]

On the elements of the basis, \( E \) cycles as follows:

\[ y_{(1,|1-\eta|)} \mapsto y_{(0,|1-\eta|)} \mapsto y_{(1,|1-2\eta|)} \mapsto \cdots \]

\[ \cdots \mapsto y_{(1,|1|)} \mapsto y_{(0,|0|)} \mapsto y_{(1,|1-\eta|)}. \]

Moreover, the derivations \( F_i \) now acts as \( y_{(s,[i])}F_i = sf_{(s+t,i)} \).

By translating the expression (7) for the nondegenerate associative form in terms of the new basis by means of the transformations (1) and the property (2), we can see that \( \lambda \) now is defined over the prime field and it reads as

\[ \lambda(y_{(s,[i])}, y_{(t,[j])}) = (s+t)\delta_{Z,/(k-1)Z}(2k - 5 - ([i] + [j]), 0) \]

\[ = (s+t)\delta([i] + [j], k - 4). \]

Similarly, the new expression for the cocycles is the following:

\[ \phi_c(y_{(s,[i])}, y_{(t,[j])}) = (s+t)\delta_{Z,/(k-1)Z}([i] + [j], 2^c - 3) \]

\[ = (s+t) \]

\[ \cdot (\delta([i] + [j], 2^{g+h} + 2^c - 4) + \delta([i] + [j], 2^c - 3)) ; \]

now, \( \alpha_{[i],[j]} = 1 \) when \( [i] + [j] = 2^c - 3 \) for \( 1 \leq c \leq g+h \), while it is zero for \( c = 0 \) and when \( [i] + [j] = 2^{g+h} + 2^c - 4 \) for \( c > 1 \); then the map \( g_c(y_{(r,[i])}) = r\delta([l], 2^c - 3) \) is a coboundary for \( 1 \leq c \leq g+h \), and hence we can take as representatives of \( H^2(B(\eta, h), F_2) \) the following 2-cocycles, which we denote as \( \psi_c \):

\[ \psi_c(y_{(s,[i])}, y_{(t,[j])}) = (s+t)\delta([i] + [j], r_c) \]

where

\[ r_c = \begin{cases} 
-2 & \text{for } c = 0 \\
2^{g+h} + 2^c - 4 & \text{for } 1 \leq c \leq g+h-1.
\end{cases} \]

So we can define, on the extension \( \hat{B}(g, h) \) the product

\[ [y_{(s,[i])}, y_{(t,[j])}] = g^{h-1} \sum_{c=0}^{g+h-1} \psi_c(y_{(s,[i])}, y_{(t,[j])}) \cdot \zeta_c. \]
Here the map $\phi_E = \psi_E$ acts on the pair $y(s,i), y(t,j)$ as
\[
(1 + s + t) ((1 - s) (\delta([i] + [j], -2) + \delta([i] + [j], 2g+h - 3)) + \\
+ s (\delta([i] + [j], \eta - 3) + \delta([i] + [j], 2g+h + \eta - 4))) ,
\]
whereas the derivation $F_1$ acts now as $y(s,i) F_1 = s y(1-s,i)$, so that the map $\phi_{F_1} = \psi_{F_1}$ in the new presentation has the following expression:
\[
\psi_{F_1}(y(s,i), y(t,j)) = s(1 + s + t) \delta([i] + [j], 2g+h - 4) .
\]

Now we study the extension of the cocycles $\psi_c$ to the derivation $E$, by using the definition already employed in the previous sections, which in the new presentation has the following shape:
\[
b_{(s,i),(t,j)}^{(c)}(y(s+t,[i]+[j]+st(1-\eta)), E) = \psi_c(y(1+s, i-1+s(1-\eta)), y(t,j)) + \psi_c(y(s,i), y(1+t, j-1+t(1-\eta))) \\
= (1 + s + t) (11) \\
\cdot (\delta([i - 1 + s(1 - \eta)] + [j], r_c) + \delta([i] + [j - 1 + t(1 - \eta)], r_c)) ,
\]
where $r_c$ is $-2$ for $c = 0$ and $2g+h+2c-4$ for $1 \leq c \leq g+h-1$. Moreover, we have that the two delta functions are not zero for the following values of $(s, [i]), (t, [j])$:

- $(0, [0]), (0, [-1])$ (for $c = 0$): in this case only one of the two deltas is not zero so we get a non-zero right-hand side of (11), while the coefficient in the left-hand side is $a_{-1,0} = 1$;
- $(1, [\eta - 1]), (1, [-1])$ (for $c = 0$): the same argument applies here and the coefficient is $b_{-1, [\eta - 1]} = 1$; furthermore, note that for any other values of $[i], [j]$ whose sum is $\eta - 2$ the left-hand side coefficient vanishes, giving a trivial identity;
- $(1, [2^c - 1]), (0, [-1])$ (for $c > 0$): same behaviour here; the coefficient is $a_{-1,2^c - 1} = 1$;
- $(0, [w]), (0, [2g+h+2c-3-w])$ with $2c \leq w \leq 2g+h-3$ (for $c > 0$): in this case and its symmetric both the delta functions are not zero and so we have a zero right-hand side; the coefficient $a_{[i], [j]}$ on the left-hand side is zero by (3);
- $(1, [w]), (1, [2g+h+2g+2c-5-w])$ with $2^c + 2^g - 2 \leq w \leq 2g+h-3$ (for $c > 0$): also here we get a trivial identity, since both the deltas are non zero and the coefficient $b$ vanish by (9);
- $(1, [w]), (1, [2^g + 2^c - 4 - w])$ with $-1 \leq w \leq 2^c - 2$ (for $1 \leq c \leq g$): in this case and its symmetric only one of the delta functions is non-zero, while the coefficient $b_{[i], [j]} = \left(\frac{2^c - 1}{w+1}\right) + \left(\frac{2^c - 1}{w+2^c - 3 - w}\right) = 1 + 0 = 1$;
The equations above read as follows:

- \( (1, [w]), (1, [2^g + 2^c - 4 - w]) \) with \( 2^c - 1 \leq w \leq 2^g - 3 \) (for \( 1 \leq c \leq g \)): here both the deltas have value one and the coefficient \( b_{ij} = \left( \frac{2^e - 1}{w + 1} \right) + \left( \frac{2^e - 1}{w + 3 - w} \right) = 0 + 0 = 0; \)
- \( (1, [w]), (1, [2^g + 2^c - 4 - w]) \) with \(-1 \leq w \leq 2^g - 3\) (for \( c > g \)): these values and their symmetric make only one of the deltas different from zero, while \( b_{ij} = \left( \frac{2^e - 1}{w + 1} \right) + \left( \frac{2^e - 1}{w + 3 - w} \right) = 1 + 0 = 1. \)

For all other values of the indices, the right-hand side of the equation is zero, while, if \( \gamma \neq 2^c - 2 \) with \(-1 \leq \gamma \leq 2^{g+h} - 4\), we have that \( b(a, [\gamma - 1], [0, [\gamma + 1]]) = a-1, \gamma+1 = 1 \) and \( a_{0, 2^{g+h}-3} = 1 \). All the above can be summarized by the position

\[
\psi_c(y(s, [i]), E) = (1 + s)\delta([i], 2^c - 2) \quad \text{for} \quad 0 \leq c \leq g + h - 1 ,
\]

which leads to the extended derivation \( \hat{E} \) defined as in (8). Note that, in the previous presentation, the 2-cocycles \( \psi_c \) differ to the \( \phi_c \) by a coboundary and read as follows:

\[
\psi_c(e_{(i, \alpha)}, e_{(j, \beta)}) = (i + j)\alpha 2^c - 1 \delta_{\alpha \beta} (\alpha + \beta) + (\alpha + \beta) 2^c - 1 ;
\]

their extension to the derivation \( E \) then becomes

\[
\psi_c(e_{(i, \alpha)}, E) = (1 + i)\alpha 2^c - 1 .
\]

As in the previous presentation (regardless of the chosen cocycles) the associative form can only be trivially extended to the derivation \( \hat{E} \) and to the central elements \( W \): we already know that \( \lambda(W, W) = \lambda(B(g, h), W) = 0; \) we need to complete the extension by using the two more relations of associativity (see [11])

\[
\lambda([a, b], \hat{E}) = \lambda(a, [b, \hat{E}]) ,
\]

\[
\lambda([a, \hat{E}], \hat{E}) = \lambda(a, [\hat{E}, \hat{E}]) .
\]

If \((s, [i])\) and \((t, [j])\) satisfy \([y(s, [i]), y(t, [j])]\) = \( \zeta_c \), then \( s + t = 1 \) and the first equations above reads as follows:

\[
\lambda(\zeta, \hat{E}) = \lambda(y(s, [i]), y(t, [j])E) \\
= \psi_E(y(s, [i]), y(t, [j])) \\
= (1 + s + t)\delta([i] + [j], -2 + s(\eta - 1)) \\
= 0 .
\]

Applying this result in the second equation

\[
\lambda(aE, \hat{E}) + \sum_{c=0}^{g+h-1} \psi_c(a, E) \cdot \lambda(\zeta_c, \hat{E}) = 0 ,
\]
we get that $\lambda(W, \bar{E}) = \lambda(B(g, h), \bar{E}) = 0$, and hence $\lambda$ can be only trivially extended to $B(g, h) \oplus \langle \bar{E} \rangle$.

5. A CYCLIC GRADING

We can now introduce a third grading for the Bi-Zassenhaus algebras. In the notations above, consider the following map, where the bar denotes the remainder modulo two:

$$
\mu: \{1, 2, \ldots, 2k - 2\} \rightarrow \mathbb{F}_2 \times G_k
\quad \mapsto (\overline{m}, [2^{g-1}(\overline{m} - m) - m]).
$$

The map $\mu$ is bijective, its inverse being the map

$$
\mu^{-1}(s, [\bar{t}]) = 2v - s,
$$

where

$$
v = 2^h (s \eta - [\bar{t}]) \pmod{k - 1};
$$

furthermore, when $-1 \leq \mu(m) + \mu(n) + \overline{m}(1 - \eta) \leq k - 3$, the map $\mu$ satisfies the property

$$
\mu(m) + \mu(n) + \overline{m}(1 - \eta) = \mu(m + n),
$$

where we used the fact that, in the integers, $\overline{m} + \overline{n} = \overline{m + n} + 2\overline{mn}$; in the case $n = 1$, we have that

$$
\mu(m) + 1 + \overline{m}(1 - \eta) + C(k - 1) = \mu(m + 1),
$$

where $C = 1$ if the second component of $\mu(m)$ is less than $\eta - 1$. These properties, together with the (9), allow to define a new basis for the Bi-Zassenhaus algebra

$$
u_m = y_{\mu(m)} \quad 1 \leq m \leq 2k - 2,
$$

such that

$$
[u_m, u_n] = b_{\mu(m), \mu(n)} u_{m+n}
$$

and for which, by (10), we have

$$
u_mE = \nu_{m+1},
$$

and then $B(g, h), E = B(g, h), E_{c+1}$ cyclically. Moreover, the derivation $F_1$ acts now as $u_m F_1 = \overline{m} u_{m+2^k+1} - 1$, where the indices are to be taken modulo the
dimension $d = 2k - 2$ of the algebra. In the basis $\{u_m\}$, the associative form reads as

$$\lambda(u_m, u_n) = (m + n)(\delta(m + n, 2^{b+2} + 1) + \delta(m + n, 2^{g+b+1} + 2^{b+2} - 1))$$

$$= \delta(m + n, 2^{b+2} + 1) + \delta(m + n, 2^{g+b+1} + 2^{b+2} - 1),$$

while the cocycles become

$$\tilde{\psi}_c(u_m, u_n) = \delta(m + n, r_c^{(1)}) + \delta(m + n, r_c^{(2)}),$$

where, for $i = 1, 2$, we set

$$r^{(i)}_c = \begin{cases} 2^{b+1} + 1 + (i - 1)d & \text{for } c = 0 \\ 2^{g+b+1} + 2^{b+2} - 2^c + 1 - 1 + (i - 1)d & \text{for } 1 \leq c \leq g - 1 \\ 2^{b+2} - 2^{-g+1} + 1 + (i - 1)d & \text{for } g \leq c \leq g + h - 1, \end{cases}$$

Note that we can write every $\tilde{\psi}_c(u_m, u_n)$ as $(1 - b_{\mu(m), \mu(n)})\tilde{\psi}_c(u_m, u_n) + b_{\mu(m), \mu(n)}\tilde{\psi}_c(u_m, u_n)$, where the second addendum is a coboundary; hence, we can take

$$\psi_c(u_m, u_n) = (1 - b_{\mu(m), \mu(n)})\left(\delta(m + n, r_c^{(1)}) + \delta(m + n, r_c^{(2)})\right)$$

as representatives of $H^2(B(g, h), F_2)$ in this presentation. Again we denote by $\zeta_c$ the central element linked to the cocycle $\psi_c$.

The map $\psi_E$ has the form

$$\psi_E(u_m, u_n) = \delta(m + n, 2^{b+2}) + \delta(m + n, 2^{g+b+1} + 2^{b+2} - 2),$$

while $\psi_{F_1}$ is now

$$\psi_{F_1}(u_m, u_n) = \mathcal{M}\left(\delta(m + n, 2^{b+1} + 2) + \delta(m + n, 2^{g+b+1} + 2^{b+1})\right).$$

Furthermore, the extension of the cocycles to the derivation $E$ gives

$$\psi_c(u_m, E) = \delta(m, r_c^{(1)}) - 1 \quad \text{for } 0 \leq c \leq g + h - 1,$$

while again the associative form admits only a trivial extension to $\hat{B}(g, h) \hat{\otimes} \langle \hat{E} \rangle$, where $\hat{E}$ is defined as usual.

### 6. THE BI-ZASSENHAUS LOOP ALGEBRAS

Now we can construct the twisted loop algebra through the following standard technique. Use the derivation $E$ to extend the simple Bi-Zassenhaus algebra, then tensor it with the polynomial ring in one variable over the prime field:

$$(B(g, h) \hat{\otimes} F_2 \cdot E) \otimes F_2[t];$$
the Bi-Zassenhaus loop algebra $B_t(g, h)$ is the subalgebra generated by $u_1 \otimes t$ and $E \otimes t$. The cyclic grading of the simple algebra and the behaviour of the derivation show that $B_t(g, h)$ is an infinite-dimensional, $\mathbb{N}$-graded algebra of maximal class generated by its first homogeneous component, since

$$
B_t(g, h) = \bigoplus_{i=1}^{\infty} B_t(g, h)_i, \quad \text{where}
B_t(g, h)_1 = F_2 \cdot (u_1 \otimes t) + F_2 \cdot (E \otimes t)
B_t(g, h)_i = F_2 \cdot (u_i \otimes t^i) \text{ for } i > 1.
$$

We now want to show that Bi-Zassenhaus loop algebras are not isomorphic to AFS-algebras. To this aim, we investigate their two-step centralizers, i.e. the one dimensional subspaces

$$C_i = C_{B_t(g, h)_i}(B_t(g, h)_i) \quad \text{for } i \geq 2,$$

since by Theorem 3.2 in [8] a graded Lie algebra of maximal class is determined up to isomorphism by its sequence $\{C_i\}$ of two-step centralizers. Moreover, by Lemma 3.3 of the same paper, the first centralizer $C_2$ occurs in contiguous subsequences, called constituents, separated by isolated occurrences of a different centralizer. Following the notation of [9], suppose that in the sequence of two-step centralizers we have a pattern of the form

$$C_i \neq C_2, \quad C_{i+m} \neq C_2,$$

$$C_{i+1} = C_{i+2} = \ldots = C_{i+m-1} = C_2,$$

then $(C_{i+1}, C_{i+2}, \ldots, C_{i+m})$ is a constituent of length $m$. By [8, Theorem 5.5], a two-step centralizer different from the first one will first occur in class $2q$ for some power $q$ of the characteristic of the underlying field; the number $q$ is called the parameter of the algebra, while by convention we say that the length of the first constituent is $2q$. As a consequence of the previous theorem, when the algebra has only two distinct two-step centralizers, the sequence of the constituents lengths settles its isomorphism type. By definition,

$$[u_1, u_m] = b_{(1, 1), \mu(m)} u_{m+1} = b_{(1, 1), \mu(m)} u_m E,$$

and then the two-step centralizers are exactly

$$C_m = F \cdot ((u_1 - b_{(1, 1), \mu(m)}) \cdot E) \otimes t), \quad (16)$$

and since $b_{(\cdot, \cdot), (\cdot, \cdot)} \in \{0, 1\}$ the loop algebra $B_t(g, h)$ has only two distinct two-step centralizers, generated by $x = u_1 \otimes t$ and $y = (u_1 - E) \otimes t$ in the standard
notations. In particular, by definition we have that
\[
b^{(1, -1)}(m, \mu(m)) = \begin{cases} 
a^{[-1], [-2^{n-1}m]} & \text{when } m \in 2\mathbb{Z} \\
b^{[-1], [2^{n-1}(1-m)-1]} & \text{when } m \in 2\mathbb{Z} + 1 ,
\end{cases}
\]
and then, from previous calculations, that
\[
b^{(1, -1)}(m, \mu(m)) = \begin{cases} 
1 - \delta_{2^{i+k}}([-1], [-2^{n-1}m]) & \text{for } m \in 2\mathbb{Z} \\
1 - \sum_{w=-1}^{n-2} \delta_{2^{i+k}}([w], [2^{n-1}(1 - m) - 1]) & \text{for } m \in 2\mathbb{Z} + 1 ,
\end{cases}
\]
Thus, to express explicitly the two-step centralizers in (16) in terms of \(m\) we have to solve the following equations in \(G_k\):
\[
[-2^{n-1}m] = [-1] \quad \text{for } m \in 2\mathbb{Z} , \quad \text{and} \\
[2^{n-1}(1 - m) - 1] = [w] \quad \text{for } m \in 2\mathbb{Z} + 1 , \quad w \in \{-1, \cdots, \eta - 2\} .
\]
The solutions are respectively, for \(n \in \mathbb{N}_0 ,
\[
m = 2q + nd \quad \text{and} \\
m = 2q(\eta - w) - 1 + nd ,
\]
where \(q = 2^k\) is called the parameter of the loop algebra and denotes the first occurrence of the second two-step centralizers \(F_{2x}\); remember that we use \(d\) to denote the dimension \(2k - 2\) of the Bi-Zassenhaus algebra. The sequence of the two-step centralizers (16) reads then
\[
C_m = \begin{cases} 
F_{2} \cdot x & \text{for } m \text{ as in (17b)} \\
F_{2} \cdot y & \text{otherwise .}
\end{cases}
\]
Then the constituent lengths sequence can be constructed: if we denote exponentially patterns’ repetition, it reads as
\[
2q, 2q - 1, (2q^{n-1}, 2q - 1^2)^\infty .
\]
Note that this implies that two Bi-Zassenhaus loop algebras with different parameters are never isomorphic.

In the paper [8] two techniques are described to build more algebras of maximal class starting from a given one \(L\): they are called inflation and deflation and the obtained algebras are respectively indicated by \(L^+\) and \(L^-\). For a complete theoretical approach to them, we refer to the cited work. For our purposes, we can rely on the results listed in the Propositions 3.3–3.6 in [9], where \(p\) is the
characteristic of the field (in our case \( p = 2 \)). Moreover, we recall the constituent lengths sequence of the Albert-Frank-Shalev algebra:

\[
\text{AFS}(a, b, n, p) = 2q, (q^{r-2}, 2q - 1, (q^{r-2})^{s-1})^\infty,
\]

where \( q = p^a, r = p^{b-a} \) and \( s = p^{a-b} \). Since an algebra is inflated if and only if all constituents have length a multiple of the \( p \), neither the Bi-Zassenhaus loop algebras nor the Albert-Frank-Shalev algebras are inflated: actually, they are the only non-inflated ones [18]. Given any algebra of maximal class \( L \), its deflation can be obtained by multiplying all its constituents lengths by \( p \): then, if we denote by \( q^\uparrow = q^{h+1} \) the parameter of the inflated algebra, we have

\[
B_i(g, h)^\uparrow = 2q^\uparrow, 2q^\uparrow - 2, (2(q^\uparrow)^{q-1}, 2q^\uparrow - 2)^\infty.
\]

Repeated inflations (say, \( \beta \) times) will change the original lengths \( 2q, 2q - 1 \) of the constituent into \( 2p^{b+\beta}, 2p^{b+\beta} - p^\beta \). When an algebra \( L \) of maximal class is inflated, deflation has the effect to divide all constituent lengths by \( p \): this follows from the fact \( (L^\downarrow)^\downarrow = L \); the converse is true only for inflated algebras. When \( L \) is not inflated, the effect of deflation can be read from the sequence of two-step centralizers by using [9, Proposition 3.3]: if for \( i \geq 2 \) all the two-step centralizers

\[
C_i, C_{i+1}, \ldots, C_{(i+1)p-1}
\]

coincide with \( C_2 = Fy \), then \( C_i = Fy \), otherwise if one of them equals \( Fu \neq Fy \), then \( C_i = Fu \). So, for \( i \geq 4 \), write down the two-step centralizers of \( B_i(g, h) \) obtained in (18), and for every \( l \geq 2 \) set

\[
C_l = \begin{cases} 
Fy & \text{when } C_{2l} = C_{2l+1} = Fy \\
Fx & \text{otherwise}. 
\end{cases}
\]

Grouping them in constituents, we get two different results, depending on the parameter of the Bi-Zassenhaus loop algebra: if \( h = 1 \) the sequence of constituent lengths of the deflated algebra reads as

\[
4, (2^{n-1}, 3)^\infty,
\]

while when \( h > 1 \), if we let \( q^\downarrow = q/2 = 2^{h-1} \), we get

\[
2q^\downarrow, 2q^\downarrow - 1, (2(q^\downarrow)^{q-1}, 2q^\downarrow - 1).
\]

Both the above patterns are recognisable as AFS algebras, for a suitable choice of the parameters: in particular, if we understand \( \text{AFS}(0, b, n, 2) = \text{AFS}(b, n, n, 2) \) we can write in every case

\[
B_i(g, h)^\downarrow = \text{AFS}(h - 1, h, h + g, 2).
\]
Note that we also have that (see [8, Proposition 7.3])

\[ \text{AFS}(h, h + 1, h + g, 2)^\ell = \text{AFS}(h - 1, h, h + g, 2); \]

so we have two non-isomorphic loop algebras whose deflation coincide: this is a consequence (see [7]) of the isomorphism \( \phi \), in characteristic two, between the Zassenhaus algebra \( Z_k \)

\[ Z_k = \langle e_\alpha; \alpha \in F^*_k \rangle, \quad [e_\alpha, e_\beta] = (\alpha + \beta)e_{\alpha + \beta} \]

and the Albert-Frank algebra \( \text{AF} \)

\[
\text{AF}(a, b, k) = \langle e'_\alpha; \alpha \in F^*_k \rangle, \quad [e'_\alpha, e'_\beta] = (\alpha^2 \beta^2 + \alpha^2 \beta^2) e'_{\alpha + \beta} \\
= \langle e''_\alpha; \alpha \in F^*_k \rangle, \quad [e''_\alpha, e''_\beta] = (\alpha + \beta)^2 e''_{\alpha + \beta}
\]

given by

\[ \phi(e''_\alpha) = e_{a^2 - 2}. \]

The last part of the paper is devoted to describe the cohomology of the loop algebras; first we deal with the cocycles coming from the simple algebras.

Consider the loop \( (\tilde{B}(g, h) \otimes \langle E \rangle) \otimes F_2[t] \) and the subalgebra \( \tilde{B}(g, h) \) generated by \( u_1 \otimes t \) and by \( \tilde{E} \otimes t \). Here the product is given, for \( u, v \in B(g, h) \otimes \langle E \rangle \) and \( a, b \in N \), by the following expression:

\[
[u \otimes t^a, v \otimes t^b] = [u, v] + \sum_{c=0}^{g+h-1} \psi_c(u, v) \cdot \zeta_c \otimes t^{a+b}.
\]

where the value \( \psi_c(u, v) \) is given by (12) and (15). In the paper [17], we employ a different notation for the central elements in \( \tilde{B}(g, h) \) by introducing the elements

\[
\theta^{f(c)}_{n_1} = \zeta_c \otimes t^{(1+n)d},
\]

where \( f(0) = 1 \) and \( f(c) = g + h + 1 - c \) for \( 1 \leq c \leq g + h - 1 \); in this notation, the weight of \( \theta^{p_1}_{n_1} \) is less than the weight of \( \theta^{p_2}_{n_2} \) if either \( n_1 < n_2 \) or \( n_1 = n_2 \) and \( s_1 < s_2 \).

As we have seen above, the associative form \( \lambda \) admits only a trivial extension to \( \tilde{B}(g, h) \otimes \langle E \rangle \); furthermore, if we define the additional loop cocycle in the standard way by multiplying the associative form either by the residue map (if \( a = 1 \), see [13]) or its analogue (if \( a = 2 \), see [10]), we obtain

\[
\lambda(u \otimes t^a, v \otimes t^b) \cdot \rho(t^a, t^b) = \lambda(u \otimes t^a, v \otimes t^b) \cdot a \cdot \delta_{Z \otimes Z}(a + b, 0) = 0,
\]
since \( \lambda \) vanishes on elements whose weights have even sum, and the parity of elements in \( \mathcal{B}(g, h) \) is the same of those in \( \hat{\mathcal{B}}(g, h) \).

Note that the centre of \( \mathcal{B}(g, h) \) is infinite-dimensional, and the quotient of \( \hat{\mathcal{B}}(g, h) \) over its centre is isomorphic to the algebra of maximal class \( \mathcal{B}_l(g, h) \); so we can apply the translation to Lie algebras of a classical group-theoretical result due to B.H. Neumann (see [24, pp. 52-53]):

**Theorem 6.1.** Let \( \rho \) be any epimorphism from a finitely generated Lie algebra \( \mathcal{L} \) to a finitely presented algebra \( \mathcal{L} \). Then \( \ker(\rho) \) is finitely generated as an ideal in \( \mathcal{L} \).

If \( \rho: \hat{\mathcal{B}}(g, h) \to \mathcal{B}_l(g, h) \) is the homomorphism induced by the canonical surjection from \( \hat{\mathcal{B}}(g, h) \) to \( \mathcal{B}(g, h) \), then \( \ker(\rho) = W = Z(\mathcal{B}(g, h)) \), so \( \mathcal{B}_l(g, h) \) cannot be finitely generated. So we can state as a direct consequence of the above theorem that

**Theorem 6.2.** The Bi-Zassenhaus loop algebras are not finitely presented.

In fact, a stronger result (proved in [17]) holds, analogous to the one proved by Carrara in [12] for AFS-algebras:

**Theorem 6.3.** For every Bi-Zassenhaus loop algebra \( \mathcal{B}_l(g, h) \), there exists a finitely presented graded Lie algebra \( M(g, h) \) such that

\[
M(g, h)/Z_2(M(g, h)) \cong \mathcal{B}_l(g, h).
\]

Now we conclude by introducing the loop cocycles: this will give a construction of the covering algebra \( M(g, h) \).

First, look at the maps \( \psi_{EF}, \psi_{F_1} \); they are not cocycles of the simple algebra, since they do not vanish on the pairs \((a, a)\). The natural way to turn them into cocycles of the loop algebras is to define

\[
\Psi_A(v \otimes t^{\alpha+\beta}, w \otimes t^{b+\gamma}) = \psi_A(v, w) \cdot \delta_{F_2}(\alpha + \beta, 1),
\]

where \( v \) and \( w \) are elements of \( \hat{\mathcal{B}}(g, h) \) of weight \( a \) and \( b \) respectively and \( A \) is either \( E \) or \( F_1 \). Now we show if they can be consistently extended to the whole of \( \hat{\mathcal{B}}(g, h) \).

First we deal with \( \Psi_{F_1} \). When \( v \) and \( w \) are elements of \( \mathcal{B}(g, h) \) we have the expression (14); actually, we can get a simpler expression by adding a "coboundary" \( b_{\mu(m), \mu(n)}(\delta(m + n, 2^{b+1} + 2) + \delta(m + n, 2^{\gamma+b+1} + 2^{\gamma+1})) \): in this way the coefficient in front at the sum of the two delta maps is \( m + b_{\mu(m), \mu(n)} \); this coefficient is always zero if \( m + n = 2^{\gamma+b+1} + 2^{\gamma+1} \), while it is always one when \( m + n = 2^{b+1} + 2 \), but in the case \( m = 1, n = 2^{b+1} + 1 \) and its symmetric.
Hence we have

\[ \Psi_F(v_m \otimes t^{m+\alpha d}, v_n \otimes t^{n+\beta d}) = (\delta(m + n, 2^h + 1) + 2) \\
-\delta(m, 1) \delta(n, 2^h + 1) \\
-\delta(n, 1) \delta(m, 2^h + 1)) \\
\delta_{F_3}(\alpha + \beta, 1), \]  

which means that \( \Psi_{F_3} \) is not zero only on the elements \( (u_m \otimes t^{m+\alpha d}, u_n \otimes t^{n+\beta d}) \) such that \( \alpha + \beta \) is odd and \( (u_m, u_n) \) is one of the \( 2^{h+1} - 1 \) couples \( \{(2, 2^{h+1}), (3, 2^{h+1} - 1), \ldots, (2^{h+1}, 2)\} \). Now we use the cocycle identity

\[ \Psi_A([a_1, a_2], a_3) + \Psi_A([a_2, a_3], a_1) + \Psi_A([a_3, a_1], a_2) = 0 \]  

to define the above maps on the remaining elements. Since \( B(g, h) \) is perfect and any central element of \( B(g, h) \) is a commutator (see [11]), we have that \( \Psi_{F_3}(v \otimes t^{a+\alpha d}, v \otimes t^{b+\beta d}) = 0 \) for any \( v \in B(g, h) \); the result comes from evaluating equation (20) with \( a_3 = \zeta \otimes t^{r^{(i)}+\beta d} \) and \( a_1, a_2 \) satisfying \( [a_1, a_2] = v \otimes t^{a+\alpha d} \). Now choose \( a_1 = u_m \otimes t^{m+\alpha d}, a_2 = u_p \otimes t^{p+\beta d} \) and \( a_3 = \widehat{E} \otimes t^{c} \). Consider first the case when \( [a, b] = \zeta \otimes t^{r^{(i)}+n d}, \widehat{E} \otimes t^{c} \), so that \( b(u_{(m),\mu(p)}) = 0 \). When \( n \) is even the cocycle equation gives \( \Psi_{F_3}(\zeta \otimes t^{r^{(i)}+nd}, \widehat{E} \otimes t^{c}) = 0 \), while when \( n \) is odd it reduces to

\[ \Psi_{F_3}(\zeta \otimes t^{r^{(i)}+nd}, \widehat{E} \otimes t^{c}) = \Psi_{F_1}(u_{p+1} \otimes t^{p+1+\beta d}, u_m \otimes t^{m+\alpha d}) \\
+ \Psi_{F_1}(u_p \otimes t^{p+\beta d}, u_{m-1} \otimes t^{m+1+\alpha d}). \]

Since \( m + p = r^{(i)} \), the right-hand side of the previous equation vanishes when \( 1 \leq c \leq g + h - 1 \). If \( c = 0 \), we know that the only couples of elements whose product is \( \zeta \) is \((y_{(0,-1)}, y_{(1,-1)})\) and its symmetric, that is, \((m, p) = (1, 2^{h+1})\). With these values, the right-hand side of the above equations is one, so we can summarize what we found by saying that \( \Psi_{F_3}(\zeta \otimes t^{r^{(i)}+nd}, \widehat{E} \otimes t^{c}) = \delta(c, 0) \delta_{F_3}(n, 1). \)

Finally, when \( b_{(m),\mu(p)} = 1 \) we find the last non-trivial case

\[ \Psi_{F_3}(u_{m+p} \otimes t^{m+p+\alpha d}, \widehat{E} \otimes t^{c}) = \Psi_{F_1}(u_{p+1} \otimes t^{p+1+\beta d}, u_m \otimes t^{m+\alpha d}) \\
+ \Psi_{F_1}(u_p \otimes t^{p+\beta d}, u_{m-1} \otimes t^{m+1+\alpha d}) \\
= 0, \]

by equation (19). Then \( \Psi_{F_3} \) is a non-trivial cocycle of \( B(g, h) \).

We can carry on a similar argument for \( \Psi_E \) as well, but we get a different result. First we reach a simpler form for this map on elements in \( B(g, h) \) by splitting its expression (13) into \((1 - b_{(\cdot, \cdot)} \psi_E) + b_{(\cdot, \cdot)} \psi_E \) and getting rid of the second
addendum, which is a coboundary: in terms of the \( y_{(s,i,d)} \) presentation, since 
\[
a_{[i], [j]} = 1 \text{ for } [i] + [j] = 2^{q+h} - 3 \text{ and } b_{[i], [j]} = 1 \text{ for } [i] + [j] = 2^{q+h} + \eta - 4,
\]
it is equivalent to express the map \( \psi_E \) as 
\[
\psi_E(y_{(s,i,d)}, y_{(t,j,d)}) = (1 + s + t)\delta([i] + [j], -2 + s(\eta - 1));
\]
in this form,
\[
\Psi_E(u_m \otimes t^{m+ad}, u_n \otimes t^{n+bd}) = (1 - b_{\mu(m), \mu(n)}) (\delta(m + n, 2^{h+2})
+ (\delta(m + n, 2^{q+h+1} + 2^{h+2} - 2)) \delta_{\mathcal{M}}(a + b, 1)
+ \sum_{(M,N) \in \mathcal{M}} \delta_{\mathcal{B}}((m,n), (M,N))
- \delta_{\mathcal{M}}(a + b, 1),
\]
where \( \mathcal{M} \) is the set 
\[
\mathcal{M} = \{(2^{h+1}, 2^{h+1}), (2^{h+2} - 1), (2^{h+2} - 1, 1),
(2^{h+1}(\eta - \alpha) - 1, 2^{h+1}(\alpha + 3) - 1), 0 \leq \alpha \leq \eta - 3\}.
\]
As in the previous case, \( \Psi_E(v \otimes t^{s+ad}, \zeta_c \otimes t^{t^{1}+\beta d}) = 0 \) for any \( v \in \hat{\mathcal{B}}(g, h) \). Moreover, from the cocycle identity (20) and from expression (21) it follows that \( \Psi_E(v \otimes t^{s+ad}, \widehat{E} \otimes t) \) can be different from zero only when \( v = u_{2^{q+2} - 1} \); furthermore, note that, when \( m + n = 2^{h+2} - 1 \) for \( 1 \leq m, n \leq 2k - 2 \) we have that \( b_{\mu(m), \mu(n)} = 1 \) if and only if either \( (m,n) = (M, N - 1) \) or \( (m,n) = (M, N) \) for \( (M,N) \in \mathcal{M} \); when this is the case, then exactly one of \( \psi_E(u_m, u_n) \) or \( \psi_E(u_m, u_{n+1}) \) is one. By using these results, if we apply the cocycle equation (20) to the elements \( a_1 = u_{2^{q+2} - 2} \otimes t^{2^{h+1} - 2 + \alpha d}, a_2 = u_{2^{q+2} - 2} \otimes t^{2^{h+2} - 1 + \beta d} \) and \( a_3 = \widehat{E} \otimes t \) first, and then to \( a_1 = u_{2^{q+2} - 1} \otimes t^{2^{h+1}(\eta - \alpha) - 2 + \alpha d}, a_2 = u_{2^{q+2} - 2} \otimes t^{2^{h+1}(\xi + 3) - 1 + \beta d} \) and \( a_3 = \widehat{E} \otimes t \) for \( \alpha + \beta \) even and \( 0 \leq \xi \leq \eta - 3 \) we get a contradiction, since we obtain that \( \Psi_E(u_{2^{q+2} - 1} \otimes t^{2^{h+2} - 1 + (\alpha + \beta + 1)d}, \widehat{E} \otimes t) \) should be one in the first case and zero in the second: then \( \Psi_E \) cannot be extended to the whole of \( \mathcal{B}(g, h) \).

Now we can construct a central extension \( M(g,h) \) of \( \mathcal{B}(g, h) \) by defining the Lie product as
\[
[u \otimes t^a, v \otimes t^b] = [u \otimes t^a, v \otimes t^b] + \Psi_{F_a}(u \otimes t^a, v \otimes t^b) \cdot \omega \otimes t^{a+b},
\]
where \( \omega \otimes t^{a+b} \) are central elements lying in classes \( 2^{\eta+1} + 2 + (2n + 1)d \); in the paper [17] we define \( \theta_a^x = \omega \otimes t^{2^{h+1}+2+(2n+1)d} \). In particular, note that the element \( \zeta_0 \otimes t^{2^{h+1}+1+nd} \) is always central in \( \mathcal{B}(g, h) \), while in \( M(g,h) \) it satisfies
\[
\lbrack \zeta_0 \otimes t^{2^{h+1}+1+nd}, \widehat{E} \otimes t \rbrack = \delta_{\mathcal{M}}(n, 1) \cdot \omega \otimes t^{2^{h+1}+2+nd}.
\]
so it lives in $Z_2(M(g,h))$ for odd $n$’s.

REFERENCES


