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IMAGINARY POWERS OF LAPLACE OPERATORS

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ABSTRACT. We show that if L is a second-order uniformly elliptic operator in divergence form on $\mathbf{R}^d$, then $C_1(1+|\alpha|)^{d/2} \le \|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2}$. We also prove that the upper bounds remain true for any operator with the finite speed propagation property.

1. Introduction

Assume that $a_{ij} \in C^\infty(\mathbf{R}^d)$, $a_{ij} = a_{ji}$ for $1 \le i, j \le d$ and that $\kappa I \le (a_{ij}) \le \tau I$ for some positive constants κ and τ. We define a positive self-adjoint operator L on $L^2(\mathbf{R}^d)$ by the formula

$$L = -\sum \partial_i a_{ij} \partial_j. \tag{1}$$

We refer readers to [8] for the precise definition and basic properties of L. In particular, L admits a spectral resolution $E(t)$ and we can define the operator $L^{i\alpha}$ by the formula

$$L^{i\alpha} = \int_0^\infty t^{i\alpha} dE(t).$$

By spectral theory $\|L^{i\alpha}\|_{L^2\to L^2} = 1$. It is well known that $L^{i\alpha}$ falls within the scope of classical Calderón-Zygmund theory (as described in [3] or [22]) and so it extends to a bounded operator on $L^p, 1 < p < \infty$, and is also weak type (1,1). The main aim of this paper is to obtain the sharp estimate for the weak type $(1,1)$ norm of $L^{i\alpha}$ in terms of α.

The study of imaginary powers of operators is an important part of the theory of operators of type ω with H^∞ functional calculus (see e.g., [6], [9] and [17]). What is perhaps more interesting and relevant from the point of view of this paper is that the weak type $(1,1)$ norm of imaginary powers of self-adjoint operators can play a central role in the theory of spectral multipliers. See [5] and [15]. Imaginary powers of Laplace operators on compact Lie groups were also investigated in [20]. Theorem 2 below applied to Laplace operators on compact Lie groups gives the sharp endpoint result of Theorem 3 in [20], pp. 58. See also Corollary 4 of [20], pp. 121.

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However, the starting point for this paper is the following observation from [2]. If we denote the weak type (1,1) norm of an operator T on a measure space (X, μ) by $\|T\|_{L^1 \to L^{1,\infty}} = \sup \lambda\ \mu(\{x \in X :\ |Tf(x)| > \lambda\})$ where the supremum is taken over $\lambda > 0$ and functions f with $L^1(X)$ norm less than one, then for the standard Laplace operator on $\mathbf{R}^d$,

$$C_1(1+|\alpha|)^{d/2} \le \|(-\Delta_d)^{i\alpha}\|_{L^1 \to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2}\log(1+|\alpha|). \tag{2}$$

The classical Hörmander multiplier theorem (see [13]) states that a multiplier operator T_m on $\mathbf{R}^d$ with multiplier m satisfies

$$\|T_m\|_{L^1 \to L^{1,\infty}} \ \le\ C_s \sup_{t>0} \|\eta(\cdot)m(t\cdot)\|_{H_s} \ \le\ A \tag{3}$$

for any $s > d/2$ and any $\eta \in C_c^\infty(\mathbf{R}_+)$ not identically zero. Here H_s is the Sobolev space of order s on $\mathbf{R}^d$. Since the Sobolev norm in (3) behaves like $(1+|\alpha|)^s$ for the multiplier $m(x) = |x|^{i\alpha}$ of $(-\Delta)^{i\alpha}$, (2) shows that the exponent $d/2$ in Hörmander's theorem is sharp. Furthermore, if (3) is satisfied with $A < \infty$, then the distribution $K = \hat{m}$ agrees with a locally integrable function away from the origin which satisfies

$$I(B) \ =\ \sup_{y \ne 0} \int_{|x| \ge B|y|} |K(x-y) - K(x)|\, dx \ \le\ A \tag{4}$$

for $B \ge 2$ and Hörmander's theorem actually shows that the weak type (1,1) norm of T_m is bounded by $I(B) + \|m\|_{L^\infty}^2 + B^d$. One can easily compute that for the convolution kernel K of $(-\Delta)^{i\alpha}$, the integral $I(B)$ is bounded above and below by $(1+|\alpha|)^{d/2}\log(1+|\alpha|/B)$. Hence Hörmander's theorem gives the upper bound in (2). The lower bound is a simple consequence of the explicit formula for the kernel K of $(-\Delta)^{i\alpha}$. See for example, [21] pp. 51-52.

The main observation of this paper is to note that there is a slight improvement of the bound $I(B)+\|m\|_{L^\infty}^2+B^d$ to $I(B)+(\|m\|_{L^\infty}^2 B^d)^{1/2}$. This can be achieved either by using C. Fefferman's ideas in [11] of exploiting more information of L^2 bounds or by varying the level of the Calderón-Zygmund decomposition and optimising. Hence we will be able to remove the *log* term in (2). We will show that this more precise estimate holds for a general class of operators.

Theorem 1. *Suppose that L is defined by* (1)*. Then*

$$C_1(1+|\alpha|)^{d/2} \le \|L^{i\alpha}\|_{L^1 \to L^{1,\infty}} \le C_2(1+|\alpha|)^{d/2} \tag{5}$$

for all $\alpha \in \mathbf{R}$.

Proof of the lower bound. We begin with some known estimates for the kernel $p_t(x,y)$ of the heat operator e^{-tL} associated to L. First, this kernel satisfies Gaussian bounds

$$C_1 \frac{1}{t^{d/2}}\, \mathrm{e}^{-b_1\rho^2(x,y)/t} \ \le\ p_t(x,y) \ \le\ C_2 \frac{1}{t^{d/2}}\, \mathrm{e}^{-b_2\rho^2(x,y)/t} \tag{6}$$

(see [8]) for some positive constants C_1, C_2, b_1 and b_2 and where $\rho(x,y)$ denotes the geodesic distance between x and y given by the Riemannian metric $(a_{i,j})$. In this setting of uniform ellipticity, $\kappa|x-y| \le \rho(x,y) \le \tau|x-y|$. Secondly, from the construction of a parametrix for the heat equation with respect to L (either via Hadamard's construction, see §17.4 of [14], or using pseudodifferential operator techniques, see chapter 7, §13 of [23]), we have for each $y \in \mathbf{R}^d$ a ball $B(y,r)$ such that for $x \in B(y,r)$ and $0 < t < 1$,

$$|p_t(x,y) - (\det a_{ij}(y))^{-1/2}(4\pi t)^{-d/2}\mathrm{e}^{-\rho^2(x,y)/4t}| \leq Ct^{1/2}t^{-d/2}. \tag{7}$$

Here we are using the fact that p_t is symmetric, $p_t(x,y) = p_t(y,x)$. From (6) and (7), we have for $x \in B(y,r)$ the bound

$$|p_t(x,y) - (\det a_{ij}(y))^{-1/2}(4\pi t)^{-d/2}\mathrm{e}^{-\rho^2(x,y)/4t}| \leq Ct^{1/4}t^{-d/2}\exp\left(-b'\rho(x,y)^2/t\right)$$

which translates into a bound for the kernel $K_{L^{i\alpha}}$ of $L^{i\alpha}$ since the functional calculus for L gives us the relationship

$$L^{i\alpha} = \Gamma(-i\alpha)^{-1}\int_0^\infty t^{-i\alpha-1}e^{-tL}dt$$

for $\alpha \neq 0$. Thus for $x \in B(y,r)$,

$$\begin{aligned} &|K_{L^{i\alpha}}(x,y) - (\det a_{ij}(y))^{-1/2}4^{i\alpha}\pi^{-d/2}\gamma(\alpha)\rho(x,y)^{-d-i2\alpha}| \\ &\qquad \leq C|\Gamma(-i\alpha)|^{-1}\rho(x,y)^{-d+1/2} \end{aligned} \tag{8}$$

where $\gamma(\alpha) = \Gamma(i\alpha + d/2)/\Gamma(-i\alpha)$. Using (8) with $y = 0$ we obtain for λ large enough

$$\begin{aligned} &\mu(\{|K_{L^{i\alpha}}(x,0)| \geq \lambda\}) \\ &\quad \geq \mu(\{C_1|\gamma(\alpha)|\rho^{-d}(x,0) \geq 2\lambda\}) - \mu(\{C_2|\Gamma(-i\alpha)|\rho^{-d+\frac{1}{2}}(x,0) \geq \lambda\}) \\ &\quad = \mu(B(0,(2C_1|\gamma(\alpha)|/\lambda)^{1/d})) - \mu(B(0,(C_2|\Gamma(-i\alpha)|/\lambda)^{1/(d-1/2)})) \\ &\quad \geq C'|\gamma(\alpha)|/\lambda. \end{aligned}$$

Here μ is Lebesgue measure and the sets above have the further restriction that $x \in B(0,r)$. Since $K_{L^{i\alpha}}$ is smooth away from the diagonal, we see that $L^{i\alpha}\phi_\delta(x)$ tends to $K_{L^{i\alpha}}(x,0)$ as $\delta \to 0$ for any $x \neq 0$ and any approximation of the identity $\{\phi_\delta\}$. Hence the above estimate shows that the weak type (1,1) norm of $L^{i\alpha}$ is bounded below by $|\gamma(\alpha)| = |\Gamma(i\alpha + d/2)/\Gamma(-i\alpha)| \sim (1+|\alpha|)^{\frac{d}{2}}$ (see [10]).

The upper bound in Theorem 1 holds in a much more general setting which we describe now. Assume that (X,μ,ρ) is a space with measure μ and metric ρ. If $\|P\|_{L^2\to L^\infty} < \infty$, then we can define the kernel K_P of the operator P by the formula

$$\langle P(\psi),\phi\rangle = \int P(\psi)\overline{\phi}d\mu = \int K_P(x,y)\psi(x)\overline{\phi(y)}d\mu(x)d\mu(y).$$

Note that $\sup_x \|K_P(x,\cdot)\|_{L^2} = \|P\|_{L^2\to L^\infty}$. Next, we say that

$$\operatorname{supp} K_P \subset \{(x,y) \in X^2 : \rho(x,y) \leq r\} \tag{9}$$

if $\langle P(\psi),\phi\rangle = 0$ for every $\phi,\psi \in L^2$ and every $r_1 + r_2 + r < \rho(x',y')$ such that $\psi(x) = 0$ for $\rho(x,x') > r_1$ and $\phi(x) = 0$ for $\rho(x,y') > r_2$. This definition (9) makes sense even if $\|P\|_{L^2\to L^\infty} = \infty$. Now if L is a self-adjoint positive definite operator acting on $L^2(\mu)$, then we say that it satisfies the finite speed propagation property of the corresponding wave equation if

$$\operatorname{supp} K_{C_t(\sqrt{L})} \subset \{(x,y) \in X^2 : \rho(x,y) \leq t\}, \tag{10}$$

where $C_t(\sqrt{L}) = \int \cos(t\sqrt{\lambda})\,dE(\lambda)$.

Theorem 2. *Suppose that L satisfies* (10)*. Next assume that*

(11) $\|\exp(-tL)\|^2_{L^2\to L^\infty} \le C_1 V_{d,D}(t^{1/2})^{-1} \le C\mu(B(x,t^{1/2}))^{-1} \le C_2 V_{d,D}(t^{1/2})^{-1}$

for all $t>0$ and $x\in X$, where $B(x,t)$ is a ball with radius t centred at x and

$$V_{d,D}(t) = \begin{cases} t^d & \text{for } t\le 1,\\ t^D & \text{for } t>1,\end{cases}$$

for $d,D\ge 0$. Then

$$\|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_2(1+|\alpha|)^{\max(d,D)/2}$$

for all $\alpha\in\mathbf{R}$.

We remark that (10) and (11) are equivalent to having Gaussian upper bounds on the heat kernel and the associated volume growth on balls. See [18]. Furthermore, the upper bound in Theorem 1 follows from Theorem 2. Indeed, if $X=\mathbf{R}^d$, $\rho(x,y)=\tau|x-y|$ and μ is Lebesgue measure, then it is well known (see e.g. [8] and [19]) that (11) and (10) hold. We are going to prove Theorem 2 only in the case $d=D$. The argument for the other cases is similar.

2. Preliminaries

The following lemma is a very simple but useful consequence of (10).

Lemma 1. *Assume that L satisfies* (10) *and that $\hat F$ is a Fourier transform of an even bounded Borel function F with* supp $\hat F\subset[-r,r]$*. Then*

$$\operatorname{supp} K_{F(\sqrt{L})} \subset \{(x,y)\in X^2 : \rho(x,y)\le r\}.$$

Proof. If F is an even function, then by the Fourier inversion formula,

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat F(t) C_t(\sqrt{L})\,dt.$$

But since supp $\hat F\subset[-r,r]$,

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-r}^{r} \hat F(t) C_t(\sqrt{L})\,dt$$

and Lemma 1 follows from (10).

Lemma 2. *Let $\phi\in C_c^\infty(\mathbf{R})$ be even, $\phi\ge 0$, $\|\phi\|_{L^1}=1$,* $\operatorname{supp}(\phi)\subset[-1,1]$*, and set $\phi_r(x)=1/r\,\phi(x/r)$ for $r>0$. Let Φ denote the Fourier transform of ϕ and Φ^r denote the Fourier transform of ϕ_r. If* (11) *and* (10) *hold, then the kernel $K_{\Phi^r(\sqrt{L})}$ of the self-adjoint operator $\Phi^r(\sqrt{L})$ satisfies*

(12) $$\operatorname{supp} K_{\Phi^r(\sqrt{L})} \subset \{(x,y)\in X^2; \rho(x,y)\le r\}$$

and

(13) $$|K_{\Phi^r(\sqrt{L})}(x,y)| \le C\,r^{-d}$$

for all $r>0$ and $x,y\in X$.

Proof. (12) follows from Lemma 1. For any $m \in \mathbf{N}$ and $r > 0$, we have the relationship

$$(I + rL)^{-m} \;=\; \frac{1}{m!} \int_0^\infty \mathrm{e}^{-rtL} \mathrm{e}^{-t} t^{m-1}\, dt$$

and so when $m > d/4$, (11) implies

$$\begin{aligned} \|(I + rL)^{-m}\|_{L^2 \to L^\infty} &\le \frac{1}{m!} \int_0^\infty \|\exp(-rtL)\|_{L^2 \to L^\infty} e^{-t} t^{m-1} dt \\ &\le C_1\, r^{-d/4} \end{aligned} \tag{14}$$

for all $r > 0$. Now $\|(I + r^2 L)^{-m}\|_{L^1 \to L^2} = \|(I + r^2 L)^{-m}\|_{L^2 \to L^\infty}$ and so

$$\|\Phi^r(\sqrt{L})\|_{L^1 \to L^\infty} \;\le\; \|(I + r^2 L)^{2m} \Phi^r(\sqrt{L})\|_{L^2 \to L^2}\, \|(I + r^2 L)^{-m}\|^2_{L^2 \to L^\infty}.$$

The L^2 operator norm of the first term is equal to the L^∞ norm of the function $(1 + r^2|t|)^{2m} \Phi(r\sqrt{|t|})$ which is uniformly bounded in $r > 0$ and so (13) follows by (14).

Next we recall the Calderón-Zygmund decomposition in the general setting of spaces of homogeneous type (see e.g. [3] or [22]).

Lemma 3. *There exists C such that, given $f \in L^1(X, \mu)$ and $\lambda > 0$, one can decompose f as*

$$f = g + b = g + \sum b_i$$

so that

1. $|g(x)| \le C\lambda$, *a.e.* x *and* $\|g\|_{L^1} \le C\|f\|_{L^1}$.
2. *There exists a sequence of balls $B_i = B(x_i, r_i)$ such that the support of each b_i is contained in B_i and*

$$\int |b_i(x)| d\mu(x) \;\le\; C\lambda\mu(B_i).$$

3. $\sum \mu(B_i) \le C\, \frac{1}{\lambda} \int |f(x)|\, d\mu(x)$.
4. *There exists $k \in \mathbf{N}$ such that each point of X is contained in at most k of the balls $B(x_i, 2r_i)$.*

We are now in a position to prove Theorem 2.

3. Proof of Theorem 2

The proof follows closely the line of argument in [1] (which of course generalises to this setting). We are attempting to prove

$$\lambda\, \mu(\{x \in X \,:\, |L^{i\alpha} f(x)| \ge \lambda\}) \;\le\; C(1 + |\alpha|)^{\frac{d}{2}}\, \|f\|_{L^1}.$$

As usual we start by decomposing f into $g + \sum b_i$ at the level of λ according to Lemma 3. We will follow the idea of C. Fefferman [11] of using more information of the L^2 operator norm (in our case, $\|L^{i\alpha}\|_{L^2 \to L^2} = 1$) by smoothing out the bad functions b_i at a scale smaller than the size of its support and considering this part of the good function where L^2 estimates can be used (see also [4]). In our case for each b_i, consider $\Phi^{s_i}(\sqrt{L}) b_i$ where $s_i = \theta r_i$, $\theta = (1 + |\alpha|)^{-\frac{1}{2}}$, and let

$G = g + \sum \Phi^{s_i}(\sqrt{L})b_i$ be the modified good function. Hence $f = G + B$ where $B = \sum(I - \Phi^{s_i}(\sqrt{L}))b_i$ and we write

$$\begin{aligned}\lambda\,\mu(\{|L^{i\alpha}f(x)| \geq \lambda\}) \leq\; & \lambda\,\mu(\{|L^{i\alpha}G(x)| \geq \lambda/2\}) \\ & + \lambda\,\mu(\{|L^{i\alpha}B(x)| \geq \lambda/2\}).\end{aligned} \tag{15}$$

The first term is less than $4/\lambda\,\|L^{i\alpha}G\|_{L^2}^2 \leq 4/\lambda\,\|G\|_{L^2}^2$. However, according to Lemma 2,

$$|\Phi^{s_i}(\sqrt{L})b_i(x)| \leq \int |K_{\Phi^{s_i}(\sqrt{L})}(x,y)b_i(y)|\,d\mu(y) \leq C\,(\theta r_i)^{-d}\|b_i\|_{L^1}\,\mathbb{1}_{B(x_i,2r_i)}$$

and therefore by Lemma 3, $|G(x)| \leq C\theta^{-d}\lambda$ for *a.e.*, x. Using Lemma 2 again which shows that the $L^p \to L^p$ operator norms of $\Phi^r(\sqrt{L})$ are uniformily bounded in $r > 0$, we also have that

$$\|G\|_{L^1} \leq \|g\|_{L^1} + C\sum \|\Phi^{s_i}(\sqrt{L})b_i\|_{L^1} \leq \|g\|_{L^1} + C\sum \|b_i\|_{L^1} \leq C\|f\|_{L^1}.$$

Therefore the first term in (15) is bounded by $(1+|\alpha|)^{\frac{d}{2}}\,\|f\|_{L^1}$.

Since $\mu(\bigcup B(x_i,\theta^{-1}r_i)) \leq C\theta^{-d}\sum\mu(B_i) \leq C(1+|\alpha|)^{\frac{d}{2}}\|f\|_{L^1}/\lambda$, then to bound the second term in (15), it suffices to show

$$\int\limits_{x\notin\cup B_i^*} |L^{i\alpha}B(x)|\,d\mu(x) \leq C(1+|\alpha|)^{\frac{d}{2}}\|f\|_{L^1}, \tag{16}$$

where $B_i^* = B(x_i,\theta^{-1}r_i)$. Let $H^\alpha(t) = |t|^{2i\alpha}$ so that

$$L^{i\alpha}B(x) = \sum H^\alpha(1-\Phi^{s_i})(\sqrt{L})b_i(x)$$

and therefore the left side of (16) is less than

$$\begin{aligned}&\sum_i \int\limits_{x\notin\cup_j B_j^*} \left|\int K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)b_i(y)\,d\mu(y)\right| d\mu(x) \\ &\qquad \leq \sum_i \int |b_i(y)| \int\limits_{x\notin B_i^*} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(x)\,d\mu(y).\end{aligned}$$

Since $F(L)^* = \overline{F}(L)$, we may interchange the roles of x and y, and so (16) will follow from Lemma 3 once we establish

$$\sup_{x,i} \int\limits_{\rho(x,y)\geq\theta^{-1}r_i} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y) \leq C\,(1+|\alpha|)^{\frac{d}{2}}. \tag{17}$$

We now fix $x \in X$ and i. Let $\eta \in C_c^\infty(\mathbf{R})$ be an even function supported in $\{t \in \mathbf{R} : 1 \leq |t| \leq 4\}$ such that

$$\sum_{n=-\infty}^{\infty} \eta(2^{-n}t) = 1 \text{ for all } t \neq 0.$$

We put $H_n^\alpha(t) = \eta(2^{-n}t)H^\alpha(t)$ so that

$$H^\alpha(1-\Phi^{s_i})(\sqrt{L}) = \sum_n H_n^\alpha(1-\Phi^{s_i})(\sqrt{L}).$$

Thus

$$\int_{y\notin B_i^*} |K_{H^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y) \ \le\ \sum_n \int_{y\notin B_i^*} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y) \tag{18}$$

and we will estimate each term in the sum on the right side in terms of n and i, uniformly in $x \in X$.

Let $k_o = [d/2]+1$ so that

$$\int_{y\notin B_i^*} (1+2^n\rho(x,y))^{-2k_o} d\mu(y) \le C \int_{\theta^{-1} r_i}^{\infty} (1+2^n r)^{-2k_0} r^{d-1} dr \le C 2^{-2nk_o}(\theta^{-1} r_i)^{d-2k_o}$$

and therefore by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\int_{y\notin B_i^*} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|\,d\mu(y)\\ &\quad\le\ C\,2^{-nk_o}(\theta^{-1}r_i)^{\frac{d}{2}-k_o}\\ &\qquad\cdot\Big(\int_{\rho(x,y)\ge\theta^{-1}r_i} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2(1+2^n\rho(x,y))^{2k_o} d\mu(y)\Big)^{1/2}.\end{aligned} \tag{19}$$

We break up the integral on the right side of (19) where $2^n\rho(x,y)$ is roughly constant and consider

$$\sum_{2^j\ge 2^n r_i\theta^{-1}} 2^{2jk_o} \int_{2^{j-1-n}<\rho(x,y)\le 2^{j-n}} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2\,d\mu(y). \tag{20}$$

Fix a nonnegative even $\varphi \in C_c^\infty(\mathbf{R})$ such that $\varphi = 1$ on $[-1/4,1/4]$ and $\varphi = 0$ on $\mathbf{R}\setminus[-1/2,1/2]$. Then the Fourier transforms of $H_n^\alpha(1-\Phi^{s_i})$ and $H_n^\alpha(1-\Phi^{s_i}) * (\delta - \hat{\varphi}_{2^{n-j}})$ agree on $\{\xi : |\xi| \ge 2^{j-1-n}\}$ and so by Lemma 1, the kernels of $H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})$ and $H_n^\alpha(1-\Phi^{s_i}) * (\delta-\hat{\varphi}_{2^{n-j}})(\sqrt{L})$ agree on the set $\{(x,y) \in X^2 : \rho(x,y) \ge 2^{j-1-n}\}$. Here δ denotes the Dirac mass at 0. For each j, the integrals in (20) satisfy the bound

$$\int_{2^{j-1-n}<\rho(x,y)\le 2^{j-n}} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)|^2\,d\mu(y) \ \le\ \|K_{F_{n,j}^\alpha(\sqrt{L})}\|^2_{L^2\to L^\infty},$$

where we are defining $F_{n,j}^\alpha(t) = H_n^\alpha(1-\Phi^{s_i}) * (\delta - \hat{\varphi}_{2^{n-j}})(t)$. So by (14), the right side of this inequality is bounded by $\|(I+2^{-2n}L)^m F_{n,j}^\alpha(\sqrt{L})\|^2_{L^2\to L^2}\, 2^{nd}$ as long as $m > d/4$. Everything then comes down to estimating the L^∞ norm of $(1+2^{-2n}t^2)^m F_{n,j}^\alpha(t)$. We make the following claim.

Claim. For each j, n and $m > d/4$,

$$(1+2^{-2n}t^2)^m |F_{n,j}^\alpha(t)| \ \le\ C_m|\alpha|^{k_o} 2^{-jk_o} \min(1,(2^n r_i\theta)^2)\,\min(1,|\alpha|2^{-j})$$

uniformly in $t \in \mathbf{R}$.

The claim shows that

$$\|K_{F_{n,j}^\alpha}(\sqrt{L})\|_{L^2\to L^\infty} \ \le\ C\,|\alpha|^{k_o} 2^{-jk_o} 2^{\frac{nd}{2}}\ \min(1,(2^n r_i\theta)^2)\,\min(1,|\alpha|2^{-j})$$

and hence the sum in (20) is bounded by

$$|\alpha|^{2k_o}2^{nd}\min^2(1,(2^n r_i\theta)^2)\sum_{2^j\geq 2^n r_i\theta^{-1}}\min^2(1,|\alpha|2^{-j})$$
$$\leq |\alpha|^{2k_o}2^{nd}\min^2(1,(2^n r_i\theta)^2)\log(2+\frac{|\alpha|}{2^n r_i\theta^{-1}}).$$

Recall that θ and α are related so that $\theta|\alpha| = |\alpha|/(1+|\alpha|)^{\frac{1}{2}} \leq \theta^{-1}$. Plugging this into (19) gives

$$\int_{y\notin B_i^*} |K_{H_n^\alpha(1-\Phi^{s_i})(\sqrt{L})}(x,y)d\mu(y) \leq \theta^{-d}(2^n r_i\theta)^{\frac{d}{2}-k_o}\ \min(1,(2^n r_i\theta)^2)\log(2+\frac{1}{2^n r_i\theta})$$

and this makes the sum in (18) bounded by $\theta^{-d} = (1+|\alpha|)^{\frac{d}{2}}$, proving (17) and hence Theorem 2.

Proof of the Claim. If $G_n(t) = H_n^\alpha(t)(1-\Phi^{s_i}(t))$, then $F_{n,j}^\alpha(t) = 2^{(n-j)k_o}G_n^{(k_o)} * \hat{\psi}_{2^{n-j}}(t)$ where $\psi(\xi) = \xi^{-k_o}(1-\varphi(\xi))$ (and so $\hat{\psi}$ is continuous, rapidly decreasing and has vanishing moments, $\int t^\ell\hat{\psi}(t)dt = 0,\ \ell = 0,1,2,\dots$). Hence

$$F_{n,j}^\alpha(t) = 2^{(n-j)k_o}\int_{\mathbf{R}}\big[G_n^{(k_o)}(t-s) - G_n^{(k_o)}(t)\big]\hat{\psi}_{2^{n-j}}(s)\,ds$$
$$= 2^{(n-j)k_o}\int_{\mathbf{R}}\big[G_n^{(k_o)}(t-2^{n-j}s) - G_n^{(k_o)}(t)\big]\hat{\psi}(s)\,ds.$$

However $G_n(t) = \eta(2^{-n}t)|t|^{2i\alpha}(1-\Phi(s_i t))$ and thereby each time we take a derivative, we gain a factor of 2^{-n}. $G_n^{(k_o)}(t)$ is thus a finite sum of terms of the form $\alpha^p 2^{-nk_o}\tilde{\eta}(2^{-n}t)|t|^{2i\alpha}\Psi(s_i t)$ where $\tilde{\eta}\in C_c^\infty(\mathbf{R})$, $\operatorname{supp}(\tilde{\eta})\subset\operatorname{supp}(\eta)$ and Ψ is a Schwartz function which is $0(t^2)$ as $t\to 0$ (note that $\Phi'(0) = \int x\phi(x)dx = 0$ since ϕ is even). The worst power p is k_o which occurs when all derivatives land on the factor $|t|^{2i\alpha}$.

Without loss of generality, let us suppose that

$$G^{(k_o)}(t) = \alpha^{k_o}2^{-nk_o}\eta(2^{-n}t)|t|^{2i\alpha}\Psi(s_i t).$$

From the above integral representation of $F_{n,j}^\alpha(t)$, we see that the main contribution to $(1+2^{-2n}t^2)^m|F_{n,j}^\alpha(t)|$ occurs when $|t|\sim 2^n$ and in this case,

$$|F_{n,j}^\alpha(t)| \leq C|\alpha|^{k_o}2^{(n-j)k_o}2^{-nk_o}\min(1,(s_i 2^n)^2) \leq C|\alpha|^{k_o}2^{-jk_o}\min(1,(2^n r_i\theta)^2).$$

However we may write

$$F_{n,j}^\alpha(t) = -2^{(n-j)k_o}2^{n-j}\int_0^1\int_{\mathbf{R}} G_n^{(k_o+1)}(t-\sigma 2^{j-n}s)s\hat{\psi}(s)\,ds\,d\sigma$$

and therefore we also have

$$|F_{n,j}^\alpha(t)| \leq C|\alpha|^{k_o+1}2^{(n-j)k_o}2^{n-j}2^{-n(k_o+1)}\min(1,(s_i 2^n)^2)$$
$$\leq C|\alpha|^{k_o}2^{-jk_o}|\alpha|2^{-j}\min(1,(2^n r_i\theta)^2),$$

establishing the claim.

Remarks. Theorem 1 holds also for Laplace-Beltrami operators on compact manifolds of dimension d. The proof is essentially the same as the proof of Theorem 1.

The hypotheses of Theorem 2 are satisfied for Laplace operators on Lie groups of polynomial growth. However, if L is a sub-Laplacian on the three dimensional Heisenberg group, then $d = 4$ but

$$C_1(1+|\alpha|)^{3/2} \le \|L^{i\alpha}\|_{L^1\to L^{1,\infty}} \le C_\epsilon(1+|\alpha|)^{3/2+\epsilon}.$$

(See [16]; see also [12].) The same estimates hold for a sub-Laplacian on SU(2) for which $d = 4$ and $D = 0$ (see [7]). Thus there are situations where the upper bound is better than the one given by Theorem 2 and where the lower bound in Theorem 1 is false. For general groups of polynomial growth Theorem 2 gives the best known estimates; however as the above examples show, these bounds are not always best possible.

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