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THE "BOADWAY PARADOX" REVISITED

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Abstract

The "Boadway paradox" identifies potential Pareto improvements in a general equilibrium setting when none exist. It finds the compensating variations (CV's) for a lump-sum redistribution of income have a positive sum. Clearly, this undermines the credibility of a conventional cost-benefit analysis. If it cannot reliably separate income and substitution effects, then it cannot be used to identify potential welfare gains in project evaluation. This paradox occurs when CV's are computed at non-market clearing prices. It is avoided by allowing prices to clear markets in the compensated equilibrium.

1. Introduction

For some time now the "Boadway paradox" has cast doubt on the reliability of compensated welfare measures in a general equilibrium setting.[1] It does this by finding the compensating variations (CV's) for a redistribution of income have a positive sum. In other words, there are potential welfare gains from transferring income between consumers and, at the same time, reversing it with compensating transfers. This undermines the conventional practice of aggregating money costs and benefits across individuals to evaluate projects. When these costs and benefits are measured using CV's they isolate potential Pareto improvements, but if the CV's are unreliable, as is suggested by the paradox, the gains may well be illusory. Indeed, Boadway observes that *"when comparing alternative projects or policies, the one with the largest net gain is not necessarily the "best" one in the compensation sense."* (Boadway, 1974, p. 926)

To see how the paradox arises, consider a redistribution of income between consumers that moves them along a (non-linear) contract curve in a two-person two-good economy. The CV's, when computed at the new relative price, have a positive sum. This occurs because markets do not clear at this price in the compensated equilibrium. However, once the relative price can change, it returns back to its initial market clearing level. And once it does, the CV's sum to zero as confirmation there is no substitution effect. Indeed, this is an application of the second fundamental theorem of welfare economics where the transfers return consumers to their initial indifference curves along the contract curve as prices change endogenously to clear markets.

On most occasions we use hypothetical CV's to isolate potential welfare gains. By measuring them at constant utility they are gains in the sense that winners could compensate losers. And they are not conditional on the compensation being actual rather than hypothetical. To be a true indication that winners can compensate losers, they must be gains that could be realised by the transfers as an actual equilibrium outcome. When these CV's are computed at market clearing prices they separate income and substitution effects in project evaluation. And this is convenient because efficiency effects are objectively agreed welfare measures, while equity effects are not; they depend on subjectively chosen distributional weights. We do not ignore the equity effects, but report them separately so that policy makers and others can make their own assessments of them.

When the CV's are computed at constant prices they are partial equilibrium welfare measures. And they coincide with CV's measured in general equilibrium if there is a single market clearing price ratio along the contract curve, which is the case when consumers have identical and homothetic preferences. Once the relative price changes along the contract curve the two measures do not coincide, and as the "Boadway paradox" demonstrates, partial equilibrium CV's are misleading measures of potential welfare gains in these circumstances.

We begin the formal analysis in the next section by demonstrating the "Boadway paradox" in an endowment economy. We then show how the paradox can be avoided when the relative price is endogenous. The analysis is extended to a tax-distorted economy and to an

[1] Smart (1999) cites this paradox as a problem for the way Peck (1998) measures the inefficiency of a poll tax. Similarly, Johansson (1998) uses it to argue that the compensating variations do not provide a way of escaping the problems arising in cost-benefit analysis from the choice of numeraire identified by Brekke (1997). See Blackorby and Donaldson (1990) for a detailed analysis of the paradox.

economy with production to demonstrate that it applies generally.

2. Welfare Changes in General Equilibrium

Consider the two-by-two endowment economy where individuals a and b consume goods x and y to maximise their respective utilities, U^a and U^b. Social welfare is defined by the function:

$$W = W\left(U^a(x^a, y^a), U^b(x^b, y^b)\right), \tag{4.1}$$

and consumer expenditures are constrained by the market value of their endowments, with:

$$\begin{aligned} p\bar{x}^a + \bar{y}^a + L &= px^a + y^a \quad \text{for a; and,} \\ p\bar{x}^b + \bar{y}^b - L &= px^b + y^b \quad \text{for b.} \end{aligned} \tag{4.2}$$

The bars denote endowments, and L is a lump sum transfer of income from b to a. Good y is chosen as numeraire, so p is the relative price of good x; it is determined in a competitive (Walrasian) equilibrium.[2]

Using the budget constraints in (2), and the first order conditions for price-taking consumers, the change in social welfare for a marginal redistribution of income will be:

$$dW = \beta^a\left((\bar{x}^a - x^a)dp + dL\right) + \beta^b\left((\bar{x}^b - x^b)dp - dL\right) \tag{4.3}$$

where $\beta^i = \dfrac{\partial W}{\partial U^i}\lambda^i$ is the marginal social utility of income for $i \in a, b$.[3]

To compute this change in social welfare we need values for the distributional weights i.e., the β's. Also, it is generally path dependent for a policy that distorts more than one relative price.[4] The conventional solution to both these problems is to compute compensated welfare changes that isolate potential gains (or losses) with CV's to offset the effects of a

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[2] For market clearing we require: $\sum_{i \in a,b} \bar{x}^i = \sum_{i \in a,b} x^i$ and $\sum_{i \in a,b} \bar{y}^i = \sum_{i \in a,b} y^i$.

[3] Equation (3) is obtained by totally differentiating the social welfare function in (1), where:

$$dW = \frac{\partial W}{\partial U^a}\frac{\partial U^a}{\partial x^a}dx^a + \frac{\partial W}{\partial U^a}\frac{\partial U^a}{\partial y^a}dy^a + \frac{\partial W}{\partial U^b}\frac{\partial U^b}{\partial x^b}dx^b + \frac{\partial W}{\partial U^b}\frac{\partial U^b}{\partial y^a}dy^b.$$

Using the first order conditions for price-taking consumers, we have:

$$\partial U^i/\partial x^i = \lambda^i p \text{ and } \partial U^i/\partial y^i = \lambda^i \quad \text{for } i \in a, b,$$

where the change in social welfare becomes:

$$dW = \beta^a(p\,dx^a + dy^a) + \beta^b(p\,dx^b + dy^b).$$

Using the consumer budget constraints, we have:

$$p\,dx^a + dy^a = (\bar{x}^a - x^a)dp + dL \quad \text{and} \quad p\,dx^b + dy^b = (\bar{x}^b - x^b)dp - dL.$$

We obtain (3) by substituting these into the change in social welfare above.

[4] Uncompensated welfare measures are path dependent when the marginal utility of income changes with the relative price. If preferences are homothetic or quasi-linear preferences, the marginal utility of income is constant (for the appropriately chosen numeraire). In general, however, the marginal utility of income changes with relative prices. This is examined in Silberberg (1972).

policy change. For the redistribution of income the CV's are obtained using (3), where:

$$dW = \beta^a\left((\bar{x}^a - \hat{x}^a)d\hat{p} + d\hat{L} - CV^a\right) + \beta^b\left((\bar{x}^b - \hat{x}^b)d\hat{p} - d\hat{L} - CV^b\right) = 0, \tag{4.4}$$

with:

$$\begin{aligned} CV^a &= (\bar{x}^a - \hat{x}^a)d\hat{p} + d\hat{L}; \text{ and,} \\ CV^b &= (\bar{x}^b - \hat{x}^b)d\hat{p} - d\hat{L}. \end{aligned} \tag{4.5}$$

The ^ denotes variables computed in the compensated equilibrium with utilities held constant at the initial levels U_l^i for $i \in a, b$. To demonstrate the paradox in an Edgeworth box diagram for a discrete redistribution of income we write the CV's in (5), as:

$$\begin{aligned} CV^a &= \int(\bar{x}^a - \hat{x}^a)d\hat{p} + \Delta\hat{L}; \text{ and,} \\ CV^b &= \int(\bar{x}^b - \hat{x}^b)d\hat{p} - \Delta\hat{L}. \end{aligned} \tag{4.6}$$

Point I in Figure 1 is the initial equilibrium for the endowment point E_I.

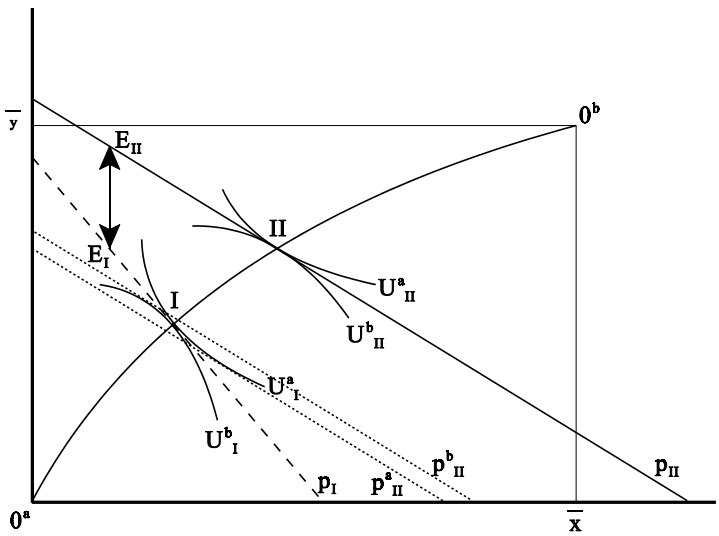

Figure 1

After income is redistributed from b to a by moving the endowment point to E_{II} on price line p_{II}, the competitive equilibrium moves along the contract curve to point II. In fact, any reallocation that moves the new endowment point onto the price line p_{II} achieves the same equilibrium outcome; it is therefore independent of the choice of numeraire.[5] The "Boadway paradox" is isolated by computing the CV's that move each consumer from point II back onto their initial indifference curves at the unchanged relative price p_{II}. At this price there is an excess demand for good x and an excess supply of good y, where this leads to $CV^a > 0$ and $CV^b < 0$, with $CV^a + CV^b > 0$.

[5] Brekke (1997) demonstrates that the numeraire does matter in economies when the marginal rates of substitutions differ across consumers. This is the case when there is a public good and consumers have different Lindahl prices. In competitive equilibrium allocations, and in the absence of externalities, the choice of numeraire is irrelevant to the welfare measures.

Now consider what happens when the compensating transfers are made in general equilibrium. As a's income falls and b's rises the relative price rises endogenously to clear the excess demand for good x and the excess supply of good y. This moves the equilibrium back along the contract curve from point II to point I. When we finally arrive at point I the market clearing price returns to p_I. With $d\hat{p} = 0$, the CV's now sum to zero, and this is confirmed using (6), where:

$$CV^a = \Delta\hat{L} \quad \text{and} \quad CV^b = -\Delta\hat{L}. \tag{4.7}$$

In Figure 1 they are the vertical distance between the two endowment points i.e., the exact reverse of the income redistribution. Equally, they are the horizontal distance between the price lines p_I and p_{II} from point E_I when good x is numeraire.

It might be argued that consumers, when asked to reveal their CV's, will do so at the current relative price p_{II}.[6] But since they observe the relative price change when they move to point II, they will also anticipate the price change when they are returned by the CV's back to point I. In general equilibrium agents anticipate market clearing prices, even when they are price takers themselves, because as traders they operate inside the institutions that set these prices. And by this logic, they will compute their CV's for the redistribution of income using the initial relative price p_I. These CV's would actually return consumers to their initial indifference curves in the compensated equilibrium. In contrast, the CV's measured at the unchanged price p_{II} cannot support an equilibrium because markets do not clear at this price.

2.1 A Tax-Distorted Exchange Economy

We now repeat the analysis with an expenditure tax on good x, where revenue is returned as a lump-sum transfer, with $R = t(\bar{x}^a - x^a)$.[7] After rewriting the budget constraints in (2), we have:

$$\begin{aligned} (p+t)\bar{x}^a + \bar{y}^a + L + R &= (p+t)x^a + y^a \quad \text{for a; and,} \\ p\bar{x}^b + \bar{y}^b - L &= px^b + y^b \quad \text{for b.} \end{aligned} \tag{4.8}$$

For a discrete redistribution of income the CV's are:

$$\begin{aligned} CV^a &= \int \left(\bar{x}^a - \hat{x}^a + t\frac{\partial \hat{x}^a}{\partial p} \right) d\hat{p} + \Delta\hat{L}; \text{ and,} \\ CV^b &= \int \left(\bar{x}^b - \hat{x}^b \right) d\hat{p} - \Delta\hat{L}. \end{aligned} \tag{4.9}$$

Once again, when the compensating transfers are made the relative price must return back to its initial market clearing level. With $d\hat{p} = 0$ the CV's in (9) will sum to zero, and this is confirmation there is no potential gain (or loss) from a policy that does not itself distort the relative price.

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[6] This is somewhat misleading. The CV's are not computed by asking consumers about them. Rather, they are the transfers that would have to made in equilibrium to return consumers to their initial indifference curves.

[7] Notice the tax falls on purchases of good x by the buyer.

2.2 A Production Economy

Finally, we include production in the economy by endowing consumers with two inputs, capital (K) and labour (N), that are used to produce goods x and y with the strictly concave technologies:

$$\text{(4.10)} \qquad \left.\begin{aligned} \tilde{x}^i &= f^i(N_x^i, K_x^i) \\ \tilde{y}^i &= g^i(N_y^i, K_y^i) \end{aligned}\right\} \quad \text{for } i \in a, b$$

Capital and labour are traded in competitive markets at their respective relative prices, r and w, where the budget constraints in (2) become:

$$\text{(4.11)} \qquad p\tilde{x}^i + \tilde{y}^i + w\bar{N}^i + r\bar{K}^i + L^i = px^i + y^i \quad \text{for } i \in a, b.$$ [8]

After totally differentiating these constraints and applying the first order conditions for price-taking consumers and firms, we have:

$$\text{(4.12)} \qquad CV^i = -\int \hat{x}^i d\hat{p} + \int \left(\bar{N}^i - \hat{N}^i\right) d\hat{w} + \int \left(\bar{K}^i - \hat{K}^i\right) d\hat{r} - \Delta\hat{L}^i \quad \text{for } i \in a, b,$$

with: $\Delta\hat{L}^a = -\Delta\hat{L}^b = \Delta\hat{L}$.

The welfare changes for the redistribution of income are illustrated in Figure 2 where production changes around the frontier PPF from 0_b to $0'_b$. The goods market equilibrium is moved from point I on contract curve $0_a 0_b$ to point II on contract curve $0_a 0'_b$.

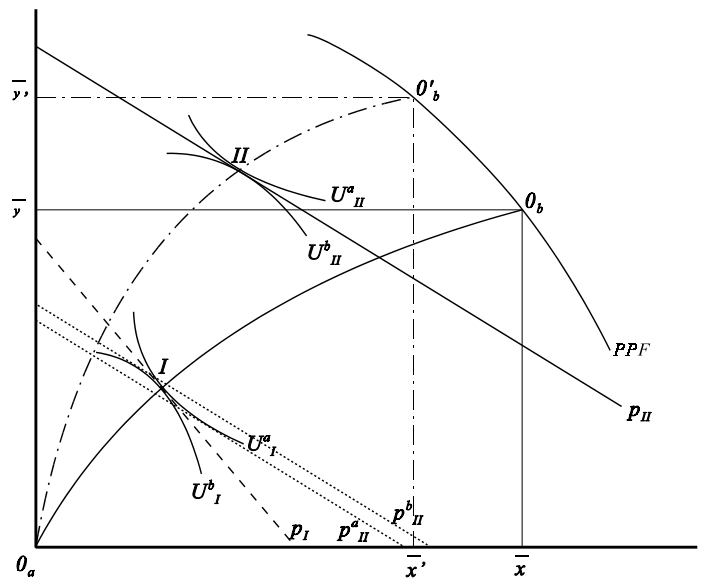

Figure 2

[8] For market clearing in the goods markets we require:

$$\sum_{i \in a,b} p\tilde{x}^i = \sum_{i \in a,b} x^i \text{ and } \sum_{i \in a,b} \tilde{y}^i = \sum_{i \in a,b} y^i,$$

and in the factor markets: $\sum_{i \in a,b} \bar{N}^i = \sum_{i \in a,b} (N_x^i + N_y^i)$ and $\sum_{i \in a,b} \bar{K}^i = \sum_{i \in a,b} (K_x^i + K_y^i)$.

This example is made more complicated by the shifts in the contract curve. When production moves up along the frontier PPF to $0'_b$, b's indifference curves rotate toward the north-west, and this shifts the contract curve to $0_a 0'_b$. As the CV's begin to reverse the income redistribution, consumers do not move along this contract curve. Instead, they move onto lower contract curves as the higher relative price drives production back to 0_b. This price rise clears the excess demand for good x and the excess supply of good y. Eventually the equilibrium returns to point I on the "old" contract curve $0_a 0_b$ where the relative price returns to p_I. It is the unique market clearing price that puts both consumers on their initial indifference curves, U_I^a and U_I^b, respectively. At this equilibrium all the relative prices have returned back to their initial values to clear the goods and input markets. And once again, with $d\hat{p} = d\hat{w} = d\hat{r} = 0$, the CV's in (12) sum to zero. It is straightforward to see that the same result also holds when the trade tax in the previous section is introduced to the production economy.

Another way to see this outcome is to invoke the CV's simultaneously with the income redistribution. This holds the equilibrium at point I with the unchanged price p_I. Now only substitution effects change the market clearing price in the compensated equilibrium. For example, introducing a new tax or changing an existing one, will cause consumers to substitute around their initial indifference curves. Income effects, however, will have no impact on the compensated price.

3. Concluding Remarks

For reasons that are well understood, we are reluctant to use actual dollar changes in utility in project evaluation. Compensated welfare measures are preferred because they are path independent and are free of any income effects. By isolating substitution effects they measure potential Pareto improvements. That is, they isolate gains that could actually be realised using compensating transfers. These gains, when transferred to consumers, become the actual dollar changes in utility. This conventional approach to cost-benefit analysis, however, is undermined in a general equilibrium setting by the "Boadway paradox". It finds potential welfare gains for a lump-sum redistribution of income. And, as Johansson (1998) observes, this leaves us with no reliable way of separating the efficiency and equity effects in project evaluation. We show the paradox is avoided by allowing relative prices to change in the compensated equilibrium. Then the compensated welfare analysis is a reliable measure of potential welfare gains in project evaluation.

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