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Weak links, good shots and other public good games: Building on BBV

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Abstract

We suggest an alternative way of analyzing the canonical Bergstrom–Blume–Varian model of non-cooperative voluntary contributions to a public good that avoids the proliferation of dimensions as the number of players is increased. We exploit this approach to analyze models in which the aggregate level of public good is determined as a more general social composition function of individual gifts — specifically, as a CES form — rather than as an unweighted sum. We also analyze Hirshleifer's weakest-link and best-shot models. In each case, we characterize the set of equilibria, in some cases establishing existence of a unique equilibrium as well as briefly pointing out some interesting comparative static properties. We also study the weakest-link and best-shot limits of the CES composition function and show how the former can be used for equilibrium selection and the latter to establish that equilibria of some better-shot games are identical to those of the much simpler best-shot game.

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1. Introduction

The canonical model of non-cooperative public good provision set out by Bergstrom, Blume and Varian (1986) — hereafter BBV — is a prominent example of a non-cooperative game with an aggregative structure. The common object of all players' preferences is a simple aggregate, the

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unweighted sum of all individual contributions g_i, and each player's preferences can be represented by a payoff function of the form $_i(g_i,G)$, where $G=\sum_j g_j$.

We study a systematic way of exploiting its aggregative structure that avoids the proliferation of dimensions as the number of players grows and thereby simplifies its analysis. This permits us to extend the model by allowing G to be a more general (social composition) function of individual contributions, rather than an unweighted sum. This extra generality allows us to consider weaker link and better shot situations, of which Hirshleifer's (1983) weakest-link and best-shot games are extreme cases. Social composition functions involving weaker-link public goods exhibit a convex technology for transforming individual contributions into the aggregate level of G, and imply a unique equilibrium in pure strategies. Situations involving better-shot public goods have non-convex social composition functions and, typically, multiple equilibria. Our approach also elucidates structural properties of equilibria and comparative statics properties. To avoid excessive length, we avoid a full treatment of these topics, confining ourselves to passing remarks.

In Sections 2 and 3, we show how "replacement functions" offer a simple and intuitive proof of existence, uniqueness and neutrality in the canonical model of BBV. The main aim of these sections is to set the scene for extensions of the canonical model which yield to modifications of this approach. For example, in Section 4, we show how easily an appropriate redefinition of replacement functions shows that Hirshleifer's weakest-link model has a continuum of Pareto ranked equilibria. A more radical extension of the replacement function can be used to analyze games with a generalized concave CES social composition function.[1] In Section 5, we prove that such games always have a unique equilibrium. The fact that the weakest-link composition function can be viewed as a limiting case of CES suggests using the limiting equilibrium to select from the continuum of equilibria under weakest link. We show that this can indeed be done, but it does not typically select the Pareto dominant equilibrium. A further extension of replacement functions (to correspondences) allows us to handle non-concave social composition functions and we illustrate this by discussing games with convex CES composition functions and Cobb–Douglas preferences in Section 6. When the (negative) elasticity of substitution is close enough to zero, we offer a complete characterization of the set of equilibria. In particular, we characterize all possible sets of players active in some equilibrium and show that, given such a set, the equilibrium is uniquely determined. A final extension of replacement correspondences allows us to characterize the set of equilibria in best-shot games and to show that Pareto-improving and coordination-resolving transfers may be available in such games. Finally, we show that when the elasticity of substitution is negative and close enough to zero in the better-shot game, the set of equilibria coincides with that in the best-shot game.

2. The canonical model

We generally follow BBV's notation and assumptions. Their model has four elements:

1. Individual preferences: Player i , $i=1,\ldots, n$, has preferences represented by the utility function $u_i(x_i,G)$, where $x_i \geq 0$ is the quantity of a private good and G the total quantity of a pure public good. BBV do not explicitly impose much structure directly on preferences. In particular, they

[1] Todd Sandler, with various coauthors, has stressed the potential relevance of weak link and good shot public good models to many situations involving regional and global public goods. See especially (Arce and Sandler, 2001; Sandler, 1998, 2003, 2004). Varian (2004) applies the weakest link model to modelling system reliability.

assume that both goods are desirable and strictly normal. We will refer to this last assumption as normal preferences, or normality. For convenience, we shall assume, without explicit statement, that u_i is continuously differentiable. Then, normality implies that the marginal rate of substitution (of private for public goods) is strictly decreasing in G and non-decreasing in x_i. Such assumptions imply strictly convex preferences.

2. Individual budget constraints: Player i's budget constraint requires that

$$x_i + g_i = w_i$$

where $g_i \geqq 0$ is her contribution to a pure public good, or her gift, and w_i is her exogenous income.

3. The social composition function: The total public good provision is the unweighted sum of individual gifts:[2]

$$G = \sum_{i=1}^{n} g_i.$$

4. The behavioral assumption: The game is a static, or simultaneous, non-cooperative game in which the strategic choice variables are the individual gifts, $(g_1, g_2, \ldots, g_n)$.

BBV demonstrate the existence of a unique Nash equilibrium in pure strategies. Existence is established by appealing to Brouwer's fixed point theorem, and uniqueness by a separate argument tailored to the public good model. Although there is no doubting its formal correctness, their uniqueness argument has not struck all readers as intuitively transparent.[3]

Their comparative static analysis proceeds by direct examination of the budget set, and establishes the well-known neutrality result associated with income redistributions amongst contributors, as well as the limits on redistributions that maintain neutrality.

3. An alternative approach: the replacement function

Cornes and Hartley (2007) formally demonstrate the existence, and explore the properties, of the replacement function r_i of player i, which expresses the player's best response as a function, not of the sum of best responses of all other players, but of the total level of public good G. That is, $g_i = r_i(G)$ if and only if the strategy choice of player i is g_i in all Nash equilibria in which aggregate public good provision is G. (This includes the player's own choice amongst the arguments of the function.) Replacement functions provide a simpler and more direct unified analysis of existence, uniqueness and comparative static properties of equilibrium in the canonical model.[4]

A central proposition in Cornes and Hartley (2007) characterizes the properties of a player's replacement function in the canonical model as follows:

Proposition 3.1. *In the canonical public good model, player i has a replacement function* $r_i(G)$ *with the following properties:*

1. *There exists a finite value,* $\bar{G}_i$*, at which* $r_i(\bar{G}_i) = \bar{G}_i$*.*
2. $r_i(G)$ *is defined for all* $G \quad \bar{G}_i$*.*

[2] The term "social composition function" was suggested by Hirshleifer (1983).

[3] Indeed, in (Bergstrom et al., 1992) they tighten up their original uniqueness argument in response to concerns voiced by Fraser (1992).

[4] An earlier treatment of public goods that exploits their aggregative nature is Okuguchi (1993).

3. *$r_i(G)$ is continuous.*
4. *$r_i(G)$ is everywhere non-increasing in G, and is strictly decreasing wherever it is strictly positive.*

Here, $\bar{G}_i$ is the level of public good that player i would prefer, if that player were the sole contributor. Replacement functions can be used to study equilibria via the following characterization, whose proof is trivial.

Characterization of a Nash equilibrium: *A strategy profile $(\hat{g}_1,\ldots,\hat{g}_n)$ is a Nash equilibrium if and only if*

$$\hat{g}_j = r_j(\hat{G}) \text{ for } j = 1,\ldots,n,$$

where $\hat{G}=\sum_{j=1}^{n}\hat{g}_j$

Note that $\hat{G}$ is an equilibrium level of the public good if and only if it is a fixed point of the *aggregate replacement function* $R(G)=\sum_{j=1}^{n} r_j(G)$. It follows that, if R has a unique fixed point, the game has a unique equilibrium. This will happen if R is continuous, strictly decreasing where positive and has a graph that crosses the 45° line. Exactly these properties follow from Proposition 3.1. Thus both existence and uniqueness of Nash equilibrium are established in a single simple line of argument.

Not only do existence and uniqueness follow directly from simple geometric considerations. So, too, do comparative static properties such as the well-known neutrality proposition. We refer readers to Cornes and Hartley (2007) for a more systematic account of such issues. In particular, they relax the assumption of a common unit cost of contributing across players, and permit player i's budget constraint to take the form $x_i+c_i g_i=w_i$. Such an extension poses no problems for the present approach, and raises interesting and surprising possibilities, which Cornes and Hartley explore, concerning the consequences of both idiosyncratic and systematic changes in unit cost levels. The same paper provides further references to earlier literature that exploits the aggregative structure of the public good model, but in a less systematic way.

In the rest of the paper, we demonstrate how to extend this approach to more general social composition functions. Specifically, we consider the CES social composition function

$$G = \left(\sum_{j=1}^{n}\beta_j g_j^{\theta}\right)^{1/\theta} \qquad (6)$$

for real values of $\theta \neq 0$[5,6] and $\beta_1,\ldots,\beta_n > 0$. When $\theta < 1$ (and $\theta \neq 0$), replacement functions always exist but may not be decreasing. However, the *share function*: $s_i(G)=\beta_i(r_i/G)^{\theta}$ is always monotonic. Since $\hat{G}$ is an equilibrium level of the public good if and only if

$$S(\hat{G}) = \sum_{j=1}^{n} s_j(\hat{G}) = 1,$$

[5] Of course, if $\theta \to 0$, the social choice function (6) approaches Cobb–Douglas (with parameters $\beta_i/\sum_{j=1}^{n}\beta_j$). We do not explicitly study this case, but the principal qualitative conclusions (existence and uniqueness of an equilibrium) are still valid.

[6] When $\theta<0$ and some $g_i = 0$, we suppose $G=0$ to ensure continuity.

where $S(G)$ is the *aggregate share function*, multiple equilibria are ruled out. The equilibrium strategy of Player i is $\hat{g}_i = \hat{G}s_i(\hat{G})$. If we can also show that the aggregate share function is continuous and takes values greater and less than 1, the game has a unique equilibrium. When >1, the social composition function is no longer concave and multiple equilibria are possible. Nevertheless, extending replacement and share functions to correspondences allows us to characterize equilibrium sets. We also examine the weakest and strongest link social composition functions, which can be viewed as limiting cases of a CES function. 0nce again replacement function methods, appropriately modified, can be used to analyze equilibria.

4. Weakest-link public goods

The approach of the previous section can be adapted to Hirshleifer's weakest-link and best-shot public good games. In this section, we discuss the weakest-link social composition function:

$$G = \min_{j=1,\dots,n} g_j;$$

deferring treatment of the best-shot case to Section 7. Recall that if r_i is the replacement function of player i, $r_i(G)$ is the unique strategy of the player in all equilibria in which the aggregate level of the public good is G.

Let $\bar{G}_i$ denote the preferred level of the public good of Player i among all allocations in which $G = g_i$, and let $G_{-i} = \min_{j \neq i} g_j$. For all contribution levels such that $G_{-i} \leq \bar{G}_i$, Player i will want to match the smallest of the other contributions, since her preferences are convex. In this event, her best response is $\hat{g}_i = G_{-i}$ and the total quantity of the public good is $G = \min\{\hat{g}_i, G_{-i}\} = \hat{g}_i = G_{-i}$. However, if $G_{-i} > \bar{G}_i$, Player i will only want to contribute up to the level $\bar{G}_i$, and no further. Her contribution then determines the value of the weakest link. In this event, $G = \min\{\hat{g}_i, G_{-i}\} = \bar{G}_i$. It follows that

$$(r_i(G), G) = (G_{-i}, G_{-i}) \text{ if } 0 \leq G_{-i} \leq \bar{G}_i$$

and

$$(r_i(G), G) = (\bar{G}_i, \bar{G}_i) \text{ if } G_{-i} > \bar{G}_i.$$

Proposition 4.1. *If Player i has convex increasing preferences, her replacement function has domain* $[0, \bar{G}_i]$ *and satisfies* $r_i(G) = G$ *in this domain.*

Nash equilibrium levels of the public good are still fixed points of the aggregate replacement function R, provided the definition of R is modified to $R(G) = min_{j=1,\dots,n} r_j(G)$. The domain of R is the intersection of the domains of individual replacement functions, so

$$R(G) = G \text{ for } 0 \leq G \leq \min_{j=1,\dots,n} \bar{G}_i.$$

We may conclude that any non-negative level of the public good not exceeding any individually preferred level is an equilibrium.

Proposition 4.2. *If all players have convex, increasing preferences,* $(\hat{g}_1, \dots, \hat{g}_n)$ *is an equilibrium strategy profile if and only if* $\hat{g}_j = \hat{g}$ *for all j for some* $\hat{g}$ *satisfying* $0 \leq \hat{g} \leq min_{j=1,\dots,n} \bar{G}_i$.

Thus, there is a continuum of Pareto ranked equilibria. Hirshleifer suggested that the salient equilibrium is precisely the value $\hat{g}_j = min_{j=1,\dots,n} \hat{G}_i$, which Pareto dominates all the others. However, experimental evidence has not supported the idea of Pareto dominance as a selection criterion. Weber (2006) provides an interesting discussion of experimental evidence in situations requiring coordination. In Section 5, we will offer an additional theoretical perspective on the coordination issue in the weakest-link model.

Vicary (1990), Sandler and Vicary (2001) and Vicary and Sandler (2002) explore weakest-link games in which players are able to make income transfers.

5. More general concave social composition functions

In this and following sections, we revisit the extension of the canonical model suggested by Cornes (1993) focussing on the CES social composition functions. Throughout this section we restrict attention to concave social composition functions, which ensures continuity of behavioral functions. In the following section, we extend the analysis to the non-concave case. There are two principal results in this section. Proposition 5.1 demonstrates that normality still ensures that the game has a unique Nash equilibrium. The second main result shows that the Pareto dominant Nash equilibrium identified by Hirshleifer in his analysis of weakest-link models may not be a reliable guide to the equilibrium of weaker link models (in which the marginal productivities of low contributors are significantly larger than those of higher contributors).

5.1. Best responses, replacement and share functions

5.1.1. Best responses

If the social composition function takes the CES form, the payoff of player i is

$$u_i\left(w_i - g_i, \left(\sum_{j=1}^{n} \beta_j g_j^{\theta}\right)^{1/\theta}\right)$$

and, in this section, we assume $\theta \le 1$ and $\theta \neq 0$. In this parameter range, the social composition function is a concave function of g_i and, since u_i is strictly quasi-concave and increasing in both arguments, the payoff is a strictly quasi-concave function of g_i. It follows that best responses can be characterized by standard first-order conditions.

The marginal payoff of Player i (holding other players' strategies fixed) is

$$-\frac{\partial u_i}{\partial x_i} + \beta_i \frac{\partial u_i}{\partial G}\left(\frac{g_i}{G}\right)^{\theta-1}.$$

So the first-order conditions can be written: $g_i \in (0, w_i]$ and

$$MRS_i(w_i - g_i, G) \ge \frac{1}{\beta_i}\left(\frac{g_i}{G}\right)^{1-\theta} \quad \text{with equality if } g_i < w_i; \tag{2}$$

where MRS_i denotes the marginal rate of substitution ($\frac{\partial g_i}{\partial G}$ for fixed u_i). Note that we cannot have a solution to the first-order conditions with $g_i = 0$ and $G > 0$[7], so this boundary value can never be a best response.

[7] If $0 < \theta < 1$, satisfaction of the first order conditions would imply $MRS_i(w_i, G) = 0$ which is inconsistent with normal demand. When $\theta < 0$, such a solution is excluded by fiat.

Defining

$$G_{-i}=\left(\sum_{j\neq i}\beta_j g_j^{\theta}\right)^{1/\theta};$$

we observe that best responses depend on individual strategies of other players only through G_{-i} and write $b_i(G_{-i})$ for the best response to G_{-i}.

5.1.2. Replacement functions

The replacement function $r_i(G)$ of player i is the strategy the player would choose in any equilibrium in which public good provision is G. That is, $g_i=r_i(G)$ satisfies $g_i=b_i(G-g_i)$, which means that (g_i,G) satisfies Eq. (2). It is necessary to establish that r_i is well-defined by showing that Eq. (2) has a unique solution in g_i for any G in a suitable domain.

The desired conclusion follows from normality, since this implies that, if we hold G fixed, $MRS_i(G,w_i-g_i)$ is non-increasing in g_i. Furthermore, the right hand side of Eq. (2) is strictly increasing in g_i and takes the value zero at $g_i=0$, since $\theta<1$. Fig. 1 illustrates these observations. The upper panel shows an interior best response and the lower panel displays a boundary case.

Fig. 1.

We deduce that Eq. (2) has a unique solution in g_i. However, for this to correspond to a strategy profile satisfying the feasibility requirement $g_j \geq 0$ for $j \neq i$, we must have $G_{-i} \geq 0$. In the case $\rho \geq 0$, $G_{-i} > 0$ implies $G > g_i$, so the solution is feasible if and only if $G \geq \bar{G}_i$, where

$$\bar{G}_i = \beta_i^{1/\rho} b_i(0)$$

is the level of the public good which player i would choose, if she were the sole contributor. If $\rho < 0$, $G_{-i} > 0$ implies $G < g_i$ and the solution is feasible if and only if $G \leq \bar{G}_i$. Formal proofs of these observations are a little delicate and given in the Appendix. The following Lemma summarizes our conclusions at this point.

Lemma 5.1. *Under increasing, normal preferences, there is a unique $g_i \in (0, w_i]$ satisfying* Eq. (2). *Furthermore, the feasibility condition $g_i \leq G$ is satisfied for $G \geq \bar{G}_i$ if $0 < \rho < 1$ and for $G \leq \bar{G}_i$ if $\rho < 0$.*

This lemma implies that replacement function $r_i(G)$, the unique g_i satisfying Eq. (2), is well-defined and has domain $[\bar{G}_i, \infty)$ if $[0 < \rho < 1$ and $[0, \bar{G}_i]$ if $\rho < 0$. Note also that $\bar{G}_i = \beta_i^{1/\rho} b_i(0)$ implies that

$$r_i(\bar{G}_i) = \bar{G}_i. \tag{3}$$

5.1.3. Share functions

In contrast to the canonical case, the replacement function need not be decreasing. We can circumvent this difficulty by using the *share function* s_i for player i, which is defined as

$$s_i(G) = \beta_i \frac{[r_i(G)]^\rho}{G^\rho}. \tag{4}$$

It follows from the definition of equilibrium that $S(G) = \sum_{j=1}^{n} s_j(G) = 1$ is a necessary and sufficient condition for G to be a Nash equilibrium level of public good provision. It follows that, if S is strictly monotonic and its graph crosses the line $S = 1$, the game has a unique equilibrium. We complete the proof by establishing properties of individual share functions entailing these properties.

In carrying out this analysis, it is convenient to rewrite the interior first-order condition (2) as

$$MRS_i\left(w_i - \beta_i^{-1/\rho}\sigma_i^{1/\rho}G; G\right) = \beta_i^{-1/\rho}\sigma_i^{(1-\rho)/\rho} = \beta_i^{-1/\rho}\left(\sigma_i^{1/\rho}\right)^{1-\rho}, \tag{5}$$

where $\sigma_i = s_i(G)$. Normality implies that, if we hold G fixed, an increase in $\sigma_i^{1/\rho}$ leads to a strict decrease in MRS_i and, if we hold σ_i fixed, an increase in G leads to a strict decrease in MRS_i. Thus, the graph of the left-hand side of Eq. (5) as a function of $\sigma_i^{1/\rho}$ slopes down and shifts upwards with G. Furthermore, the right hand side is strictly increasing in $\sigma_i^{1/\rho}$ and independent of G. This is illustrated in Fig. 2 and shows that the intersection at $s_i^{1/\rho}(G)$ is decreasing in G.

Fig. 2.

If 0b b1, this means that $s_i(G)$ is a strictly decreasing function defined on the domain $[\overline{G}_i,\infty)$ (from Lemma 5.1). Furthermore, Eq. (3) implies that $s_i(\overline{G}_i)=1$ and the constraint $r_i(G)\leq w_i$ implies that $s_i(G)$ 0 as G ∞. Continuity of s_i follows from continuity of both MRS_i and the right-hand side of Eq. (5).

This share function is shown in the upper graph in Fig. 3 and is one of two possible forms. The second form is shown in the bottom panel and applies when b0. In this case the domain is $[0,\overline{G}_i]$ and $s_i(\overline{G}_i)=1$. Furthermore, s_i is strictly increasing in G, since $s_i^{1/}$ is decreasing and therefore $s_i(G)$ has a limit $\bar{}_i\geq 0$ as G 0. To ease the exposition, we make the additional assumption: MRS_i $(w_i,0)$. (This holds when the indifference map is asymptotic to the axes, as with Cobb–Douglas preferences for example, and implies that $\bar{}_i=0$. This follows directly by taking the limit G 0 in Eq. (5).

There is a simple intuition for the sensitivity of the slope of the share function to the sign of the parameter . In the canonical model, an increase of one unit in G implies an increase of one unit in i's full income. Normality implies that i's preferred quantity of public good also increases, but by less than one unit. Player i will therefore prefer to reduce her contribution. The sole mechanism at work is an income effect.

However, when b0, an increase in G has a second effect, in addition to the pure income effect identified in the paragraph above. At any given level of g_i, it also increases the marginal productivity of player i's contribution. This 'substitution effect' by itself will encourage player i to contribute more, since a given increase in G generated by her contribution now has a lower opportunity cost in terms of private consumption. Thus the fact that the public good is of the weaker link type at least moderates the negative slope of the replacement function and, if is sufficiently small, the substitution effect dominates the overall behavioral response, making it positive.

In either of the two possible cases illustrated in Fig. 3, there are values of G at which the aggregate share function S is at least one ($\max_j\overline{G}_j$ for N0 and $\min_j\overline{G}_j$ for b0) as well as values at which it is less than one (large enough if N0 and close to zero if b0). Since S is

Fig. 3.

continuous and monotonic in either case, S has a unique fixed point, implying a unique equilibrium. The following proposition summarizes these results.

Proposition 5.1. *Suppose all players have increasing, normal preferences and the social composition function takes the CES form (1) with parameter ρ. If $0<\rho<1$ or if $\rho<0$ and $MRS_i(w_i, 0)^{-1}=0$, the game has a unique equilibrium.*

This uniqueness result holds for a wider class of social composition functions than CES. Specifically, consider the following generalization:

$$G=\left(\sum_{j=1}^{n}\beta_j g_j^{\rho_j}\right)^{1/\rho}; \tag{6}$$

where $\beta_1,\ldots,\beta_n>0$. If the parameters $\rho_1,\ldots,\rho_n$ and ρ satisfy

$$0<\rho_i\le\rho\le 1 \text{ and for } i=1,\ldots,n \text{ and } \max_{j=1,\ldots,n}\rho_j<1; \tag{7}$$

the argument given above for $0<\rho<1$ may be modified to prove that share functions continue to take the form in the upper panel of Fig. 3. Details may be found in (Cornes and Hartley, 2006).

Alternatively, if

$$\rho\le\rho_i<0 \text{ for } i=1,\ldots,n;$$

share functions take the form displayed in case (b). In either case, we have a unique equilibrium.

5.2. The weakest-link limit

When $\rho \to -\infty$, the symmetric CES social composition function: $(\sum_{j=1}^{n} g_j^{\rho})^{1/\rho}$ approaches the weakest-link social composition function. We have seen that there is typically a continuum of equilibria in the weakest-link case. Under conditions on preferences that ensure a unique equilibrium when $\rho < 0$, we might hope that limits of equilibria as $\rho \to -\infty$ would select a unique member of the continuum, perhaps the Pareto dominant equilibrium studied by Hirshleifer. In the rest of this section, we investigate this issue for symmetric games and verify that a unique equilibrium of the weakest-link game is selected, but it is not the Pareto dominant equilibrium.

Throughout this subsection we assume a symmetric CES social composition function with $\rho < 0$ and identical preferences that are strictly increasing, normal and satisfy $MRS_i(w_i,0)^{-1}$ for all i. This guarantees a unique equilibrium by Proposition 5.1 and we shall write $\hat{G}(\rho)$ for the level of public good provision in this equilibrium. Symmetry of payoffs implies a unique equilibrium in which $\hat{g}_i = n^{-1/\rho}\hat{G}$ for all i and substitution in Eq. (2) gives

$$MRS\left(w - n^{-1/\rho}\hat{G}(\rho), \hat{G}(\rho)\right) \geq n^{1-1/\rho}; \tag{8}$$

with equality if $\hat{G}(\rho) < n^{1/\rho} w$ (dropping subscripts). Since $0 \leq \hat{G}(\rho) \leq n^{1/\rho} w$ for all $\rho < 0$, the set $\{\hat{G}(\rho): \rho < 0\}$ has limit points. Let $\hat{G}(-\infty)$ be such a limit point and take the limit in Eq. (8) to obtain $\hat{G}(-\infty) \leq w$ and

$$MRS\left(w - \hat{G}(-\infty), \hat{G}(-\infty)\right) \geq n; \tag{9}$$

with equality if $\hat{G}(-\infty) < w$. Normality implies that there is a unique $\hat{G}(-\infty)$ satisfying these conditions and therefore $\hat{G}(\rho) \to \hat{G}(-\infty)$ as $\rho \to -\infty$. We now show that for an interior solution $\hat{G}(-\infty)$ is never the Pareto dominant level of the public good in the best-shot game.

As we saw in Section 4, the equilibrium values of G in the symmetric weakest-link game are $[0,\bar{G}]$, where $\bar{G}$ is the level each player would choose as sole contributor. Since $\bar{G}$ maximizes $u(w-G,G)$, it satisfies $\bar{G} \leq w$ and

$$MRS(w - \bar{G}, \bar{G}) \geq 1; \tag{10}$$

with equality if $\bar{G} < w$. Comparison of Eqs. (9) and (10) shows that $\hat{G}(-\infty) \leq \bar{G}$ and this inequality is strict unless $\hat{G}(-\infty) = w$, in which case $\bar{G} = w$. Indeed, the equilibrium depends on the number of players and as this increases, $\hat{G}(-\infty)$ decreases, reducing payoffs of all players. Thus with symmetric, increasing, normal preferences and a symmetric CES social composition function, we have the following observations.

- The limiting weaker link equilibrium is Pareto dominated by the weakest-link equilibrium with $\hat{G} = \bar{G}$.
- This dominance is strict if there is any private consumption in the weakest-link equilibrium.
- Under these conditions, an increase in the number of players leads to a Pareto inferior limiting weaker link equilibrium.

As an example, suppose every player has a Cobb–Douglas utility function

$$u(x, G) = x^{\alpha} G$$

where $\alpha > 0$. It can also be shown in this example that the replacement function for $\theta \le 0$ approaches the weakest-link replacement function as $\theta \to -\infty$. However, since

$$\mathrm{MRS}(w-g; G) = \frac{w-g}{\alpha G};$$

we have

$$\phi(-\infty) = \frac{w}{\alpha n + 1} \le \bar{G} = \frac{w}{\alpha + 1}.$$

In this case, $\phi(-\infty)$ decreases to zero as $n \to \infty$.

6. Convex CES social composition function

6.1. Best responses and the replacement correspondence

When the social composition function is not concave, even qualitative results are more sensitive to the specific form of the social composition function and preferences. However, the approach of the preceding sections may still be adapted to handle such cases, provided we extend replacement and share functions to correspondences. To illustrate, we focus on Cobb–Douglas preferences:

$$u_i(x_i; G) = x_i^{\alpha_i} G;$$

with $\alpha_i > 0$ for all i and a CES social composition function with $\theta > 1$. Our approach uses a set-valued extension of replacement and share functions and we start by examining best responses. The payoff function of Player i is

$$\pi_i = (w_i - g_i)^{\alpha_i} \left(\sum_{j=1}^{n} \beta_j g_j^{\theta} \right)^{1/\theta}.$$

Recalling the definition

$$G_{-i} = \left(\sum_{j \ne i} \beta_j g_j^{\theta} \right)^{1/\theta};$$

we can write $B_i(G_{-i})$ for the set of best responses:

$$B_i(G_{-i}) = \arg\max_{0 \le g_i \le w_i} \left\{ (w_i - g_i)^{\alpha_i} \left(\beta_i g_i^{\theta} + G_{-i}^{\theta} \right)^{1/\theta} \right\}.$$

If the contributions of the other players are sufficiently large, a given player will free ride. Specifically, there is a critical value of G_{-i} above which zero is the unique best response for player

i. Below this value, there is a single interior best response, which is a stationary point of π_i. At the critical value, both strategies become alternative best responses.

Lemma 6.1. *There exists $G^*_{-i} > 0$ and a positive real-valued function b_i on $[0, G^*_{-i}]$ such that*

$$B_i(G_{-i}) = \begin{cases} \{b_i(G_{-i})\} & \text{if } G_{-i} < G^*_{-i}; \\ \{0, b_i(G_{-i})\} & \text{if } G_{-i} = G^*_{-i}; \\ \{0\} & \text{if } G_{-i} > G^*_{-i}. \end{cases}$$

A detailed proof of this lemma is given in the Appendix. However, the intuition behind the result is straightforward and displayed in Fig. 4.

As a function of own strategy, each player's payoff π_i can take one of three forms. For large enough G_{-i}, the payoff decreases strictly as g_i goes from 0 to w_i, (see top panel) and $g_i = w_i$ is the unique maximizer. For smaller G_{-i}, as g_i increases from 0 to w_i, the payoff initially decreases,

Fig. 4.

then increases to a local maximum and finally decreases to zero at $g_i = w_i$. The local maximum must be at the unique stationary point and therefore satisfies

$$\alpha_i(\beta_i g_i^{\rho} + G^{\rho}_{-i}) = (w_i - g_i)\beta_i g_i^{\rho-1}. \tag{11}$$

This is also the global maximum if the payoff at the stationary point exceeds the payoff at $g_i = 0$:

$$(w_i - g_i)^{\alpha_i}(\beta_i g_i^{\rho} + G^{\rho}_{-i})^{1/\rho} \geq w_i^{\alpha_i} G_{-i}. \tag{12}$$

The value of G^*_{-i} (and the corresponding stationary point g_i^*) are determined by the requirement that payoffs at $g_i = 0$ and $g_i = g_i^*$. Hence, (g_i^*, G^*_{-i}) satisfies Eqs. (11) and (12) with equality. In the proof, we show that, if G^*_{-i}, the payoff at the local maximum exceeds that at $g_i = 0$ and is therefore the global maximum (see middle panel), whereas for $G_{-i} > G^*_{-i}$ the maximum is higher at $g_i = 0$ (see bottom panel).

The nature of the best-responses does not permit a well-defined replacement function. However, we can define the replacement correspondence $\mathcal{R}_i$ of Player i by

$$\mathcal{R}_i(G) = \left\{ g_i : g_i \in B_i\left[(G^{\rho} - \beta_i g_i^{\rho})^{1/\rho}\right] \right\}.$$

It follows from the lemma that $g_i \in \mathcal{R}_i(G)$ if and only if either (i) $g_i = 0$ and $G \geq G^*_{-i}$ or (ii) $g_i > 0$ and

$$\alpha_i G^{\rho} = \beta_i (w_i - g_i) g_i^{\rho-1}; \tag{13}$$

$$\beta_i^{1/\rho} g_i \leq G \leq \beta_i^{1/\rho} w_i^{\alpha_i} g_i \left[w_i^{\alpha_i \rho} - (w_i - g_i)^{\alpha_i \rho}\right]^{-1/\rho}. \tag{14}$$

Eq. (3) and the right-hand inequality in Eq. (14) are simply restatements of Eqs. (11) and (12) respectively. The left-hand inequality is equivalent to the requirement $G_{-i} \geq 0$.

6.2. Properties of the replacement and share correspondences

We have shown that the graph of the replacement $\mathcal{R}_i$ has two components. One runs along the axis to the right of G^*_{-i}. From Eq. (13), the second *positive component* can be viewed as the reflection in the 45° line of that portion of the graph of the function

$$G = (\beta_i/\alpha_i)^{1/\rho}\left[w_i g_i^{\rho-1} - g_i^{\rho}\right]^{1/\rho}, \tag{15}$$

which also satisfies Eq. (14). This is illustrated in the upper panel of Fig. 5, which graphs the right hand side of Eq. (15) as well as shading the regions which fail to satisfy the bounds on G imposed by Eq. (14).

The right hand side of Eq. (15) vanishes at both $g_i = 0$ and $g_i = w_i$ and has a unique maximum at $g_i = (1 - 1/\theta)w_i$. The boundary line of the left-hand inequality in Eq. (14): $G = \beta_i^{1/\theta} g_i$, crosses the graph of Eq. (15) at the point

$$(\bar{g}_i, \bar{G}_i) = \left(\frac{w_i}{1+\alpha_i}, \frac{\beta_i^{1/\theta} w_i}{1+\alpha_i} \right).$$

By definition of g_i^* and G_{-i}^* the boundary curve of the right-hand inequality in Eq. (14) crosses the graph Eq. (15) to the left of this point, at $(g_i, G) = (g_i^*, G_i^*)$, where

$$G_i^* = (\beta_i g_i^{*\theta} + G_{-i}^{*\theta})^{1/\theta} \equiv G_i^*.$$

If $\theta \geq 1 + (1/\alpha_i)$, both $(\bar{g}_i, \bar{G}_i)$ and (g_i^*, G_i^*) lie to the left of (or at) the maximum of the function (15). This puts both points on the increasing portion of this function which implies that the

Fig. 5.

positive component of the replacement correspondence is single-valued on its domain: $[G_i^*, \bar{G}_i]$. This is the case illustrated in the two panels of Fig. 5.

In characterizing equilibria it proves convenient to use shares and so we define the *share correspondence* of player i as

$$\mathcal{S}_i(G) = \left\{ \frac{\beta_i g^\theta}{G^\theta} : g \in \mathcal{R}_i(G) \right\}.$$

Since the positive component of $\mathcal{R}_i$ can be viewed as the graph of a function, the same is true of the positive component of $\mathcal{S}_i$. Furthermore, this function, which we write $s_i(G)$, is strictly increasing. To see this note that, if $\rho_i = s_i(G)$, Eq. (13) can be written after some manipulation as:

$$G = \frac{\beta_i^{1/\theta} w_i \rho_i^{(\theta-1)/\theta}}{\alpha_i + \rho_i} \tag{16}$$

and therefore $s_i' = (dG/d\rho_i)^{-1}$ where

$$\frac{dG}{d\rho_i} = \frac{\beta_i^{1/\theta} w_i \rho_i^{-1/\theta}[\alpha_i(\theta-1) - \rho_i]}{\theta(\alpha_i + \rho_i)^2} > 0$$

for $\rho_i < 1$. The inequality uses the fact that $\alpha_i(\theta - 1) \geq 1 > \rho_i$. Furthermore, s_i is the inverse of a continuous function (16) on a compact domain and therefore itself continuous. The following proposition summarizes these observations and is illustrated in Fig. 6.

Fig. 6.

Proposition 6.1. *If* $\ \geq 1+1/{}_i$, *the graph of* S_i *is the disjoint union of two sets:* $\{(G,0){:}G \geq G_{-i}^*\}$ *and* $\{(G,s_i(G)){:}G_i^* \leq G \leq \bar{G}_i\}$. *Furthermore,* s_i *is continuous, strictly increasing and satisfies*

$$s_i(G_i^*) = \rho_i^* = \beta_i \frac{g_i^{*q}}{G_i^{*q}} > 0$$

and $s_i(\bar{G}_i) = 1$.

6.3. Equilibria

The results of the preceding subsection allow us to characterize equilibria when $\ \geq 1+1/{}_i$ for all i. Share correspondences can be used to study equilibria by exploiting the fact that $\hat{G}$ is an equilibrium level of the public good if and only if there is $\beta_i \in S_i(\hat{G})$ for all i such that $\sum_{j=1}^{n} \beta_j = 1$. In the associated equilibrium strategy profile $\hat{g}_i = (\beta_i/{}_i)^{1/}\hat{G}$.

Equilibria are not typically unique. However, it follows from Proposition 6.1 that if J is the set of players making a positive contribution in some equilibrium, the corresponding level of the public good G satisfies

$$\sum_{j \in J} s_j(G) = 1; \tag{17}$$

since $\hat{g}_j = 0$ for all $j \notin J$. Since each s_j is strictly increasing, this holds for at most one G and we write $\hat{G}(J)$ for this value. Note that the corresponding equilibrium is also uniquely defined by taking $\beta_j = s_j(\hat{G}(J))$ for $j \in J$ and $\beta_j = 0$ for $j \notin J$. We shall call such a J an *equilibrium set*.

Hence, characterizing equilibria is equivalent to characterizing equilibrium sets and necessary conditions for a non-empty subset of players to be a such a set follow from Proposition 6.1, for we must have

$$G_j^* \leq \hat{G}(J) \leq \bar{G}_j \quad \text{for all } j \in J;$$
$$G_{-j}^* \leq \hat{G}(J) \quad \text{for all } j \notin J.$$

Hence, if we define

$$G^*(J) = \max\left\{\max_{j \in J} G_j^*; \max_{j \notin J} G_{-j}^*\right\}; \tag{18}$$

we must have $G^*(J) \leq \hat{G}(J) \leq \min_{j \in J}\{\bar{G}_j\}$. Furthermore, since each s_j is increasing,

$$\sum_{j \in J} s_j(G^*(J)) \leq \sum_{j \in J} s_j\left(\hat{G}(J)\right) = 1.$$

We may conclude that

$$G^*(J) \leq \min_{j \in J}\{\bar{G}_j\}; \tag{19}$$

$$\sum_{j \in J} s_j(G^*(J)) \leq 1. \tag{20}$$

Conversely, if Eqs. (19) and (20) hold, $\sum_{j \in J} s_j(G)$ takes values not exceeding 1 (at $G^*(J)$) and not less than one (at $\bar{G} = \min_{j \in J}\{\bar{G}_j\}$). Since G^* and $\bar{G}$ lie in the domain of $\sum_{j \in J} s_j$ and this function is continuous, Eq. (17) holds for some (unique) G.

The following proposition summarizes these conclusions.

Proposition 6.2. *Suppose that* $\ \geq 1+1/\ _i$ *for* $i=1,\ldots,n$. *Non-empty* $J\subseteq\{1,\ldots,n\}$ *is an equilibrium if and only if it satisfies Eqs.* (19) *and* (20).

It follows from this proposition that non-empty subsets of equilibrium sets are themselves equilibrium sets. If $K\subset J$,

$$\max_{j\in K} G_j^* \leq \max_{j\in J} G_j^* \leq G^*(J),$$
$$\max_{j\notin K} G^*_j \leq \max\{\max_{j\notin J} G^*_j, \max_{j\in J\setminus K} G^*_j\} \leq \max\{\max_{j\notin J} G^*_j, \max_{j\in J\setminus K} G_j^*\} \leq G^*(J).$$

So $G^*(K)\leq G^*(J)$ and therefore

$$G^*(K) \leq G^*(J) \leq \min_{j\in J}\{\bar{G}_j\} \leq \min_{j\in K}\{\bar{G}_j\};$$
$$\sum_{j\in K} s_j(G^*(K)) \leq \sum_{j\in K} s_j(G^*(J)) \leq \sum_{j\in J} s_j(G^*(J)) \leq 1;$$

where we have used the fact that the s_j are increasing functions. This shows that K is an equilibrium set. Noting that $\hat{G}(J)\geq G^*(J)\geq G^*(K)$, we also have

$$\sum_{j\in K} s_j\left(\hat{G}(J)\right) \leq \sum_{j\in J} s_j(G^*(J)) = 1 = \sum_{j\in K} s_j\left(\hat{G}(K)\right);$$

which shows that $\hat{G}(K)\geq\hat{G}(J)$.

Corollary 6.1. *If* $\ \geq 1+1/\ _i$ *for* $i=1,\ldots,n$ *and* J *is an equilibrium set, any* $K\subset J$ *(proper inclusion) is also an equilibrium set and* $\hat{G}(K)\geq\hat{G}(J)$.

Note that under the supposition of the corollary, all players not in K prefer K to J as equilibrium set as they free ride on a larger quantity of the public good. Indeed, if $j\in J$ all players except j prefer the equilibrium in which j is the sole positive contributor. In the next section, we shall show, *inter alia*, that for large enough , all equilibrium sets are of this form.

7. Best-shot public goods

7.1. Equilibria

The strongest link, or best-shot social composition function is $G=\max_{j=1,\ldots,n} g_j$. In this case, replacement correspondences need not be well-defined. For example, suppose $\bar{G}_i$ denotes the preferred level of the public good of Player i as sole contributor and $G_{-i}=\max_{j\neq i} g_j$. Then $g_i=\bar{G}_i$ is the best response to $G_{-i}=0$, whereas $g_i=0$ is the best response to $G_{-i}=\bar{G}_i$ and in both cases $G=\max\{g_i,G_{-i}\}=\bar{G}_i$. This means that there is no strategy that is a best response to all and every G_{-i} satisfying $\max\{g_i,G_{-i}\}=G$.

Instead, we use the *upper replacement correspondence* $\bar{R}_i$, which puts $g_i\in\bar{R}_i(G)$ if and only if there is some G_{-i} such that g_i is a best response to G_{-i} and $\max\{g_i,G_{-i}\}=G$. In our example,

Fig. 7.

$\bar{R}_i(\bar{G}_i)=\{0,\bar{G}_i\}$. Such correspondences offer a necessary, but not sufficient, condition for equilibrium. In particular, if $(\hat{g}_1,\ldots,\hat{g}_n)$ is a Nash equilibrium, then

$$\hat{g}_i \in \bar{R}_i(\hat{G}), \text{ where } \hat{G}=\max_{j=1,\ldots,n} \hat{g}_j. \qquad (21)$$

Thus, the solutions of Eq. (21) provide a superset of the Nash equilibria and it is necessary to test each strategy profile in the superset to eliminate members that are not equilibria.

To determine $\bar{R}_i$, let G^*_{-i} denote the level of G_{-i} at which player i is indifferent between not contributing and being the sole contributor:

$$u_i(0, G^*_{-i}) = u_i(\bar{G}_i, \bar{G}_i).$$

If $G_{-i} < G^*_{-i}$, player i will prefer to contribute $\bar{G}_i$ and $G=\bar{G}_i$. However, if $G_{-i} > G^*_{-i}$, player i is better off being a free rider and contributing nothing. In this case, $G=G_{-i}$. If $G_{-i}=G^*_{-i}$, the player is indifferent between the two strategies and $G=G_{-i}=G^*_{-i}$. This is clear from the indifference map in Fig. 7. The form of the upper replacement correspondence is evident and described in the following proposition.

Proposition 7.1. *If Player i has convex, increasing preferences, her upper replacement function* $\bar{R}_i$ *has domain* $[G^*_{-i},\infty)$ *and satisfies*

$$\bar{R}_i(G) = \{0\} \quad \text{if } G > \bar{G}_i; \{0, \bar{G}_i\} \quad \text{if } G = \bar{G}_i.$$

To find equilibria, we look for solutions of Eq. (21) and observe first that $\hat{G}$ must lie in the intersection of the domains of all $\bar{R}_i$, which means $\hat{G} \geq \max_j G^*_{-i}$. This precludes $\hat{G}=0$, and the proposition implies that $\hat{G}=\hat{g}_i$ for at least one i. However, we cannot have $\hat{G}=\bar{G}_i=\bar{G}_j$ for $i \neq j$, since i and j would not be choosing best responses. If we label every player i for which $\bar{G}_i \geq \max_j G^*_{-j}$ as *potentially active*, every solution of Eq. (21) has $\hat{g}_i=\bar{G}_i$ and $\hat{g}_j$ for $j \neq i$ each potentially active player. It is trivial to verify that each such strategy profile is indeed a Nash equilibrium, so we have characterized the set of Nash equilibria.

Proposition 7.2. *If all players have convex, increasing preferences, Nash equilibria are in 1Š1 correspondence with the set of potentially active players. In each such equilibrium the potentially active player contributes her preferred level of the public good and no other player contributes.*

7.1.1. An example

To illustrate the approach, consider two players with identical Cobb–Douglas preferences: $u_i = x_i G$ and incomes w_i for $i = 1, 2$. Then, $\bar{G}_i = w_i/2$ and G^*_{-i}. If incomes do not differ too much, specifically $w_2 \leq 2w_1 \leq 4w_2$, both players are potentially active and there are two equilibria in each of which one player contributes half their income and the other free rides. If w_2 N$2w_1$, only player 2 is potentially active and there is a unique equilibrium in which only player 2 contributes $w_2/2$. Similarly, if w_1 N$2w_2$, player 1 is sole contributor. This raises the possibility of Pareto-improving transfers.

If $w_1 = w_2 = 8$, there is an equilibrium: $\hat{g}_1 = 4, \hat{g}_2 = 0$ for which $\hat{G} = 4$ and payoffs are $_1 = 16$, $_2 = 32$, as well as an alternative equilibrium with player 2 as sole contributor. If $w_1 = 4$ and $w_2 = 12$, there is a unique equilibrium: $\hat{g}_1 = 0, \hat{g}_2 = 6$, in which $\hat{G} = 6$ and payoffs are $_1 = 24$, $_2 = 36$. It follows that, if player 1 offers a transfer of 4 units of income to player 2, the latter does better to accept and the former also benefits. Such a transfer also resolves the coordination problem arising from multiple equilibria. It can be shown further that, if w_2 b $2w_1$ b $4w_2$ a player anticipating an equilibrium in which they are sole contributor can always find a transfer that results in a unique equilibrium with the other player as sole contributor and in which *both* players have strictly greater payoffs than the anticipated equilibrium. Indeed, in some simple modifications of the two-player game with two equilibria, there will be transfers such that the post-transfer game has a unique equilibrium which strictly Pareto dominates both the equilibria in the original game. Clearly, the topic of transfers in best-shot games deserves further investigation.

7.2. Best-shot and better-shot games

In this subsection, we return to the better-shot game of Section 6 and consider what happens as ∞. In this limit, the CES social composition function approaches the best-shot case and we investigate whether the same is true of the sets of equilibria. To keep the exposition simple, we shall assume $_i = 1$ for all i, though our conclusions remain valid without this assumption. The next result shows that the CES replacement correspondence approaches the best-shot upper replacement correspondence. First, consider player i, recall that preferences are Cobb–Douglas and from Fig. 5 that the replacement correspondence for finite has two components. The first runs along the axis from $G^*_{-i}(\ )$ to infinity and there is a positive component from G_i^* to $\bar{G}_i$. Here, $\bar{G}_i$ is the level of public good provision that Player i would provide if she were the sole contributor and $G^*_{-i} = G^*_{-i}(\ )$ and $g_i^* = g_i^*(\ )$ satisfy Eq. (13) and equality in Eq. (14). We have made the dependence on explicit to avoid confusion with the best-shot values, which we will continue to write as G_i^* etc. In the Appendix, we prove the following lemma.

Lemma 7.1. *As increases, so do g_i^* and G_i^*. Furthermore, G_i^* $\bar{G}_i$ as ∞.*

This shows that the positive component of the replacement correspondence shrinks to the point of tangency between the 45° line and the indifference curve in the (G_i, g_i) plane in Fig. 7. Several consequences from this lemma and the figure. We have (i) $g_i^*(\ )$ $\bar{G}_i$ and so (ii) $_i^*(\ ) = g_i^*(\ )/G_i^*(\ )$ 1. Furthermore, equality in Eq. (14) implies that $(G^*_{-i}, 0)$ and $(G_i^*(\ ), g_i^*(\ ))$ lie on the same indifference curve, so (iii) $G^*_{-i}(\theta)$ is decreasing and (iv) $G^*_{-i}(\theta) \to G^*_{-i}$. It follows from (ii) and Proposition 7.2 that, if θ is large enough, all equilibrium sets are singletons and that $\{i\}$ is such a set if and only if

$$\bar{G}_i > \max_{j=1,\ldots,n}\{G^*_{-j}(\theta)\} \qquad (22)$$

It follows (iii) that player i is potentially active in the best-shot game. Conversely, Proposition 7.2 shows that in any equilibrium of the best-shot game, in which player i is the sole contributor $\bar{G}_i \geq \max_j G^*_{-j}$ and we call this equilibrium *strict* if this inequality holds strictly. If i is such a player, (iv) implies that Eq. (22) is valid for all large enough θ.

Proposition 7.3. *Under Cobb–Douglas preferences and a CES social composition function, there is a* b *such that, if* $\theta > b$*, the set of equilibria coincides with the set of strict equilibria of the best-shot game.*

Thus the best-shot game is more than just a limiting approximation to the better-shot game. Equilibria of the latter can be analyzed by studying the (simpler) former game. Barring the coincidence that there exist $i \neq j$ such that $\bar{G}_i = G^*_{-j} \geq G^*_{-k}$ for all $k \neq i,j$, the two games have the same equilibria for large enough θ. For example, the discussion of transfers in the previous subsection is also applicable to better-shot games. We conjecture that this conclusion holds for a much wider class of preferences than Cobb–Douglas.

8. Conclusion

The pure public good provision model of BBV is an outstandingly tractable model of reciprocal positive externalities. Its usefulness prompts one to enquire whether, and in what ways, its scope can be extended with minimal sacrifice of tractability. The present paper has explored extensions that modify the form of social composition function while retaining the game's aggregative structure. For reasons of space, we have concentrated on existence and uniqueness and limited ourselves to a few observations on comparative statics such as income redistribution. A more complete treatment of these issues is a subject for future research. Further extensions can also be envisaged that incorporate this aggregative structure — for example, the joint characteristics model of Cornes and Sandler may be revisited. We have explored circumstances under which our approach can be exploited to finesse what Richard Bellman once called, in another context, the "curse of dimensionality". The time seems ripe for further consideration of the range of interpretations and applications on which the model, and our method of analysis, may shed useful light.

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Appendix A

In this Appendix, we give several proofs displaced from the main text.

Proof of second assertion in Lemma 5.1. We start by defining

$$\tilde{\phi}_i(G_{-i}) = (b_i[r_i(G_{-i})]^{\theta} + G^{\theta}_{-i})^{1/\theta}$$

and observing that continuity of b_i implies continuity of $\tilde{\phi}_i$. Note also that, if $g_i = r_i(G)$ and $G_{-i} = G - \sum_{j \neq i} g_j$, then $G = \tilde{\phi}_i(G_{-i})$. In the case of Parameter Set 1, suppose that we had a G' satisfying Eq. (2) with $0 < G' < \bar{G}_i$. By Lemma 5.1, there is a unique g' satisfying Eq. (2) and this implies a unique G'_{-i} and $\tilde{\phi}_i(G'_{-i}) = G' < \bar{G}_i$. Now choose any $G''_{-i} > \bar{G}_i$ and note that $G'' = \tilde{\phi}_i(G''_{-i}) \geq G''_{-i} > \bar{G}_i$. By continuity, there would be a $G'''_{-i} \in (G'_{-i}, G''_{-i})$ such that $\tilde{\phi}_i(G'''_{-i}) = \bar{G}_i$ and $G'''_{-i} > 0$ giving two distinct solutions to (2) with $G = \bar{G}_i$ and contradicting Lemma 5.1.

This argument requires some modification with Parameter Set 2. Suppose that $G' > \bar{G}_i$, let G'_{-i} satisfy $\tilde{\phi}_i(G'_{-i}) = G'$ and note that, as $g_j \to 0$ for some $j \neq i$, it follows from Eq. (6) that $G \to 0$. This means that we can choose G''_{-i} such that $\tilde{\phi}_i(G''_{-i}) < \bar{G}_i$. We use the fact that, for any $\varepsilon > 0$, there is a $\delta > 0$ and a G''' within δ of $\bar{G}_i$ such that Eq. (2) holds, which means that $\tilde{\phi}_i(G'''_{-i}) = G'''$. By continuity, there is a $G''' \in (G'_{-i}, G''_{-i})$ such that $\tilde{\phi}_i(G'''_{-i}) = G'''$, giving two distinct solutions to Eq. (2) and contradicting Lemma 5.1.

The following lemma is used in both the remaining proofs.

Lemma A. For any $i = 1,\ldots,n$ and $\theta > 1$, there is a unique $x \in (0,1)$ satisfying

$$\psi_i(x;\theta) = (\theta + \alpha_i)x + (1 - x)^{\alpha_i \theta + 1} = 1. \quad (23)$$

Writing $\tilde{x}(\theta)$ for this solution, $\tilde{x}(\theta)$ is strictly increasing in θ and $\tilde{x}(\theta) \to (1 + \alpha_i)^{-1}$ as $\theta \to \infty$. Note that $x = 0$ is always a solution of Eq. (23): the lemma is concerned with positive solutions.

Proof of Lemma A. Observe that

$$\frac{\partial \psi_i}{\partial x}(0;\theta) = \alpha_i(1 - \theta) < 0;$$

and $\partial^2 \psi_i / \partial x^2 > 0$ for $0 < x < 1$. It follows that $\psi_i(x;\theta)$ is strictly convex and strictly decreasing at $x = 0$, as well as satisfying

$$\psi_i(0;\theta) = 1 < \psi_i(1;\theta).$$

This establishes the existence of a unique $\tilde{x}(\theta)$ satisfying $\psi_i(\tilde{x}(\theta);\theta) = 1$. Furthermore, $\psi_i(x;\theta)$ is increasing in x and strictly decreasing in θ at $x = \tilde{x}(\theta)$. Consequently, $\tilde{x}(\theta)$ is strictly increasing in θ and, since it is bounded above (by 1), it has a limit as $\theta \to \infty$. Letting $\theta \to \infty$ in Eq. (23) shows that this limit is $(1 + \alpha_i)^{-1}$.

Proof of Lemma 6.1. The payoff π_i takes the value $w_i{}^{\prime}G_{-i}$ *at* $g_i = 0$ and zero at $g_i = w_i$. A little algebraic manipulation shows that stationary points of π_i satisfy Eq. (11). Writing

$$\Lambda_i = (w_i - g_i)b_i g_i^{\theta - 1} - \alpha_i b_i g_i^{\theta};$$

it is straightforward to verify that $\Lambda_i(0) = 0$ (since $\theta > 1$) and $\Lambda_i(w_i) = -\alpha_i b_i w_i^{\theta} < 0$. Further, Λ_i has a unique stationary point $g'_i \in (0, w_i)$ satisfying $\Lambda_i(g'_i) > 0$. We conclude that g'_i maximizes π_i over

$[0,w_i]$, which means that Eq. (11) has a solution if and only if $_iG_{-i} \leq \;_i(g'_i)$, so this is a necessary and sufficient condition for stationary points. Thus $_i$ is a strictly decreasing in $[0,w_i]$ if $_iG_{-i} \geq \;_i$ (g'_i), has an interior local minimum and local maximum if 0 b $_iG_{-i}$ b $_i(g'_i)$ and just an interior local (and global) maximum for $G_{-i}=0$. It follows that there are two cases.

A. A stationary point is the global maximum if the value of $_i$ at that point is not exceeded by that at $g_i=0$; these two conditions are expressed in Eqs. (11) and (12) holding with equality.

B. The global maximum is at $g_i=0$ if either $_iG_{-i} \geq \;_i(g'_i)$ or the value of $_i$ at all stationary points does not exceed that at $g_i=0$.

The proof is completed by showing that G^*_{-i} is well-defined and Case A holds for $G_{-i} \leq G^*_{-i}$ and Case B for $G_{-i} \geq G^*_{-i}$. This can be achieved by showing that there is a unique $G_{-i}=G^*_{-i}$ for which both A and B hold and appealing to continuity[8] together with the fact that Case B holds for large enough G_{-i}. Note that both A and B hold if and only if marginal payoff is zero and the payoff equals that at $g_i=0$; these conditions are equivalent to Eqs. (11) and (12) holding with equality.

Raising Eq. (12) to the power , multiplying by $_i$, substituting for G_{-i} from Eq. (11) and dividing by $w_i^{\;_i\;+1}$ shows that Eq. (12) holding with equality is equivalent to $_i(g_i/w_i;\;)=1$, where u is defined in Lemma A. Choose $g^*_i = w_i\tilde{x}(\;)$ and G^*_{-i} to satisfy

$$\mathrm{a}_i G^{*\mathrm{q}}_i \text{ ¼ } \mathrm{b}_i w_i g_i^{*\mathrm{q}\;\;1} \;\; 1 \;\; \text{ð } 1 \text{ þ } \mathrm{a}_i\text{Þ}\frac{g_i^*}{w_i} \;:$$

By construction, (g^*_i, G^*_{-i}) satisfies Eqs. (11) and (12) with equality. Lemma A implies that 0 b $\tilde{x}(\;)$ b $(1+\;_i)^{-1}$ and therefore G^*_{-i} N0 (and g^*_i N0).

Proof of Lemma 7.1. In the proof immediately above we showed that $_i[g^*_i\; w_i;\;]=1$, where is defined in Lemma A. Recalling that $\bar{g}_i=w_i(1+\;_i)^{-1}$, this lemma implies that $G^*_i \quad \bar{G}_i$ as ∞.

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[8] Alternatively, one can show that the payoff at g_i ¼ 0 increases faster than the payoff at any stationary point, using an envelope theorem.

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