Truncated-Bloch-wave solitons in optical lattices

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We study self-trapped localized nonlinear states in the form of truncated Bloch waves in one-dimensional optical lattices, which appear in the gaps of the linear band-gap spectrum. We demonstrate the existence of families of such localized states which differ by the number of intensity peaks. These families do not bifurcate from the band edge, and their power curves exhibit double branches. Linear-stability analysis demonstrates that in deep lattice potentials, the states corresponding to the lower branches are stable, whereas those corresponding to the upper branches are unstable, independently of the number of peaks.

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The theoretical study of localized states in periodic potentials is driven by many experimental demonstrations in nonlinear optics [1] and the physics of Bose-Einstein condensates (BECs) [2–5]. One of the most striking effects is the possibility of spatial localization due to an interplay between periodicity and nonlinearity even when the nonlinearity is repulsive. The corresponding localized states have long been known in nonlinear optics as *gap solitons* [6] whose frequencies lie inside the gaps of the linear band-gap spectrum of a periodic structure. Similar objects have been observed in atomic physics, where the experimental generation of BEC gap solitons proved to be a challenge due to the requirement of low atom numbers and densities [5]. The physics of such localized states is well understood, and usually the gap solitons appear through bifurcations of the localized states from the edges of the bands of periodic solutions [7].

Seemingly different localized states with steep edges and a large number of atoms populated evenly in a number of adjacent lattice sites were observed a few years ago in BEC loaded into an optical lattice [8]. It was observed that in a deep one-dimensional optical lattice the BEC wave packet does not diffuse but instead its initial expansion stops and the width remains finite. This effect has been attributed to the self-trapping mechanism of energy localization and the existence of broad nonlinear localized states in the form of truncated nonlinear Bloch waves in the gaps of the matter-wave linear spectrum [9]. In particular, Alexander *et al.* [9] revealed that these robust nonlinear localized states may exist in all dimensions and can have arbitrary extension within the lattice. They seem to provide an important missing link between gap solitons and nonlinear Bloch waves and can be termed as *truncated-Bloch-wave solitons*.

In this paper, we consider a model for the Bose-Einstein condensate in a one-dimensional optical lattice under repulsive nonlinearity, and we systematically study the existence and stability of the truncated-Bloch-wave solitons. On the question of existence, we reveal that these solitons exist as countable families which are characterized by the number of

intensity peaks. These solution families do not bifurcate from the edges of Bloch bands, and their power curves exhibit double branches. If the depth of the lattice potential falls below a certain threshold value, these truncated-Bloch-wave solitons cease to exist. We also demonstrate that in a deep lattice potential these solitons on the lower branches of the power curves are stable, while those on the upper branches of the power curves are unstable, independent of the number of intensity peaks. In a shallow lattice potential, both branches of these solitons are linearly unstable. These results provide a solid background for the earlier experimental observations of broad localized states in BECs [8] as well as the corresponding numerical studies [9], and they reveal the specific role that these types of self-trapped localized states play as a link between nonlinear Bloch waves and gap solitons in periodic potentials.

The physical system we consider is that of a Bose-Einstein condensate loaded into a one-dimensional optical lattice under repulsive nonlinearity [5,8]. The mathematical model for this system is the one-dimensional Gross-Pitaevskii equation with a lattice potential [10]

$$iU_t + U_{xx} - (V_0 \sin^2 x)U - |U|^2 U = 0, \qquad (1)$$

where U is the mean-field wave function of the BEC and the lattice potential is π periodic with potential depth V_0. We have used the same normalization as that found in Ref. [9] to make the model dimensionless with all the quantities in units of characteristic scales of the lattice. In particular x and t are in units of $a_L = d/\pi$ and $\omega_L^{-1} = \hbar/E_{rec}$, respectively, where d is the period of the lattice and $E_{rec} = \hbar^2/2ma_L^2$ is the recoil energy for an atom of mass m absorbing a lattice photon. The depth of the lattice, V_0, is in units of E_{rec}. We have assumed that the condensate is tightly confined in the transverse dimensions to use the standard dimensionality reduction procedure to reduce the dynamics to an effectively one-dimensional model.

Solitons in this system are sought in the form of $U(x,t) = u(x)e^{-i\mu t}$, where μ is the chemical potential and the spatial function $u(x)$ is real valued and satisfies the reduced equation
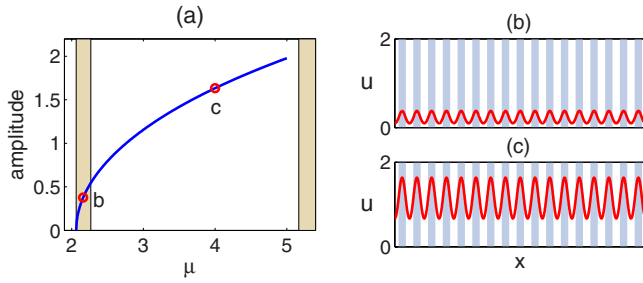
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FIG. 1. (Color online) Nonlinear Bloch waves bifurcated from the lower edge of the first Bloch band under repulsive nonlinearity ($V_0=6$). (a) The amplitude versus the chemical-potential diagram; [(b) and (c)] nonlinear Bloch waves at two chemical potentials marked in (a). The shaded stripes represent low-potential regions.

$$u_{xx} - (V_0 \sin^2 x)u - u^3 = -\mu u. \tag{2}$$

In this paper, we first consider a deep lattice potential with $V_0=6$, where the existence and linear stability of truncated-Bloch-wave solitons will be determined in detail. These solitons and their stability in shallower potentials will be addressed toward the end of the paper.

To understand the origin of the truncated-Bloch-wave solitons, we first examine periodic nonlinear Bloch waves which originate from linear Bloch waves at the band edges. In a deep potential with $V_0=6$, the linear dispersion curves of Eq. (2) can be found in Fig. 2(a) of Ref. [11]. At the lower edge of the first Bloch band [which is 2.0632 in Fig. 1(a)], Eq. (2) admits an infinitesimal (linear) Bloch wave which is π periodic, and its adjacent intensity peaks are in phase with each other. This linear Bloch wave (in arbitrary units) is shown in Fig. 2(1) of Ref. [11] and appears similar to the profile shown in Fig. 1(b) here. When μ increases from this

band edge (into the first Bloch band), this linear Bloch wave bifurcates into a nonlinear π-periodic Bloch wave with a finite amplitude (under the present repulsive nonlinearity). Numerically we can easily compute these nonlinear Bloch waves, and their amplitudes versus the chemical potential μ are displayed in Fig. 1(a). Here the amplitude of a nonlinear Bloch wave $u(x)$ is defined as the maximum value of $u(x)$. At two representative chemical potentials $\mu=2.16$ and 4 (the former lies inside the first Bloch band and the latter lies inside the first band gap), the corresponding nonlinear Bloch waves are displayed in Figs. 1(b) and 1(c), respectively. As can be seen, as μ increases from the lower band edge, the amplitude of the nonlinear Bloch wave also increases.

Now we truncate these infinitely extended nonlinear Bloch waves to a finite number of intensity peaks (with the μ value fixed) and ask whether the truncated Bloch waves can develop exponentially decaying tails at large $|x|$ values so that they can become localized soliton solutions of Eq. (2). When the chemical potential lies inside the first Bloch band [see Fig. 1(b)], the truncated Bloch waves will still possess continuous-wave tails and are thus still infinitely extended. But when the chemical potential lies inside the first band gap [see Fig. 1(c)], then due to repeated Bragg reflections, the truncated Bloch waves *can* decay exponentially at large $|x|$ values and thus become localized solitons. To confirm this, we numerically compute the soliton solutions of Eq. (2) by the squared-operator iteration methods developed in Ref. [12] using the truncated nonlinear Bloch waves as initial conditions. When truncating the Bloch wave of Fig. 1(c) to three and seven peaks, we indeed obtain soliton solutions, these are displayed in Figs. 2(b) and 2(d) respectively. These solitons closely resemble the corresponding nonlinear Bloch wave [see Fig. 2(d) for comparison], but their tails are now exponentially decaying rather than infinitely extended. If the Bloch wave of Fig. 1(c) is truncated to any other number of peaks, the corresponding soliton state can be obtained as well. Thus a countable number of solitons can be found from different truncations of a nonlinear Bloch wave (at the same μ value). We have thus termed these solitons truncated-Bloch-wave solitons. They were first observed in BEC experiments [8] and then found numerically in Ref. [9].

When the chemical potential changes, the truncated-Bloch-wave solitons will deform. Hence each truncated-Bloch-wave soliton generates a continuous family of such solitons parametrized by the chemical potential. We will call the solution family of the three-peak truncated-Bloch-wave soliton in Fig. 2(b) the three-peak family and that of the seven-peak soliton in Fig. 2(d) the seven-peak family, and so on. Important questions which have not been addressed before are the following: what solutions are contained in each family? What do their power diagrams look like? Here the power of a soliton is defined as $P = \int_{-\infty}^{\infty} |u|^2 dx$ as usual. To address these questions, we have numerically obtained the entire families of two-peak to seven-peak solitons. The power diagrams of three-peak and seven-peak families are displayed in Fig. 3 (power curves for other-peak families are similar). These power diagrams show three important common features about truncated-Bloch-wave solitons. The first feature is that these solitons exist only inside the band gap but not inside a Bloch band. This contrasts with the associ-
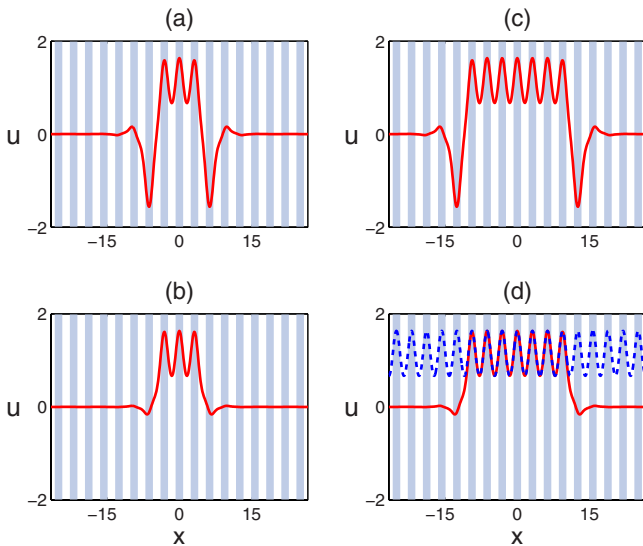

FIG. 2. (Color online) Truncated-Bloch-wave solitons on the upper and lower power branches of the three-peak (left column) and seven-peak (right column) families in the first band gap ($V_0=6$). The chemical potentials of these solitons are marked in Fig. 3. In (d), the corresponding nonlinear Bloch wave (dashed curve) is also shown for comparison [see also Fig. 1(c)].
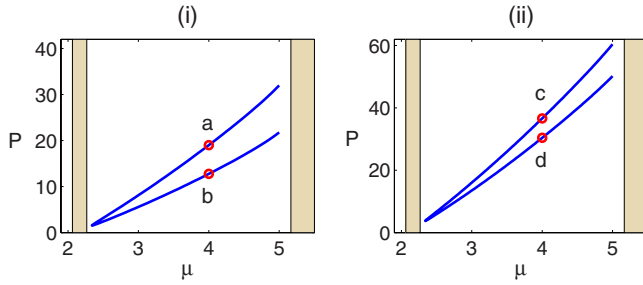
FIG. 3. (Color online) Power curves of the (a) three-peak and (b) seven-peak families of truncated-Bloch-wave solitons ($V_0=6$). The solutions at marked points are displayed in Fig. 2.
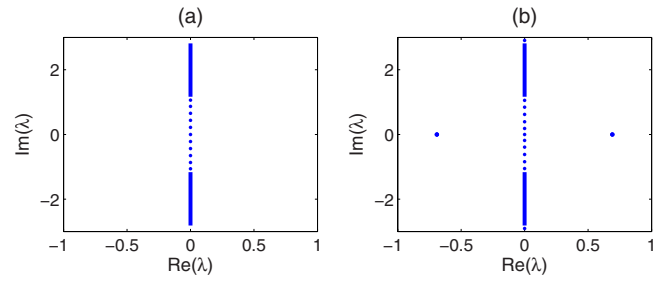

FIG. 4. (Color online) Linear-stability analysis of the seven-peak family of truncated-Bloch-wave solitons in Figs. 1 and 2 ($V_0=6$): (a) stability spectrum of the lower-branch soliton shown in Fig. 2(d); (b) stability spectrum of the upper-branch soliton shown in Fig. 2(c).

ated nonlinear Bloch waves which exist in both the band gap and the Bloch band [see Fig. 1(a)]. The second feature is that these solitons do not bifurcate from the (upper) edge of the (first) Bloch band; thus they are fundamentally different from the gap solitons studied before [7] which do bifurcate from edges of Bloch bands. The reason why these solitons cannot bifurcate from the upper edge of the first Bloch band is obviously because adjacent peaks in these solitons are in phase, which does not match the Bloch wave at the band edge (where adjacent peaks are out of phase). Note that these solitons are closely related to nonlinear Bloch waves which bifurcate from the lower edge of the first Bloch band (see Fig. 1). But unlike those nonlinear Bloch waves, these solitons cannot exist inside the first Bloch band; thus they cannot bifurcate from the lower edge of the first Bloch band. The third common feature of the truncated-Bloch-wave solitons is that their power diagrams exhibit double branches. This double-branch phenomenon has been reported for dipole and vortex solitons in periodic potentials recently [13,14]. Now we know that this phenomenon will always occur for any soliton family which does not bifurcate from the edges of Bloch bands (or edges of the continuous spectrum) as is the case for truncated-Bloch-wave solitons. The three-peak and seven-peak solitons shown in Figs. 2(b) and 2(d) lie on the lower branches of their solution families (more specifically at points b and d marked in Fig. 3). On the upper branches, the solitons will develop two extra intensity peaks at the two edges of the solitons, and these extra peaks are out of phase with the other intensity peaks in the interior. Two examples on the upper branches of the three-peak and seven-peak families are displayed in Figs. 2(a) and 2(c), and their positions on the power diagrams are marked in Fig. 3. It is noted that these solution families of various peaks are distinct from each other, with no s-shaped bifurcations connecting them near the band edge in the power diagram. When the chemical potential μ approaches the right end of the first band gap, all solutions become less localized. If μ crosses this right end into the (second) Bloch band, solutions become delocalized due to resonance with the Bloch modes, and the powers of the solutions become infinite.

Stability of these truncated-Bloch-wave solitons is clearly an important issue, and this issue has not been considered before in the literature. Here we investigate the linear stability of these solitons. Following the standard procedure, we perturb the soliton as $U(x,t)$ $=e^{-i\mu t}\{u(x)+[v(x)-w(x)]e^{\lambda t}+[v^*(x)+w^*(x)]e^{\lambda^* t}\}$, where

$v,w \ll 1$ are normal-mode perturbations, λ is the eigenvalue of the normal mode, and the superscript * represents complex conjugation. Inserting this perturbed soliton into the evolution Eq. (1) and dropping higher-order terms in (v,w), a linear eigenvalue problem $L(v,w)^T=\lambda(v,w)^T$ is obtained. Here L is the linearization operator, and the superscript T represents the transpose of a vector. Linear stability of a soliton is determined by the spectrum of the linearization operator L, and the existence of any eigenvalue with a positive real part implies linear instability of the soliton. We have computed this spectrum for truncated-Bloch-wave solitons of various families in the deep potential of $V_0=6$ (see Figs. 2 and 3) by the Fourier collocation method [15]. We have found that for all these solution families, solitons on the lower branches are always stable and those on the upper branches always unstable. To illustrate, we pick the two solitons in Figs. 2(c) and 2(d), which lie on the upper and lower branches of the seven-peak family in Fig. 3(b). The linear-stability spectra of these two solitons are shown in Figs. 4(b) and 4(a), respectively. We see that the eigenvalues of the lower-branch soliton all lie on the imaginary axis, indicating that this soliton is linearly stable. But the spectrum for the upper-branch soliton contains real positive eigenvalues, indicating its linear instability. Intuitively, the linear instability of this upper-branch soliton is easy to understand. Indeed, this soliton has two intensity peaks on the two edges which are out of phase with the interior peaks. Thus the structures near the edges of this soliton can be viewed as out-of-phase dipoles. It is well known that in lattices under attractive nonlinearity, in-phase dipoles are linearly unstable (see, e.g., Ref. [16]). If the nonlinearity is repulsive as it is in model (1), dipole stability is switched, and out-of-phase dipoles become linearly unstable (see, e.g., Ref. [17]). Consequently, this upper-branch soliton in Fig. 2(c) should be linearly unstable. Regarding the linear stability of the lower-branch soliton, however, it is less obvious intuitively, and this result is one of the main findings of this paper.

The above linear-stability result of lower-branch truncated-Bloch-wave solitons in a deep lattice potential shows that these solitons should also be nonlinearly stable under weak perturbations. To confirm this, we take the seven-peak soliton of Fig. 2(d) and modulate its seven-peak amplitudes by 10% random perturbations. Specifically we add to this soliton a perturbation which is a superposition of
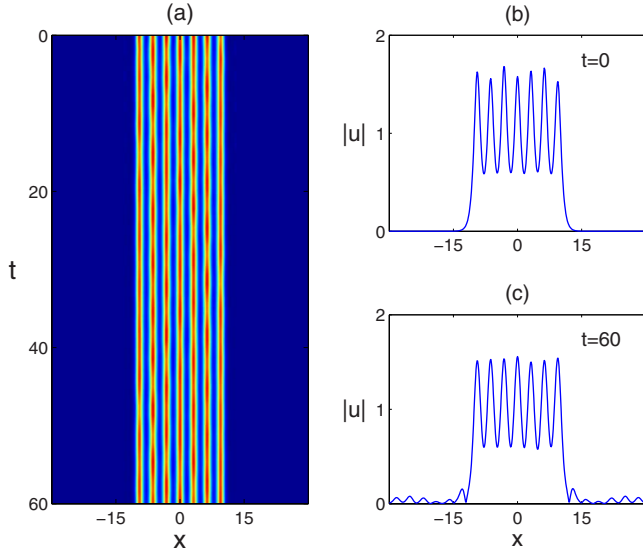
FIG. 5. (Color online) Nonlinear evolution of the seven-peak truncated-Bloch-wave soliton of Fig. 2(d) under 10% perturbations: (a) intensity ($|u|^2$) contours in the (x,t) plane; (b) initial perturbed state of $|u|$; (c) final state of $|u|$ after 60 time-unit evolution.
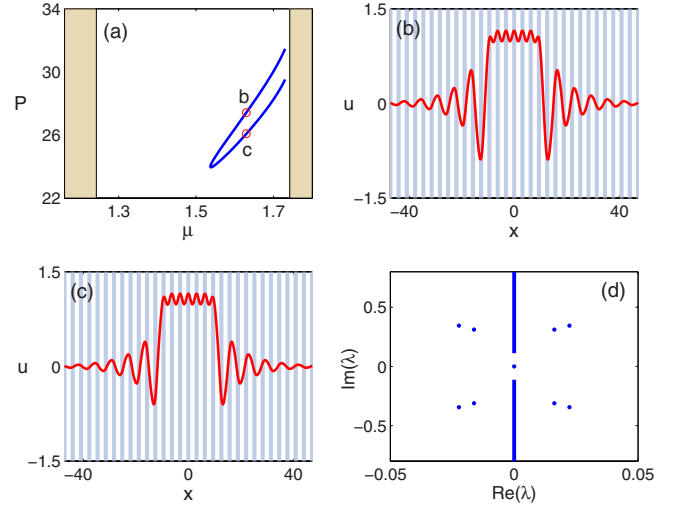

FIG. 6. (Color online) Truncated-Bloch-wave solitons in a shallow lattice potential ($V_0=1$): (a) the power diagram; [(b) and (c)] solitons on the upper and lower branches as marked in (a); (d) linear-stability spectrum for the lower-branch soliton in (c).

seven Gaussian humps of the shape $\exp(-x^2)$ centered at the seven lattice sites of the soliton. The amplitudes of these Gaussian humps are taken randomly at the level of about 10% of the soliton amplitude. For the realization of amplitude values $(0,-0.08,0.06,-0.06,0.01,0.05,-0.11)$ for these Gaussian humps, the nonlinear evolution of the perturbed soliton after 60 time-unit evolution is displayed in Fig. 5. We see that the evolution is stable as expected, showing the robustness of these types of soliton structures in lattice potentials. Evolution with other realizations of the perturbations is qualitatively similar.

These theoretical results appear very similar to the experimental observation of nonlinear self-trapping of Bose-condensed ^{87}Rb atoms in a one-dimensional deep periodic potential [8]. In the experiment of Ref. [8], it was observed that with a higher number of atoms, the repulsive atom-atom interaction led to the formation of a self-trapped broad localized state with steep edges and roughly uniform interior, closely resembling the truncated-Bloch-wave soliton shown in Fig. 2(d). We note however that the lattice depth used in the experiment corresponds to $V_0 \sim 11$, much deeper than the $V_0=6$ considered in Fig. 2. Furthermore, as noted in Ref. [8], the experiment was not carried out in the one-dimensional regime, with the higher-order transverse effects playing an important role. To directly compare our results with those of experiment we may obtain the physical density and atom number from our normalized model by multiplying the normalized density $|U|^2$ by $(2 \times 10^{-6})/8\pi a_L^2 |a_s|$, where a_s is the s-wave scattering length, to obtain the density in atoms cm^{-3} and multiply the power P by $\hbar/(4|a_s|m\omega_\perp a_L)$, where $\omega_\perp$ is the transverse trap frequency, to obtain the total number of atoms. Inherent in our one-dimensional approximation is the assumption that the nonlinear energy at the peak density is much less than the transverse trap energy, i.e., that $|u_{\text{peak}}|^2\hbar/(ma_L^2\omega_\perp)\ll 1$, and as such we consider the experimental parameters $d=3.5$ μm and $\omega_\perp=320$ Hz. For a ^{87}Rb

BEC the seven-peak solution shown in Fig. 2(d) then contains approximately 400 atoms, with $\sim$60 atoms per lattice period and a peak density of $\sim 2.8 \times 10^{-13}$ atoms cm^{-3}, all experimentally reasonable values.

The above theoretical results pertain to truncated-Bloch-wave solitons in a deep lattice potential. What will happen to these solitons if the lattice potential becomes shallow? To address this question, we take a shallow potential with $V_0 =1$ and investigate the existence and stability properties of the truncated-Bloch-wave solitons. For the seven-peak family, the results are summarized in Fig. 6 (results for other-peak families are qualitatively similar). In this shallow potential, we see from Fig. 6(a) that these solitons exist in a smaller region of the first gap. From Figs. 6(b) and 6(c) we see that the edges of these solitons become less steep, their tails become longer, and the solution profiles on the upper and lower branches become more similar. From Fig. 6(d), we see that the soliton on the lower branch now becomes linearly unstable due to oscillatory instabilities caused by complex unstable eigenvalues (this holds for other solitons on the lower branch as well). The solitons on the upper branch are certainly also unstable. Thus in a shallow potential, both branches of truncated-Bloch-wave solitons are linearly unstable, which contrasts with the deep-potential case.

By comparing truncated-Bloch-wave solitons in the deep and shallow potentials of $V_0=6$ and $V_0=1$, we see that when the potential becomes shallow, the existence region of these solitons shrinks toward the upper part of the (first) band gap (see Figs. 3 and 6). This shrinkage continues until the potential depth drops below a critical value of $V_0=0.65$, where these truncated-Bloch-wave solitons all disappear (see also Ref. [9]). Thus very shallow lattice potentials cannot support truncated-Bloch-wave solitons. Regarding the linear stability of these solitons, the results above show that in deep lattice potentials, lower branches of truncated-Bloch-wave solitons are stable, but in shallow potentials, both branches of these solitons are unstable. As the potential depth decreases, the

stable region of truncated-Bloch-wave solitons gradually shrinks toward the middle part of the lower branch and then vanishes when $V_0 \approx 2.3$. Thus whole families of these solitons become unstable when V_0 drops below approximately 2.3. The experiment of Ref. [8], which used $V_0 \sim 11$, was thus far above this instability threshold. As V_0 is defined in the unit of the lattice recoil energy E_{rec}, the dimensionless instability threshold of $V_0 \approx 2.3$ thus corresponds to the dimensional instability threshold of $V_0 \approx 11.35 \hbar^2 / md^2$, which is inversely proportional to the atom mass and the square of the lattice period.

In this paper, we only analyzed truncated-Bloch-wave solitons in the first band gap under repulsive nonlinearity. These solitons were obtained by truncating the nonlinear Bloch waves originating from the lower band edge of the first Bloch band. It should be noted that this branch of nonlinear Bloch waves (see Fig. 1) continues into higher Bloch bands and band gaps. In higher band gaps, by truncating these nonlinear Bloch waves, we may obtain higher-gap truncated-Bloch-wave solitons. It is also noted that from every edge of a Bloch band, a branch of nonlinear Bloch waves bifurcates out for both attractive and repulsive nonlinearities (the bifurcation goes rightward [see Fig. 1(a)] under repulsive nonlinearity and leftward under attractive nonlinearity). Whenever a branch of nonlinear Bloch waves enters a band gap, we may obtain the corresponding truncated-Bloch-wave solitons. These other truncated-Bloch-wave solitons can be similarly analyzed, but this lies outside the scope of the present paper.

In conclusion, we have analyzed the existence and stability of truncated-Bloch-wave solitons existing in the gaps of the linear band-gap spectra of matter waves. We have demonstrated that such self-trapped structures form families of broad but localized solutions which can be identified by the number of peaks. These localized states do not bifurcate from the band edge, and their power diagrams exhibit double branches. Linear-stability analysis demonstrates that in a deep lattice potential, the solutions on the lower branches are stable regardless of the number of peaks, but in a shallow potential, solutions on both branches are unstable. When the lattice potential becomes weak enough, these truncated-Bloch-wave solitons cease to exist.

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