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Increasing and Decreasing Risk Aversion for Generalized Preferences

John Quiggin
School of Economics
Faculty of Economics and Commerce
Australian National University
Canberra, ACT, 0200, Australia

and

Robert G. Chambers
Department of Agricultural and Resource Economics
University of Maryland
College Park MD 20742

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Abstract
In this paper, concepts of comparative risk aversion are developed for generalized risk averse preferences. These measures are applied to the development of general concepts of increasing and decreasing risk aversion. A range of comparative static results for expected utility theory are generalized and extended.

1 Introduction

There is a large literature on the comparative statics of choice under uncertainty, beginning with the work of Arrow (1965) and Pratt (1964). The archetypal result is Pratt's demonstration, for the portfolio problem with one safe asset and one risky asset, that an expected-utility maximizer displays decreasing absolute risk aversion if and only if the amount of the risky asset in the optimal portfolio increases with an increase in base wealth. There is a sense in which this result may be viewed as trivial. If the concepts '(absolute) risk aversion declining in wealth' and '(absolute) riskiness' have been defined correctly, an increase in wealth must, by definition, lead to the choice of an (absolutely) less risky portfolio. This idea is explored by Yaari (1969), who considers the concept of decreasing absolute risk aversion separately from its characterization in the expected-utility model. Most subsequent work on the comparative statics of choice under uncertainty, however, has confined attention to the expected-utility model. However, it is generally agreed that choices under uncertainty systematically deviate from those predicted by the expected-utility model. Hence, the validity of comparative-static results derived under the expected-utility model is open to question.

It is desirable, then, to consider to what extent concepts such as decreasing absolute risk aversion can be characterized in purely behavioral terms, without unduly restrictive assumptions, such as expected utility, about the functional form of preferences under uncertainty. Some progress has already been made in this direction. Quiggin and Chambers (1998) and Chambers and Quiggin (2000) show that the standard tools for the analysis of economic problems involving uncertainty, including risk premiums, certainty equivalents as well as the notions of absolute and relative risk aversion, can be developed and applied without making specific assumptions on functional form beyond the basic requirements of monotonicity, transitivity, and continuity. The approach relies on the distance and benefit functions (Shephard 1970; Luenberger 1992) to characterize preferences relative to a given state-contingent vector of outcomes, and then derives results directly from the properties of these functions. The expected-utility hypothesis is redundant in this context.

In this paper, the same approach is applied to develop tools for comparisons of risk aversion. Such comparisons involve two crucial elements. First, there must exist a given partial or total ordering of riskiness. Second, there must exist a notion of compensation. For any specification of these two elements, we may make statements of the form 'A is more risk-averse than B, if for any increase in risk, A requires more compensation than B'.

If the risk ordering displays appropriate invariance properties, this concept may be extended to comparisons between the risk attitudes of an individual with a given initial state-contingent income vector and the same individual with a different initial position. In particular, any partial ordering of riskiness which is preserved under translations of the state-contingent incomes vector (that is, increases in base wealth) gives rise to a notion of decreasing (or increasing) absolute risk aversion.

The paper is organized as follows. We first specify a general representation of preferences over state-contingent income vectors. We then consider partial orderings of riskiness and notions of compensation, and develop associated invariance concepts. These concepts are used to make precise the idea that one decision-maker is more risk-averse than another, and that a particular decision-maker displays decreasing or increasing risk-aversion.

Having developed these analytical tools, we consider the implications for comparative-static analysis. We first formalize the idea of a risk–return trade-off, that is a choice problem in which all options can be ranked in terms of risk (as evaluated by the risk ordering) and return (as evaluated by the compensation measure). For such choice problems, the more risk-averse of two decision-makers will always choose an option that is less risky, but has a lower expected return, than the option chosen by the less risk-averse decision-maker. We next consider the comparative statics of changes in the problem parameters. Jointly sufficient conditions for a change in the problem parameters to lead to a more risky choice are derived and illustrated using the simple portfolio problem with one safe asset and one risky asset.

2 Notation and Assumptions

We concern ourselves with preferences over state-contingent income or consumption vectors represented as mappings from a state space Ω to an outcome space $Y \subseteq \Re$. We focus on the case where Ω is a finite set $\{1...S\}$. The space of state-contingent income vectors is $\mathbf{Y} \subseteq Y^S \subseteq \Re^S$. Particular use is made of the unit vector $\mathbf{1} = (1, 1, ...1)$. For any $\mathbf{y}$, define the increasing rearrangement of $\mathbf{y}$ as the permutation $\mathbf{y}_{[]}$ of Ω, such that $\mathbf{y}_{[1]} \leq \mathbf{y}_{[2]} \leq ... \leq \mathbf{y}_{[S]}$ with ties broken arbitrarily (for example, by preserving the original ordering of Ω).

Preferences over $\mathbf{Y}$ are given by a total ordering with an associated indifference ordering that may be represented by a strictly increasing and continuous function $W : \mathbf{Y} \rightarrow \Re$. W can also be represented in terms of its certainty equivalent, $e : \mathbf{Y} \rightarrow \Re$:

$$e(\mathbf{y}) = \sup\{c : W(c\mathbf{1}) \leq W(\mathbf{y})\}.$$

The certainty equivalent is the canonical cardinal representation of preferences upon which we rely in the following. The certainty equivalent is a complete function representation of preferences. The certainty equivalent always maps points on the equal-incomes vector, $c\mathbf{1}$, into c, that is,

$$e(c\mathbf{1}) = c, \quad c \in \Re.$$

Following Yaari (1969) and Quiggin and Chambers (1998), we define probabilities and risk aversion simultaneously.

Definition A decision-maker is *weakly risk-averse* if there exists a vector $\pi \in \Re_+^S$, with $\Sigma \pi_s = 1$ and

$$e(\mathbf{y}) \leq \mu(\mathbf{y}), \quad \forall\ \mathbf{y}$$

where $\mu(\mathbf{y}) = \sum_{s \in \Omega} \pi_s y_s$. A decision maker is *risk-neutral* if equality holds for every $\mathbf{y}$ and *risk-averse* otherwise. A decision maker is *strictly risk-averse* if strict inequality holds whenever $\mathbf{y} \neq \mu(\mathbf{y})\mathbf{1}$.

As Yaari and Quiggin and Chambers observe, the vector $\boldsymbol{\pi}$ need not be unique. More properly, therefore, both the mean and the risk premium are functions of the specific probability vector chosen. The analysis in this paper does not rely on the uniqueness of $\boldsymbol{\pi}$. Hence, in general, the risk premium and definitions derived from it will depend on the choice of $\boldsymbol{\pi}$. For notational simplicity, this functional dependence is not recognized. Since we do not consider changes in $\boldsymbol{\pi}$, there is no loss of generality.

2.0.1 Examples

(i) Expected utility. For a strictly monotone-increasing utility function $u : Y \to \Re$

$$e(\mathbf{y}) = u^{-1}\left(\sum \pi_s y_s\right)$$

(ii) The dual model of Yaari (1987). For a strictly increasing probability transformation $q : [0, 1] \to [0, 1]$

$$e(\mathbf{y}) = \sum h_{[s]} y_{[s]}$$

where

$$h_{[s]} = q\left(\sum_{t=1}^{s} \pi_{[t]}\right) - q\left(\sum_{t=1}^{s-1} \pi_{[t]}\right)$$

is a probability weight.

(iii) Rank-dependent expected utility

$$e(\mathbf{y}) = u^{-1}\left(\sum h_{[s]} u\left(y_{[s]}\right)\right).$$

(iv) Mean-standard deviation preferences

$$e(\mathbf{y}) = \phi(\mu(\mathbf{y}), \sigma(\mathbf{y}))$$

where ϕ is increasing in its first argument and decreasing in its second and $\sigma(\mathbf{y})$ denotes the standard deviation.

2.1 Risk orderings

The fundamental building block for a notion of risk aversion, which goes beyond a strict preference for certainty, is a definition of what it means for one state-contingent income vector to be 'riskier' than another. Many different definitions of the statement '$\mathbf{y}'$ is riskier than $\mathbf{y}$' have been offered. No one definition is best for all purposes. In this section, we develop the idea of a risk ordering and show how invariance properties of the risk ordering relate to properties of preferences and risk premiums.

We begin by confining attention to pairs $(\mathbf{y}, \mathbf{y}')$ with the same mean, so that $\mu(\mathbf{y}) = \mu(\mathbf{y}')$. A *risk ordering* is a partial order $\preceq$ on $\mathbf{Y}$, confined to pairs with the same mean, and satisfying

(i) transitivity $\mathbf{y} \preceq \mathbf{y}'$ and $\mathbf{y}' \preceq \mathbf{y}'' \Longrightarrow \mathbf{y} \preceq \mathbf{y}''$;

(ii) reflexivity $\mathbf{y} \preceq \mathbf{y}$, $\mathbf{y} \in \mathbf{Y}$; and

(iii) risk-aversion $\mu(\mathbf{y})\mathbf{1} \preceq \mathbf{y}, \qquad \mathbf{y} \in \mathbf{Y}$.

In words, we say that $\mathbf{y}'$ *is derived from* $\mathbf{y}$ *by a mean-preserving increase in risk whenever* $\mathbf{y} \preceq \mathbf{y}'$. If $\mathbf{y} \preceq \mathbf{y}' \Rightarrow e(\mathbf{y}) \geq e(\mathbf{y}')$, so that a mean-preserving increase in risk always reduces the certainty equivalent, we say that e *is risk-averse with respect to* $\preceq$ *and that* $\preceq$ *is consistent with* e.

2.1.1 Examples

Some examples may prove useful in what follows. They are arranged in increasing order of the number of pairs $(\mathbf{y}, \mathbf{y}')$ ordered in terms of increasing risk.

(i) The minimal risk ordering consistent with risk aversion, requiring that receipt of mean income with certainty is preferred to the corresponding risky state-contingent income vector is denoted $\preceq_0$. The only risk-ordering relationships implied by $\preceq_0$ are of the form $\mu(\mathbf{y})\mathbf{1} \preceq_0 \mathbf{y}$.

(ii) Given quasi-concavity, $\preceq_0$ implies the stronger partial ordering $\preceq_1$,described by

$$\lambda \mathbf{y} + (1-\lambda)\mu(\mathbf{y})\mathbf{1} \preceq_1 \ \mathbf{y}, 0 \leq \lambda \leq 1\}$$

and stated as $\mathbf{y}'$ *is derived from* $\mathbf{y}$ *by a multiplicative spread about the mean.*

(iii) Suppose that $\mathbf{y}$, $\mathbf{y}'$ and $\varepsilon = \mathbf{y} - \mathbf{y}'$ are comonotonic, in the sense that $(y_s - y_t)(y'_s - y'_t) \geq 0$ $\forall s, t$, and that $\mu(\varepsilon) = 0$. Then we write $\mathbf{y} \preceq_m \mathbf{y}'$, stated as $\mathbf{y}'$ *is derived from* $\mathbf{y}$ *by a monotone spread* (Quiggin 1991; Chateauneuf and Cohen 1991, Chateauneuf, Cohen and Meilijson 1997). Observe that if $\mathbf{y} \preceq_1 \mathbf{y}'$, $\mathbf{y} - \mathbf{y}' = (1-\lambda)(\mathbf{y} - \mu(\mathbf{y})\mathbf{1})$ which satisfies the stated conditions.

(iv) We will use the notation $\mathbf{y} \preceq_\pi \mathbf{y}'$ to mean '$\mathbf{y}'$ is derived from $\mathbf{y}$ by a mean-preserving increase in risk in the sense of Rothschild and Stiglitz (1970).[1] Quiggin (1991) shows that $\mathbf{y} \preceq_m \mathbf{y}' \Rightarrow \mathbf{y} \preceq_\pi \mathbf{y}'$, but the reverse implication does not hold

(v) Let σ be the standard deviation operator

$$\sigma(\mathbf{y}) = \sqrt{(\mathbf{y} - \mu(\mathbf{y})\mathbf{1})'(\mathbf{y} - \mu(\mathbf{y})\mathbf{1})}.$$

We write $\mathbf{y} \preceq_\sigma \mathbf{y}'$ if $\sigma(\mathbf{y}) \preceq \sigma(\mathbf{y}')$. The results of Rothschild and Stiglitz show that $\mathbf{y} \preceq_\pi \mathbf{y}' \Rightarrow \mathbf{y} \preceq_\sigma \mathbf{y}'$, but the reverse implication does not hold.

The stronger is the risk ordering $\preceq$, the more restrictive is the class of preferences consistent with $\preceq$. All risk-averse preferences are consistent with $\preceq_0$. All quasi-concave risk-averse preferences are consistent with $\preceq_1$. All risk-averse expected-utility preferences (those with a concave utility function) are consistent with $\preceq_\pi$. More generally, suitably smooth preferences are consistent with $\preceq_\pi$ if and only if all local utility functions are concave (Machina 1982). Only mean-standard deviation preferences, including expected utility with a quadratic utility function, are consistent with $\preceq_\sigma$. As noted, for expected utility preferences, consistency with any of (i)–(iv) implies consistency with all. This is not true more generally. In particular, consider the dual model of Yaari (1987). For this model, (i)-(iii) are equivalent to each other and to the requirement that the probability transformation function should lie above the identity, that is, overweight bad outcomes (Quiggin 1991). Consistency with $\preceq_\pi$ is equivalent to the stronger requirement that the probability transformation function should be concave (Chew, Karni and Safra 1987).

[1] See also Hadar and Russell (1969) and Hanoch and Levy (1969).

2.2 Risk compensation measures and risk premiums

Consider any continuous $r : \mathbf{Y} \times \Re \rightarrow \mathbf{Y}$ strictly increasing in its first argument and strictly decreasing in its second, and such that

$$\mu(\mathbf{y}) = \mu(\mathbf{y}') \Leftrightarrow \mu(r(\mathbf{y},\theta)) = \mu(r(\mathbf{y}',\theta)) \quad \forall\theta$$

Note that it is important to specify the domain $\mathbf{Y}$ correctly, and in particular, that the specification of r must ensure that $\mathbf{Y} \times \Re$ is mapped into $\mathbf{Y}$. In some cases, the second argument may be restricted to a subset of $\Re$, such as $\Re_+$.

For any r, define $\hat{r} : \mathbf{Y} \times \Re \rightarrow \mathbf{Y}$

$$\hat{r}(\mathbf{y},e) = sup\{\theta : e(r(\mathbf{y},\theta)) \geq e\}.$$

Any function $\hat{r}$ derived from r in this fashion is described as a *risk compensation measure.*

We have:

(a)$\hat{r}(\mathbf{y},e)$ is nondecreasing in $\mathbf{y}$ and nonincreasing in e; and

b) $\hat{r}(\mathbf{y},e)$ is jointly continuous in $\mathbf{y}$ and e in the interior of the region $\mathbf{Y} \times \Re$ where $\hat{r}(\mathbf{y},w;r)$ is finite

Proof: All proofs are in the Appendix

A special case of the risk compensation measure $\hat{r}$ is the risk premium

$$v(\mathbf{y};\hat{r}) = \hat{r}(\mu(\mathbf{y})\mathbf{1},e(\mathbf{y}))$$

More generally, when $\mathbf{y} \preceq \mathbf{y}'$, the *conditional risk premium* for $\mathbf{y}'$ relative to $\mathbf{y}$ may be expressed as $v : \mathbf{Y} \times \mathbf{Y} \rightarrow \Re$

$$v(\mathbf{y}', \mathbf{y}; \hat{\mathbf{r}}) = \hat{r}(\mathbf{y},e(\mathbf{y}')).$$

Finally, it is useful to define

$$\hat{\mathbf{y}}(\mathbf{y},e) = r(\mathbf{y},\hat{r}(\mathbf{y},e))$$

Examples A number of examples may be derived from the literatures on consumer choice under certainty and on income distribution. For any $\mathbf{Y} \subseteq \Re^S$ satisfying free disposal, the *translation function* (Blackorby and Donaldson 1980) is defined as

$$B(\mathbf{y},e) = \max\{\beta \in \Re : e(\mathbf{y} - \beta\mathbf{1}) \geq e\}$$

if $e(\mathbf{y}-\beta\mathbf{1}) \geq e$ for some β and $-\infty$ otherwise . By a slight abuse of notation, we will also refer to the function $r(\mathbf{y},\beta) = \mathbf{y}-\beta\mathbf{1}$ as a translation. Setting $\hat{r}$ equal to B in the definition of the risk premium v, the translation function gives rise to the *absolute risk premium*:

$$\begin{aligned} v(\mathbf{y};B) &= B\left(\mu\left(\mathbf{y}\right)\mathbf{1},e\left(\mathbf{y}\right)\right) \\ &= \mu\left(\mathbf{y}\right)-e\left(\mathbf{y}\right). \end{aligned}$$

Quiggin and Chambers (1998) show that this measure corresponds to the ordinary notion of the absolute risk premium for expected utility theory.

The translation function is a special case of the *benefit function* (Luenberger 1992)

$$B(\mathbf{y},e,\mathbf{g}) = \max\{\beta \in \Re : e(\mathbf{y}-\beta\mathbf{g}) \geq e\}$$

Using the benefit function, we can define, for an arbitrary direction $\mathbf{g}$ and an appropriate choice of $\mathbf{Y}$, the risk compensation measure

$$\hat{r}_{\mathbf{g}}(\mathbf{y},e) = \max\{\beta \in \Re : W(\mathbf{y}-\beta\mathbf{g}) \geq e\}.$$

For any cone $\mathbf{Y} \subseteq \Re^S_{++}$, the (Shephard 1953, Malmquist 1953) *distance function* $D : \mathbf{Y}\times\Re \rightarrow \Re_+$ is defined as :

$$D(\mathbf{y},e) = \sup\{\lambda > 0 : e(\mathbf{y}/\lambda) \geq e\} \qquad y \in \Re^S_{++}\ .$$

We will refer to $r(\mathbf{y},\lambda) = \mathbf{y}/\lambda$ as a radial contraction. The distance function gives rise to the *relative risk premium*

$$\begin{aligned} v(\mathbf{y};D) &= D\left(\mu\left(\mathbf{y}\right)\mathbf{1},w\left(\mathbf{y}\right)\right) \\ &= \frac{\mu\left(\mathbf{y}\right)}{e\left(\mathbf{y}\right)} \end{aligned}$$

2.3 Invariance

For a given $r : \mathbf{Y}\times\Re \rightarrow \mathbf{Y}$, preferences are invariant with respect to r if

$$e\left(\mathbf{y}\right) \geq e\left(\mathbf{y}'\right) \Leftrightarrow e\left(r\left(\mathbf{y},\theta\right)\right) \geq e\left(r\left(\mathbf{y}',\theta\right)\right) \quad \forall\theta$$

Quiggin and Chambers (1998) define *constant absolute risk aversion* (CARA) as invariance with respect to translations and *constant relative risk aversion* (CRRA) as invariance with respect to radial contractions. For the case of expected utility, these definitions coincide with the usual definitions due to Arrow (1965) and Pratt (1964).

The requirement that preferences be invariant with respect to r is often more restrictive than is necessary. Strong results can often be obtained if comparisons are confined to equal-mean pairs, in which case preferences are characterized by the risk ordering. We say a risk ordering $\preceq$ is invariant with respect to r if:

$$\mathbf{y} \preceq \mathbf{y}' \Leftrightarrow r(\mathbf{y},\theta) \preceq r(\mathbf{y}',\theta) \quad \forall\theta.$$

Clearly invariance of preferences e implies invariance of any risk ordering consistent with r. The converse is not true, as may be shown by counterexample. It is easy to see that the ordering of random variables by their standard deviation is invariant with respect to translations and radial contractions. However, this risk ordering is consistent with all mean-variance preferences, and mean-variance preferences do not, in general, display either CARA or CRRA.

When a risk ordering $\preceq$ is invariant with respect to some r, we can compare the riskiness of state-contingent income vectors $\mathbf{y}, \mathbf{y}'$ without the requirement of a common mean $\mu(\mathbf{y}) = \mu(\mathbf{y}')$. The function r may be used to map $\mathbf{y}$ into some $\tilde{\mathbf{y}}$ such that $\mu(\tilde{\mathbf{y}}) = \mu(\mathbf{y}')$. The risk ordering $\preceq$ may then be used to rank $\tilde{\mathbf{y}}$ and $\mathbf{y}'$. If $\tilde{\mathbf{y}} \preceq \mathbf{y}'$, we say that $\mathbf{y} \preceq^r \mathbf{y}'$. Invariance of $\preceq$ with respect to r ensures that the resulting risk-ordering has all the standard properties of a partial ordering.

More precisely, we assume that for all $\mathbf{y},\mu'$ there exists a unique $\theta(\mathbf{y},\mu')$ such that

$$\mu(r(\mathbf{y},\theta(\mathbf{y},\mu'))) = \mu'.$$

Let

$$\tilde{\mathbf{y}}(\mu';r) = r\left(\mathbf{y},\theta\left(\mathbf{y},\mu'\right)\right).$$

Then we say that $\mathbf{y}'$ represents an $r-$mean-adjusted increase in risk on $\mathbf{y}$, denoted $\mathbf{y} \preceq^r \mathbf{y}'$ if

$$\tilde{\mathbf{y}}(\mu(\mathbf{y}');r) \preceq \mathbf{y}'.$$

Note that the invariance of $\preceq$ with respect to r and the mean-preserving property of r imply the converse property

$$\mathbf{y} \preceq \tilde{\mathbf{y}}'(\mu(\mathbf{y});r).$$

If $\mathbf{y} \preceq^r \mathbf{y}'$ and $e(\mathbf{y}') = e(\mathbf{y})$, we say that $\mathbf{y}'$ represents an $r-$compensated increase in risk on $\mathbf{y}$. In particular, in the case where r is a translation, we say that $\mathbf{y}'$ represents an absolutely compensated increase in risk on $\mathbf{y}$, denoted $\mathbf{y} \preceq^a \mathbf{y}'$.

The restriction to appropriately invariant risk-orderings is essential. If $\preceq$ is not invariant with respect to r, there must exist $\mathbf{y}, \mathbf{y}', \theta > 0$ such that $\mathbf{y} \preceq \mathbf{y}'$, $r(\mathbf{y}', \theta) \preceq r(\mathbf{y}, \theta)$. It is then possible to find $\mathbf{y}''$ such that $\mathbf{y}''$ represents an $r-$mean-adjusted increase in risk on $\mathbf{y}$, but also such that $\mathbf{y}$ represents an $r-$mean-adjusted increase in risk on $\mathbf{y}''$.

3 Comparative risk aversion

To any risk ordering $\preceq$ and risk compensation measure $\hat{r}$ there corresponds a notion of comparative risk aversion.[2] More precisely, for any certainty-equivalent functions e^1, e^2, risk-averse with respect to common subjective probabilities $\boldsymbol{\pi}$, we say that e^1 is *more risk-averse* than e^2 with respect to $\preceq, \hat{r}$ if for all $\mathbf{y}', \mathbf{y}$, with $\mathbf{y} \preceq \mathbf{y}'$,

$$v^1(\mathbf{y}', \mathbf{y}; \hat{r}) \geq v^2(\mathbf{y}', \mathbf{y}; \hat{r})$$

that is, if e^1 has higher conditional risk premiums for $\preceq, \hat{r}$ than e^2. When the risk compensation measure r is a translation (radial contraction), we say that e^1 is absolutely (relatively) more risk-averse than e^2 with respect to $\preceq, \hat{r}$.

It is apparent that, if e^1 is more risk-averse than e^2 with respect to $\preceq, \hat{r}$, then, whenever $\mathbf{y}'$ is a $r-$ compensated increase in risk on $\mathbf{y}$ for e^1 with respect to $\preceq$, $e^2(\mathbf{y}') \geq e^2(\mathbf{y})$. Conversely, whenever $\mathbf{y}'$ is an $r-$compensated increase in risk on $\mathbf{y}$ for e^2 with respect to $\preceq$, $e^1(\mathbf{y}) \geq e^1(\mathbf{y}')$. The stronger is the risk-ordering $\preceq$, the more restrictive are the requirements to determine that e^1 is more risk-averse than e^2. More precisely, we have

Result 1 Let $\preceq$ and $\preceq'$ be risk-orderings such that :

(i) for all $\mathbf{y}, \mathbf{y}^o$, $\mathbf{y} \preceq \mathbf{y}^o \Rightarrow \mathbf{y} \preceq' \mathbf{y}^o$.

Then for any preference orderings e^1 , e^2, e^1 is more risk-averse than e^2 with respect to $\preceq', r \Rightarrow e^1$ is more risk-averse than e^2 with respect to $\preceq, r$

Condition (i) in Result 1 is stated as 'the risk-ordering $\preceq'$ is stronger than $\preceq$'.

The standard Arrow–Pratt interpretation of the statement 'e^1 is *more risk-averse* than e^2' is obtained for the case when $\preceq$ is the risk-ordering $\preceq_0$ and r is the translation function B, so that e^1 has higher absolute risk premiums than e^2. It is easy to see that the same definition is obtained

[2] For example, Diamond and Stiglitz explore the case of comparative risk aversion for the ordering $\preceq_\pi$ and the absolute risk compensation measure.

when the criterion is expressed in terms of radial contractions, leading to the requirement that e^1 should have higher relative risk premiums than e^2. This equivalence arises because, in both cases the premium is calculated with respect to $e(\mathbf{y})\mathbf{1}$, that is, $\hat{\mathbf{y}}(\mu(\mathbf{y})\mathbf{1},e(\mathbf{y})) = e(\mathbf{y})\mathbf{1}$.

Some further results may be obtained under specific assumptions about the structure of preferences. In particular, the results of Arrow and Pratt show that e^1 is more risk-averse than e^2 for $\preceq_0$ if and only if u^1 is a concave transform of u^2. Segal (1987)shows that, in the case of Yaari dual-model functionals , e^1 is more risk-averse than e^2 for $\preceq_0$ if and only if $q^1(p) \geq q^2(p) \quad \forall p$. These results may be extended as follows:

Lemma 2 Suppose e^1 is more risk-averse than e^2 for $\preceq_0$and B. Then

(i) If $e^1$and e^2 are expected-utility functionals displaying CARA, e^1 is more risk-averse than e^2 for $\preceq_m$and B.

(ii) If $e^1$and e^2 are Yaari dual-model functionals , e^1 is more risk-averse than e^2 for $\preceq_m$and B.

As will be shown below, the content of Lemma 2 plays a crucial, but normally implicit, role in the derivation of a wide range of comparative static results.

3.1 Comparative risk aversion at different starting points

As well as comparing different preference orderings, risk aversion may be compared at different initial state-contingent income vectors. Suppose r is such that $r(\mathbf{y},0) = \mathbf{y} \quad \forall \mathbf{y}$. We say that preferences e are more risk-averse at $\mathbf{y}$ than at $\mathbf{y}'$ if for all $\theta \geq 0$

$$\hat{r}(r(\mathbf{y},\theta);e(\mathbf{y})) \leq \hat{r}(r(\mathbf{y}',\theta);e(\mathbf{y}')).$$

A global characterization is available in the case of risk-orderings invariant to the chosen r. Suppose that e is a certainty-equivalent, that $\preceq$ is a risk-ordering consistent with e, and let r define a risk compensation measure such that $\preceq$ is invariant with respect to r. Suppose r is such that $r(\mathbf{y},0) = \mathbf{y} \quad \forall \mathbf{y}$. The certainty-equivalent function e displays *decreasing risk-aversion* with respect to $r, \preceq$ if, for all $\mathbf{y} \preceq^r \mathbf{y}',\theta \geq 0$

$$\hat{r}(\mathbf{y}';e(\mathbf{y})) \geq \hat{r}(r(\mathbf{y}',\theta),e(r(\mathbf{y},\theta))).$$

Similarly, e displays *increasing risk-aversion* with respect to $r, \preceq$ if, for all $\mathbf{y} \preceq^r \mathbf{y}',\theta \geq 0$

$$\hat{r}(\mathbf{y}';e(\mathbf{y})) \leq \hat{r}(r(\mathbf{y}',\theta),e(r(\mathbf{y},\theta))).$$

Notice, in particular, that the definition of decreasing (or increasing) risk-aversion depends on which risk ordering is being used. The stronger the concept of comparative risk aversion, the more restrictive the requirements for decreasing risk-aversion.

As was noted above, invariance of preferences is a stronger condition that the invariance of the risk ordering required by these definitions. Indeed, we have:

Lemma 3 Preferences are invariant with respect to r if and only if they display both decreasing and increasing risk aversion with respect to $r, \preceq$ for any $\preceq$ consistent with e

As above when r is a translation, we refer to decreasing (increasing) absolute risk aversion (DARA, IRRA). DARA (IRRA) therefore requires that, for $\mathbf{y} \preceq^r \mathbf{y}'$

$$B\left(\mathbf{y}',e\left(\mathbf{y}\right),\mathbf{1}\right) \geq (\leq) \, B\left(\mathbf{y}'+\delta\mathbf{1},e\left(\mathbf{y}+\delta\mathbf{1}\right),\mathbf{1}\right) \quad \delta \geq 0.$$

Hence, Lemma 2 implies, as expected that preferences display CARA if and only if they display DARA and IRRA. Similarly,when r is a radial contraction we refer to decreasing relative risk aversion (DRRA), and require

$$D\left(\mathbf{y}',e\left(\mathbf{y}\right)\right) \geq (\leq) \, D\left(t\mathbf{y}',e\left(t\mathbf{y}\right)\right) \quad t \geq 1.$$

When applied to $\preceq_0$, our definition gives rise to the usual characterization of decreasing absolute risk aversion in terms of the requirement that the absolute risk premium for a risky prospect should decline as a result of an increase in base wealth. Using Lemma 2, it is possible to extend this characterization to more general risk orderings, including the monotone spread ordering $\preceq_m$.

Result 2 For expected-utility preferences, DARA with respect to $\preceq_0$ implies DARA with respect to $\preceq_1$and $\preceq_m$.

This analysis may be extended to the case of rank-dependent expected utility preferences using arguments similar to those of Quiggin (1991). We begin by observing that Yaari preferences display CARA with respect to any risk-ordering that ranks only comonotonic vectors $\mathbf{y}, \mathbf{y}'$, that is those for which, for any s, t,

$$\left(y_s - y_t\right)\left(y_s' - y_t'\right) \geq 0.$$

This is because, holding the ordering of states constant, Yaari preferences are equivalent to risk-neutral preferences taken with respect to a transformed probability distribution

$$e\left(\mathbf{y}\right) = \sum \tilde{\pi}_s y_s,$$

which is a CARA certainty equivalent. This observation leads directly to

Result 3 For rank-dependent expected-utility preferences, DARA with respect to $\preceq_0$ implies DARA with respect to $\preceq_1$and $\preceq_m$.

However, even in the special case of expected utility, stronger risk orderings may give rise to different notions of decreasing absolute risk aversion. The results of Ross (1981) imply that,under expected utility,, DARA with respect to $\preceq_0$ does not imply DARA with respect to the Rothschild–Stiglitz ordering $\preceq_\pi$.

4 Comparative-static problems

A wide range of comparative static problems involving choice under uncertainty have been analyzed. Any problem involving comparisons of risk aversion between individuals must have the same basic form. To see the nature of this problem, consider a choice problem in which a decision-maker risk-averse with respect to $\preceq$ must choose an element of a choice set $Y \subseteq \mathbf{Y}$. An element $\mathbf{y} \in Y$ is described as absolutely undominated for $\preceq$ if there exists no $\mathbf{y}'$ such that $\mathbf{y}' \preceq^a \mathbf{y}$ and $\mu[\ \mathbf{y}] \leq \mu[\mathbf{y}']$ (with at least one of these inequalities strict). A choice problem is an *absolute risk–return trade-off for* $\preceq$, if, for any undominated $\mathbf{y}$, $\mathbf{y}' \in Y$, either $\mathbf{y}' \preceq^a \mathbf{y}$ (implying that $\mu[\mathbf{y}] \leq \mu[\mathbf{y}']$) or $\mathbf{y} \preceq^a \mathbf{y}'$ (implying that $\mu[\mathbf{y}] \geq \mu[\mathbf{y}']$). The choices between $\mathbf{y}$ and $\mathbf{y}'$ will depend on risk attitudes.

This definition may be generalized to arbitrary risk compensation measures r. An element $\mathbf{y} \in Y$ is described as undominated for $\preceq^r$ if there exists no $\mathbf{y}'$ such that $\mathbf{y}' \preceq {}^r\mathbf{y}$ and $\theta\left(\mathbf{y},\mu\left(\mathbf{y}'\right);r\right) \geq 0$ (with at least one of these inequalities strict). A choice problem is a *risk–return trade-off for* $\preceq^r$ if, for any undominated $\mathbf{y}$, $\mathbf{y}' \in Y$, either $\mathbf{y}' \preceq^r \mathbf{y}$ (implying that $\theta\left(\mathbf{y},\mu\left(\mathbf{y}'\right);r\right) \leq 0$) or $\mathbf{y} \preceq^r \mathbf{y}'$ (implying that $\theta\left(\mathbf{y},\mu\left(\mathbf{y}'\right);r\right) \geq 0$).

The foregoing analysis implies that if a choice problem is a risk–return trade off for $\preceq^r$, e^1 is more risk-averse than e^2 with respect to $\preceq$ and r, and

$$\begin{aligned}
\mathbf{y}^1 &\in \arg\max\{e^1(\mathbf{y}) : \mathbf{y} \in Y\} \\
\mathbf{y}^2 &\in \arg\max\{e^2(\mathbf{y}) : \mathbf{y} \in Y\},
\end{aligned}$$

it must be true that $\mathbf{y}^1 \preceq^r \mathbf{y}^2$ and $\mu[\mathbf{y}^2] \geq \mu[\mathbf{y}^1]$. That is, the less risk-averse individual will choose an element with lower mean and less risk than that chosen by the more risk-averse individual.

Conversely, risk–return trade-offs for $\preceq$ are the only choice problems in which, given that e^1 is more risk-averse than e^2 with respect to $\preceq$, the less risk-averse individual will always choose an element with lower mean than that chosen by the more risk-averse individual. Otherwise, taking elements $\mathbf{y}$, $\mathbf{y}' \in Y$ unrelated by $\preceq^r$ we can choose preference orderings e^1 and e^2 risk-averse with respect to $\preceq$ such that e^1 is more risk-averse than e^2 with respect to $\preceq$ and such that

$$\begin{aligned} \mathbf{y} &\in \arg\max\{e^1(\mathbf{y}) : \mathbf{y} \in Y\} \\ \mathbf{y}' &\in \arg\max\{e^2(\mathbf{y}) : \mathbf{y} \in Y\} \\ & \theta\left(\mathbf{y}, \mu\left(\mathbf{y}'\right); r\right) \leq 0. \end{aligned}$$

This point is often obscured by the additional structure that may be obtained through assumptions about the functional form of e, for example the assumption that e is an expected-utility functional satisfying CARA. In this case, comparative risk-aversion may be characterized by the requirement that e^1 should be more risk-averse than e^2 with respect to $\preceq_0$, that is, that risk premiums should be greater for 1 than for 2. No explicit references is generally made to stronger risk orderings such as $\preceq_m$. However, clear-cut results may be obtained for choice problems that are not risk–return trade-offs for $\preceq_0$, but which are (absolute) risk–return trade-offs for $\preceq_m$. This is because, by Lemma 2, the fact that e^1 is more risk-averse than e^2 with respect to $\preceq_0$ implies that e^1 is also more risk-averse than e^2 with respect to $\preceq_m$. By contrast, the same implication does not hold for $\preceq_\pi$, except under restrictive conditions on the utility function. As a result, comparative static results for $\preceq_\pi$, such as those derived by Rothschild and Stiglitz (1971) are more restrictive than for $\preceq_0$ and $\preceq_0$.

Now consider a comparison between choice problems parametrized by an exogenous variable $z \in Z$ and a choice variable $\alpha \in A$ (either of which may be a vector or a scalar) such that an increase in α implies an increase in risk and return in the sense that $\alpha' \geq \alpha \Leftrightarrow \mathbf{y}(\alpha, z) \preceq^r \mathbf{y}(\alpha', z)$. Suppose that, for any given value of z, the choice set Y generated by considering all possible values of α is a risk–return trade-off for $\preceq^r$. Let the optimal value for a given individual, conditional on z, be denoted $\mathbf{y}\left(\alpha^*(z), z\right)$. Now consider a shift from z to z'.

We first consider conditions under which the optimal value of α will remain unchanged following a shift from z to z'. Suppose that there exists θ such that for all α

$$\mathbf{y}\left(\alpha, z'\right) = r\left(\mathbf{y}\left(\alpha, z\right), \theta\right).$$

In this case, we say that the risk–return trade-off is unaffected by the shift from z to z'. Then if preferences are invariant with respect to r, the optimality of $\alpha^*(z)$ means that for any α'

$$e\left(\mathbf{y}\left(\alpha^*\left(z\right),z\right)\right) \geq e\left(\mathbf{y}\left(\alpha',z\right)\right)$$

and therefore

$$\begin{aligned}
e\left(\mathbf{y}\left(\alpha^*\left(z\right),z'\right)\right) &= e\left(r\left(\mathbf{y}\left(\alpha^*\left(z\right),z\right),\theta\right)\right) \\
&\geq e\left(r\left(\mathbf{y}\left(\alpha',z\right),\theta\right)\right) \\
&= e\left(\mathbf{y}\left(\alpha',z'\right)\right)
\end{aligned}$$

so that

$$\alpha^*\left(z\right) = \alpha^*\left(z'\right).$$

The canonical version of this result is that if an individual displays constant absolute risk aversion, changes in wealth will have no effect on their optimal holdings of risky assets.

It is straightforward to extend this result to more general risk attitudes. If $\theta > 0$ and the individual displays decreasing risk aversion with respect to $\preceq^r$, the argument above still applies to any case when $\alpha' \leq \alpha^*\left(z\right)$ and therefore corresponds to a less risky choice than $\alpha^*\left(z\right)$. Hence, the optimal choice must be $\alpha^*\left(z'\right) \geq \alpha^*\left(z\right)$.

We can also address the case when the risk–return trade-off changes as a result of the shift from z to z'. Suppose that

$$\mathbf{y}\left(\alpha^*,z'\right) = r\left(\mathbf{y}\left(\alpha^*,z\right),\theta\right),$$

and consider the sets

$$Y\left(z'\right) = \left\{\mathbf{y}\left(\alpha,z'\right) : \alpha \in A\right\}$$

and

$$Y\left(z,\theta\right) = \left\{r\left(\mathbf{y}\left(\alpha,z\right),\theta\right) : \alpha \in A\right\}.$$

In the case when the risk–return trade-off is unaffected by the shift from z to z', $Y\left(z'\right) = Y\left(z,\theta\right)$, we say that the risk–return trade off is improved by the shift from z to z' if for every $\alpha \geq \alpha^*\left(z\right)$, $r\left(\mathbf{y}\left(\alpha,z\right),\theta\right)$ is dominated by an element $\mathbf{y}\left(\alpha',z'\right) \in Y\left(z'\right)$, and for every $\alpha \leq \alpha^*\left(z\right)$, $\mathbf{y}\left(\alpha,z'\right)$ is dominated by an element $r\left(\mathbf{y}\left(\alpha,z\right),\theta\right) \in Y\left(z,\theta\right)$. In the case where α is a scalar, this condition may be interpreted as saying that the curve $Y\left(z'\right)$ 'cuts $Y\left(z,\theta\right)$ from below.' Under these circumstances, if preferences are invariant with respect to r, the shift from z to z' must lead to an increase in the optimal value of α. A special case arises when, for $\theta\left(\alpha\right)$ increasing in α,

$$\mathbf{y}\left(\alpha^*, z'\right) = r\left(\mathbf{y}\left(\alpha^*, z\right), \theta\left(\alpha\right)\right).$$

Finally, we can relax the assumption that

$$\mathbf{y}\left(\alpha, z'\right) = r\left(\mathbf{y}\left(\alpha, z\right), \theta\right)$$

and suppose instead that for some monotonic $g : A \to A$,

$$\mathbf{y}\left(g\left(\alpha\right), z'\right) = r\left(\mathbf{y}\left(\alpha, z\right), \theta\right).$$

Under the assumptions that preferences are invariant and that the risk–return trade-off is unchanged, we have

$$\alpha^*\left(z'\right) = g\left(\alpha^*\left(z\right)\right)$$

and hence a sufficient condition for the shift from z to z' to lead to an increase in the optimal value of α is that $g\left(\alpha\right) \geq \alpha$, $\forall \alpha \in A$.

Summarizing, we have the following characterization of comparative static problems involving risk:

Result 4 Consider any choice problem of the form

$$\alpha^*\left(z\right) = \arg max_{\alpha}\left\{e\left(\mathbf{y}\left(\alpha, z\right)\right)\right\}$$

which is, for every z, a risk–return trade off for $\preceq^r$. For any comparative static shift from z to z' such that

$$\mathbf{y}\left(g\left(\alpha\right), z'\right) = r\left(\mathbf{y}\left(\alpha, z\right), \theta\right) \forall \alpha \in A.$$

the following are joint sufficient conditions to ensure $\alpha^*\left(z'\right) \geq \alpha^*\left(z\right)$:

(i) Preferences display constant or decreasing risk-aversion;

(ii) The risk–return trade off is unaffected or improved; and

(iii) $g\left(\alpha\right) \geq \alpha$

If any of the inequalities implied by (i)-(iii) is strict, $\alpha^*\left(z'\right) > \alpha^*\left(z\right)$.

4.1 Example: The portfolio problem

The classic example of a comparative static problem under uncertainty is that of the portfolio problem with one safe asset and one risky asset. Let M represent initial wealth, α the amount of the risky asset held, $\mathbf{r} \in \Re^S$ the vector of state-contingent returns on the risky asset and p the price of the asset. Thus $\mathbf{z} = (M, \mathbf{r}, p)$ is the vector of exogenous parameters. Define the vector of state-contingent returns contingent upon the choice of α by:

$$\mathbf{y}(\alpha, \mathbf{z}) = \alpha\mathbf{r} + (M - p\alpha)\mathbf{1}.$$

The individual's choice problem is:

$$\max_{\alpha}\{e\left(\mathbf{y}(\alpha, \mathbf{z})\right).$$

Note that an interior optimum will exist only if $p < \mu\left(\mathbf{r}\right)$, since otherwise a risk-averse investor will always set $\alpha = 0$.

This problem is an absolute risk–return trade-off for $\preceq_1$. Consider first the change from $\mathbf{z}$ to $\mathbf{z}'$ in which M is increased to $M + \theta$. Then

$$\mathbf{y}(\alpha, \mathbf{z}') = \mathbf{y}(\alpha, \mathbf{z}) + \theta\mathbf{1} \quad \forall\alpha \in A$$

and the risk–return trade-off is unchanged. Hence, provided preferences display DARA with respect to $\preceq_1$, Result 4 shows that $\alpha^*\left(\mathbf{z}'\right) \geq \alpha^*\left(\mathbf{z}\right)$.

Next, consider a reduction in the price of the risky asset from p to $p - \delta$. Then

$$\mathbf{y}(\alpha, \mathbf{z}') = \mathbf{y}(\alpha, \mathbf{z}) + \alpha\delta\mathbf{1}$$

and the risk–return trade-off is improved relative to

$$\theta = \alpha^*\left(\mathbf{z}\right)\delta \quad \forall\alpha \in A.$$

Hence, assuming DARA, Result 4 shows that $\alpha^*\left(\mathbf{z}'\right) \geq \alpha^*\left(\mathbf{z}\right)$. The same argument applies if the return vector $\mathbf{r}$ is increased to $\mathbf{r}+\delta\mathbf{1}$. Finally, consider the case of a multiplicative reduction in the riskiness of $\mathbf{r}$, so that

$$\mathbf{r}' = \lambda\mathbf{r} + (1 - \lambda)\,\mu\left(\mathbf{r}\right)\mathbf{1} \quad 0 < \lambda < 1.$$

Letting

$$g\left(\alpha\right) = \frac{\alpha}{\lambda},$$

$$
\begin{aligned}
\mathbf{y}\left(g\left(\alpha\right),\mathbf{z}'\right) &= \alpha\mathbf{r}+\alpha\frac{(1-\lambda)}{\lambda}\mu\left(\mathbf{r}\right)\mathbf{1}-\left(M-p\frac{\alpha}{\lambda}\right)\mathbf{1} \quad \forall\alpha\in A \\
&= \mathbf{y}(\alpha,\mathbf{z})+\alpha\frac{(1-\lambda)}{\lambda}\left(\mu\left(\mathbf{r}\right)-p\right)\mathbf{1},
\end{aligned}
$$

which yields an improved risk–return trade-off. Hence, assuming DARA, $\alpha^*\left(\mathbf{z}'\right) \geq \alpha^*\left(\mathbf{z}\right)$. To summarize:

Result 5 . If e displays DARA with respect to $\preceq_1$then any of

(i) an increase in initial wealth;

(ii) a increase in the mean rate of return;

(iii) a reduction in price; and

(iv) a multiplicative reduction in the riskiness of $\mathbf{r}$

all lead to an increase in the optimal level of α.

Using Lemma 2 and the discussion of DARA presented above , we may derive the following corollaries to Result 5:

Corollary 5.1 (Sandmo 1971, Coes 1977, Feder 1977) For expected-utility preferences if the utility function displays DARA with respect to $\preceq_0$, (i) through (iv) in Result 5 all lead to an increase in the optimal level of α.

Corollary 5.2 (Quiggin 1991) For rank-dependent expected utility preferences if the utility function displays DARA with respect to $\preceq_0$, (i) through (iv) in Result 5 all lead to an increase in the optimal level of α.

Corollary 5.3 (Meyer, Sinn) For mean-variance preferences, $e(\mathbf{y}) = f\left(\mu(\mathbf{y}), \sigma(\mathbf{y})\right)$, if

$$
f_{12}\left(\mu(\mathbf{y}), \sigma(\mathbf{y})\right) \geq 0 \quad \forall\mathbf{y},
$$

(i) through (iv) in Result 5 all lead to an increase in the optimal level of α.

The ease with which these results are derived contrasts with the complexity of the original proofs of Corollaries 5.1 and 5.2. In the original proofs, the crucial role of Lemma 2 was not recognised. Instead, the results were derived directly from the characterization of DARA with respect to $\preceq_0$ in terms of the derivatives of the utility function.

To illustrate the flexibility of the approach adopted here, suppose that there is no safe asset, but instead that one asset is a government bond with a payoff distribution $\mathbf{g} \neq c\mathbf{1}$. The stochastic nature of the payoff might result from unanticipated inflation or from the possibility of repudiation. Suppose that the other asset is relatively risky in the sense that for some risk-ordering $\preceq$, and the natural compensation measure $r = \hat{r}_{\mathbf{g}}$, $\alpha \leq \alpha' \Rightarrow \mathbf{y}(\alpha, \mathbf{z}) \preceq^r \mathbf{y}(\alpha, \mathbf{z}')$. Then the preceding analysis applies with the unit vector $\mathbf{1}$ replaced everywhere by $\mathbf{g}$.

A large number of comparative static analyses have involved stochastic functions of the form $f : A \times \Omega \times Z \to Y$, where f is supermodular in $\alpha \in A$ and $s \in \Omega$. In the differentiable case when Ω is an interval and f is differentiable, this is equivalent to the requirement that $f_{\alpha s} \geq 0$. Examples include problems of the firm under uncertainty (Milgrom 1994) and self-protection against environmental hazards (Lewis and Nickerson 1989). The assumption that $f_{\alpha s} \geq 0$ implies that, for given z and for $\alpha' \geq \alpha$, the random variables $f(\alpha), f(\alpha')$ satisfy

$$f(\alpha) \preceq_m^a f(\alpha').$$

Hence, under DARA, comparative static properties analogous to those of Result 5 apply to this problem. Moreover, as noted for the portfolio problem, the absolute risk premium can be replaced by a risk premium derived from an alternative compensation measure r. In the firm problem, for example, this would be equivalent to making the cost of production stochastic rather than deterministic.

4.2 Concluding comments

The expected-utility model has been one of the most fruitful innovations in the history of economics. A vast range of issues from the theory of insurance and futures markets to the problems of principal–agent relationships has been examined using the model. Yet there is considerable evidence that individual decisions are not, in general, consistent with expected-utility theory. Results derived on the basis of the expected-utility hypothesis are, therefore, open to question. In this paper, it has been shown that tools for comparisons of risk and of risk aversion can be developed without relying on the expected-utility hypothesis, and used to derive extensions of the standard comparative-static results for the expected-utility model. A valuable by-product has been an improved understanding of the role of concepts like decreasing absolute risk aversion in the expected-utility model.

This paper has focused on the duality between notions of increasing risk and notions of comparative risk aversion. The central tool has been the characterization of risk attitudes directly in terms of the benefit function, rather than indirectly in terms of curvature properties of utility and certainty-equivalent functions. Using this characterization it is possible to develop notions of absolute and relative risk aversion, linear risk tolerance and constant risk aversion for general preferences and risk-orderings. A wide range of comparative static results for asset demand and similar problems may then be derived.

5 Appendix:Proof of Results

5.1 Lemma 1

(i) $\mathbf{y} \geq \mathbf{y}' \Rightarrow r(\mathbf{y},\theta) \geq r(\mathbf{y}',\theta) \Rightarrow e(r(\mathbf{y},\theta)) \geq e(r(\mathbf{y}',\theta))$ and the result is proved.

(ii) Follows directly from the continuity of e and r.

5.2 Result 1

Suppose (i) holds, and that e^1 is more risk-averse than e^2 with respect to $\preceq', r$. Now for any $\mathbf{y}, \mathbf{y}^o$, $\mathbf{y} \preceq \mathbf{y}^o \Rightarrow \mathbf{y} \preceq' \mathbf{y}^o$ and therefore

$$v^1(\mathbf{y}^o, \mathbf{y}; \hat{r}) \geq v^2(\mathbf{y}^o, \mathbf{y}; \hat{r}),$$

as required.

5.3 Lemma 2

1(i) Under expected utility theory, for differentiable u

$$\frac{\partial e}{\partial y_s} = \frac{\pi_s u'(y_s)}{\sum_t \pi_t u'(y_t)}.$$

If e^1 is more risk-averse than e^2 for $\preceq_0$, u^1 must be a concave transform of u^2. Hence, in the case of differentiable preferences,

$$\frac{\left(\frac{\partial e^2}{\partial y_s} - \frac{\partial e^1}{\partial y_s}\right)}{\pi_s}$$

is increasing in y_s. For a shift from $\mathbf{y}$ to $\mathbf{y}' = \mathbf{y} + \boldsymbol{\varepsilon}$, with $\boldsymbol{\varepsilon}$ comonotonic with $\mathbf{y}$ and $\mathbf{y}'$, and $\mu(\varepsilon) = 0$, we have

$$\begin{aligned} e^2(\mathbf{y}') - e^1(\mathbf{y}') &\approx e^2(\mathbf{y}) - e^1(\mathbf{y}) + \sum_s \left(\frac{\partial e^2}{\partial y_s} - \frac{\partial e^1}{\partial y_s}\right)\varepsilon_s \\ &= e^2(\mathbf{y}) - e^1(\mathbf{y}) + \sum_s \frac{\left(\frac{\partial e^2}{\partial y_s} - \frac{\partial e^1}{\partial y_s}\right)}{\pi_s}\pi_s\varepsilon_s \end{aligned}$$

$$
\begin{aligned}
&\geq e^2(\mathbf{y}) - e^1(\mathbf{y}) + \mu(\varepsilon)\sum_s \frac{\left(\frac{\partial e^2}{\partial y_s} - \frac{\partial e^1}{\partial y_s}\right)}{\pi_s} \\
&= e^2(\mathbf{y}) - e^1(\mathbf{y}).
\end{aligned}
$$

The extension of the argument to cover the case where u is piecewise differentiable is notationally tedious, but otherwise straightforward.

For 1(ii) consider $\boldsymbol{\varepsilon}_{[]}$, the increasing rearrangement of $\boldsymbol{\varepsilon}$. There exists s, such that for $t \leq s$, $\boldsymbol{\varepsilon}_{[t]} \leq 0$ and for $t > s$, $\boldsymbol{\varepsilon}_{[t]} > 0$. Now if e^1 is more risk-averse than e^2,

$$q^1\left(\sum_{t=1}^{s} \pi_{[s]}\right) \geq q^2\left(\sum_{t=1}^{s} \pi_{[s]}\right)$$

5.4 Lemma 3

If e displays CARA, then for any $\mathbf{y} \preceq \mathbf{y}', \delta \in \Re$

$$v(\mathbf{y}'; \mathbf{y}) = v(\mathbf{y}' + \delta\mathbf{1}; \mathbf{y} + \delta\mathbf{1}).$$

so that DARA and IARA are both satisfied.

For the converse, observe that, since $\preceq$ is a risk ordering,

$$\mu(\mathbf{y})\mathbf{1} \preceq \mathbf{y}.$$

Hence, if e displays both DARA and IARA,

$$
\begin{aligned}
r(\mathbf{y}) &= B(e(\mathbf{y}), \mu(\mathbf{y})\mathbf{1}) \\
&= B(e^{\beta}(\mathbf{y}), (\mu(\mathbf{y}) + \beta)\mathbf{1}) \\
&= B(e(\mathbf{y} + \beta\mathbf{1}), (\mu(\mathbf{y}) + \beta\ )\mathbf{1}) \\
&= r(\mathbf{y} + \beta\mathbf{1}),
\end{aligned}
$$

so that e displays CARA.

5.5 Result 2

Let

$$e^{\delta}(\mathbf{y}) = e(\mathbf{y} + \delta\mathbf{1}) - \delta.$$

Then e displays DARA with respect to $\preceq$ if and only if e^{δ} is absolutely less risk-averse than e with respect to $\preceq$ for every $\delta > 0$. By Lemma 2, assuming expected utility e^{δ} is absolutely less risk-averse than e with respect to $\preceq^0$ if and only if e^δ is absolutely less risk-averse than e with respect to $\preceq_1$and $\preceq_m$.

5.6 Result 3

Proved in text.

5.7 Result 4

Proved in text.

5.8 Result 5

Proved in text.

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