Lett Math Phys (2015) 105:447–461
DOI 10.1007/s11005-015-0747-0

Tetrahedron Equation and Quantum $\boldsymbol{R}$ Matrices for Modular Double of $\boldsymbol{U_q(D^{(2)}_{n+1})}$, $\boldsymbol{U_q(A^{(2)}_{2n})}$ and $\boldsymbol{U_q(C^{(1)}_n)}$

ATSUO KUNIBA[1], MASATO OKADO[2] and SERGEY SERGEEV[3]
[1]*Institute of Physics, University of Tokyo, Komaba, Tokyo 153-8902, Japan. e-mail: atsuo@gokutan.c.u-tokyo.ac.jp*
[2]*Department of Mathematics, Osaka City University, 3-3-138, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan. e-mail: okado@sci.osaka-cu.ac.jp*
[3]*Faculty of Education, Science, Technology, Engineering and Mathematics, University of Canberra, Canberra, ACT 2106, Australia. e-mail: Sergey.Sergeev@canberra.edu.au*

Received: 17 September 2014 / Revised: 20 January 2015 / Accepted: 21 January 2015
Published online: 3 February 2015 –

Abstract. We introduce a homomorphism from the quantum affine algebras $U_q(D^{(2)}_{n+1}), U_q(A^{(2)}_{2n})$, $U_q(C^{(1)}_n)$ to the n-fold tensor product of the q-oscillator algebra $\mathcal{A}_q$. Their action commutes with the solutions of the Yang–Baxter equation obtained by reducing the solutions of the tetrahedron equation associated with the modular and the Fock representations of $\mathcal{A}_q$. In the former case, the commutativity is enhanced to the modular double of these quantum affine algebras.

Mathematics Subject Classification. 81R50, 17B37, 16T25.

Keywords. Tetrahedron equation, q-oscillator algebra, Yang–Baxter equation, modular double.

1. Introduction

The tetrahedron equation [24] is a three-dimensional (3D) generalization of the Yang–Baxter equation [1]. Among its several versions, the basic one adapted to homogeneous 3D vertex models has the form

$$R_{1,2,4}R_{1,3,5}R_{2,3,6}R_{4,5,6} = R_{4,5,6}R_{2,3,6}R_{1,3,5}R_{1,2,4}, \tag{1.1}$$

where R is a linear operator on the tensor cube of some vector space. The equality holds for the operators on its sixfold tensor product, where the indices specify the components on which R acts nontrivially. We call a solution to the tetrahedron equation a 3D R.

Several solutions have been found until now with some important clues to the relevant algebraic structures such as the quantized coordinate ring of SL_3 [14], PBW basis of the nilpotent subalgebra of $U_q(sl_3)$ [20], the q-oscillator algebra $\mathcal{A}_q$ [3,5] and so forth. It is known [16] that the 3D R associated with the Fock representation of $\mathcal{A}_q$ [5] coincides with the one in [14].

The tetrahedron equation reduces to the Yang–Baxter equation

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$$

if the spaces $4, 5, 6$ are evaluated away suitably [15,21]. In [17,18], such a reduction was achieved by taking the matrix elements with respect to certain boundary vectors. The resulting solutions to the Yang–Baxter equation were identified with the quantum R matrices of various quantum affine algebras $U_q(\mathfrak{g})$ and their representations.

In this paper, we exploit further aspects of the 3D R associated with what we call the modular and the Fock representations of the q-oscillator algebra $\mathcal{A}_q$. We use the two boundary vectors to generate four families of solutions to the Yang–Baxter equation for each representation. Our first result, Theorem 4.1, is that they commute with the quantum affine algebras $U_q(\mathfrak{g})$ with $\mathfrak{g} = D^{(2)}_{n+1}, C^{(1)}_n, A^{(2)}_{2n}$ and $\tilde{A}^{(2)}_{2n}$ (see Section 2.1 for the definition). The essential ingredient for this statement is a new homomorphism from $U_q(\mathfrak{g})$ to $\mathcal{A}_q^{\otimes n}$ in Proposition 2.1. The two boundary vectors correspond to the short and the long simple roots of $\mathfrak{g}$ at the two ends of the Dynkin diagram.

Our second result, Theorem 5.3, is obtained by applying Theorem 4.1 to the modular representation of the pair $(\mathcal{A}_q, \mathcal{A}_{\tilde{q}})$ such that $(\log q)(\log \tilde{q}) = -\pi^2$. We find that the symmetry of the relevant solutions of the Yang–Baxter equation is enhanced naturally from $U_q(\mathfrak{g})$ to its modular double $U_q(\mathfrak{g}) \otimes U_{\tilde{q}}({}^L\mathfrak{g})$ where ${}^L\mathfrak{g}$ is the Langlands dual of $\mathfrak{g}$. The key to this result is Proposition 5.2 showing that the two boundary vectors interchange their role when passing to the modular dual. An analogous feature has been observed in [11]. For general background on modular double, we refer to [8,9].

The layout of the paper is as follows. In Section 2, the algebra homomorphism from $U_q(\mathfrak{g})$ to $\mathcal{A}_q^{\otimes n}$ is presented in Proposition 2.1. In Section 3, the 3D R [3,5] and the characterization of the boundary vectors [18] are recapitulated. In Section 4, reduction to the Yang–Baxter equation [17,18] is explained and the symmetry of the consequent solution is described in Theorem 4.1. All the arguments until this point are valid either for the modular or the Fock representations of $\mathcal{A}_q$. They systematize the proof of the commutativity significantly. In Section 5, the general construction in the preceding sections are embodied in the modular representation. The boundary vectors in this representation are new and described explicitly in terms of their wave functions. They lead to our main result, Theorem 5.3. In Section 6, it is explained how the specialization of the general results in Sections 2, 3, 4 to the Fock representation covers an essential part of the earlier result [17]. Appendix A contains identities involving the boundary wave function $\chi_b(\sigma)$ and the quantum dilogarithm.

2. Quantum Affine Algebras and q-Oscillator Algebra

2.1. QUANTUM AFFINE ALGEBRAS

We assume that q is generic except in Section 5 and Appendix A. (The basic parameter is $q^{\frac{1}{2}}$ rather than q in our convention.) The Drinfeld–Jimbo quantum affine algebras (without derivation operator) $U_q = U_q(A^{(2)}_{2n})$, $U_q(\tilde{A}^{(2)}_{2n})$, $U_q(C^{(1)}_n)$ and $U_q(D^{(2)}_{n+1})$ are the Hopf algebras generated by $e_i, f_i, k_i^{\pm 1}$ $(0 \le i \le n)$ satisfying the relations [7,12]

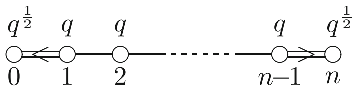
$$\begin{aligned} &k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad [k_i, k_j] = 0, \\ &k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \\ &\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu e_i^{(1-a_{ij}-\nu)} e_j e_i^{(\nu)} = 0, \quad \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu f_i^{(1-a_{ij}-\nu)} f_j f_i^{(\nu)} = 0 \quad (i \ne j), \end{aligned} \tag{2.1}$$

where $e_i^{(\nu)} = e_i^\nu / [\nu]_{q_i}!$, $f_i^{(\nu)} = f_i^\nu / [\nu]_{q_i}!$ and $[m]_q! = \prod_{k=1}^m [k]_q$ with $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$. The Cartan matrix $(a_{ij})_{0 \le i,j \le n}$ [13] is given by

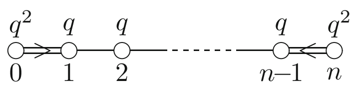
$$a_{i,j} = 2\delta_{i,j} - \max\big((\log q_j)/(\log q_i), 1\big)\delta_{|i-j|,1}.$$

The data q_i are specified above the corresponding vertex i $(0 \le i \le n)$ in the Dynkin diagrams:

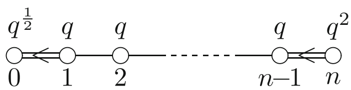
$\mathfrak{g}^{1,1} = D^{(2)}_{n+1}$



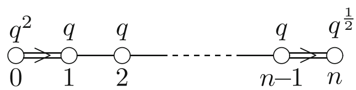
$\mathfrak{g}^{2,2} = C^{(1)}_n$



$\mathfrak{g}^{1,2} = A^{(2)}_{2n}$



$\mathfrak{g}^{2,1} = \tilde{A}^{(2)}_{2n}$



We also let $\mathfrak{g}^{s,t}$ $\big(s, t \in \{1, 2\}\big)$ denote the relevant affine Lie algebras as above. $\mathfrak{g}^{2,1} = \tilde{A}^{(2)}_{2n}$ is isomorphic to $\mathfrak{g}^{1,2} = A^{(2)}_{2n}$ and their difference is only the enumeration of vertices. The Langlands dual of $\mathfrak{g}^{s,t}$ is given by ${}^L\mathfrak{g}^{s,t} = \mathfrak{g}^{3-s,3-t}$. Note that $q_0 = q^{s^2/2}$, $q_n = q^{t^2/2}$ and $q_i = q$ for $0 < i < n$.

The coproduct Δ has the form

$$\Delta k_i^{\pm 1} = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta e_i = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i. \tag{2.2}$$

The opposite coproduct is denoted by $\Delta' = P \circ \Delta$, where $P(u \otimes v) = v \otimes u$ is the exchange of the components.

2.2. HOMOMORPHISM FROM U_q TO q-OSCILLATOR ALGEBRA

Let $\mathcal{A}_q$ be the algebra over $\mathbb{C}(q^{\frac{1}{2}})$ generated by $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ and $\mathbf{k}^{-1}$ obeying the relations

$$\mathbf{k}\,\mathbf{k}^{-1} = \mathbf{k}^{-1}\,\mathbf{k} = 1, \quad \mathbf{k}\,\mathbf{a}^{\pm} = q^{\pm 1}\mathbf{a}^{\pm}\,\mathbf{k}, \quad \mathbf{a}^{\pm}\,\mathbf{a}^{\mp} = 1 - q^{\mp 1}\mathbf{k}^2. \tag{2.3}$$

The algebra $\mathcal{A}_q$, which we call the q-oscillator algebra, plays a central role in this paper. Set

$$d = \frac{q}{(q - q^{-1})^2}, \quad d_1 = d|_{q \to q^{1/2}}, \quad d_2 = d|_{q \to q^2}.$$

In what follows an element $\mathbf{a}^+ \otimes 1 \otimes \mathbf{k} \otimes \mathbf{a}^- \in \mathcal{A}_q^{\otimes 4}$ for example will be denoted by $\mathbf{a}_1^+\mathbf{k}_3\mathbf{a}_4^-$ etc. Thus the q-oscillator generators with different indices are commuting.

PROPOSITION 2.1. *For a parameter z the following map defines an algebra homomorphism $\pi_z : U_q(\mathfrak{g}^{s,t}) \to \mathcal{A}_q^{\otimes n}[z, z^{-1}]$. (On the left-hand side, $\pi_z(g)$ is denoted by g for simplicity.)*

$$\begin{aligned}
&e_0 = z^s d_s (\mathbf{a}_1^+)^s, && f_0 = z^{-s} \mathrm{i}^{s^2} (\mathbf{a}_1^-)^s \mathbf{k}_1^{-s}, && k_0 = (\mathrm{i}\mathbf{k}_1)^s,\\
&e_i = d\, \mathbf{a}_i^- \mathbf{a}_{i+1}^+ \mathbf{k}_i^{-1}, && f_i = \mathbf{a}_i^+ \mathbf{a}_{i+1}^- \mathbf{k}_{i+1}^{-1}, && k_i = \mathbf{k}_i^{-1}\mathbf{k}_{i+1} \quad (0 < i < n),\\
&e_n = \mathrm{i}^{t^2} d_t (\mathbf{a}_n^-)^t \mathbf{k}_n^{-t}, && f_n = (\mathbf{a}_n^+)^t, && k_n = (-\mathrm{i}\mathbf{k}_n^{-1})^t.
\end{aligned}$$

The proposition can be shown by directly checking the relations (2.1). The convention $z^{\pm s}$ rather than $z^{\pm 1}$ is just to avoid $z^{\mathbf{h}_3/s}$ in the forthcoming formula (4.1).

Remark 2.2. If the formulas for e_i, f_i, k_i with $0 < i < n$ are interpreted with $i \in \mathbb{Z}_n$, then Proposition 2.1 gives an algebra homomorphism $U_q(A_{n-1}^{(1)}) \to \mathcal{A}_q^{\otimes n}[z, z^{-1}]$. For this case and $U_q(\mathfrak{g}^{s,t})$ without e_0, f_0 and k_0, the homomorphism π_z was essentially known in [10]. For type A with q roots of unity, similar homomorphisms and their applications have been studied in [2,4,6,10,22,23].

3. 3 Dimensional *R* and Boundary Vectors

In Sections 5 and 6, we will consider the modular representation and the Fock representation of the q-oscillator algebra $\mathcal{A}_q$. Let M uniformly denote the left $\mathcal{A}_q$ module therein. Then there is a unique (up to sign) involutive operator $R \in \mathrm{End}(M^{\otimes 3})$ [3,5] such that

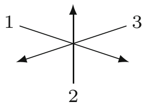
$$R\,\mathbf{k}_2\mathbf{a}_1^+ = (\mathbf{k}_3\mathbf{a}_1^+ + \mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-)R, \qquad R\,\mathbf{k}_2\mathbf{a}_1^- = (\mathbf{k}_3\mathbf{a}_1^- + \mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+)R, \tag{3.1}$$

$$R\,\mathbf{a}_2^+ = (\mathbf{a}_1^+\mathbf{a}_3^+ - \mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+)R, \qquad R\,\mathbf{a}_2^- = (\mathbf{a}_1^-\mathbf{a}_3^- - \mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-)R, \tag{3.2}$$

$$R\,\mathbf{k}_2\mathbf{a}_3^+ = (\mathbf{k}_1\mathbf{a}_3^+ + \mathbf{k}_3\mathbf{a}_1^-\mathbf{a}_2^+)R, \qquad R\,\mathbf{k}_2\mathbf{a}_3^- = (\mathbf{k}_1\mathbf{a}_3^- + \mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_2^-)R, \tag{3.3}$$

$$R\,\mathbf{k}_1\mathbf{k}_2 = \mathbf{k}_1\mathbf{k}_2R, \qquad R\,\mathbf{k}_2\mathbf{k}_3 = \mathbf{k}_2\mathbf{k}_3R. \tag{3.4}$$

Moreover, it satisfies the tetrahedron equation (1.1) in $\mathrm{End}(M^{\otimes 6})$. We simply call it the 3D R. It is customary to depict $R = R_{1,2,3}$ as the intersection of the three arrows $1, 2$ and 3:

$R_{1,2,3} =$ 

Let M (resp. M^*) be a left (resp. right) $\mathcal{A}_q$ module. Suppose they are equipped with the bilinear pairing $\langle\ |\ \rangle : M^* \times M \to \mathbb{C}$ such that $\langle \tilde{m}'|m\rangle = \langle m'|\tilde{m}\rangle$ ($\langle \tilde{m}'| := \langle m'|g$, $|\tilde{m}\rangle := g|m\rangle$) for any $g \in \mathcal{A}_q$.

Consider the vectors $|\chi^{(s)}\rangle \in M$ and $\langle \chi^{(s)}| \in M^*$ for $s = 1, 2$ satisfying

$$\mathbf{a}^\pm|\chi^{(1)}\rangle = (1 \mp q^{\mp\frac{1}{2}}\mathbf{k})|\chi^{(1)}\rangle, \quad \langle\chi^{(1)}|\mathbf{a}^\pm = \langle\chi^{(1)}|(1 \pm q^{\pm\frac{1}{2}}\mathbf{k}), \tag{3.5}$$

$$\mathbf{a}^+|\chi^{(2)}\rangle = \mathbf{a}^-|\chi^{(2)}\rangle, \qquad \langle\chi^{(2)}|\mathbf{a}^+ = \langle\chi^{(2)}|\mathbf{a}^-. \tag{3.6}$$

PROPOSITION 3.1. [18, Prop. 4.1] *The following equalities in $M^{\otimes 3}$ and $M^{*\otimes 3}$ are valid for $s = 1, 2$:*

$$R\Big(|\chi^{(s)}\rangle \otimes |\chi^{(s)}\rangle \otimes |\chi^{(s)}\rangle\Big) = |\chi^{(s)}\rangle \otimes |\chi^{(s)}\rangle \otimes |\chi^{(s)}\rangle,$$

$$\Big(\langle\chi^{(s)}| \otimes \langle\chi^{(s)}| \otimes \langle\chi^{(s)}|\Big)R = \langle\chi^{(s)}| \otimes \langle\chi^{(s)}| \otimes \langle\chi^{(s)}|.$$

We call these vectors boundary vectors. In the representations considered in Sections 5 and 6, the generator $\mathbf{k}$ is expressed as $\mathrm{const}\cdot q^{\mathbf{h}}$ using some operator $\mathbf{h}$. It satisfies $[\mathbf{h}, \mathbf{a}^\pm] = \pm\mathbf{a}^\pm$ according to (2.3). Moreover, the relation (3.4) implies

$$[R, \mathbf{h}_1 + \mathbf{h}_2] = [R, \mathbf{h}_2 + \mathbf{h}_3] = 0 \tag{3.7}$$

for $R = R_{1,2,3}$. It is an analog of the ice rule for the 6 vertex model [1] and will be referred to as the conservation law.

4. Solution of Yang–Baxter Equation

4.1. GENERAL CONSTRUCTION

The 3D R and the boundary vectors enable one to construct a family of solutions of the Yang–Baxter equation labeled with $n \geq 1$ [17,18]. Consider $3n+3$ copies of

M labeled with $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,\gamma_1,\dots,\gamma_n$ and 4, 5, 6. Composing the tetrahedron equation (1.1) with the spaces 1, 2, 3 relabeled as $\alpha_i,\beta_i,\gamma_i$, we get

$$x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}\big(R_{\alpha_1,\beta_1,4}R_{\alpha_1,\gamma_1,5}R_{\beta_1,\gamma_1,6}\big)\cdots\big(R_{\alpha_n,\beta_n,4}R_{\alpha_n,\gamma_n,5}R_{\beta_n,\gamma_n,6}\big)R_{4,5,6}$$
$$=R_{4,5,6}x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}y^{\mathbf{h}_6}\big(R_{\beta_1,\gamma_1,6}R_{\alpha_1,\gamma_1,5}R_{\alpha_1,\beta_1,4}\big)\cdots\big(R_{\beta_n,\gamma_n,6}R_{\alpha_n,\gamma_n,5}R_{\alpha_n,\beta_n,4}\big),$$

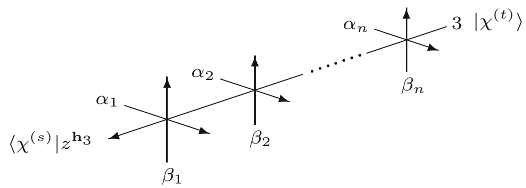
where we have multiplied $x^{\mathbf{h}_4+\mathbf{h}_5}y^{\mathbf{h}_5+\mathbf{h}_6}$ from the left and used (3.7) on the right-hand side. Regard the boundary vectors in Proposition 3.1 as belonging to the spaces 4, 5, 6. Then evaluating the above relation between the boundary vectors one obtains the Yang–Baxter equation

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x)S_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(xy)S_{\boldsymbol{\beta},\boldsymbol{\gamma}}(y)=S_{\boldsymbol{\beta},\boldsymbol{\gamma}}(y)S_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(xy)S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x),$$

where $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)$ etc. The solution $S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)$ takes a matrix product form with the boundary "magnetic field" $z^{\mathbf{h}_3}$:

$$S_{\boldsymbol{\alpha},\boldsymbol{\beta}}(z)=\langle\chi^{(s)}|z^{\mathbf{h}_3}R_{\alpha_1,\beta_1,3}R_{\alpha_2,\beta_2,3}\cdots R_{\alpha_n,\beta_n,3}|\chi^{(t)}\rangle. \tag{4.1}$$

The composition of the 3D R and the evaluation by bra and ket vectors in (4.1) are taken with respect to the space M signified by 3. Plainly $S(z)\in\mathrm{End}(M^{\otimes n}\otimes M^{\otimes n})$ suppressing the dummy labels. We will denote $S(z)$ by $S^{s,t}(z)$ when the dependence on $s,t\in\{1,2\}$ should be emphasized. The formula (4.1) is depicted as



4.2. QUANTUM GROUP SYMMETRY

We supplement the q-oscillator algebra $\mathcal{A}_q^{\otimes n}$ or its representation $\mathrm{End}(M^{\otimes n})$ with an invertible element K satisfying the relations

$$K\mathbf{k}_j=\mathbf{k}_jK,\quad K\mathbf{a}_j^{\pm}=(\mathrm{i}q^{\frac{1}{2}})^{\pm1}\mathbf{a}_j^{\pm}K\quad(1\le j\le n). \tag{4.2}$$

Introduce a slightly modified $S^{s,t}(z)$ by the so-called "zig-zag transformation":

$$\hat{S}^{s,t}(z)=(K\otimes1)S^{s,t}(z)(1\otimes K^{-1}). \tag{4.3}$$

In view of $\mathbf{k}\,\mathbf{a}^{\pm}=q^{\pm1}\mathbf{a}\,\mathbf{k}$ in (2.3) one can formally realize K as $K=(\mathbf{k}_1\dots\mathbf{k}_n)^{\nu}$ for some ν. From this fact and (3.4), it follows that $[S^{s,t}(z),K\otimes K]=0$. Using these properties one can show that $\hat{S}^{s,t}(z)$ also satisfies the Yang–Baxter equation.

Given parameters x,y and $g\in U_q$, let $\Delta'(g)$ and $\Delta(g)$ simply mean the image of $(\pi_x\otimes\pi_y)\big(\Delta'(g)\big)$ and $(\pi_x\otimes\pi_y)(\Delta(g))$ in $\mathrm{End}(M^{\otimes n}\otimes M^{\otimes n})$ by the representation of $\mathcal{A}_q$. The following theorem, which is the main result of this section, shows the

U_q-symmetry of $\hat{S}^{s,t}(z)$. The statement and the proof given in Section 4.3 are valid irrespectively of the representations of $\mathcal{A}_q$.

THEOREM 4.1. *With the choice $z=x/y$, the following commutativity holds for $s,t\in\{1,2\}$:*

$$\Delta'(g)\hat{S}^{s,t}(z)=\hat{S}^{s,t}(z)\Delta(g) \qquad \forall g\in U_q(\mathfrak{g}^{s,t}). \tag{4.4}$$

4.3. PROOF OF THEOREM 4.1

It suffices to show it for $g=k_i, e_i$ and f_i $(0\le i\le n)$. The case $g=k_i$ follows easily from the first relation of (2.2), Proposition 2.1 and (3.4). Let us present a proof for $g=e_i$. In terms of $S^{s,t}(z)$ the relation (4.4) with $g=e_i$ takes the form

$$(\tilde{e}_i\otimes 1+k_i\otimes e_i)S^{s,t}(z)-S^{s,t}(z)(1\otimes\tilde{e}_i+e_i\otimes k_i)=0. \tag{4.5}$$

Here $\tilde{e}_i=K^{-1}e_iK$ and the symbol $\pi_x\otimes\pi_y$ is again omitted.

(i) Case $0<i<n$. From Proposition 2.1 and (4.2) we find $\tilde{e}_i=e_i$. Moreover e_i and k_i act nontrivially only on the factors $R_{\alpha_i,\beta_i,3}R_{\alpha_{i+1},\beta_{i+1},3}$ constituting $S^{s,t}(z)$ in (4.1). We rename the spaces $\alpha_i,\alpha_{i+1},\beta_i,\beta_{i+1}$ as $1,1',2,2'$, respectively. Accordingly $R_{\alpha_i,\beta_i,3}R_{\alpha_{i+1},\beta_{i+1},3}=R_{1,2,3}R_{1',2',3}$ will simply be denoted by RR' with the product to be understood in the space 3. From Proposition 2.1 the proof of (4.5) is reduced to showing

$$(\mathbf{a}^-_1\mathbf{a}^+_{1'}\mathbf{k}^{-1}_1+\mathbf{k}^{-1}_1\mathbf{k}_{1'}\mathbf{a}^-_2\mathbf{a}^+_{2'}\mathbf{k}^{-1}_2)RR'-RR'(\mathbf{a}^-_2\mathbf{a}^+_{2'}\mathbf{k}^{-1}_2+\mathbf{a}^-_1\mathbf{a}^+_{1'}\mathbf{k}^{-1}_1\mathbf{k}^{-1}_2\mathbf{k}_{2'})=0.$$

All the terms here are transformed into the form $R(\cdots)R'$ using the defining relations (3.1), (3.2), (3.3), (3.4) and their alternative forms via $R=R^{-1}$ as follows.

$$\begin{aligned}
&\mathbf{a}^-_1\mathbf{a}^+_{1'}\mathbf{k}^{-1}_1RR'=\mathbf{k}_2\mathbf{a}^-_1R\mathbf{k}^{-1}_1\mathbf{k}^{-1}_2\mathbf{a}^+_{1'}R'=R(\mathbf{k}_3\mathbf{a}^-_1+\mathbf{k}_1\mathbf{a}^-_2\mathbf{a}^+_3)\mathbf{k}^{-1}_1\mathbf{k}^{-1}_2\mathbf{a}^+_{1'}R',\\
&\mathbf{k}^{-1}_1\mathbf{k}_{1'}\mathbf{a}^-_2\mathbf{a}^+_{2'}\mathbf{k}^{-1}_2RR'=\mathbf{a}^-_2R\mathbf{k}^{-1}_1\mathbf{k}^{-1}_2\mathbf{k}_{1'}\mathbf{a}^+_{2'}R'=R(\mathbf{a}^-_1\mathbf{a}^-_3-\mathbf{k}_1\mathbf{k}_3\mathbf{a}^-_2)\mathbf{k}^{-1}_1\mathbf{k}^{-1}_2\mathbf{k}_{1'}\mathbf{a}^+_{2'}R',\\
&RR'\mathbf{a}^-_2\mathbf{a}^+_{2'}\mathbf{k}^{-1}_2=R\mathbf{a}^-_2\mathbf{k}^{-1}_2R'\mathbf{a}^+_{2'}=R\mathbf{a}^-_2\mathbf{k}^{-1}_2(\mathbf{a}^+_{1'}\mathbf{a}^+_3-\mathbf{k}_{1'}\mathbf{k}_3\mathbf{a}^+_{2'})R',\\
&RR'\mathbf{a}^-_1\mathbf{a}^+_{1'}\mathbf{k}^{-1}_1\mathbf{k}^{-1}_2\mathbf{k}_{2'}=R\mathbf{a}^-_1\mathbf{k}^{-1}_1\mathbf{k}^{-1}_2R'\mathbf{k}_{2'}\mathbf{a}^+_{1'}=R\mathbf{a}^-_1\mathbf{k}^{-1}_1\mathbf{k}^{-1}_2(\mathbf{k}_3\mathbf{a}^+_{1'}+\mathbf{k}_{1'}\mathbf{a}^+_{2'}\mathbf{a}^-_3)R'.
\end{aligned}$$

To see the cancelation of these terms is now straightforward.

(ii) Case $i=0$ and $s=1$. e_0 and k_0 act nontrivially only on the factor $R=R_{\alpha_1,\beta_1,3}$ in (4.1). From Proposition 2.1 and (4.2), we find $\tilde{e}_0=-\mathrm{i}q^{-\frac{1}{2}}e_0$. Renaming the spaces α_1 and β_1 as 1 and 2, we see that (4.5) is reduced to $0=\langle\chi^{(1)}|z^{\mathbf{h}_3}(-\mathrm{i}q^{-\frac{1}{2}}x\mathbf{a}^+_1+y\mathrm{i}\mathbf{k}_1\mathbf{a}^+_2)R-\langle\chi^{(1)}|z^{\mathbf{h}_3}R(-\mathrm{i}q^{-\frac{1}{2}}y\mathbf{a}^+_2+x\mathbf{a}^+_1\mathrm{i}\mathbf{k}_2)$. Up to an overall factor the last quantity is calculated as

$$\begin{aligned}&\langle\chi^{(1)}|z^{\mathbf{h}_3}\left(q^{-\frac{1}{2}}z\mathbf{a}_1^+R-\mathbf{k}_1\mathbf{a}_2^+R-q^{-\frac{1}{2}}R\mathbf{a}_2^++zR\mathbf{k}_2\mathbf{a}_1^+\right)\\&\quad=\langle\chi^{(1)}|z^{\mathbf{h}_3}\left(q^{-\frac{1}{2}}z\mathbf{a}_1^+-\mathbf{k}_1\mathbf{a}_2^+-q^{-\frac{1}{2}}(\mathbf{a}_1^+\mathbf{a}_3^+-\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+)+z(\mathbf{k}_3\mathbf{a}_1^++\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-)\right)R.\end{aligned}$$

Due to $\langle\chi^{(1)}|z^{\mathbf{h}_3}\mathbf{a}_3^\pm=z^{\pm1}\langle\chi^{(1)}|z^{\mathbf{h}_3}(1\pm q^{\pm\frac{1}{2}}\mathbf{k}_3)$ by (3.5), this vanishes.

(iii) Case $i=0$ and $s=2$. We have $\tilde{e}_0=-q^{-1}e_0$ and (4.5) is reduced to

$$\begin{aligned}0&=\langle\chi^{(2)}|z^{\mathbf{h}_3}\left(q^{-1}z^2(\mathbf{a}_1^+)^2R+\mathbf{k}_1^2(\mathbf{a}_2^+)^2R-q^{-1}R(\mathbf{a}_2^+)^2-z^2R(\mathbf{a}_1^+)^2\mathbf{k}_2^2\right)\\&=\langle\chi^{(2)}|z^{\mathbf{h}_3}\left(q^{-1}z^2(\mathbf{a}_1^+)^2+\mathbf{k}_1^2(\mathbf{a}_2^+)^2-q^{-1}(\mathbf{a}_1^+\mathbf{a}_3^+-\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+)^2\right.\\&\quad\left.-z^2(\mathbf{k}_3\mathbf{a}_1^++\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-)^2\right)R.\end{aligned}$$

Due to (2.3) and $\langle\chi^{(2)}|z^{\mathbf{h}_3}\mathbf{a}_3^+=z^2\langle\chi^{(2)}|z^{\mathbf{h}_3}\mathbf{a}_3^-$ by (3.6), this vanishes. We remark that z appearing in π_z is linked to the z-commuting relation $z^{\mathbf{h}_3}\mathbf{a}_3^\pm=z^{\pm1}\mathbf{a}_3^\pm z^{\mathbf{h}_3}$ relevant to $i=0$, i.e., Case (ii) and Case (iii).

(iv) Case $i=n$ and $t=1$. e_n and k_n act nontrivially only on the factor $R=R_{\alpha_n,\beta_n,3}$ in (4.1). From Proposition 2.1 and (4.2), we find $\tilde{e}_n=\mathrm{i}q^{\frac{1}{2}}e_n$. Renaming the spaces α_n and β_n as 1 and 2, we see that (4.5) is reduced to

$$\begin{aligned}0&=\left(q^{\frac{1}{2}}\mathbf{a}_1^-\mathbf{k}_1^{-1}-\mathbf{k}_1^{-1}\mathbf{a}_2^-\mathbf{k}_2^{-1}\right)R|\chi^{(1)}\rangle-R\left(q^{\frac{1}{2}}\mathbf{a}_2^-\mathbf{k}_2^{-1}-\mathbf{a}_1^-\mathbf{k}_1^{-1}\mathbf{k}_2^{-1}\right)|\chi^{(1)}\rangle\\&=\left(q^{\frac{1}{2}}\mathbf{k}_2\mathbf{a}_1^--\mathbf{a}_2^-\right)R\mathbf{k}_1^{-1}\mathbf{k}_2^{-1}|\chi^{(1)}\rangle-R\left(q^{\frac{1}{2}}\mathbf{a}_2^-\mathbf{k}_2^{-1}-\mathbf{a}_1^-\mathbf{k}_1^{-1}\mathbf{k}_2^{-1}\right)|\chi^{(1)}\rangle\\&=R\left(\left(q^{\frac{1}{2}}(\mathbf{k}_3\mathbf{a}_1^-+\mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+)-(\mathbf{a}_1^-\mathbf{a}_3^--\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-)\right)\mathbf{k}_1^{-1}\mathbf{k}_2^{-1}\right.\\&\quad\left.-q^{\frac{1}{2}}\mathbf{a}_2^-\mathbf{k}_2^{-1}+\mathbf{a}_1^-\mathbf{k}_1^{-1}\mathbf{k}_2^{-1}\right)|\chi^{(1)}\rangle.\end{aligned}$$

Due to $\mathbf{a}_3^\pm|\chi^{(1)}\rangle=(1\mp q^{\mp\frac{1}{2}}\mathbf{k}_3)|\chi^{(1)}\rangle$ by (3.5), this vanishes.

(v) Case $i=n$ and $t=2$. We have $\tilde{e}_n=-qe_n$ and (4.5) is reduced to

$$\begin{aligned}0&=\left(q(\mathbf{a}_1^-)^2\mathbf{k}_1^{-2}+\mathbf{k}_1^{-2}(\mathbf{a}_2^-)^2\mathbf{k}_2^{-2}\right)R|\chi^{(2)}\rangle-R\left((\mathbf{a}_2^-)^2\mathbf{k}_2^{-2}+(\mathbf{a}_1^-)^2\mathbf{k}_1^{-2}\mathbf{k}_2^{-2}\right)|\chi^{(2)}\rangle\\&=R\left(\left(q(\mathbf{k}_3\mathbf{a}_1^-+\mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+)^2+(\mathbf{a}_1^-\mathbf{a}_3^--\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-)^2\right)\mathbf{k}_1^{-2}\mathbf{k}_2^{-2}\right.\\&\quad\left.-(\mathbf{a}_2^-)^2\mathbf{k}_2^{-2}-(\mathbf{a}_1^-)^2\mathbf{k}_1^{-2}\mathbf{k}_2^{-2}\right)|\chi^{(2)}\rangle.\end{aligned}$$

Due to (2.3) and $\mathbf{a}_3^+|\chi^{(2)}\rangle=\mathbf{a}_3^-|\chi^{(2)}\rangle$ by (3.6), this vanishes. This completes the proof of (4.4) for all $g=e_i$. The case $g=f_i$ can be verified similarly. □

5. Example: Modular Representation

5.1. MODULAR REPRESENTATION OF q-OSCILLATOR ALGEBRA

Let $\boldsymbol{\sigma}, \boldsymbol{p}$ be the generators of the Heisenberg algebra $[\boldsymbol{\sigma},\boldsymbol{p}]=\frac{\mathrm{i}}{2\pi}$. We introduce a modular pair of the Weyl algebras, the exponential form of the Heisenberg algebra, by

$$\begin{aligned} &\mathbf{k} = -\mathrm{i}e^{\pi b\boldsymbol{\sigma}}, \quad \mathbf{w} = e^{2\pi b\boldsymbol{p}}, \quad \mathbf{kw} = q\mathbf{wk}, \quad q = e^{\mathrm{i}\pi b^2}, \\ &\tilde{\mathbf{k}} = -\mathrm{i}e^{\pi b^{-1}\boldsymbol{\sigma}}, \quad \tilde{\mathbf{w}} = e^{2\pi b^{-1}\boldsymbol{p}}, \quad \tilde{\mathbf{k}}\tilde{\mathbf{w}} = \tilde{q}\tilde{\mathbf{w}}\tilde{\mathbf{k}}, \quad \tilde{q} = e^{\mathrm{i}\pi b^{-2}}. \end{aligned} \tag{5.1}$$

It also satisfies $\mathbf{k}\tilde{\mathbf{w}} = -\tilde{\mathbf{w}}\mathbf{k}$, $\tilde{\mathbf{k}}\mathbf{w} = -\mathbf{w}\tilde{\mathbf{k}}$, $[\mathbf{k}, \tilde{\mathbf{k}}] = [\mathbf{w}, \tilde{\mathbf{w}}] = 0$. The “tilde” transformation just means the replacement $b \to b^{-1}$. We set $\eta = \frac{1}{2}(b + b^{-1})$ and concentrate on the so-called strong coupling regime $0 < \eta < 1$ in this section. This implies that $|b| = 1, \mathrm{Re}(b) = \mathrm{Re}(b^{-1}) = \eta$.

Recall that $\mathcal{A}_q = \langle \mathbf{a}^{\pm}, \mathbf{k}^{\pm}\rangle$ is the q-oscillator algebra (2.3). We call the q-oscillator algebra $\mathcal{A}_{\tilde{q}} = \langle \tilde{\mathbf{a}}^{\pm}, \tilde{\mathbf{k}}^{\pm}\rangle$ the modular dual of $\mathcal{A}_q$. Identifying the generators $\mathbf{k}, \tilde{\mathbf{k}}$ in $\mathcal{A}_q, \mathcal{A}_{\tilde{q}}$ with those in the Weyl algebras, it is easy to see that

$$\begin{aligned} &\mathbf{a}^+ = (1 - q^{-1}\mathbf{k}^2)^{1/2}\mathbf{w}, \quad \mathbf{a}^- = (1 - q\mathbf{k}^2)^{1/2}\mathbf{w}^{-1}, \\ &\tilde{\mathbf{a}}^+ = (1 - \tilde{q}^{-1}\tilde{\mathbf{k}}^2)^{1/2}\tilde{\mathbf{w}}, \quad \tilde{\mathbf{a}}^- = (1 - \tilde{q}\tilde{\mathbf{k}}^2)^{1/2}\tilde{\mathbf{w}}^{-1} \end{aligned} \tag{5.2}$$

satisfy the defining relations of $\mathcal{A}_q$ and $\mathcal{A}_{\tilde{q}}$. The modular pair of the Heisenberg/Weyl algebras has the coordinate representations on the bra and ket vectors[1] as

$$\begin{aligned} &\langle \sigma|\boldsymbol{\sigma} = \sigma\langle\sigma|, \quad \langle\sigma|\mathbf{w} = \langle\sigma - \mathrm{i}b|, \quad \langle\sigma|\tilde{\mathbf{w}} = \langle\sigma - \mathrm{i}b^{-1}|, \\ &\boldsymbol{\sigma}|\sigma\rangle = \sigma|\sigma\rangle, \quad \mathbf{w}|\sigma\rangle = |\sigma + \mathrm{i}b\rangle, \quad \tilde{\mathbf{w}}|\sigma\rangle = |\sigma + \mathrm{i}b^{-1}\rangle \quad \big(\langle\sigma|\sigma'\rangle = \delta(\sigma - \sigma')\big). \end{aligned} \tag{5.3}$$

The composition of (5.2) and (5.3) will be referred to as *modular representation* of the pair $(\mathcal{A}_q, \mathcal{A}_{\tilde{q}})$.

The relevant 3D R is given by the integral kernel [3]

$$\begin{aligned} \langle\sigma_1, \sigma_2, \sigma_3|R|\sigma'_1, \sigma'_2, \sigma'_3\rangle &= \delta_{\sigma_1+\sigma_2, \sigma'_1+\sigma'_2}\delta_{\sigma_2+\sigma_3, \sigma'_2+\sigma'_3}\sqrt{\frac{\varphi(\sigma_1)\varphi(\sigma_2)\varphi(\sigma_3)}{\varphi(\sigma'_1)\varphi(\sigma'_2)\varphi(\sigma'_3)}} \\ &\times e^{-\mathrm{i}\pi(\sigma_1\sigma_3 - \mathrm{i}\eta(\sigma_1+\sigma_3-\sigma'_2))}\int_{\mathbb{R}} du\, e^{2\pi\mathrm{i}u(\sigma'_2 - \mathrm{i}\eta)}\frac{\varphi(u + \frac{\sigma'_1+\sigma'_3+\mathrm{i}\eta}{2})\varphi(u + \frac{-\sigma_1-\sigma_3+\mathrm{i}\eta}{2})}{\varphi(u + \frac{\sigma_1-\sigma_3-\mathrm{i}\eta}{2})\varphi(u + \frac{-\sigma_1+\sigma_3-\mathrm{i}\eta}{2})}, \end{aligned} \tag{5.4}$$

where $\delta_{\sigma,\sigma'} = \delta(\sigma - \sigma')$ and $\langle\sigma_1, \sigma_2, \sigma_3| = \langle\sigma_1| \otimes \langle\sigma_2| \otimes \langle\sigma_3|$ etc. The integral is convergent for $\sigma_i, \sigma'_i \in \mathbb{R}$ [3]. The function φ is the quantum dilogarithm

$$\varphi(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0}\frac{e^{-2\mathrm{i}zw}}{\sinh(wb)\sinh(w/b)}\frac{dw}{w}\right),$$

which is manifestly symmetric under the exchange $b \leftrightarrow b^{-1}$. Its main difference property is

$$\frac{\varphi(z - \mathrm{i}b^{\pm1}/2)}{\varphi(z + \mathrm{i}b^{\pm1}/2)} = 1 + e^{2\pi z b^{\pm1}}.$$

[1]In terms of wave functions, it is a representation in the space of square integrable functions of $\sigma \in \mathbb{R}$ admitting an analytical continuation into an appropriate horizontal strip. See [19] for further details.

In fact this enables one to establish the formula (5.4) by checking the defining relations (3.1), (3.2), (3.3), (3.4).

Remark 5.1. The defining relations (3.1), (3.2), (3.3), (3.4) of R are based on $\mathcal{A}_q$ but do not involve q explicitly. Moreover (5.4) is symmetric under $b \leftrightarrow b^{-1}$. It follows that the R also satisfies (3.1), (3.2), (3.3), (3.4) with $\langle \mathbf{a}_i^{\pm}, \mathbf{k}_i^{\pm}\rangle$ replaced by $\langle \tilde{\mathbf{a}}_i^{\pm}, \tilde{\mathbf{k}}_i^{\pm}\rangle$. In this sense (5.4) is the 3D R for the modular representation of $(\mathcal{A}_q, \mathcal{A}_{\tilde{q}})$. This fact will further be utilized in Theorem 5.3.

5.2. BOUNDARY VECTORS

Consider the boundary ket vectors $|\chi^{(s)}\rangle$ satisfying the left conditions in (3.5) and (3.6). In the spirit of quantum mechanics we denote its wave function $\langle\sigma|\chi^{(s)}\rangle$ by $\chi^{(s)}(\sigma)$. Then the conditions read

$$\frac{\chi^{(1)}(\sigma - \mathrm{i}\frac{b}{2})}{\chi^{(1)}(\sigma + \mathrm{i}\frac{b}{2})} = \sqrt{\frac{1+\mathrm{i}e^{\pi b\sigma}}{1-\mathrm{i}e^{\pi b\sigma}}}, \qquad \frac{\chi^{(2)}(\sigma - \mathrm{i}b)}{\chi^{(2)}(\sigma + \mathrm{i}b)} = \sqrt{\frac{1+e^{2\pi b(\sigma+\mathrm{i}\frac{b}{2})}}{1+e^{2\pi b(\sigma-\mathrm{i}\frac{b}{2})}}}. \tag{5.5}$$

We may also set $\chi^{(s)}(\sigma) = \langle\chi^{(s)}|\sigma\rangle$ for the boundary bra vectors $\langle\chi^{(s)}|$ since $\langle\chi^{(s)}|\sigma\rangle$ obeys the same difference equations as (5.5).

As we are concerned with the modular representation of $(\mathcal{A}_q, \mathcal{A}_{\tilde{q}})$, it is natural to also consider the boundary vectors $\langle\tilde{\chi}^{(s)}|$ and $|\tilde{\chi}^{(s)}\rangle$ obeying (3.5) and (3.6) with $\mathcal{A}_q = \langle \mathbf{a}^{\pm}, \mathbf{k}^{\pm}\rangle$ replaced by $\mathcal{A}_{\tilde{q}} = \langle \tilde{\mathbf{a}}^{\pm}, \tilde{\mathbf{k}}^{\pm}\rangle$. Their wave functions $\tilde{\chi}^{(s)}(\sigma) = \langle\sigma|\tilde{\chi}^{(s)}\rangle = \langle\tilde{\chi}^{(s)}|\sigma\rangle$ are to satisfy (5.5) with b replaced by b^{-1}:

$$\frac{\tilde{\chi}^{(1)}(\sigma - \mathrm{i}\frac{b^{-1}}{2})}{\tilde{\chi}^{(1)}(\sigma + \mathrm{i}\frac{b^{-1}}{2})} = \sqrt{\frac{1+\mathrm{i}e^{\pi b^{-1}\sigma}}{1-\mathrm{i}e^{\pi b^{-1}\sigma}}}, \qquad \frac{\tilde{\chi}^{(2)}(\sigma - \mathrm{i}b^{-1})}{\tilde{\chi}^{(2)}(\sigma + \mathrm{i}b^{-1})} = \sqrt{\frac{1+e^{2\pi b^{-1}(\sigma+\mathrm{i}\frac{b^{-1}}{2})}}{1+e^{2\pi b^{-1}(\sigma-\mathrm{i}\frac{b^{-1}}{2})}}}. \tag{5.6}$$

Introduce the function

$$\chi_b(\sigma) = \exp\left(\frac{1}{8}\int_{\mathbb{R}+\mathrm{i}0} \frac{e^{-2\mathrm{i}\sigma w}}{\sinh(wb)\cosh(w/b)}\frac{dw}{w}\right). \tag{5.7}$$

It is analytic in the strip $-\eta < \mathrm{Im}(\sigma) < \eta$ but *not* symmetric under the exchange $b \leftrightarrow b^{-1}$. See Appendix A for more properties. Now we present our key observation.

PROPOSITION 5.2. *The following provides a solution to (5.5) and (5.6):*

$$\chi^{(1)}(\sigma) = \tilde{\chi}^{(2)}(\sigma) = \chi_b(\sigma), \quad \chi^{(2)}(\sigma) = \tilde{\chi}^{(1)}(\sigma) = \chi_{b^{-1}}(\sigma).$$

The statement can be verified by a direct calculation. For example for $\chi^{(1)}(\sigma)$, taking the log of the difference equation (5.5), making Fourier transformation assuming analyticity in the strip $-\frac{1}{2}\mathrm{Re}(b) < \mathrm{Im}(\sigma) < \frac{1}{2}\mathrm{Re}(b)$ leads to the result. Proposition 5.2 demonstrates a curious feature of the vectors $|\chi^{(1)}\rangle$ and $|\chi^{(2)}\rangle$. They are transformed to each other simply by the interchange $b \leftrightarrow b^{-1}$.

5.3. MODULAR DOUBLE FOR QUANTUM GROUPS

The operator $\mathbf{h}$ in (3.7) and its modular dual $\tilde{\mathbf{h}}$ satisfying $[\mathbf{h}, \mathbf{a}^{\pm}] = \pm\mathbf{a}^{\pm}$ and $[\tilde{\mathbf{h}}, \tilde{\mathbf{a}}^{\pm}] = \pm\tilde{\mathbf{a}}^{\pm}$ can be taken as $-ib^{-1}\boldsymbol{\sigma}$ and $-ib\boldsymbol{\sigma}$. See (5.1) and (5.2). Thus in (4.1) we may choose $z^{\mathbf{h}} = \mathrm{e}^{2\pi\mathrm{i}\lambda\boldsymbol{\sigma}}$ so that $\langle\sigma|z^{\mathbf{h}}|\sigma'\rangle = \delta_{\sigma,\sigma'}\mathrm{e}^{2\pi\mathrm{i}\lambda\sigma}$ with λ being the (additive) spectral parameter. Let us denote the resulting object (4.1) by $S^{s,t}(\lambda)$. It is given by the integral kernel

$$\langle\boldsymbol{\alpha}, \boldsymbol{\beta}|S^{s,t}(\lambda)|\boldsymbol{\alpha}', \boldsymbol{\beta}'\rangle = \int \prod_{k=0}^{n} d\sigma_k \chi^{(s)}(\sigma_0)\mathrm{e}^{2\pi\mathrm{i}\lambda\sigma_0}\left(\prod_{k=1}^{n}\langle\alpha_k, \beta_k, \sigma_{k-1}|R|\alpha'_k, \beta'_k, \sigma_k\rangle\right)\chi^{(t)}(\sigma_n), \tag{5.8}$$

where $\langle\boldsymbol{\alpha}, \boldsymbol{\beta}| = \langle\boldsymbol{\alpha}| \otimes \langle\boldsymbol{\beta}|$ with $\langle\boldsymbol{\alpha}| = \langle\alpha_1| \otimes \cdots \otimes \langle\alpha_n|$, $\langle\boldsymbol{\beta}| = \langle\beta_1| \otimes \cdots \otimes \langle\beta_n|$ $(\alpha_i, \beta_i \in \mathbb{R})$ and similarly for $|\boldsymbol{\alpha}', \boldsymbol{\beta}'\rangle$.[2] The boundary wave function $\chi^{(s)}(\sigma)$ is the one specified in Proposition 5.2. Due to the conservation law [δ factors in (5.4)], (5.8) is proportional to $\prod_{k=1}^{n}\delta_{\alpha_k+\beta_k, \alpha'_k+\beta'_k}$ and the integrals over $\sigma_0, \ldots, \sigma_n \in \mathbb{R}$ actually reduce to a single one.

Introduce $\hat{S}^{s,t}(\lambda) = (K_d \otimes 1)S^{s,t}(\lambda)(1 \otimes K_d^{-1})$ generalizing (4.3). The operator K_d simultaneously obeying (4.2) and its modular counterpart, i.e., $K_d\tilde{\mathbf{k}}_j = \tilde{\mathbf{k}}_j K_d$, $K_d\tilde{\mathbf{a}}_j^{\pm} = (\mathrm{i}\tilde{q}^{\frac{1}{2}})^{\pm 1}\tilde{\mathbf{a}}_j^{\pm} K_d$ $(1 \le j \le n)$, is realized as $\langle\boldsymbol{\alpha}|K_d|\boldsymbol{\alpha}'\rangle = \prod_{k=1}^{n}\delta_{\alpha_k, \alpha'_k}\mathrm{e}^{\pi\eta\alpha_k}$.

Let us describe the quantum group symmetry of the solution $\hat{S}^{s,t}(\lambda)$ of the Yang–Baxter equation. We prepare the representation ρ_λ of $U_q(\mathfrak{g}^{s,t})$ obtained by combining $\pi_z : U_q(\mathfrak{g}^{s,t}) \to \mathcal{A}_q^{\otimes n}$ in Proposition 2.1 with $z = \mathrm{e}^{-2\pi b\lambda}$ and the modular representation of $(\mathcal{A}_q, \mathcal{A}_{\tilde{q}})$. This choice of z stems from $z^{\mathbf{h}}\mathbf{a}^{\pm} = \mathrm{e}^{2\pi\mathrm{i}\lambda\boldsymbol{\sigma}}\mathbf{a}^{\pm} = \mathrm{e}^{\mp 2\pi b\lambda}\mathbf{a}^{\pm}z^{\mathbf{h}}$ and the remark at the end of (iii) in the proof of Theorem 4.1.

Similarly let $\tilde{\rho}_\lambda$ be the representation of $U_{\tilde{q}}({}^L\mathfrak{g}^{s,t})$ consisting of $\pi_{\tilde{z}} : U_{\tilde{q}}({}^L\mathfrak{g}^{s,t}) \to \mathcal{A}_{\tilde{q}}^{\otimes n}$ with $\tilde{z} = \mathrm{e}^{-2\pi b^{-1}\lambda}$ and the modular representation of $(\mathcal{A}_q, \mathcal{A}_{\tilde{q}})$. Here ${}^L\mathfrak{g}^{s,t} = \mathfrak{g}^{3-s,3-t}$ is the Langlands dual of $\mathfrak{g}^{s,t}$ as mentioned in Section 2.1.

Given parameters μ, ν, let $\Delta(g)$ and $\Delta'(g)$ simply mean $(\rho_\mu \otimes \rho_\nu)(\Delta(g))$ and $(\rho_\mu \otimes \rho_\nu)(\Delta'(g))$ for $g \in U_q(\mathfrak{g}^{s,t})$. Similarly let they mean $(\tilde{\rho}_\mu \otimes \tilde{\rho}_\nu)(\Delta(g))$ and $(\tilde{\rho}_\mu \otimes \tilde{\rho}_\nu)(\Delta'(g))$ for $g \in U_{\tilde{q}}({}^L\mathfrak{g}^{s,t})$. The following theorem, which is the main result in this section, states that the symmetry of $\hat{S}^{s,t}(\lambda)$ implied by Theorem 4.1 is enhanced naturally to the modular double.

[2]The symbols $\boldsymbol{\alpha}$ etc. that appeared as labels of the spaces in (4.1) are used here as variables.

THEOREM 5.3. *For μ, ν such that $\lambda = \mu - \nu$, the following commutativity is valid:*

$$\Delta'(g)\hat{S}^{s,t}(\lambda) = \hat{S}^{s,t}(\lambda)\Delta(g) \qquad \forall g \in U_q(\mathfrak{g}^{s,t}),$$
$$\Delta'(g)\hat{S}^{s,t}(\lambda) = \hat{S}^{s,t}(\lambda)\Delta(g) \qquad \forall g \in U_{\tilde{q}}({}^L\mathfrak{g}^{s,t}).$$

Proof. The first relation is due to Theorem 4.1. The second relation is due to the first relation, Remark 5.1 and Proposition 5.2. □

Let $U_{q,\tilde{q}}(\mathfrak{g}^{s,t})$ denote the modular double of quantum affine algebras in the sense of $U_{q,\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ [11, eq. (1.8)] with $\mathfrak{g}_{\mathbb{R}}$ formally replaced by $\mathfrak{g}^{s,t}$.[3] Set $\check{S}^{s,t}(\lambda) = P\hat{S}^{s,t}(\lambda)$, where P is defined after (2.2). By writing the coproduct action $\Delta(U_{q,\tilde{q}})$ just as $U_{q,\tilde{q}}$, Theorem 5.3 may be rephrased symbolically as

$$[\check{S}^{1,1}(\lambda), U_{q,\tilde{q}}(D^{(2)}_{n+1})] = 0, \quad [\check{S}^{2,2}(\lambda), U_{q,\tilde{q}}(C^{(1)}_n)] = 0,$$
$$[\check{S}^{1,2}(\lambda), U_{q,\tilde{q}}(A^{(2)}_{2n})] = 0, \quad [\check{S}^{2,1}(\lambda), U_{q,\tilde{q}}(\tilde{A}^{(2)}_{2n})] = 0.$$

6. Example: Fock Representation

Here we explain that the specialization of the results in Proposition 2.1–Theorem 4.1 to the Fock representation of the q-oscillator algebra $\mathcal{A}_q$ reproduces the earlier result in [17]. We assume q is generic. By the Fock representation of $\mathcal{A}_q$ (2.3) we mean the following on $F = \oplus_{m\geq 0}\mathbb{C}(q^{\frac{1}{2}})|m\rangle$:

$$\mathbf{a}^{+}|m\rangle = |m+1\rangle, \quad \mathbf{a}^{-}|m\rangle = (1-q^{2m})|m-1\rangle, \quad \mathbf{k}^{\pm 1}|m\rangle = q^{\pm(m+\frac{1}{2})}|m\rangle.$$

Combining this with π_z in Proposition 2.1 yields an irreducible representation of $U_q(\mathfrak{g}^{s,t})$ on $F^{\otimes n}[z, z^{-1}]$ except $s = t = 2$. In the latter case, $F^{\otimes n}[z, z^{-1}]$ splits into two irreducible $U_q(C^{(1)}_n)$ modules. They were obtained in [17, Prop. 1–3] without factoring through $\mathcal{A}_q^{\otimes n}$ via π_z, and called "q-oscillator representations".

The 3D R associated with the Fock representation is given by

$$R(|i\rangle\otimes|j\rangle\otimes|k\rangle) = \sum_{a,b,c\geq 0} R^{a,b,c}_{i,j,k}|a\rangle\otimes|b\rangle\otimes|c\rangle,$$

$$R^{a,b,c}_{i,j,k} = \delta_{a+b,i+j}\delta_{b+c,j+k}\sum_{\lambda+\mu=b}(-1)^{\lambda}q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)}\frac{(q^2)_{c+\mu}}{(q^2)_c}\binom{i}{\mu}_{q^2}\binom{j}{\lambda}_{q^2},$$

where $\binom{m}{k}_q = \frac{(q)_m}{(q)_k(q)_{m-k}}$, $(q)_m = (q;q)_m$ and $(z;q)_m = \prod_{k=1}^{m}(1-zq^{k-1})$. The sum over λ, μ is taken under the conditions $\lambda + \mu = b$, $0 \leq \mu \leq i$ and $0 \leq \lambda \leq j$.

This solution of the tetrahedron equation was obtained as the intertwiner of the quantum coordinate ring $A_q(sl_3)$ [14].[4] It was also found from a quantum

[3] $U_q(\mathfrak{g}^{s,t})$ and $U_{\tilde{q}}({}^L\mathfrak{g}^{s,t})$ actually commute only up to sign in general. See e.g., [11, Prop. 9.1] for the positive principal series representations. The so-called transcendental relations therein are not considered here.

[4] The formula for it on p194 in [14] contains a misprint unfortunately.

geometry consideration in a different gauge including square roots [3,5]. They were shown to be the same object constituting a solution of the 3D reflection equation in [16]. The 3D R can also be identified with the transition matrix of the PBW bases of the nilpotent subalgebra of $U_q(sl_3)$ [20].

Let $F^* = \oplus_{m\geq 0}\mathbb{C}(q^{\frac{1}{2}})\langle m|$ be the right $\mathcal{A}_q$ module defined by

$$\langle m|\mathbf{a}^+ = (1-q^{2m})\langle m-1|, \quad \langle m|\mathbf{a}^- = \langle m+1|, \quad \langle m|\mathbf{k}^{\pm 1} = q^{\pm(m+\frac{1}{2})}\langle m|.$$

The bilinear pairing of F^* and F is specified by $\langle m|m'\rangle = (q^2)_m\delta_{m,m'}$. The operator $\mathbf{h}$ argued around (3.7) can be defined by $\langle m|\mathbf{h} = m\langle m|$ and $\mathbf{h}|m\rangle = m|m\rangle$.

The boundary vectors satisfying the postulates in Section 3 are given by [18]

$$\langle\chi^{(s)}| = \sum_{m\geq 0}\frac{1}{(q^{s^2})_m}\langle sm|, \qquad |\chi^{(s)}\rangle = \sum_{m\geq 0}\frac{1}{(q^{s^2})_m}|sm\rangle \quad (s=1,2).$$

In [17], Theorem 4.1 was proved for the present setting of the Fock representation. It was also shown that $F^{\otimes n}[x,x^{-1}]\otimes F^{\otimes n}[y,y^{-1}]$ with generic x,y is an irreducible $U_q(\mathfrak{g}^{s,t})$ module except $s=t=2$. (In the latter case it splits into four irreducible $U_q(C_n^{(1)})$ modules.) Thus the commutativity with U_q provides a characterization of $S^{s,t}(z)$ up to normalization.

Acknowledgements

This work is supported by Australian Research Council and Grants-in-Aid for Scientific Research No. 23340007 and No. 24540203 from JSPS.

Appendix A. Properties of $\chi_b(\sigma)$

Let us collect some properties of the function $\chi_b(\sigma)$ in (5.7) which are derived by residue analyses. In what follows, we will also use the Jacobi modular partner $\overline{q} = \mathrm{e}^{-\mathrm{i}\pi b^{-2}}$ to q.

$$\frac{\chi_b(\sigma)}{\chi_b(-\sigma)} = \mathrm{e}^{-\frac{1}{2}\pi b^{-1}\sigma}, \qquad \chi_b(\sigma)\chi_{b^{-1}}(\sigma) = \frac{\varphi(\frac{\sigma+\mathrm{i}\eta}{2})}{\varphi(\frac{\sigma-\mathrm{i}\eta}{2})}.$$

Set $w = \mathrm{e}^{2\pi b\sigma}$, $\overline{w} = \mathrm{e}^{2\pi b^{-1}\sigma}$. In the regime $|q|<1$ one has the infinite product representations

$$\chi_b(\sigma) = \sqrt{\frac{(-\mathrm{i}q^{1/2}w^{1/2};q)_\infty}{(\mathrm{i}q^{1/2}w^{1/2};q)_\infty}\frac{(-\overline{q}^3\overline{w};\overline{q}^4)_\infty}{(-\overline{q}\,\overline{w};\overline{q}^4)_\infty}},$$

$$\sqrt{\varphi(\sigma)}\chi_b(\sigma) = \frac{(-\mathrm{i}q^{1/2}w^{1/2};q)_\infty}{(-\overline{q}\,\overline{w};\overline{q}^4)_\infty}, \qquad \frac{1}{\sqrt{\varphi(\sigma)}}\chi_b(\sigma) = \frac{(-\overline{q}^3\overline{w};\overline{q}^4)_\infty}{(\mathrm{i}q^{1/2}w^{1/2};q)_\infty}.$$

The following integral identities hold:

$$\int_{\mathbb{R}} \chi_b(\sigma)^2 e^{-2\pi i\sigma\lambda} d\sigma = e^{-\frac{\pi}{2}b^{-1}(2\lambda - i\eta)} \frac{\chi_b(i\eta - 2\lambda)^2}{\cosh \pi b\lambda},$$
$$\int_{\mathbb{R}} \chi_b(\sigma)\chi_{b^{-1}}(\sigma) e^{-2\pi i\sigma\lambda} d\sigma = 2e^{-i\pi\eta^2/2} \frac{\varphi(2\lambda)}{\varphi(2\lambda - i\eta)} e^{2\pi\lambda\eta}.$$

Remark A.1. In [18], reduction of the 3D L operators was studied by boundary vectors in the Fock representation leading to the quantum R matrices for the spin representations of $U_q(D^{(2)}_{n+1})$, $U_q(B^{(1)}_n)$ and $U_q(D^{(1)}_n)$. Formulas in this appendix enable one to calculate the reduction in the modular representation and give exactly the same R-matrices.

References

1. Baxter, R.J.: Exactly solved models in statistical mechanics, Dover (2007)
2. Bazhanov, V.V., Baxter, R.J.: New solvable lattice models in three dimensions. J. Stat. Phys **69**, 453–485 (1992)
3. Bazhanov, V.V., Mangazeev, V.V., Sergeev, S.M.: Quantum geometry of 3-dimensional lattices. J. Stat. Mech. P07004 (2008)
4. Bazhanov, V.V., Kashaev, R.M., Mangazeev, V.V., Stroganov, Yu.G.: $(Z_N\times)^{n-1}$ generalization of the chiral Potts model. Commun. Math. Phys **138**, 393–408 (1991)
5. Bazhanov, V.V., Sergeev, S.M.: Zamolodchikov's tetrahedron equation and hidden structure of quantum groups. J. Phys. A Math. Theor **39**, 3295–3310 (2006)
6. Date, E., Jimbo, M., Miki, K., Miwa, T.: Generalized chiral Potts models and minimal cyclic representations of $U_q(gl(n,\mathbb{C}))$. Commun. Math. Phys **137**, 133–147 (1991)
7. Drinfeld, V.G.: Quantum groups. In: Proceedings of the ICM, Vols. 1, 2 (Berkeley, Calif., 1986), pp. 798–820. Am. Math. Soc., Providence (1987)
8. Faddeev, L.: Modular double of a quantum group. Conférence Moshé Flato 1999, **1**, pp. 149–156. Math. Phys. Stud. Kluwer Acad. Publ. (2000)
9. Frenkel, I.B., Ip, I.C.-H.: Positive representations of split real quantum groups and future perspectives. arXiv:1111.1033
10. Hayashi, T.: Q-analogues of Clifford and Weyl algebras–spinor and oscillator representations of quantum enveloping algebras, Comm. Math. Phys **127**, 129–144 (1990)
11. Ip, I.C.-H.: Positive representations of split real quantum groups of type B_n, C_n, F_4, and G_2. arXiv:1205.2940
12. Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang–Baxter equation. Lett. Math. Phys **10**, 63–69 (1985)
13. Kac, V.G.: Infinite dimensional Lie algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
14. Kapranov, M.M., Voevodsky, V.A.: 2-Categories and Zamolodchikov tetrahedron equations. Proc. Symp. Pure Math **56**, 177–259 (1994)
15. Kashaev, R.V., Volkov, A.Yu.: From the tetrahedron equation to universal R-matrices. Am. Math. Soc. Transl. Ser. **2**(201), 79–89 (2000)
16. Kuniba, A., Okado, M.: Tetrahedron and 3D reflection equations from quantized algebra of functions. J. Phys. A Math.Theor. **45**, 465206, pp. 27 (2012)
17. Kuniba, A., Okado, M.: Tetrahedron equation and quantum R matrices for q-oscillator representations of $U_q(A^{(2)}_{2n})$, $U_q(C^{(1)}_n)$ and $U_q(D^{(2)}_{n+1})$. Commun. Math. Phys in press

18. Kuniba, A., Sergeev, S.: Tetrahedron equation and quantum R matrices for spin representations of $B_n^{(1)}, D_n^{(1)}$ and $D_{n+1}^{(2)}$. Commun. Math. Phys **324**, 695–713 (2013)
19. Schmüdgen, K.: Integrable operator representations of $\mathbb{R}_q^2$, $X_{q,\gamma}$ and $\mathrm{SL}_q(2,\mathbb{R})$. Commun. Math. Phys **159**, 217–237 (1994)
20. Sergeev, S.M.: Tetrahedron equations and nilpotent subalgebras of $\mathcal{U}_q(sl_n)$. Lett. Math. Phys **83**, 231–235 (2008)
21. Sergeev, S.M.: Two-dimensional R-matrices–descendants of three-dimensional R-matrices. Mod. Phys. Lett. A **12**, 1393–1410 (1997)
22. Sergeev, S.M., Mangazeev, V.V., Stroganov Yu, G.: The vertex formulation of the Bazhanov–Baxter model. J. Stat. Phys **82**, 31–50 (1994)
23. Tarasov, V.: Cyclic monodromy matrices for $sl(n)$ trigonometric R-matrices. Commun. Math. Phys **158**, 459–483 (1993)
24. Zamolodchikov, A.B.: Tetrahedra equations and integrable systems in three-dimensional space, Soviet Phys. JETP **79**, 641–664 (1980)