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A Continued Fractions Approach to a Result of Feit

John P. Robertson and Keith R. Matthews

1. INTRODUCTION. For primes that can be written as a sum of integer squares, $p = a^2 + (2b)^2$, Kaplansky [4] asked whether the binary quadratic form $F = x^2 - py^2$ always represents a and $4b$ (that is, whether there are integer solutions to $x^2 - py^2 = a$ and $x^2 - py^2 = 4b$). Feit [1] and Mollin [4] proved that F does always represent a and $4b$ using the theory of ideals and the class group structure of quadratic orders. In this MONTHLY, Walsh [7] proved a more general result using only elementary methods. He showed that if $D > 1$ is a nonsquare odd integer, $D = a^2 + (2b)^2$, and $x^2 - Dy^2$ represents -1 , then there is a factorization of D into positive integers r and s so that $rx^2 - sy^2$ represents a , and a possibly different factorization so that $rx^2 - sy^2$ represents $4b$.

For any nonsquare positive integer D , odd or even, for which $x^2 - Dy^2$ represents -1 , we use the continued fraction algorithm to generate particular a and b so that $D = a^2 + b^2$, where a is always odd and the parity of b is opposite that of D . We also give explicit solutions to $x^2 - Dy^2 = \pm a$ and $x^2 - Dy^2 = \pm 2b$. This shows that standard continued fraction methods give a more elementary answer to Kaplansky's question than the solutions by Feit and Mollin. While this solution is not as elementary as Walsh's, it always uses the trivial factorization of D . We begin with some background.

2. THE CONTINUED FRACTION ALGORITHM. Any irrational real number ξ can be written as an infinite (simple) continued fraction

$$\xi = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

where $a_0 = \lfloor \xi \rfloor$ (with $\lfloor * \rfloor$ denoting the greatest integer function) and the a_i are positive integers for $i > 0$. For D a positive integer, not a square, the following well-known algorithm computes the continued fraction expansion of $\sqrt{D}$ [6, p. 76] [5, p. 358] [3, p. 251] and some related variables. Let $P_0 = 0$, $Q_0 = 1$, and $a_0 = \lfloor \sqrt{D} \rfloor$. For $i \geq 1$, define

- (A) $P_i = a_{i-1}Q_{i-1} - P_{i-1}$,
- (B) $Q_i = (D - P_i^2)/Q_{i-1}$, and
- (C) $a_i = \lfloor (P_i + \sqrt{D})/Q_i \rfloor$.

Also set $A_{-2} = 0$, $A_{-1} = 1$, $A_i = a_i A_{i-1} + A_{i-2}$ for $i \geq 0$, $B_{-2} = 1$, $B_{-1} = 0$, and $B_i = a_i B_{i-1} + B_{i-2}$ for $i \geq 0$. The a_i are the *partial quotients*, the ratios

$(P_i + \sqrt{D})/Q_i$ are the *complete quotients*, and the A_i/B_i are the *convergents* related to the continued fraction expansion of $\sqrt{D}$.

The sequences $\{P_i\}$, $\{Q_i\}$, and $\{a_i\}$ are periodic; denote the length of the minimal period by ℓ . For the continued fraction expansion of $\sqrt{D}$, for $i, k > 0$, $P_{i+k\ell} = P_i$ and similarly for $\{Q_i\}$ and $\{a_i\}$. As an example, in Table 1 we give the continued fraction expansion of $\sqrt{58}$, which has $\ell = 7$.

Table 1. Continued fraction expansion of $\sqrt{58}$

i	-2	-1	0	1	2	3	4	5	6	7	8
P_i			0	7	2	4	3	4	2	7	7
Q_i			1	9	6	7	7	6	9	1	9
a_i			7	1	1	1	1	1	1	14	1
A_i	0	1	7	8	15	23	38	61	99	1447	1546
B_i	1	0	1	1	2	3	5	8	13	190	203

We will need the following facts:

- (i) $x^2 - Dy^2$ represents -1 if and only if ℓ is odd [6, p. 93] [5, p. 353] [3, p. 249].
- (ii) $A_{i-1}B_i - A_iB_{i-1} = (-1)^i$ for $i \geq 0$ [6, p. 14] [5, p. 330] [3, p. 225]. In particular, $\gcd(A_i, A_{i-1}) = 1$.
- (iii) $A_{i-1}^2 - DB_{i-1}^2 = (-1)^i Q_i$ for $i \geq 0$ [6, p. 92] [5, p. 351] [3, p. 246].
- (iv) $Q_i = Q_{\ell-i}$ for $0 \leq i \leq \ell$ [6, p. 81] [3, p. 253].
- (v) $P_{i+1}B_i = A_i - Q_{i+1}B_{i-1}$ for $i \geq -1$ [6, p. 70].
- (vi) $Q_{i+2} = Q_i - a_{i+1}(P_{i+2} - P_{i+1})$ for $i \geq 0$ [6, p. 70] [5, p. 358].
- (vii) $DB_{i-1} = A_{i-1}P_i + A_{i-2}Q_i$ for $i \geq 0$ [6, p. 70].

3. EXPLICIT REPRESENTATIONS. In what follows, D is a positive integer, not a square, $x^2 - Dy^2$ represents -1 (so by (i) ℓ is odd), and $n = (\ell + 1)/2$. We prove the claims made in the second paragraph of the introduction for $a = Q_n$ and $b = P_n$. The following lemma establishes representations of $\pm Q_n$.

Lemma 1.

$$A_{n-1}^2 - DB_{n-1}^2 = (-1)^n Q_n, \quad (1)$$

$$A_{n-2}^2 - DB_{n-2}^2 = (-1)^{n-1} Q_n, \quad (2)$$

and $\gcd(A_{n-1}, B_{n-1}) = \gcd(A_{n-2}, B_{n-2}) = 1$.

Proof. Equation (1) is an immediate consequence of (iii). Equation (2) follows from (iii) with $i = n - 1$ and (iv) which gives $Q_{n-1} = Q_n$. By (ii), $\gcd(A_{n-1}, B_{n-1}) = \gcd(A_{n-2}, B_{n-2}) = 1$. ■

The next theorem shows that D is the sum of the squares of the claimed a and b .

Theorem 1. $D = Q_n^2 + P_n^2$, where Q_n is odd and $\gcd(P_n, Q_n) = 1$.

Proof. That $D = Q_n^2 + P_n^2$ is well known [6, p. 83], but the proof is short, so we include it: by (B) $D - P_n^2 = Q_n Q_{n-1}$, by (iv) $Q_n = Q_{n-1}$, and the result follows.

Next we show that two consecutive Q_i cannot both be even. Using (A) we substitute $Q_{i+1}a_{i+1} - P_{i+1}$ for P_{i+2} in (vi) to get

$$Q_i = Q_{i+1}a_{i+1}^2 + Q_{i+2} - 2P_{i+1}a_{i+1}.$$

If Q_{i+1} and Q_{i+2} were both even, Q_i would also be even, and continuing this, all Q_j with $0 \leq j \leq i+2$ would be even. But $Q_0 = 1$ is odd. It follows that $Q_n = Q_{n-1}$ is odd.

Because $D = Q_n^2 + P_n^2$, if $g = \gcd(Q_n, P_n)$, then g^2 divides D . By (1) and (2) g then divides A_{n-1}^2 and A_{n-2}^2 , so by (ii), $g = 1$. ■

Now we can establish a theorem that gives surprisingly simple explicit representations of $\pm 2b$.

Theorem 2. If $(T_1, U_1) = (A_{n-1} - A_{n-2}, B_{n-1} - B_{n-2})$, then

$$T_1^2 - DU_1^2 = (-1)^n 2P_n. \quad (3)$$

Similarly if $(T_2, U_2) = (A_{n-1} + A_{n-2}, B_{n-1} + B_{n-2})$, then

$$T_2^2 - DU_2^2 = (-1)^{n-1} 2P_n. \quad (4)$$

Finally, $\gcd(T_1, U_1) = \gcd(T_2, U_2) = 1$.

Proof.

$$\begin{aligned} T_1^2 - DU_1^2 &= (A_{n-1} - A_{n-2})^2 - D(B_{n-1} - B_{n-2})^2 \\ &= (A_{n-1}^2 - DB_{n-1}^2) + (A_{n-2}^2 - DB_{n-2}^2) \\ &\quad - 2A_{n-1}A_{n-2} + 2DB_{n-1}B_{n-2} \\ &= -2A_{n-1}A_{n-2} + 2DB_{n-1}B_{n-2} \text{ by (1) and (2).} \end{aligned} \quad (5)$$

We now use (vii) with $i = n$ to substitute $DB_{n-1} = A_{n-1}P_n + A_{n-2}Q_n$ into (5) and get:

$$\begin{aligned} T_1^2 - DU_1^2 &= 2(-A_{n-1}A_{n-2} + B_{n-2}(A_{n-1}P_n + A_{n-2}Q_n)) \\ &= 2(B_{n-2}A_{n-1}P_n + A_{n-2}(-A_{n-1} + Q_nB_{n-2})) \\ &= 2(B_{n-2}A_{n-1}P_n + A_{n-2}(-P_nB_{n-1})) \text{ by (v)} \\ &= 2(B_{n-2}A_{n-1} - A_{n-2}B_{n-1})P_n \\ &= 2(-1)^n P_n \text{ by (ii).} \end{aligned}$$

There is a similar proof for (4), or alternatively one can use

$$(T_2 + U_2\sqrt{D}) = (T_1 + U_1\sqrt{D})(Q_n + \sqrt{D})/P_n$$

and take norms of both sides.

Finally, let $g = \gcd(T_1, U_1) = \gcd(A_{n-1} - A_{n-2}, B_{n-1} - B_{n-2})$. Then

$$A_{n-2} \equiv A_{n-1} \pmod{g} \quad \text{and} \quad B_{n-1} \equiv B_{n-2} \pmod{g},$$

so g divides $A_{n-2}B_{n-1} - A_{n-1}B_{n-2}$. By (ii)

$$A_{n-2}B_{n-1} - A_{n-1}B_{n-2} = (-1)^{n-1},$$

so g divides 1 and hence $g = 1$.

The proof that $\gcd(T_2, U_2) = 1$ is similar. ■

Examples for Lemma 1 and Theorems 1 and 2 can be drawn from Table 1, which gives the continued fraction expansion of $\sqrt{58}$. Here $\ell = 7$ and $n = 4$. Lemma 1 then says that $23^2 - 58 \cdot 3^2 = (-1)^4 \cdot 7$ and $15^2 - 58 \cdot 2^2 = (-1)^3 \cdot 7$, both of which can be verified by direct computation. Theorem 1 says that $58 = 7^2 + 3^2$. Equation (3) says that $(23 - 15)^2 - 58(3 - 2)^2 = (-1)^4 \cdot 2 \cdot 3$, or $8^2 - 58 \cdot 1^2 = 6$, and (4) says that $(23 + 15)^2 - 58(3 + 2)^2 = (-1)^3 \cdot 2 \cdot 3$, or $38^2 - 58 \cdot 5^2 = -6$.

For primes $p > 0$ it is well known that $x^2 - py^2$ represents -1 (and so the lemma and theorems above apply) if and only if $p = 2$ or $p \equiv 1 \pmod{4}$ [5, p. 357]. The fifteen smallest composite D so that $x^2 - Dy^2$ represents -1 are $D = 10, 26, 50, 58, 65, 74, 82, 85, 106, 122, 125, 130, 145, 170$, and 185 .

An apparently open problem is to characterize those D that are a sum of two relatively prime squares but for which $x^2 - Dy^2$ does not represent -1 . Such D include $34, 146, 178, 194, 205, 221, 305, 377, 386$, and 410 . Grytczuk, Luca, and Wojtowicz [2] prove that $x^2 - Dy^2 = -1$ has a solution if and only if there is a primitive Pythagorean triple (A, B, C) and positive integers a, b so that $D = a^2 + b^2$ and $|aA - bB| = 1$. In this case, $x = |aB + bA|$ and $y = C$ give a solution.

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