FAILURE OF AMALGAMATION IN HILBERT LATTICES

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We show that Bruns and Harding's counterexample (see [1]) to amalgamation in orthomodular lattices also works for Hilbert lattices. The argument is based on the example of a non-modular Hilbert lattice devised by von Neumann in a letter to Birkhoff (see [2] for an extensive quotation from that letter). Consider the real sequence space $\ell_2 = \{f\colon \mathbb{N}^+ \to \mathbb{R}\colon \Sigma_{i=1}^{\infty} f(i)^2 < \infty\}$, where $\mathbb{N}^+$ stands for $\mathbb{N} \setminus \{0\}$. Let $\langle e_n\colon n \in \mathbb{N}^+\rangle$ be the standard orthonormal base of ℓ_2, i.e., $e_n(n) = 1$ and $e_n(m) = 0$ for $m \neq n$. It follows from Satz 15 in [3] that there are two unbounded self-adjoint operators X and Y such that $\mathsf{dom}(X) \cap \mathsf{dom}(Y) = \{0\}$ and moreover Y can be chosen to be the "multiplication by n" operator $Yf = \big\langle n \cdot f(n)\colon n \in \mathbb{N}^+\big\rangle$. Thus, Y is represented over the standard base as the $\mathbb{N}^+ \times \mathbb{N}^+$ matrix with entries $y_{ij} = i \cdot e_i(j)$. Further, the operators $X^2 + 2I$ and $Y^2 + 2I$ are self-adjoint and invertible. Define $A = (X^2 + 2I)^{-1}$ and $B = (Y^2 + 2I)^{-1}$, where I is the identity operator. The following lemma spells out some properties of A, B and C.

Lemma 1. *The operators A and B are bounded self-adjoint, with $||A|| \leq \frac{1}{2}$ and $||B|| \leq \frac{1}{2}$. Further, $C = \sqrt{I - A^2 - B^2}$ is also bounded and self-adjoint. Moreover the following conditions hold:*

1. $A^2 + B^2 + C^2 = I$,
2. A^{-1} *and* B^{-1} *exist,*
3. $Af + Bg = 0$ *implies* $f = g = 0$,
4. $Cf = 0$ *implies* $f = 0$,
5. *if* B^{-1} *is defined for* f *and* $B^{-1}f = 0$, *then* $f = 0$,
6. *if* B^{-1} *is defined for* f *and* $CB^{-1}f = 0$, *then* $f = 0$.

Let now $\overline{H}$ be the direct sum $\ell_2 \oplus \ell_2 \oplus \ell_2$. From (1) it follows that the map $f \mapsto (Af, Bf, Cf)$ is an embedding of ℓ_2 into $\overline{H}$. Let $\overline{L}$ be the image of ℓ_2 under this embedding. Clearly, $\overline{L}$ is a proper subspace of $\overline{H}$. Let $\overline{K} = \{(0, 0, h)\colon h \in \ell_2\}$ and $\overline{M} = \{(0, g, h)\colon g, h \in \ell_2\}$.

Lemma 2. *The spaces $\overline{K}$, $\overline{L}$ and $\overline{M}$ satisfy the following conditions:*

- $\overline{K} \subset \overline{M}$,
- $\overline{L} \vee \overline{K} = \overline{1}$,
- $\overline{L} \cap \overline{M} = \overline{0}$.

where $\overline{1}$ stands for $\overline{H}$ and $\overline{0}$ for the zero subspace.

In other words, $\overline{1}$, $\overline{K}$, $\overline{L}$, $\overline{M}$, and $\overline{0}$ form a sublattice of the Hilbert lattice of $\overline{H}$ isomorphic to the pentagon.

To proceed, we will need certain derived operators, in particular $CB^{-2}C$. Since C in bounded self-adjoint, it is represented over the standard base by a Hermitian

$\mathbb{N}^+ \times \mathbb{N}^+$ matrix, say

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots \\ c_{21} & c_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

with $c_{ij} = c_{ji}$. The operator $B^{-2} = B^{-1}B^{-1} = (Y^2 + 2I)(Y^2 + 2I)$ can also be represented by a Hermitian (unbounded, diagonal) matrix

$$\begin{pmatrix} 9 & 0 & 0 & \dots \\ 0 & 36 & 0 & \dots \\ 0 & 0 & 121 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with main diagonal entries $p_{ii} = (i^2 + 2)^2$. Therefore, the operator $CB^{-2}C$ is represented by the (non-Hermitian, unbounded) matrix

$$M = \begin{pmatrix} m_{11} & m_{12} & \dots \\ m_{21} & m_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where $m_{ij} = \sum_{n=1}^{\infty} (n^2 + 2)^2 c_{in} c_{nj}$.

Lemma 3. *The space $\overline{L}^{\perp} \cap \overline{M}$ is a non-zero subspace of $\overline{H}$.*

Let $\overline{N}$ be $\overline{L}^{\perp} \cap \overline{M}$. Since $\overline{N}$ is a non-zero proper subspace of $\overline{M}$, by properties of orthomodular lattices it follows that $\overline{N}$ and $\overline{M}$ generate an 8-element Boolean subalgebra of (the Hilbert lattice of) $\overline{H}$. In particular, putting $\overline{U} = \overline{N} \vee \overline{M}^{\perp}$ we get that $\overline{U}$ is a proper subspace of $\overline{H}$.

Lemma 4. *The space $\overline{U}$ is precisely $\{(a, b, c) \colon Bb + Cc = 0\}$, with $a, b, c \in \ell_2$. Moreover, the following equalities hold:*

1. $\overline{K} \cap \overline{U} = 0$,
2. $\overline{L} \cap \overline{U} = 0$,
3. $\overline{U} \cap \overline{K}^{\perp} = \overline{M}^{\perp}$,
4. $\overline{U}^{\perp} \vee \overline{L} = \overline{N}^{\perp}$.

We now preceed to define a subspace of $\overline{U}$. Let P be the projection on first coordinate, i.e., an operator on ℓ_2 defined by

$$Pf(n) = \begin{cases} f(n) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly the subspace $\overline{P}$ corresponding to P is a proper, non-zero subspace of ℓ_2. Thus, $\overline{W}$ defined as $\{(a, b, Pc) \colon Bb + Cc = 0\}$ is a proper, non-zero subspace of $\overline{U}$.

Lemma 5. *We have $(\overline{L} + \overline{W})^{\perp} = 0 = (\overline{K} + \overline{W})^{\perp}$.*

Lemma 6. *The subalgebra of the Hilbert lattice of $\overline{H}$ generated by $\overline{L}$, $\overline{K}$ and $\overline{W}$ is isomorphic to $MO3$ (i.e., the Chinese lantern with 6 atoms).*

Now let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two copies of the Hilbert lattice of $\overline{H}$. We will use subscripts to differentiate between their elements (subspaces of the two copies of

$\overline{H}$). Thus, for instance $\overline{L}_1$ will stand for $\overline{L} \subset \overline{H}$ as an element of $\mathbf{L}_2$, while $\overline{L}_2$ will stand for $\overline{L}$ as an element of $\mathbf{L}_2$.

By Lemma 6, $MO3$ is a sublattice of both. Let p, q, r be three distinct atoms of $MO3$. Define embeddings $e_1 \colon MO3 \longrightarrow \mathbf{L}_1$ and $e_2 \colon MO3 \longrightarrow \mathbf{L}_2$ by putting:

- $e_1(p) = \overline{L}_1$, $e_1(q) = \overline{K}_1$, $e_1(r) = \overline{W}_1$, and
- $e_2(p) = \overline{L}_2^{\perp}$, $e_2(q) = \overline{K}_2^{\perp}$, $e_1(r) = \overline{W}_2^{\perp}$.

Notice that intuitively this amounts to *identifying* the copies of $\overline{L}$, $\overline{K}$ and $\overline{W}$ in $\mathbf{L}_1$ respectively with the copies of $\overline{L}^{\perp}$, $\overline{K}^{\perp}$ and $\overline{W}^{\perp}$ in $\mathbf{L}_2$.

Suppose that the V-formation $\langle MO3, \mathbf{L}_1, \mathbf{L}_2, e_1, e_2\rangle$ can be amalgamated by a Hilbert lattice $\mathbf{L}$. We will use $\overline{1}$ and $\overline{0}$ respectively for the top and bottom elements of our lattices. In $\mathbf{L}$ we can then carry out the following calculations.

$$\overline{1} = \overline{U}_2^{\perp} \vee \overline{K}_2^{\perp} = \overline{U}_2^{\perp} \vee q \leq \overline{U}_2^{\perp} \vee q \vee \overline{U}_1^{\perp} = \overline{U}_2^{\perp} \vee \overline{K}_1 \vee \overline{U}_1^{\perp} = \overline{U}_2^{\perp} \vee \overline{M}_1 \tag{1}$$

where the inequality in the middle is trivial, the equalities neighbouring it follow from the embeddings, and the first and last equalities in the row both come from Lemma 4 (from (1) and (3) respectively). Since $\overline{N}_1 \subseteq \overline{M}_1$, by orthomodularity we get

$$\overline{N}_1 \vee \overline{U}_1^{\perp} = \overline{N}_1 \vee (\overline{N}_1^{\perp} \cap \overline{M}_1) = \overline{M}_1 \tag{2}$$

which together with (1) yields

$$\overline{1} = \overline{U}_2^{\perp} \vee \overline{N}_1 \vee \overline{U}_1^{\perp} \tag{3}$$

Then, as $\overline{N}_1 \subseteq \overline{U}_1$ and $\overline{U}_2^{\perp} \subseteq \overline{W}_2^{\perp} = r = \overline{W}_1 \subseteq \overline{U}_1$, using orthomodularity again we obtain

$$\overline{N}_1 \vee \overline{U}_2^{\perp} = \overline{U}_1. \tag{4}$$

Orthomodularity also yields

$$\overline{L}_1^{\perp} = \overline{N}_1 \vee (\overline{L}_1^{\perp} \cap \overline{N}_1^{\perp}). \tag{5}$$

Therefore

$$\begin{aligned}
\overline{N}_2^{\perp} &= \overline{U}_2^{\perp} \vee \overline{L}_2 = \overline{U}_2^{\perp} \vee p^{\perp} = \overline{U}_2^{\perp} \vee \overline{L}_1^{\perp} \\
&= \overline{U}_2^{\perp} \vee (\overline{N}_1 \vee (\overline{L}_1^{\perp} \cap \overline{N}_1^{\perp})) = (\overline{U}_2^{\perp} \vee \overline{N}_1) \vee (\overline{L}_1^{\perp} \cap \overline{N}_1^{\perp}) \\
&= \overline{U}_1 \vee (\overline{L}_1^{\perp} \cap \overline{N}_1^{\perp}) = (\overline{M}_1^{\perp} \vee \overline{N}_1) \vee (\overline{L}_1^{\perp} \cap \overline{N}_1^{\perp}) \\
&= \overline{M}_1^{\perp} \vee (\overline{N}_1 \vee (\overline{L}_1^{\perp} \cap \overline{N}_1^{\perp})) = \overline{M}_1^{\perp} \vee \overline{L}_1^{\perp} \\
&= (\overline{M}_1 \cap \overline{L}_1)^{\perp} = \overline{0}^{\perp} = \overline{1}
\end{aligned}$$

where the first equality follows by Lemma 4(4) and the last row employs Lemma 2. So we obtain $\overline{N}_2^{\perp} = \overline{1}$, which contradicts Lemma 3.

References

[1] G. Bruns, J. Harding, Amalgamation of ortholattices, *Order*, 14, 193–209, 1998.

[2] M. Redei, The birth of quantum logic, *manuscript*.

[3] J. von Neumann, Zur Theorie der unbeschränkter Matrizen, Collected Works, vol. II, no. 3, 144–172, Pergamon Press 1961.