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Many-body properties of quasi–one-dimensional boson gas across a narrow CIR

RAN QI[1] and XIWEN GUAN[2,3]

[1] *Institute for Advanced Study, Tsinghua University - Beijing, 100084, China*
[2] *State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences - Wuhan 430071, China*
[3] *Department of Theoretical Physics, Research School of Physics and Engineering, the Australian National University Canberra ACT 0200, Australia*

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Abstract – We study strong interaction effects in a one-dimensional (1D) boson gas across a narrow confinement-induced resonance (CIR). In contrast to the zero-range potential, the 1D two-body interaction in the narrow CIR can be written as a polynomial of derivative δ-function interaction on many-body level. Using the asymptotic Bethe ansatz, we find that the low-energy physics of this many-body problem is described by the Tomonaga-Luttinger liquid where the Luttinger parameters are essentially modified by an effective finite-range parameter v. This parameter drastically alters quantum criticality and universal thermodynamics of the gas. In particular, it drives the Tonks-Girardeau (TG) gas from non-mutual Fermi statistics to mutual statistics or to a more exclusive super-TG gas. This novel feature is further discussed in terms of the breathing mode which is experimentally measurable.

Introduction. – Over the past few decades, experiments on ultracold bosonic and fermionic atoms confined to one-dimensional (1D) geometry [1–3] have provided a better understanding of significant quantum statistical and strong interaction effects in quantum many-body systems. The experimental measurements to date are seen to be in good agreement with results obtained from exactly solved models [4–6]. In particular, Haller *et al.* [7] made an experimental breakthrough in 2009 by realizing a stable highly excited gas-like phase —called the super Tonks-Girardeau (TG) gas— in the strongly attractive regime of bosonic cesium atoms. Such a gas was predicted earlier by Astrakharchik *et al.* [8] on a gas of hard rods and by Batchelor *et al.* [9] from the Bethe ansatz (BA) integrable Bose gas with an attractive interaction.

So far most quasi-1D confined systems have been experimentally realized by the wide Feshbach resonance with combination of confinement-induced resonance (CIR). At low energies, the quasi-1D atom-atom interaction can be written as a δ-function potential $V(x) = g_{1d}\delta(x)$ with an effective interacting strength $g_{1d} = c_0/m$, where $c_0 = 2[a_\perp(a_\perp/a_s - C)]^{-1}$ is determined by the 3D scattering length a_s. Here $a_\perp$ is the confining length and $C = 1.0326$ is a constant [10]. Across a CIR ($a_s \sim a_\perp$), g_{1d} can be tuned from $+\infty$ to $-\infty$. Consequently, the quantum dynamics, correlations and thermodynamics of few quasi-1D systems have been experimentally elucidated.

However, the resonance width has an essential effect which affects interacting properties on many-body level. It has been recently pointed out [11] that a 3D narrow Feshbach resonance may produce strong interaction effects due to the resonance structure of its phase shift. In the narrow resonance, many-body properties are not only determined by the scattering length a_s but also by an effective range that introduces a strong energy-dependent scattering amplitude of two colliding atoms [12,13]. In this paper, we find that the 1D two-body interaction across a narrow CIR leads to significant finite-range–dependent interaction effects in many-body properties of the model, including universal Tomonaga-Luttinger liquid (TLL) physics, thermodynamics, quantum criticality, super-TG phase and collective modes, etc.

Two-body scattering property and quasi-1D confined system. – In narrow resonances, the two-body scattering physics is well described by an energy-dependent scattering length $a_s^{-1}(E)=a_s^{-1}(0)-mr_0E$, where $a_s(0)$ is the zero-energy scattering length and E is the scattering energy. Here r_0 denotes the effective range and it could be large and negative in the narrow Feshbach resonances. By solving the two-body problem under a tight harmonic confinement in two radial directions, one obtains the effective 1D scattering amplitude [13]

$$f_k=-\frac{1}{1-i2k/c(k)}, \tag{1}$$

where $c(k)=(c_0^{-1}+4vk^2)^{-1}$, $c_0^{-1}=a_\perp[a_\perp/a_s(\omega_\perp)-C]/2$ and $v=-a_\perp^2 r_0/8$. The confining length $a_\perp=\sqrt{1/(m\omega_\perp)}$ is determined by the trapping frequency $\omega_\perp$. The CIR corresponds to the limit $c_0\to\infty$. The only difference between narrow and wide CIRs is that the interacting strength $c(k)$, instead of being a constant, depends on the relative momentum of two collision atoms. Under this circumstance, the energy-dependent scattering amplitude is characterized by the parameter v which has a dimension of L^3. For an infinite wide resonance, we have $r_0\to 0$ then $v=0$, $c(k)\equiv c_0$ and the scattering amplitude f_k becomes identical to the zero-range one [10].

Remarkably, we prove that for $v\neq 0$ the two-body scattering amplitude leads to the following Hamiltonian with a polynomial of derivative δ-function interaction on many-body level:

$$\hat{H}=-\frac{1}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\frac{c_0}{m}\sum_{i<j}\delta(x_i-x_j)\left[1+\sum_{\ell=1}^{\infty}(c_0v)^\ell\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)^{2\ell}\right], \tag{2}$$

where m is atom mass and N is the total particle number. To justify eq. (2) as a proper description of our narrow CIR system, we will show that the Hamiltonian (2) leads to the correct scattering amplitude given in eq. (1) by solving the two-body scattering wave function. In a two-body problem, the Hamiltonian (2) becomes

$$\hat{H}=-\frac{1}{2m}\frac{\partial^2}{\partial x_1^2}-\frac{1}{2m}\frac{\partial^2}{\partial x_2^2}+\frac{c_0}{m}\delta(x_1-x_2)\left[1+\sum_{\ell=1}^{\infty}(c_0v)^\ell\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)^{2\ell}\right]. \tag{3}$$

Since the interaction is still contact in nature, we can choose the following form of two-body wave function in the region $x_1<x_2$:

$$\psi(x_1,x_2)=e^{ik_1x_1+ik_2x_2}+A_{k_1,k_2}e^{ik_2x_1+ik_1x_2}. \tag{4}$$

The coefficient A_{k_1,k_2} is determined by the boundary condition [4]

$$\frac{1}{m}\left(\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_1}\right)\psi|_{x_1=x_2}=\frac{c\left(\frac{k_1-k_2}{2}\right)}{m}\psi|_{x_1=x_2}, \tag{5}$$

where $c(k)$ was defined below eq. (1) and we have used the relation $(1+x)^{-1}=1+\sum_{\ell=1}^{\infty}(-x)^\ell$. This boundary condition gives

$$A_{k_1,k_2}=-\frac{c\left(\frac{k_1-k_2}{2}\right)+i(k_1-k_2)}{c\left(\frac{k_1-k_2}{2}\right)-i(k_1-k_2)}. \tag{6}$$

Finally, the two-body scattering amplitude is given as

$$f(k)=\frac{1}{2}(A_{k,-k}-1)=-\frac{1}{1-i2k/c(k)} \tag{7}$$

which is exactly the same as eq. (1). As a result, we have proved that the Hamiltonian (2) indeed reproduces the correct two-body scattering amplitude of a narrow CIR thus justifies the using of this model.

Below, we will try to solve the full many-body Hamiltonian (2) analytically. As mentioned above, the interactions in model (2) is still contact in nature. Therefore, in the knowledge of asymptotic BA [14], for the domain $x_1<\cdots<x_N$, the wave function of the model (2) can be written as as super positions of plane waves [15,16], *i.e.*,

$$\Psi(x_1,x_2,\cdots,x_N)=\sum_P A_P e^{i\sum_j k_{P_j}x_j}, \tag{8}$$

where P runs over all $N!$ permutations, the coefficients $A_P=(-1)^P\prod_{j<\ell}^{N}\frac{1+i(k_{P_\ell}-k_{P_j})[c_0^{-1}+v(k_{P_\ell}-k_{P_j})^2]}{\sqrt{1+(k_{P_\ell}-k_{P_j})^2[c_0^{-1}+v(k_{P_\ell}-k_{P_j})^2]^2}}$. The wave functions in other domains can be obtained from the interchange symmetry of bosons. The eigenvalue of the model (2) is given by $E=\sum_{j=1}^{N}k_j^2$ where the wave numbers $\{k_j\}$ satisfy the following transcendental equations:

$$k_jL=2\pi I_j-\sum_{\ell=1}^{N}\theta(k_j-k_\ell). \tag{9}$$

Here we denoted $\theta(p)=2\arctan[p(c_0^{-1}+vp^2)]$. I_j is an integer for odd N and a half odd integer for even N. In this work, we only consider the repulsive scattering branch corresponding to the solutions with all k_j being real.

Ground-state and low-energy physics. – In the thermodynamic limit N, $L\to\infty$ with finite particle density $n=N/L$, we can define a quasi-momentum distribution ρ_k. At the ground state, ρ_k is non-zero only in a finite interval $[-Q,Q]$, where Q is the "Fermi momentum" of the 1D interacting bosons. From eq. (9) we have ρ_k satisfying

$$\rho_k=\frac{1}{2\pi}\left[1+\int_{-Q}^{Q}K(k-q)\rho_q\mathrm{d}q\right], \tag{10}$$

where $K(p)=\frac{\partial\theta(p)}{\partial p}=\frac{2(c_0^{-1}+3vp^2)}{1+p^2(c_0^{-1}+vp^2)^2}$ and the particle density $n=\int_{-\infty}^{+\infty}\rho_k\mathrm{d}k$. Thus, the ground-state energy E_0 and chemical potential μ are given by

$$E_0=L\int_{-Q}^{Q}k^2\rho_k\mathrm{d}k=Ln^3e(\gamma_0,\tilde{v}), \tag{11}$$

$$\mu=\frac{\partial E_0}{\partial N}=n^2\left(3e-\gamma_0\frac{\partial e}{\partial\gamma_0}+3\tilde{v}\frac{\partial e}{\partial\tilde{v}}\right), \tag{12}$$

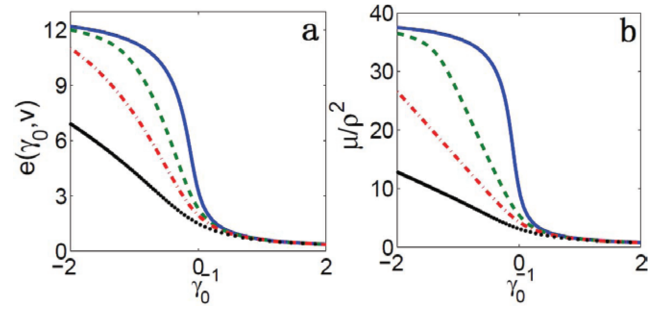

Fig. 1: (Colour on-line) The energy density and chemical potential *vs.* zero-energy interacting strength r_0^{-1} for the gas-like phase. The curves stand for the values of $\tilde{v}=0,\ 0.01,\ 0.02,\ 0.05$ (from top to bottom).

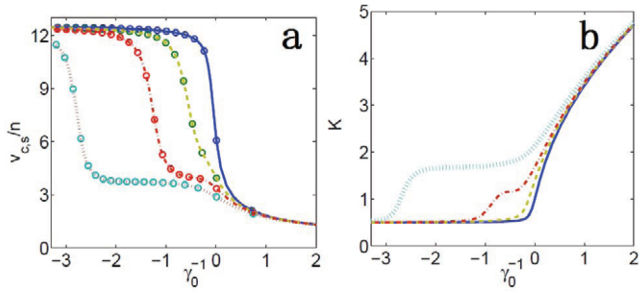

Fig. 2: (Colour on-line) Sound velocity and Luttinger parameter K *vs.* r_0^{-1} for the values of $\tilde{v}=0,\ 0.005,\ 0.01,\ 0.02$ (solid, dashed, dash-dotted and dotted lines). In panel (a), all lines stand for the result of $\nu_s=\sqrt{\nu_J\nu_N}$, whereas the open circles present the ν_c from the elementary excitation spectrums (13).

where $\gamma_0=c_0/n$, $\tilde{v}=vn^3$ are the dimensionless counterpart of c_0 and v ($v<c_0$). By analysis of the BA equation (10), we see that the finite-range effect does not play a role in weak-coupling regime. However, for a strong interacting regime, v drastically changes the ground energy. For $\gamma_0\to\infty$, the kernel function becomes $K(p)=\frac{6vp^2}{1+v^2p^6}$. This means that the effective finite range v can drive the the TG gas with non-mutual Fermi statistics [17] into the gas-like phase with mutual statistics either less or more exclusive than the free-fermion statistics, *i.e.*, super-TG gas. Figure 1 shows strong interaction effects in $e(\gamma_0,\tilde{v})$ and μ at different values of $\tilde{v}$ [18]. One sees that the energy and chemical potential are significantly reduced as $\tilde{v}$ increases in the TG and super-TG gas-like phases (in accordance with the wide CIR, we here refer the TG and super-TG regimes for $\gamma_0\to\infty$ and $\gamma_0\to-\infty$). This indicates that the zero-range repulsive interaction strength is reduced by the positive v at the collision.

The elementary excitations of the model (2) are characterized by the quantum numbers $\{I_j\}$ for the BA equations (9), including moving a particle close to the right or left Fermi points outside the Fermi sea and moving a particle from the left Fermi point to the right [4]. Here we find that the elementary spectra are phonon-like in the long-wavelength limit and have the same sound velocity. The sound velocity ν_c can be obtained from the BA equations (9) by

$$\nu_c=\lim_{p\to0}\frac{\omega_p}{p}=\frac{Q}{m}\frac{1+\frac{1}{Q}\int_{-Q}^{Q}kf_k\mathrm{d}k}{1+\int_{-Q}^{Q}f_k\mathrm{d}k},\tag{13}$$

where f_k satisfies the equation $f_k=\left[\int_{-Q}^{Q}K(k-q)f_q\mathrm{d}q+K(Q-k)\right]/(2\pi)$ and ω_p refers to the energy of the elementary particle-hole excitations.

On the other hand, the low-energy physics of a wide class of 1D interacting models can be captured by the TLL described by the following effective Hamiltonian [19,20]:

$$H_{eff}=\frac{1}{2\pi}\int_0^L\mathrm{d}x[\nu_J(\partial_x\phi(x))^2+\nu_N(\partial_x\theta(x))^2],\tag{14}$$

where $\phi(x)$ and $\theta(x)$ correspond to phase and density fluctuations. For the Galilean invariance, $\nu_J=\nu_sK=n\pi/m$ and $\nu_N=\nu_s/K$ are phase and density stiffness, respectively. Here K is the Luttinger parameter which determines the critical exponents of correlation functions. The Hamiltonian (14) also has a linear spectrum $\omega_p=\nu_sp$ with $\nu_s=\sqrt{\nu_J\nu_N}$. The velocity $\nu_N=\frac{1}{\pi}\frac{\partial\mu}{\partial n}$ can be obtained from our exact solution (12). In fig. 2, we verify from $\nu_s=\nu_c$ that the effective model (14) indeed captures the collective low-energy physics of model (2). This conclusion will be further identified from the universal leading-order finite-temperature corrections to the free energy. Panel (b) in fig. 2 shows influences of the effective finite range v on the Luttinger parameter $K=\nu_s/\nu_N$.

Universal thermodynamics. – At finite temperatures, the true physical states can be determined from the minimization of the Gibbs free energy. Following the Yang-Yang method [21], the equation of state can be obtained from the pressure

$$P=\frac{T}{2\pi}\int_{-\infty}^{+\infty}\ln(1+e^{-\epsilon_k/T})\mathrm{d}k\tag{15}$$

where the "dressed energy" ϵ_k satisfies the so-called thermodynamic BA (TBA) equation

$$\epsilon_k=k^2-\mu-\frac{T}{2\pi}\int_{-\infty}^{+\infty}K(k-q)\ln(1+e^{-\epsilon_q/T})\mathrm{d}q.\tag{16}$$

This equation provides an analytical way to access full thermodynamics of the many-body system (2).

At high temperatures, the fugacity $z=e^{\mu/T}\ll1$, thus we can perform high-temperature expansions from the pressure (15), namely,

$$P=P^{(0)}+\frac{T^{3/2}}{\sqrt{2\pi}}\Delta b_2z^2+O(z^3),\tag{17}$$

$$\Delta b_2=-\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{d}\delta(k)}{\mathrm{d}k}e^{-2k^2/T}\mathrm{d}k,\tag{18}$$

where $\delta(k) = \arctan[2k/c(k)]$ is the two-body scattering phase shift. The non-interacting part is given by $P^{(0)} = -\frac{T}{2\pi}\int_{-\infty}^{+\infty}\ln(1-ze^{-k^2/T})\mathrm{d}k$. The second virial coefficient Δb_2 in (18) fully agrees with the result obtained from the pure two-body physics [13]. In fact, the second term in (17) contains subtle corrections from quantum statistical and dynamical interactions as well as the finite-range effect to the pressure of the Boltzmann gas.

Furthermore, we may accurately calculate the thermodynamics of the model from (16) in analytic fashion using the polylog function. For capturing quantum criticality of finite range v, we consider the Tonks limit $c_0 \to \infty$. Following the approach [22], we obtain an analytic form of the equation of state,

$$P = -\frac{T^{\frac{3}{2}}}{2\sqrt{\pi}} Li_{\frac{3}{2}}\left(-e^{\frac{\mu}{T}+A}\right)\left[1 - \frac{3vT^{\frac{3}{2}}}{2\sqrt{\pi}} Li_{\frac{3}{2}}\left(-e^{\frac{\mu}{T}+A}\right)\right],$$
$$A = -\frac{3vT^{\frac{3}{2}}}{2\sqrt{\pi}} Li_{\frac{5}{2}}\left(-e^{\frac{\mu}{T}}\right)\left[1 - \frac{6vT^{\frac{3}{2}}}{\sqrt{\pi}} Li_{\frac{3}{2}}\left(-e^{\frac{\mu}{T}}\right)\right] \qquad (19)$$

which is valid for finite v. This analytical equation of state is very convenient for experimentalists to evaluate various thermodynamical quantities without involving numerical calculation of the integral equation (16). From panel (a) in fig. 3, we see a good agreement between the analytical result (19) and the numerical result from the TBA equation (16). The significance of this result is that the equation of state (19) allows the exploration of TLL physics and quantum criticality in a highly precise manner.

At low temperatures, from (19) we further find the universal field theory of TTL, *i.e.*, up to v^2,

$$P(\mu, T) = P_0 + \beta(\mu)\frac{\pi T^2}{6} + O(T^4), \qquad (20)$$
$$\beta(\mu) = \frac{1}{2\sqrt{\mu}}\left[1 + \frac{48}{5\pi}v\mu^{3/2} + \frac{2856}{25\pi^2}\left(v\mu^{3/2}\right)^2\right],$$

where pressure $P_0 = \frac{2\mu^{3/2}}{3\pi}\left[1 + \frac{16}{5\pi}v\mu^{\frac{3}{2}} + \frac{616}{25\pi^2}(v\mu^{\frac{3}{2}})^2\right]$. For fixed density n, the free energy can be written as

$$F(n, T) = E_0 - \beta(n)\frac{\pi T^2}{6} + O(T^4) \qquad (21)$$

where E_0 is the ground-state energy and $\beta(n)$ is given as

$$\beta(n) = \frac{1}{2n\pi}\left(1 + 16\pi^2\tilde{v} - \frac{288}{5}\pi^4\tilde{v}^2\right). \qquad (22)$$

Thus, we have obtained the Gaussian conformal field theory with the sound velocity $\nu_c = 1/\beta(n)$ and central charge $C = 1$. We further prove that this sound velocity does coincide with the calculation from the excitation spectrum (13).

Quantum criticality. – By analyzing the TBA equation (16) in the zero-temperature limit, we see that the quantum phase transition from vacuum to TLL occurs as the chemical potential is varied through the critical point $\mu_c = 0$. Near the quantum critical point, this many-body system is expected to show universal scaling behavior in the thermodynamic quantities due to the collective nature of the many-body effect. Taking $c_0 \to \infty$, the quantum critical phenomena of the model reveal subtle dependence of the finite-range parameter v. The universal scaling functions of density and compressibility κ near this critical point can be obtained from the equation of state (19), *i.e.*,

$$n(\tilde{\mu}, \tilde{T}) - n_0(\tilde{\mu}, \tilde{T}) = \tilde{T}^{\frac{d}{z}+1-\frac{1}{vz}}\mathcal{F}\left(\frac{\tilde{\mu}-\tilde{\mu}_c}{\tilde{T}^{\frac{1}{vz}}}\right), \qquad (23)$$

$$\kappa(\tilde{\mu}, \tilde{T}) - \kappa_0(\tilde{\mu}, \tilde{T}) = \tilde{T}^{\frac{d}{z}+1-\frac{2}{vz}}\mathcal{Q}\left(\frac{\tilde{\mu}-\tilde{\mu}_c}{\tilde{T}^{\frac{1}{vz}}}\right), \qquad (24)$$

where $\mathcal{F}(x) = -Li_{\frac{1}{2}}(-e^x)/(2v^{1/3}\sqrt{\pi})$, $\mathcal{Q}(x) = -\frac{v^{1/3}}{2\sqrt{\pi}} \times Li_{-\frac{1}{2}}(-e^x)$, $\tilde{T} = Tv^{2/3}$, $\tilde{\mu} = \mu v^{2/3}$ with the background $n_0 = \kappa_0 = 0$ read off the critical exponents $z = 2$, $\nu = 1$. This scaling form looks similar to the one of the Lieb-Liniger model [22]. However, in contrast to the scaling behavior of the hardcore Bose gas and the Lieb-Liniger model [22–26] in the wide CIR, here the scalings are rescaled by the effective finite-range parameter v. Figure 3(b) shows the universal critical behavior of the entropy density for the $c_0 \to \infty$ limit. The well-pronounced fan-shape–like phase diagram of criticality indicates a universal crossover temperature line that separates the relativistic TLL from the non-relativistic free-fermion theory.

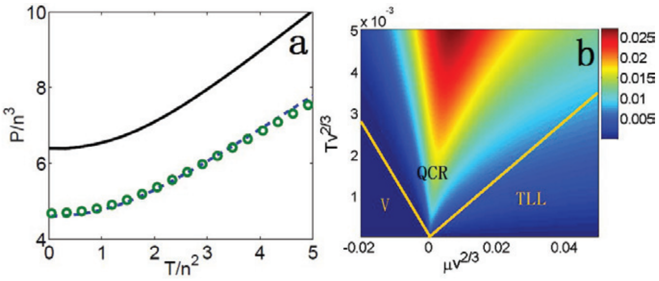

Fig. 3: (Colour on-line) Equation of state of the TG gas ($c_0 \to \infty$) plotted by numerically solving eq. (16). Left panel: pressure *vs.* temperature at fixed density for $\tilde{v} = 0$ (solid line), $\tilde{v} = 0.003$ (dash line). The open circles shows the result from the analytical result (19). Right panel: entropy in the (T, μ)-plane, where V denotes the vacuum and QCR stands quantum critical regime. The right solid line indicates a crossover temperature at which the linear-temperature–dependent entropy breaks down.

Experimental signatures. – The oscillation of fundamental modes in a shallow harmonic trap (often called breathing mode) is of particular interest in realistic experiments. The breathing mode frequency is sensitive to various interaction regimes. Therefore, it can be used to precisely determine different quantum phases in current experiments [18,27]. Despite the fact that for a narrow resonance one always has negative effective range such

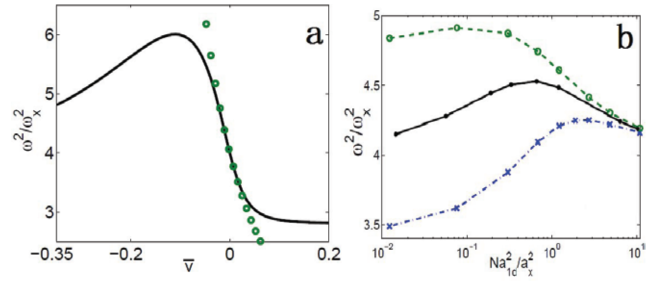

Fig. 4: (Colour on-line) Breathing mode frequency across a narrow CIR. Left panel: ω as a function of $\bar{v}$ in the Tonk limit $c_0 \to \infty$ where the solid line is from the numerical calculation while open circles refer to the small-v expansion. Right panel: ω as a function of zero-energy scattering strength in the super-TG regime where solid, dash-dotted, dashed lines refer to $\bar{v} = 0,\ 0.021,\ -0.021$, respectively.

that $v > 0$, it is also possible to create a large positive effective range $v < 0$ like the case of dipolar interaction-induced resonance [28]. We will show that the finite effective ranges $v > 0$ or $v < 0$ have dramatic influences on the breathing mode frequency ω across a narrow CIR.

Suppose the 1D system is confined in a shallow harmonic trap $V_{ext}(x) = \frac{1}{2} m\omega_x^2 x^2$ along the axial direction. The breathing mode frequency ω from sum rule is given by $\omega^2/\omega_x^2 = -2\langle x^2\rangle/(\partial\langle x^2\rangle/\partial\omega_x^2)$, where $\langle x^2\rangle = \int x^2 \rho(x)\mathrm{d}x/N$ and $\rho(x)$ is the density distribution along the trap. Here $\rho(x)$ can be obtained from the BA equation (10) within the local density approximation. Following the convention used in the experiment [7], we prefer the quantity $A^2 = (a_{1d}/\ell_x)^2$ to character the zero-energy interaction strength and $\bar{v} = v/\ell_x^3$ to character the effective range effect in the trap. In these equations, we have defined $\ell_x = a_x/\sqrt{N}$, $a_{1d} = -2/c_0$ and $a_x = \sqrt{1/(m\omega_x)}$. In fig. 4(b), we plot the ratio of ω^2/ω_x^2 as a function of A^2 in the super-TG regime with $|c_0| \gg 1$ at zero temperature. One can see that both the value and position of the peak change dramatically for the values of $\bar{v} \simeq 0, \pm 0.02$. This signature is capable of being measured in sufficiently narrow resonances. We also note that although for a given resonance the effective range r_0 is fixed, one can tune the value of $\bar{v}$ in a wide range by changing the confining length a_x, where $\bar{v} \propto 1/a_x^3$.

Moreover, in the "Tonk limit" $c_0 \to \infty$, by taking small-v expansions with eq. (20) at zero temperature, we obtain $\omega^2/\omega_x^2 \approx 4/(1 + a\bar{v} + b\bar{v}^2)$, where $a = 2048\sqrt{2}/(35\pi^2)$ and $b = (17827425\pi^2 - 153092096)/(91875\pi^4)$. We compare this asymptotic expression with the numerical result in fig. 4 and see an excellent agreement between the two results for small values of $\bar{v}$. We find a peak structure on the negative side of v around $\bar{v} \simeq -0.1$. This signature indicates that in the $c_0 \to \infty$ limit, a negative v gives an effective super-TG gas-like phase at the ground state.

Before concluding, we would like to compare our work to another related paper by Gurarie [29]. In ref. [29], the author also considered this many-body problem of a one-dimensional boson gas across a narrow resonance. In ref. [29], the author started with a pure one-dimensional two-channel mode and identified some interesting phases in different parameter regimes. It seems that their analysis does not further indicate whether these regimes are accessible or not in current cold-atom experiments. Here we start with a two-body amplitude which is obtained from a realistic two-body calculation of narrow CIR. Using the Bethe ansatz solution, we have shown that the underlying physics is likely to be accessible in current CIR experiments. Also, under particular choices of the parameters, which are based on realistic experimental conditions, we have obtained interesting results for the thermodynamic properties and quantum critical behaviors.

Summary. – In conclusion, using exact BA solution we have studied significantly different strong interaction effects in many-body properties of the quasi-1D boson gas across the narrow CIR. The universal TLL physics, equation of state, quantum criticality and high-temperature thermodynamics have been obtained in terms of the effective finite-range parameter v. It turns out that this parameter essentially changes the quantum statistics of the TG gas from free Fermi statistics to mutual statistics or to the more exclusive super-TG gas. Measuring the breathing mode enables us to capture these striking strong interaction effects in current cold-atom experiments.

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