A Hypersequent System for Gödel-Dummett Logic with Non-constant Domains

Alwen Tiu

College of Engineering and Computer Science
The Australian National University

Abstract. Gödel-Dummett logic is an extension of first-order intuitionistic logic with the linearity axiom $(A \supset B) \vee (B \supset A)$, and the so-called "quantifier shift" axiom $\forall x(A \vee B(x)) \supset A \vee \forall x B(x)$. Semantically, it can be characterised as a logic for linear Kripke frames with constant domains. Gödel-Dummett logic has a natural formalisation in hypersequent calculus. However, if one drops the quantifier shift axiom, which corresponds to the constant domain property, then the resulting logic has to date no known hypersequent formalisation. We consider an extension of hypersequent calculus in which eigenvariables in the hypersequents form an explicit part of the structures of the hypersequents. This extra structure allows one to formulate quantifier rules which are more refined. We give a formalisation of Gödel-Dummett logic without the assumption of constant domain in this extended hypersequent calculus. We prove cut elimination for this hypersequent system, and show that it is sound and complete with respect to its Hilbert axiomatic system.

1 Introduction

Gödel logics refer to a family of intermediate logics (i.e., logics between intuitionistic and classical logics) that can be characterised by the class of rooted linearly ordered Kripke models, or alternatively, as many-valued logics whose connectives are interpreted as functions over subsets of the real interval $[0, 1]$. Its conception dates back to the seminal work by Gödel on the (non-existence of) finite matrix characteristic for propositional intuitionistic logic [14]. Dummett [12] gives an axiomatisation of a (propositional) Gödel logic over an infinite set of truth values, by extending intuitionistic logic with the *linearity axiom* $(A \supset B) \vee (B \supset A)$. This logic is called **LC**, but also known as Gödel-Dummett logic. In the first-order case, Gödel logics (viewed as logics of linear Kripke frames) are usually formalised with the assumption of constant domain, i.e., it assumes the same domain of individuals for all worlds, which is captured via the *quantifier shift* axiom $(\forall x.A \vee B) \supset \forall x.(A \vee B)$, where x is not free in B.

Traditional cut-free sequent calculi for **LC** have been studied in several previous works [21,10,1,13]. Due to the linearity axiom, the formalisation of **LC** in traditional sequent calculi requires a non-standard form of introduction rule for implication, e.g., in Corsi's calculus [10], the introduction rule for $\supset$ involves a simultaneous introduction of several $\supset$-formulae (see Section 6). Avron proposed

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another proof system for Gödel-Dummett logic in the framework of *hypersequent calculus* [3] (see also [5] for related work on Gödel logics in hypersequent calculi). A hypersequent is essentially a multiset of sequents. In Avron's notation, a hypersequent with n member sequents is written as $\Gamma_1 \Rightarrow C_1 \mid \cdots \mid \Gamma_n \Rightarrow C_n$. The structural connective $\mid$ here is interpreted as disjunction.

In contrast to Corsi's sequent calculus, the introduction rules for the hypersequent calculus for Gödel-Dummett logic are the standard ones. Linearity is captured, instead, using a structural rule, called the *communication* rule:

$$\frac{G \mid \Gamma, \Delta \Rightarrow C \quad G \mid \Gamma, \Delta \Rightarrow D}{G \mid \Gamma \Rightarrow C \mid \Delta \Rightarrow D}\ com$$

In the first-order case, a standard right-introduction rule for $\forall$ is

$$\frac{G \mid \Gamma \Rightarrow A(y)}{G \mid \Gamma \Rightarrow \forall x.A(x)}\ \forall r$$

where y is not free in the conclusion. Notice that in the premise of the rule, implicitly the scope of y is over the entire hypersequent. If the structural connective $\mid$ is interpreted as disjunction and eigenvariables are interpreted as universally quantified variables, the rule essentially encapsulates the quantifier shift axiom. It would seem therefore that in the traditional hypersequent calculus, one is forced to accept the quantifier shift axiom as part of the logic.

In this paper, we are interested in seeing whether there is a way to formalise intermediate logics in hypersequent calculus in which the quantifier shift axiom may not hold. We study a particular logic with this property, i.e., what is called *quantified* **LC** in [11] (we shall refer to it as **QLC** below) , which is an extension of the first-order intuitionistic logic with the linearity axiom (but without the quantifier shift axiom). In semantic terms, this logic is just a logic of linearly ordered frames with nested domains. A sequent calculus for this logic was first considered by Corsi [10], and later Avellone, et. al. [2], where, again as in the propositional case, there is a simultaneous introduction rule for *both* $\forall$ and $\supset$.

The key idea to the hypersequent formalisation of **QLC** here is to explicitly represent eigenvariables as part of the structure of a hypersequent and to use that extra structure to control the use of eigenvariables. The idea of explicit representation of eigenvariables in sequent calculus is not new and has been considered in the abstract logic programming literature, see e.g., [17]. An intuitionistic sequent in this case is a structure of the form $\Sigma; \Gamma \Rightarrow C$ where Σ here is a set of eigenvariables, called the *signature* of the sequent. The introduction rules for $\forall$ would then be of the forms:

$$\frac{\Sigma \vdash t : \text{term} \quad G \mid \Sigma; A(t), \Gamma \Rightarrow C}{G \mid \Sigma; \forall x.A(x), \Gamma \Rightarrow C}\ \forall l \qquad \frac{G \mid \Sigma, y; \Gamma \Rightarrow A(y)}{G \mid \Sigma; \Gamma \Rightarrow \forall x.A(x)}\ \forall r$$

Notice that in $\forall r$, the eigenvariable y is explicitly added to Σ (reading the rule upward). Notice also that in instantiating a universal quantifier on the left, one needs to be able to form the term t given the signature Σ. This is enforced by the judgment $\Sigma \vdash t$: term in the premise. In the simplest case, it just means that the free variables of t need to be already in Σ. It is easy to see that the quantifier shift axiom may not be immediately provable using the above rules.

An immediate problem with the explicit representation of eigenvariables in hypersequents is that there seems to be no way to interpret rules in the hypersequent calculus as valid formulae in **QLC**. This complicates the proof of soundness of the hypersequent calculus via an encoding in the Hilbert system for **QLC** (see the discussion in Section 5). For example, if one were to interpret the signature Σ as universal quantifiers whose scope is over the sequent it is attached to, then the *com*-rule (see Section 3) turns out to be unsound for **QLC**. The solution attempted here is to interpret eigenvariables in a signature as an encoding of an *existence predicate*, that was first introduced in Scott's existence logic [19]. Intuitively, a sequent such as $x; A(x) \Rightarrow B(x)$ can be interpreted as the formula $E(x) \wedge A(x) \supset B(x)$, where E here is an existence predicate. Using this interpretation, however, we cannot directly prove soundness of our hypersequent system with respect to **QLC**, as **QLC** does not assume such an existence predicate. To overcome this problem, we use a result by Iemhoff [15] relating Gödel logics with non-constant domains with Gödel logics with constant domains extended with the existence predicate. Section 5 gives more details of this correspondence.

The remainder of this paper is structured as follows. Section 2 reviews the syntax and the semantics of Corsi's **QLC**. Section 3 presents a hypersequent calculus for **QLC**, called **HQLC**, and Section 4 shows how cut elimination can be proved for **HQLC**. The hypersequent system **HQLC** actually captures a richer logic than **QLC**, as it permits a richer term language than that allowed in **QLC**. What we aim to show here is that cut elimination still holds provided the term formation judgment ($\Sigma \vdash t$: term) satisfies four abstract properties, so our cut elimination result applies to extensions of **QLC** with richer term structures. Section 5 shows that **HQLC**, restricted to a term language containing only constant symbols as in **QLC**, is sound and complete w.r.t. **QLC**. Section 6 discusses related work and some directions for future work.

2 Semantics and an Axiomatic System of Quantified LC

The language GD of first-order Gödel-Dummett logic with nested domain is as in first-order intuitionistic logic. We consider here the connectives $\bot$ ('false'), $\top$ ('true'), $\wedge$, $\vee$, $\supset$, $\exists$ and $\forall$. We assume some given constants, but do not assume any function symbols. First-order variables are ranged over by x, y and z; constants by a,b,c; and predicate symbols by p and q. Given a formula C, $FV(C)$ denotes the set of free variables in C. This notation generalizes straightforwardly to sets of formulas, e.g., if Γ is a (multi)set of formulas, then $FV(\Gamma)$ is the set of free variables in all the formulas in Γ.

A Kripke model $\mathcal{M}$ for GD is a quadruple $\langle W, R, D, I \rangle$ where

- W is a non-empty set of worlds;
- R is a binary relation on W;

– D is the domain function assigning each $w \in W$ a non-empty set D_w such that $D_v \subseteq D_w$ whenever vRw;
– I is an interpretation function such that for each $w \in W$, for each constant c, $I_w(c) \in D_w$, and for each n-ary predicate p, $I_w(p) \subseteq D_w^n$, such that $I_v(c) = I_w(c)$ and $I_v(p) \subseteq I_w(p)$ whenever vRw.

The forcing relation $\mathcal{M}, w \models A$ is defined as in the usual definition in first-order intuitionistic logic (see, e.g., [24]). Let **QLC** [11] be the axiomatic system extending Hilbert's system for first-order intuitionistic logic (see, e.g., [23]) with the linearity axiom $(A \supset B) \vee (B \supset A)$.

Theorem 1 ([11]). QLC *is sound and complete with respect to GD, for the class of Kripke models based on linearly ordered frames (reflexive, transitive, connected and antisymmetric) with nested domains.*

3 The Hypersequent System HQLC

Definition 2. *A* sequent *is a syntactic expression of the form $\Sigma; \Gamma \Rightarrow C$ where Σ is a set of eigenvariables, Γ is a multiset of formulae and C is a formula. Σ here is called the* signature *of the sequent. A* hypersequent *is a multiset of sequents. When writing hypersequents, we shall use the symbol $\mid$ to separate individual sequents in the multiset. Thus, the following is a hypersequent with n members: $\Sigma_1; \Gamma_1 \Rightarrow C_1 \mid \cdots \mid \Sigma_n; \Gamma_n \Rightarrow C_n$.*

A *substitution* θ is a mapping from variables to terms such that the *domain* of θ, i.e., $\{x \mid \theta(x) \neq x\}$ is finite. We denote with $dom(\theta)$ the domain of θ, and with $ran(\theta)$ the *range* of θ, i.e., the set $\{\theta(x) \mid x \in dom(\theta)\}$. When we want to be explicit about the domain and range of a substitution, we enumerate them as a list of mappings, e.g, $[t_1/x_1, \ldots, t_n/x_n]$ denotes a substitution which maps x_i to t_i, with domains $\{x_1, \ldots, x_n\}$. A *renaming substitution* is a substitution which is an injective map between variables.

Substitutions are extended to mappings from terms to terms or formulae to formulae in the obvious way, taking care of avoiding capture of free variables in the range of substitutions. Given a multiset Γ and a substitution θ, $\Gamma\theta$ denotes the multiset resulting from applying θ to each element of Γ. The result of applying a substitution θ to a signature Σ is defined as follows:

$$\Sigma\theta = \bigcup\{FV(\theta(x)) \mid x \in \Sigma\}.$$

For example, if $\Sigma = \{x, y\}$ and $\theta = [a/x, y/z]$ then $\Sigma\theta = \{y\}$. The result of applying a substitution θ to a sequent $\Sigma; \Gamma \Rightarrow C$ is the sequent $\Sigma\theta; \Gamma\theta \Rightarrow C\theta$. Application of a substitution to a hypersequent is defined as applications of the substitution to its individual sequents.

We shall assume a given relation $\vdash$ between a signature Σ and a term t, capturing a notion of wellformedness of terms. We shall write $\Sigma \vdash t : \text{term}$ to denote that the term t is wellformed under Σ. The judgment $\Sigma \vdash t : \text{term}$ can

Cut and identity:

$$\frac{}{\Sigma; \Gamma, p(\vec{t}) \Rightarrow p(\vec{t}) \mid H}\ id \qquad \frac{\Sigma; \Gamma \Rightarrow B \mid H_1 \quad \Sigma; B, \Delta \Rightarrow C \mid H_2}{\Sigma; \Gamma, \Delta \Rightarrow C \mid H_1 \mid H_2}\ cut$$

Structural rule:

$$\frac{\Sigma_1, \Sigma_2; \Gamma, \Delta \Rightarrow B \mid \Sigma_2; \Delta \Rightarrow C \mid H \qquad \Sigma_1; \Gamma \Rightarrow B \mid \Sigma_1, \Sigma_2; \Gamma, \Delta \Rightarrow C \mid H}{\Sigma_1; \Gamma \Rightarrow B \mid \Sigma_2; \Delta \Rightarrow C \mid H}\ com$$

Logical rules:

$$\frac{}{\Sigma; \bot, \Gamma \Rightarrow A \mid H}\ \bot l \qquad \frac{}{\Sigma; \Gamma \Rightarrow \top}\ \top r$$

$$\frac{\Sigma; \Gamma, A \supset B \Rightarrow A \mid \Sigma; \Gamma, A \supset B \Rightarrow C \mid H \qquad \Sigma; \Gamma, A \supset B, B \Rightarrow C \mid H}{\Sigma; \Gamma, A \supset B \Rightarrow C \mid H}\ \supset l$$

$$\frac{\Sigma; \Gamma, A \Rightarrow B \mid H}{\Sigma; \Gamma \Rightarrow A \supset B \mid H}\ \supset r$$

$$\frac{\Sigma; \Gamma, A_1, A_2 \Rightarrow B \mid H}{\Sigma; \Gamma, A_1 \wedge A_2 \Rightarrow B \mid H}\ \wedge l \qquad \frac{\Sigma; \Gamma \Rightarrow A \mid H \qquad \Sigma; \Gamma \Rightarrow B \mid H}{\Sigma; \Gamma \Rightarrow A \wedge B \mid H}\ \wedge r$$

$$\frac{\Sigma; \Gamma, A \Rightarrow C \mid H \qquad \Sigma; \Gamma, B \Rightarrow C \mid H}{\Sigma; \Gamma, A \vee B \Rightarrow C \mid H}\ \vee l \qquad \frac{\Sigma; \Gamma \Rightarrow A_1 \mid \Sigma; \Gamma \Rightarrow A_2 \mid H}{\Sigma; \Gamma \Rightarrow A_1 \vee A_2 \mid H}\ \vee r$$

$$\frac{\Sigma \vdash t : \text{term} \quad \Sigma; \Gamma, A[t/x], \forall x.A \Rightarrow B \mid H}{\Sigma; \Gamma, \forall x.A \Rightarrow B \mid H}\ \forall l \qquad \frac{\Sigma, y; \Gamma \Rightarrow A[y/x] \mid H}{\Sigma; \Gamma \Rightarrow \forall x.A \mid H}\ \forall r$$

$$\frac{\Sigma, y; \Gamma, A[y/x] \Rightarrow B \mid H}{\Sigma; \Gamma, \exists x.A \Rightarrow B \mid H}\ \exists l \qquad \frac{\Sigma \vdash t : \text{term} \quad \Sigma; \Gamma \Rightarrow A[t/x] \mid \Sigma; \Gamma \Rightarrow \exists x.A \mid H}{\Sigma; \Gamma \Rightarrow \exists x.A \mid H}\ \exists r$$

Fig. 1. The hypersequent system **HQLC**. In the rules $\exists l$ and $\forall r$, the eigenvariable y is not free in the conclusion.

be seen as a typing judgment familiar from programming languages. It can also be thought as a formalisation of a form of *existence predicate* from existence logic [19] (see Section 5).

For the purpose of proving cut-elimination, the particular definition of the judgment $\vdash$ is not important, as long as it satisfies the following properties:

P1 If $\Sigma \vdash t : \text{term}$ then $FV(t) \subseteq \Sigma$.
P2 If $\Sigma \vdash t : \text{term}$ and $x \notin \Sigma$, then $\Sigma, x \vdash t : \text{term}$.
P3 If $\Sigma \vdash t : \text{term}$ and θ is a renaming substitution, then $\Sigma\theta \vdash t\theta : \text{term}$.
P4 If $\Sigma, x \vdash t : \text{term}$, where $x \notin \Sigma$, and $\Sigma \vdash s : \text{term}$ then $\Sigma \vdash t[s/x] : \text{term}$.

A consequence of **P1** is that ($\Sigma \vdash x$: term) implies $x \in \Sigma$. The converse does not hold in general, e.g., consider the case where $\vdash$ is the empty relation.

In the case of **QLC**, since we assume no function symbols, the term formation rule is very simple; $\Sigma \vdash t$: term holds iff either t is a constant or it is a variable in Σ. It is obvious that **P1** – **P4** hold in this case. But in general, any term formation judgments that satisfy the above four properties can be incorporated in our proof system and cut elimination will still hold. For example, one can have a term language based on Church's simply typed λ-calculus, e.g., as in the (first-order/higher-order) intuitionistic systems in [17]. The term formation judgment in this case would be the usual typing judgment for simply typed λ-calculus.

The hypersequent system **HQLC** is given in Figure 1. Notice that unlike traditional hypersequent calculi for Gödel-Dummett logic, **HQLC** does not have external weakening or external contraction rules. Both contraction and weakening are absorbed into logical rules. But as we shall see in Section 4, weakening and contraction are admissible in **HQLC**. Admissibility of contraction allows one to simplify slightly the Gentzen style cut elimination for **HQLC** (see Section 4). The $\vee r$ rule is non-standard, but is needed to absorb contraction. If one ignores the signature part of the hypersequents and the term formation judgments, the inference rules of **HQLC** are pretty standard for a hypersequent calculus.

Given a formula A, we say that A *is provable in* **HQLC** if the sequent $FV(A); . \Rightarrow A$ is derivable in **HQLC**.

4 Cut Elimination for HQLC

The cut elimination proof presented here is a variant of a Gentzen style cut elimination procedure for Gödel-Dummett logic [3,5]. But we note that it is also possible to extend the Schütte-Tait style of cut elimination in [4,9] to **HQLC**. We use a different form of multicut (see Section 4.3) to simplify slightly the main argument in cut elimination, but the proof is otherwise a fairly standard Gentzen style cut elimination proof. Before we proceed to the main cut elimination proof, we first establish some properties of derivations and rules in **HQLC**. Some proofs are omitted here, but they can be found in an extended version of this paper.

In a derivation of a hypersequent, one may encounter eigenvariables which are not free in the root hypersequent; we call these internal eigenvariables of the derivation. The names of those eigenvariables are unimportant, so long as they are chosen to be sufficiently fresh, in the context of the rules in which they are introduced. It is easy to prove by induction on the length of derivation and the fact that $\vdash$ is closed under renaming (property **P3**) that given a derivation of a hypersequent, there is an isomorphic derivation of the same hypersequent which differs only in the choice of naming of the internal eigenvariables. In the following proofs, we shall assume implicitly such a renaming is carried out during an inductive step so as to avoid name clash. Given a derivation Π, we denote with $|\Pi|$ its length.

4.1 Signature Weakening and Substitution

Lemma 3 (Signature weakening). *Let Π be a derivation of the hypersequent $\Sigma; \Gamma \Rightarrow \Delta \mid H$. Then for every variable x, there is a derivation Π' of $\Sigma, x; \Gamma \Rightarrow \Delta \mid H$ such that $|\Pi'| = |\Pi|$.*

Proof. By simple induction on $|\Pi|$ and property **P2**. □

Definition 4. *A substitution $[t/x]$ is said to* respect *a sequent $\Sigma; \Gamma \Rightarrow C$ if $x \in \Sigma \cup FV(\Gamma, C)$ implies $\Sigma \setminus \{x\} \vdash t : \text{term}$. It is said to respect a hypersequent H if it respects every sequent in H.*

Lemma 5 (Substitution.). *Let Π be a derivation of a hypersequent H and let $[t/x]$ be a substitution respecting H. Then there exists a derivation Π' of $H[t/x]$ such that $|\Pi'| = |\Pi|$.*

Proof. By induction on Π. The only non-trivial cases are when Π ends with $\forall l$ or $\exists r$. We show the former here, the latter can be dealt with analogously. So suppose Π is as shown below (we assume w.l.o.g. y is not free in H):

$$\frac{\Sigma \vdash s : \text{term} \quad \overset{\Pi_1}{\Sigma; \Gamma, A[s/y] \Rightarrow B \mid H'}}{\Sigma; \Gamma, \forall y.A \Rightarrow B \mid H'}\ \forall l$$

Then Π' is the derivation below

$$\frac{\Sigma' \vdash s[t/x] : \text{term} \quad \overset{\Pi_1'}{\Sigma'; \Gamma[t/x], A[t/x][s[t/x]/y] \Rightarrow B[t/x] \mid H'[t/x]}}{\Sigma'; \Gamma[t/x], \forall y.A[t/x] \Rightarrow B[t/x] \mid H'[t/x]}\ \forall l$$

where $\Sigma' = \Sigma[t/x]$ and Π_1' is obtained from the induction hypothesis. Note that as y is not free in t, we have $A[t/x][s[t/x]/y] = A[s[t/x]/y][t/x]$, so the induction hypothesis is indeed applicable to Π_1.

We still need to make sure that Π' is indeed a well-formed derivation, i.e., that the judgment ($\Sigma' \vdash s[t/x] : \text{term}$) is valid. There are two subcases to consider. The first is when $x \in \Sigma$, i.e., $\Sigma = \Sigma' \cup \{x\}$. The fact that $\Sigma \vdash s[t/x] : \text{term}$ holds follows from property **P4** of the relation $\vdash$.

Otherwise, $x \notin \Sigma$. In this case we have $\Sigma' = \Sigma$ and $\Sigma \vdash s : \text{term}$. The latter, together with **P1**, implies that $x \notin FV(s)$, and therefore $s[t/x] = s$ and $\Sigma' \vdash s[t/x] : \text{term}$ holds. □

4.2 Invertible and Admissible Rules

A rule ρ is said to be *strictly invertible* if for every instance of ρ, whenever its conclusion is cut-free derivable, then there is a cut-free derivation with the same or smaller length of for each of its premises. We say that ρ is *invertible* if it satisfies the same condition, except for the proviso on the length of derivations.

Lemma 6. *Suppose Π is a derivation of $\Sigma; \Gamma \Rightarrow C \mid H$. Then for any A, there exists a derivation Π' of $\Sigma; A, \Gamma \Rightarrow C \mid H$ such that $|\Pi| = |\Pi'|$.*

Lemma 7. *The rules* $\wedge l$, $\wedge r$, $\vee l$, $\supset r$, $\exists l$ *and* $\forall r$ *are strictly invertible.*

Proof. Straightforward by induction on the length of derivation. In the cases of $\exists l$ and $\forall r$, we need Lemma 3, and in the case of $\supset r$, we need Lemma 6. □

Lemma 8. *The rule* $\vee r$ *is invertible.*

A rule ρ is said to be *admissible* if whenever all its premises are cut-free derivable, then its conclusion is also cut-free derivable, without using ρ. It is said to be *height-conserving admissible* if given the derivations of its premises, one can derive the conclusion with less or the same length as the maximum length of its premise derivations. Using the invertibility of rules in Lemma 7 and Lemma 8, it can be proved that the following structural rules are admissible:

$$\frac{\Sigma; \Gamma \Rightarrow B \mid H}{\Sigma; A, \Gamma \Rightarrow B \mid H}\ wl \qquad \frac{\Sigma; A, A, \Gamma \Rightarrow B \mid H}{\Sigma, A, \Gamma \Rightarrow B \mid H}\ cl \qquad \frac{H}{H \mid G}\ ew \qquad \frac{H \mid H}{H}\ ec$$

Lemma 9. *The rules wl and cl are height-conserving admissible.*

Lemma 10. *The rules ew and ec are admissible.*

4.3 Cut Elimination

We first generalise the cut rule to the following *multicut*

$$\frac{\Sigma_1; \Delta_1 \Rightarrow A \mid H_1 \qquad \cdots \qquad \Sigma_n; \Delta_n \Rightarrow A \mid H_n \qquad G}{\Sigma_1; \Delta_1, \Gamma_1 \Rightarrow B_1 \mid \cdots \mid \Sigma_n; \Delta_n, \Gamma_n \Rightarrow B_n \mid H_1 \mid \cdots \mid H_n \mid H}\ mc$$

where G is the hypersequent $\Sigma_1; A, \Gamma_1 \Rightarrow B_1 \mid \cdots \mid \Sigma_n; A, \Gamma_n \Rightarrow B_n \mid H$. The formula A is called the *cut formula* of the *mc* rule. The premise $\Sigma_i; \Delta_i \Rightarrow A \mid H_i$ is called a *minor premise* of the *mc* rule. The premise G is called the *major premise.* Notice that each minor premise pairs with only one sequent in the major premise, unlike Avron's extended multicut [3] where the minor premise can pair with more than one sequent in the major premise.

A version of multicut similar to *mc* was proposed by Slaney [20] and was later used by McDowell and Miller in a cut-elimination proof for an intuitionistic logic [16]. In permutation of *mc* over (implicit) contraction in logical rules in the major premise, there will be no need to contract the hypersequents in the corresponding minor premise as is typical in Gentzen's multicuts, but one would instead duplicate the derivation of the minor premise (see the cut elimination proof below for more details). The use of *mc* allows one to simplify slightly the structure of the main arguments in the cut elimination proof (i.e., permutation of *mc* rule over other rules). In the cut elimination proof in [5], which uses Avron's multicuts [3], one needs to prove a lemma concerning cut-free admissibility of certain generalised $\exists l$ and $\vee r$ rules, e.g.,

$$\frac{H \mid \Gamma_1, A(a) \Rightarrow C_1 \mid \cdots \mid \Gamma_n, A(a) \Rightarrow C_n}{H \mid \Gamma_1, \exists x.A(x) \Rightarrow C_1 \mid \cdots \mid \Gamma_n, \exists x.A(x) \Rightarrow C_n}\ \exists l^*$$

where a is not free in the conclusion. The need to prove admissibility of this rule (and a similar version for $\vee l$) does not arise in here, due to the admissibility of (both internal and external) contraction and the *mc* rule.

The *cut rank* of an instance of an *mc* rule, with minor premise derivations $\Pi_1, \dots, \Pi_n$, major premise derivation Π, and cut formula A, is the triple

$$\langle |A|, |\Pi|, \{|\Pi_1|, \dots, |\Pi_n|\} \rangle$$

of the size of the cut formulae, the length of the major premise derivation, and the multiset of lengths of the minor premise derivations. Cut ranks are ordered lexicographically, where the last component of the triple is ordered according to multiset ordering. It can be shown that this order is wellfounded.

Theorem 11. *Cut elimination holds for* **HQLC**.

Proof. Suppose we have a derivation Ξ ending with an *mc*, with minor premise derivations $\Pi_1, \dots, \Pi_n$ and major premise derivation Π, and the cut formula A. We assume w.l.o.g. that all Π_i and Π are cut free. Cut elimination is proved by induction on the cut rank, by removing the topmost cuts in succession.

We show that, we can reduce the *mc* rule to one with a smaller cut rank. In the reduction of *mc*, the last rule applied to the major premise derivation will determine which of the minor premises is selected for reduction. Note that since the order of the sequents in the major premise and the order of the minor premises in *mc* do not affect the cut rank, w.l.o.g., we assume that the derivation Π ends with a rule affecting $\Sigma_1; A, \Gamma_1 \Rightarrow C_1$, and/or $\Sigma_2; A, \Gamma_2 \Rightarrow C_2$ (if it ends with a *com* rule). So, in the case analysis on the possible reductions we shall only look at the first and/or the second minor premises.

If Π ends with any rule affecting only H, then the *mc* rule can be easily permuted up over the rule, and it is eliminable by the induction hypothesis.

Otherwise, Π must end with a rule affecting one (or two, in case of *com*) of the sequents used in the cut rule. Without loss of generality, assume it is either the first and/or the second sequent. There are several cases to consider. We show a case involving the $\exists$ quantifier to illustrate the use of the substitution lemma.

Suppose Π_1 ends with $\exists r$ and Π ends with $\exists l$, both with the cut formula as the principal formula, i.e., they are of the following forms, respectively:

$$\frac{\Sigma_1 \vdash t : \text{term} \quad \begin{array}{c}\Pi_1' \\ \Sigma_1; \Delta_1 \Rightarrow A'(t) \mid \Sigma_1; \Delta_1 \Rightarrow \exists x.A'(x) \mid H_1\end{array}}{\Sigma_1; \Delta_1 \Rightarrow \exists x.A'(x) \mid H_1}\ \exists r$$

$$\frac{\begin{array}{c}\Psi \\ \Sigma_1, x; A'(x), \Gamma_1 \Rightarrow B_1 \mid \cdots\end{array}}{\Sigma_1; \exists x.A'(x), \Gamma_1 \Rightarrow B_1 \mid \cdots}\ \exists l$$

Let Ξ_1 be

$$\dfrac{\begin{matrix} \Pi_2 \\ \Sigma_2; \Delta_2 \Rightarrow A \mid H_2 \end{matrix} \quad \cdots \quad \begin{matrix} \Pi_n \\ \Sigma_n; \Delta_n \Rightarrow A \mid H_2 \end{matrix} \quad \begin{matrix} \Psi[t/x] \\ \Sigma_1; A'(t), \Gamma_1 \Rightarrow B_1 \mid \cdots \end{matrix}}{\Sigma_1; A'(t), \Gamma_1 \Rightarrow B_1 \mid \Sigma_2; \Delta_2, \Gamma_2 \Rightarrow B_1 \mid \cdots \mid \Sigma_n; \Delta_n, \Gamma_n \Rightarrow B_n \mid \cdots}\ mc$$

where $\Psi[t/x]$ is the result of substituting x by t in Ψ (see Lemma 5). Let Ξ_2 be

$$\dfrac{\begin{matrix} \Pi_1' \\ \Sigma_1; \Delta_1 \Rightarrow A'(t) \mid \Sigma_1; \Delta_1 \Rightarrow A \mid H_1 \end{matrix} \quad \begin{matrix} \Pi_2 \\ \cdots \end{matrix} \quad \cdots \quad \begin{matrix} \Pi_n \\ \cdots \end{matrix} \quad \begin{matrix} \Pi \\ \cdots \end{matrix}}{\Sigma_1; \Delta_1 \Rightarrow A'(t) \mid \Sigma_1; \Delta_1, \Gamma_1 \Rightarrow B_1 \mid \cdots}\ mc$$

Then Ξ reduces to the derivation:

$$\dfrac{\dfrac{\begin{matrix} \Xi_1 \\ \Sigma_1; \Delta_1 \Rightarrow A'(t) \mid \Sigma_1; \Delta_1, \Gamma_1 \Rightarrow B_1 \mid \cdots \end{matrix} \quad \begin{matrix} \Xi_2 \\ \Sigma_1; A'(t), \Gamma_1 \Rightarrow B_1 \mid \cdots \end{matrix}}{\Sigma_1; \Delta_1, \Gamma_1 \Rightarrow B_1 \mid \Sigma_1; \Delta_1, \Gamma_1 \Rightarrow B_1 \mid \cdots}\ mc}{\Sigma_1; \Delta_1, \Gamma_1 \Rightarrow B_1 \mid \cdots \mid \Sigma_n; \Delta_n, \Gamma_n \Rightarrow B_n \mid \cdots}\ ec$$

where the double lines indicate multiple applications of the rule *ec* (which is cut-free admissible by Lemma 10). It is clear that the cut ranks in the reduct of Ξ are smaller than the cut rank of Ξ so by the induction hypothesis, all the cuts in the reduct can be eliminated. □

5 Soundness and Completeness of HQLC

One way of proving soundnesss of **HQLC** would be to interpret hypersequents as formulae, and show that the formula schemes corresponding to inference rules can be proved in the Hilbert system **QLC**. Unfortunately, there does not seem to be an easy way to interpret hypersequent rules as valid formulae in **QLC**. The main problem is in the interpretation of eigenvariables in hypersequents, in particular, their intended scopes within the hypersequents. If a variable, say x, appears in the signatures of two different sequents in a hypersequent, then there is a question of what should be the scope of that variable. There are two possible encodings: one in which the signature in a sequent is seen as implicitly universally quantified over the sequent, and the other in which the scope of signatures is over the entire hypersequent. Consider for example the following hypersequent:

$$x, y; A(x, y) \Rightarrow B(x, y) \mid x, z; C(x, z) \Rightarrow D(x, z).$$

The standard encoding of hypersequents without signatures is to interpret $\Rightarrow$ as implication and $\mid$ as disjunction. If we interpret signatures as having local scopes over the individual sequents, then the above hypersequent would be encoded as

$$(\forall x \forall y.(A(x, y) \supset B(x, y)) \vee (\forall x \forall z.(C(x, z) \supset D(x, z)).$$

If we interpret signatures as having global scopes, then the two occurrences of x in the signatures will be identified as a single universal quantifier:

$$\forall x \forall y \forall z.(A(x,y) \supset B(x,y)) \vee (C(x,z) \supset D(x,z)).$$

Under the second interpretation, the $\forall r$ rule is obviously invalid in **QLC** as its validity would entail the quantifier shift axiom. Under the first interpretation, the *com*-rule would be unsound. To see why, consider the hypersequent:

$$x; p(x) \Rightarrow q(x) \mid x; q(x) \Rightarrow p(x).$$

This hypersequent is provable in **HQLC**. But if we follow the first interpretation, then it would entail that $(\forall x.p(x) \supset q(x)) \vee (\forall x.q(x) \supset p(x))$ is valid in **QLC**, which is wrong, as it is not valid even classically. It might be possible to reformulate the *com*-rule to avoid this problem, but it is at present not clear how this could be done.

The approach followed in this paper is to interpret the rules of **HQLC** in a Gödel-Dummett logic extended with an existence predicate [15]. The existence predicate was introduced by Scott [19] in an extension to intuitionistic logic, and has recently been formalised in sequent calculus [6]. We first extend the language of intuitionistic logic with a unary predicate E representing the existence predicate. The semantics of the extended logic is as in intuitionistic logic, but with the interpretation of the existence predicate satisfying: $I_w(c) \in I_w(E)$ for every $w \in W$ and every constant symbol c in **QLC**.

Let GD^c be the standard Gödel-Dummett logic with constant domain, and let GD^{ce} be GD^c extended with the existence predicate. The key to the soundness proof is a result by Iemhoff [15] which relates GD and GD^{ce}. This is achieved via a function $\lceil . \rceil$ encoding an intuitionistic formula without the existence predicate into one with the existence predicate, satisfying:

- $\lceil p(\vec{t}) \rceil = p(\vec{t})$, for any predicate symbol p,
- $\lceil . \rceil$ commutes with all propositional connectives,
- $\lceil \exists x.A \rceil = \exists x.E(x) \wedge \lceil A \rceil$, and
- $\lceil \forall x.A \rceil = \forall x.E(x) \supset \lceil A \rceil$.

The following is a corollary of a result by Iemhoff (see Lemma 4.3 in [15]).

Theorem 12. *A closed formula A is valid in GD iff $\lceil A \rceil$ is valid in GD^{ce}.*

The soundness proof below uses the following interpretation of sequents and hypersequents. Given a multiset $\Gamma = \{A_1, \ldots, A_n\}$, we denote with $\lceil \Gamma \rceil$ the formula $\lceil A_1 \rceil \wedge \cdots \wedge \lceil A_n \rceil$. Given a set of eigenvariable $\Sigma = \{x_1, \ldots, x_n\}$, we write $E(\Sigma)$ to denote the formula $E(x_1) \wedge \cdots \wedge E(x_n)$. Let τ_s be a function from sequents to formulaes such that

$$\tau_s(\Sigma; \Gamma \Rightarrow C) = E(\Sigma) \wedge \lceil \Gamma \rceil \supset \lceil C \rceil.$$

The function τ_h mapping a hypersequent to a formula is defined as:

$$\tau_h(S_1 \mid \cdots \mid S_n) = \forall \vec{x}. \bigvee_i \tau_s(S_i)$$

where $\vec{x}$ is the list of all variables in the hypersequent and each S_i is a sequent. We shall overload the symbol τ_h to denote the translation function for the term formation judgment, which is defined as follows (where $\Sigma = \{\vec{x}\}$):

$$\tau_h(\Sigma \vdash t : \text{term}) = \forall \vec{x}.E(\Sigma) \supset E(t).$$

For the remainder of this section, we shall assume that the term formulation judgment $\vdash$ is defined as follows:

$$\frac{x \in \Sigma}{\Sigma \vdash x : \text{term}} \qquad \frac{}{\Sigma \vdash c : \text{term}}\ c \text{ is a constant}$$

Since the domains are assumed to be always non-empty, we assume that there is at least one constant symbol. It is easy to show that properties **P1** – **P4** hold for this definition of $\vdash$. Additionally, we also have the following lemma.

Lemma 13. *For any term t, $\Sigma \vdash t : \text{term}$, provided that $FV(t) \subseteq \Sigma$.*

Lemma 14. *If H is provable in* **HQLC** *then $\tau_h(H)$ is valid in GD^{ce}.*

Proof. Given a rule ρ with premises $H_1, \ldots, H_n$ and conclusion H, we show that if $\tau_h(H_1), \ldots, \tau_h(H_n)$ are valid in GD^{ce} then $\tau_h(H_{n+1})$ is valid in GD^{ce}. It is not difficult to verify that (e.g., using the hypersequent system in [5])

$$\tau_h(H_1) \wedge \cdots \wedge \tau_h(H_n) \supset \tau_h(H_{n+1})$$

is a tautology in GD^c, hence it is also valid in GD^{ce}. Therefore a cut-free derivation in **HQLC** can be simulated by a chain of modus ponens using the tautologies encoding its rule instances, with two assumptions: the encodings of the identity rule and the term formation judgment. The former is obviously valid, so we show the latter. That is, $\tau_h(\Sigma \vdash t : \text{term})$ is valid in GD^{ce}, whenever $\Sigma \vdash t : \text{term}$ holds. This is straightforward from the definition of $\vdash$. □

Theorem 15. *If A is provable in* **HQLC** *then A is provable in* **QLC**.

Proof. We first show that for every *closed* formula A, if A is provable in **HQLC** then A is provable in **QLC**. By Lemma 14, $\tau_h(FV(A); . \Rightarrow A)$ is valid in GD^{ce}. Since A is closed, $FV(A) = \emptyset$, therefore $\tau_h(FV(A); . \Rightarrow A) = \lceil A \rceil$. Then, by Theorem 12, A is valid in GD, and by Theorem 1, A is provable in **QLC**.

Now, if A is not closed, i.e., $FV(A) \neq \emptyset$, then we have that $\forall \vec{x}.A$, where $\{\vec{x}\} = FV(A)$, is also provable in **HQLC**, hence by the above result, $\forall \vec{x}.A$ is provable in **QLC**. To show that A is also provable in **QLC**, we do a detour through Corsi's sequent calculus for **QLC** [10], where it is easily shown that $\forall \vec{x}.A$ is provable iff A is provable in the sequent calculus. □

Theorem 16. *If A is provable in* **QLC** *then A is provable in* **HQLC**.

Proof. We first show that whenever A is provable in **QLC** then the sequent $\Sigma; . \Rightarrow A$ is derivable in **HQLC** for some $\Sigma \supseteq FV(A)$. This is done by induction on the length of derivation in **QLC**.

It is enough to show that every instance of the axioms of **QLC** and its inference rules, modus ponens and the quantifier introduction (i.e., generalisation), are derivable in **HQLC**. The generalisation rule is trivially derivable. To derive modus ponens, in addition to using cut, we need Lemma 3 and Lemma 7 (invertibility of $\supset$). The linearity axiom is easily derived using the *com*-rule. The non-trivial part is the derivations of the following axioms that involve quantifiers:

$$(Ax1) \qquad \forall x.A \supset A[t/x] \qquad (Ax2) \qquad A[t/x] \supset \exists x.A$$

We show here a derivation of (any instance of) $(Ax1)$; the other is similar. In this case, we let Σ be the set of all free variables in $(Ax1)$. Then we have:

$$\cfrac{\cfrac{\Sigma \vdash t : \text{term} \quad \cfrac{}{\Sigma; \forall x.A, A[t/x] \Rightarrow A[t/x]}\ id}{\Sigma; \forall x.A \Rightarrow A[t/x]}\ \forall l}{\Sigma; . \Rightarrow \forall x.A \supset A[t/x]}\ \supset r$$

Note that the judgment $\Sigma \vdash t : \text{term}$ is valid, by Lemma 13.

Now, we need to show that A is provable in **HQLC**, i.e., that the sequent $FV(A); . \Rightarrow A$ is derivable. By the above result, we have a derivation Π of the sequent $\Sigma; . \Rightarrow A$ for some $\Sigma \supseteq FV(A)$. Note that Σ may contain more variables than $FV(A)$. But since we assume that the domains are non-empty, we have at least one constant, say c, in the language. Let $\vec{y}$ be the variables in $\Sigma \setminus FV(A)$. Then by applying the substitution lemma (Lemma 5), i.e., substituting all $\vec{y}$ with c, to Π, we get a derivation of $FV(A); . \Rightarrow A$. $\square$

The derivation of $\forall x.A \supset A[t/x]$ in the completeness proof above relies on the underlying assumption that all closed terms denote existing objects (see Lemma 13). A similar completeness result is shown in [6], where Gentzen's LJ is shown to be equivalent to a specific existence logic called $\mathbf{LJE}(\Sigma_{\mathcal{L}})$. The intuitionistic fragment of **HQLC** can be seen as the equivalent of $\mathbf{LJE}(\Sigma_{\mathcal{L}})$.

6 Related and Future Work

A cut-free sequent calculus for **QLC** was first introduced by Corsi in [10]. Avellone, et. al., gave a tableau calculus for the same logic [2], and showed how their tableau calculus can also be converted into a cut-free sequent calculus. Both sequent calculi are multiple-conclusion calculi and use a simultaneous introduction rule for $\supset$ and $\forall$:

$$\frac{\Gamma, A_1 \Rightarrow B_1, \Delta_1 \quad \cdots \quad \Gamma, A_m \Rightarrow B_m, \Delta_m \quad \Gamma \Rightarrow C_1(a), \Lambda_1 \quad \cdots \quad \Gamma \Rightarrow C_n(a), \Lambda_n}{\Gamma \Rightarrow \Delta}$$

where $\Delta = \{A_1 \supset B_1, , \cdots, A_m \supset B_m, \forall x.C_1(x), \cdots, \forall x.C_n(x)\}$, $\Delta_i = \Delta \setminus \{A_i \supset B_i\}$ and $\Lambda_i = \Delta \setminus \{\forall x.C_i(x)\}$.

The idea of giving quantifiers an explicit structural component in sequents has been considered in a number of previous work. Wansing [25] studies substructural quantifiers in modal logic, in which the Barcan formula may or may not hold, using the display calculus framework. The treatment of quantifiers as structural connectives has also been explored in the calculus of structures, e.g., [7,22], and in nested sequent calculi [8].

As should be clear from the soundness proof in Section 5, the existence predicate is implicit in our notion of hypersequents. One could also consider an approach where the existence predicate is an explicit part of the language of hypersequents and extends the methods in [6] to prove cut elimination. In this setting, the term formation judgment would be encoded as a set of axioms governing the derivability of the existence predicate, and a form of cut elimination can be proved following [6], showing that cuts can be restricted to a simple form where the cut formula contains only the existence predicate.

We note that although we prove soundness w.r.t. a logic without function symbols, the soundness proof shown here can be generalised straightforwardly to logics with function symbols.

From a proof theoretic perspective, the solution proposed here is not entirely satisfactory, due to the lack of a clear formula-interpretation of hypersequents, hence the inability to get a direct encoding of **HQLC** into the Hilbert axiomatic system **QLC**. It would seem more natural to treat the signature in a sequent as *binders*, along the line of the framework proposed in [8]. This would entail a move from hypersequent to a sort of nested hypersequent (or perhaps a variant of tree-hypersequent [18]) and deep-inference rules. Our reliance on the semantic correspondence in Theorem 12 means that the current approach is difficult to generalize to other logics where the existence predicate is not so well understood.

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