PINCHING ESTIMATES AND MOTION OF HYPERSURFACES BY CURVATURE FUNCTIONS

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ABSTRACT. Second derivative pinching estimates are proved for a class of elliptic and parabolic equations, including motion of hypersurfaces by curvature functions such as quotients of elementary symmetric functions of curvature. The estimates imply convergence of convex hypersurfaces to spheres under these flows, improving earlier results of B. Chow and the author. The result is obtained via a detailed analysis of gradient terms in the equations satisfied by second derivatives.

1. INTRODUCTION

The aim of this paper is to provide some insights into second-derivative estimates for fully nonlinear elliptic and parabolic equations. In particular, the paper will explore the nonlinear terms which arise in the equations satisfied by second derivatives of solutions, and introduce some tools for understanding these. The result is a new pinching estimate for second derivatives, which improves several previously known results. The estimate has a number of applications, but this paper emphasises particularly the implications for the behaviour of convex hypersurfaces moving with speed given by a function of principal curvatures.

1.1. **Second derivative estimates.** To motivate the main result, consider as a model problem fully nonlinear scalar parabolic equations of the form

$$\frac{\partial u}{\partial t} = F(D^2 u) \tag{1.1}$$

where F is a smooth function of the components of $D^2 u$.

Differentiation of Equation (1.1) yields a very nice system of equations satisfied by the first derivatives of a solution:

$$\frac{\partial}{\partial t} D_j u = \dot{F}^{kl} D_k D_l \left(D_j u \right) \tag{1.2}$$

where we sum over repeated indices, and $\{\dot{F}^{kl}\}$ is the matrix of partial derivatives of F with respect to the components of its argument:

$$\frac{\partial}{\partial s} F(A + sB)\Big|_{s=0} = \dot{F}^{kl}(A) B_{kl}.$$

However, after differentiating once more, the resulting system of equations for the second derivatives is more complicated:

$$(1.3) \qquad \frac{\partial}{\partial t}\left(D_i D_j u\right) = \dot{F}^{kl} D_k D_l \left(D_i D_j u\right) + \ddot{F}^{kl,pq} D_i D_k D_l u D_j D_p D_q u$$

where $\ddot{F}$ is the second derivative of F:

$$\left.\frac{\partial^2}{\partial s^2} F(A + sB)\right|_{s=0} = \ddot{F}^{kl,pq}(A) B_{kl} B_{pq}.$$

The second term on the right-hand side of the system (1.3) is an obstacle to simple applications of the maximum principle to control the behaviour of second derivatives of solutions, since it is difficult to obtain useful information on its sign. This applies both to arguments using the classical maximum principle and to those using other tools such as the Aleksandrov-Bakelman maximum principle. I will restrict the discussion here to the classical setting.

The main result of the paper is essentially the following: If F is concave as a function of the second derivatives, and also 'inverse-concave', meaning that the function F^* defined by $F^*(A) = -F(A^{-1})$ is concave, then the ratio of minimum eigenvalue to trace of $D^2 u$ never decreases below its initial minimum. The precise statement is given in Theorem 4.1. The proof requires a detailed understanding of the nonlinear terms arising in Equation (1.3), and includes several useful tools for understanding these. Also important in the application of the main result is a new maximum principle for tensors which is given in Theorem 3.2.

1.2. **Evolving hypersurfaces.** There has been considerable previous work on convex hypersurfaces moving by curvature flows, and the most relevant here is where the speed function is a homogeneous degree one, monotone increasing function of the principal curvatures. The first such flow considered was the flow by mean curvature, which was treated by G. Huisken [10]. He proved that convex hypersurfaces contract to points in finite time under this flow, with spherical limiting shape. B. Chow proved a similar result for the motion of an n-dimensional hypersurface by the nth root of the Gauss curvature [5]. He also proved a result for motion by the square root of the scalar curvature [6], but in that case a stronger assumption than convexity was required for the initial hypersurface. The author considered a very general class of homogeneous degree one flows in [1], and proved the general result if the speed is a convex function of principal curvatures (as is the mean curvature) or if it is concave in the principal curvatures and vanishes when any principal curvature approaches zero (as in the case of the nth root of the Gauss curvature). More generally, if the speed is concave in the principal curvatures, the result holds as long as we assume a strong enough pinching condition on the initial hypersurface. The pinching estimate proved in this paper is aimed at removing the latter restriction for a wide class of flows of interest, including the flow by square root of scalar curvature treated by Chow.

The new result for contracting hypersurfaces is as follows:

Theorem 1.1. *Let f be a smooth symmetric function defined on the positive cone $\Gamma_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, \ i = 1, \ldots, n\}$ in $\mathbb{R}^n$, $n \geq 2$, which is homogeneous of degree one and strictly monotone increasing in each argument. Suppose that either*

(1) *$n = 2$, or*
(2) *f is convex, or*
(3) *f is concave on Γ_+ and zero on the boundary of Γ_+, or*
(4) *both f and the function f^* given by $f^*(x_1, \ldots, x_n) = -f(x_1^{-1}, \ldots, x_n^{-1})$ are concave on Γ_+.*

Let $x_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth, strictly convex embedding. Then there exists a unique maximally extended solution $x : M \times [0, T) \to \mathbb{R}^{n+1}$ of

$$(1.4) \qquad \frac{\partial x(p,t)}{\partial t} = -f(\kappa_1(p,t), \ldots, \kappa_n(p,t))\nu(p,t), \quad (p,t) \in M \times [0,T);$$
$$x(p,0) = x_0(p), \quad p \in M,$$

where $\kappa_1(p,t), \ldots, \kappa_n(p,t)$ are the principal curvatures of the embedding $x_t(.) = x(.,t)$ at the point $x(p,t)$, and $\nu(p,t)$ is the outward-pointing unit normal vector to $x_t(M)$ at $x(p,t)$. The map x_t converges to a constant $z \in \mathbb{R}^{n+1}$ as t approaches T, and the rescaled embeddings $\tilde{x}_t = \frac{x_t - z}{\sqrt{T - t}}$ converge in C^∞ to a limit with image a sphere of radius $\sqrt{2f(1, \ldots, 1)}$ centred at the origin.

The same result holds if f is merely concave, provided the initial embedding is such that

$$\sup_{p \in M} \frac{\kappa_1(p,0) + \cdots + \kappa_n(p,0)}{f(\kappa_1(p,0), \ldots, \kappa_n(p,0))} < \liminf_{(x_1, \ldots, x_n) \to \partial\Gamma_+} \frac{x_1 + \cdots + x_n}{f(x_1, \ldots, x_n)}.$$

The new ingredient is the last of the four conditions allowed for f (the first is treated in a recent paper by the author [3] making use of some new regularity results special to two dimensions proved in [2], and the second and third cases were proved in [1]).

2. A CLASS OF SYMMETRIC FUNCTIONS

The statement of Theorem 1.1 brings interest to a certain class of symmetric functions defined on the positive cone Γ_+ in $\mathbb{R}^n$. In this section I will discuss this class in some detail.

The class of interest, denoted $\mathcal{C}_n$, consists of functions on Γ_+ which are

- smooth (C^∞);
- homogeneous of degree one: $f(cx) = cf(x)$ for $c > 0$;
- strictly monotone increasing: $\frac{\partial f}{\partial x_i} > 0$ for each i;
- concave; and
- inverse-concave: $f^*(x_1, \ldots, x_n) = -f(x_1^{-1}, \ldots, x_n^{-1})$ defines a concave function on Γ_+.

Also important is the subclass $\mathcal{S}_n$ consisting of functions in $\mathcal{C}_n$ which are symmetric: $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$ for any permutation σ. These are precisely the functions which satisfy the fourth condition in Theorem 1.1. Note that the main result, Theorem 4.1, does not require any homogeneity condition, and so applies to a somewhat larger class than $\mathcal{S}_n$.

Before giving examples of functions in $\mathcal{S}_n$, I will give some useful methods of constructing new examples from old.

Theorem 2.1. *A homogeneous degree one function $f : \Gamma_+ \to \mathbb{R}$ is in $\mathcal{C}_n$ if and only if the following conditions hold everywhere on Γ_+:*

(1) $\dot{f}^i = \frac{\partial f}{\partial x_i} > 0$ *for each i;*

(2) $\ddot{f}^{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ *is a non-positive matrix;*

(3) $\ddot{f}^{ij} + 2\frac{\dot{f}^i}{x_i}\delta_{ij}$ *is a non-negative matrix.*

Proof. The only non-trivial point is that the third condition is equivalent to the concavity of f^*. To see this, compute the derivatives of f^* at $(z_1, \ldots, z_n)$, where $z_i = x_i^{-1}$:

$$\frac{\partial f^*}{\partial z_i} = \frac{\partial f}{\partial x_i} x_i^2;$$

$$\frac{\partial^2 f^*}{\partial z^i \partial z^j} = -\frac{\partial^2 f}{\partial x_i \partial x_j} x_i^2 x_j^2 - 2\frac{\partial f}{\partial x_i} x_i^3 \delta_{ij}.$$

Multiplying the last identity by $x_i^{-2} x_j^{-2}$ gives the result. $\qquad\square$

Corollary 2.2. *The algebraic mean $H = \frac{1}{n}\sum_i x_i$ is in $\mathcal{S}_n$.*

Proof. In this case $\ddot{f} = 0$ and $\dot{f} > 0$. $\qquad\square$

Theorem 2.3. *If $f \in \mathcal{C}_n$ and $r \in [-1, 1]\backslash\{0\}$, then the function f_r given by*

$$f_r(x_1, \ldots, x_n) = (f(x_1^r, \ldots, x_n^r))^{\frac{1}{r}}$$

is in $\mathcal{C}_n$.

Proof. Compute the first and second derivatives of f_r at $z_i = x_i^{1/r}$:

$$\frac{\partial f_r}{\partial z_i} = f^{\frac{1}{r}-1}\dot{f}^i z_i^{r-1} > 0;$$

$$\frac{\partial^2 f_r}{\partial z_i \partial z_j} = r f^{\frac{1}{r}-1}\ddot{f}^{ij} z_i^{r-1} z_j^{r-1} + (1-r)f^{\frac{1}{r}-2}\dot{f}^i \dot{f}^j z_i^{r-1} z_j^{r-1}$$

$$+ (r-1)f^{\frac{1}{r}-1}\dot{f}^i z_i^{r-2}\delta_{ij}$$

$$= r f^{\frac{1}{r}-1} z_i^{r-1} z_j^{r-1}\left(\ddot{f}^{ij} - \left(\frac{r-1}{r}\right)\frac{\dot{f}^i \dot{f}^j}{f} + \left(\frac{r-1}{r}\right)\frac{\dot{f}^i}{x_i}\delta_{ij}\right)$$

Since f_r is homogeneous of degree one, the Euler relation implies that $\sum_i \frac{\partial^2 f_r}{\partial z_i \partial z_j} z_i = 0$, so the radial vector is a null eigenvector. Therefore to prove concavity of f_r it suffices to consider the restriction of the second

derivatives to the transversal subspace $S = \{\xi : \dot{f}^i\xi_i = 0\}$. There the bracket becomes

$$\ddot{f}^{ij} + \frac{r-1}{r}\frac{\dot{f}^i}{x_i}\delta_{ij}.$$

There are two cases to consider: If $0 < r \leq 1$, then $r - 1 \leq 0$ and the bracket is non-positive. If $-1 \leq r < 0$, then $\frac{r-1}{r} \geq 2$, and the bracket is non-negative by the third point in Theorem 2.1. In the latter case the coefficient is negative, so in both cases f_r is concave.

To establish the inequality in the third part of Theorem 2.1 for the function f_r, it suffices to show the stronger inequality

$$(2.1) \qquad \ddot{f}_r^{ij} + 2\frac{\dot{f}_r^i}{z_i}\delta_{ij} - 2\frac{\dot{f}_r^i\dot{f}_r^j}{f_r} \geq 0.$$

The expressions above give

$$\ddot{f}_r^{ij} + 2\frac{\dot{f}_r^i}{z_i}\delta_{ij} - 2\frac{\dot{f}_r^i\dot{f}_r^j}{f_r}$$

$$= rf^{\frac{1}{r}-1}z_i^{r-1}z_j^{r-1}\left(\ddot{f}^{ij} - \frac{r+1}{r}\frac{\dot{f}^i\dot{f}^j}{f} + \frac{r+1}{r}\frac{\dot{f}^i}{x_i}\delta_{ij}\right).$$

As before the bracket has the radial vector as a null eigenvector, and on the subspace S it becomes $\ddot{f}^{ij} + \frac{r+1}{r}\frac{\dot{f}^i}{x_i}\delta_{ij}$, which is bounded below by $\ddot{f}^{ij} + 2\frac{\dot{f}^i}{x_i}\delta_{ij}$ if $0 < r \leq 1$, and bounded above by $\ddot{f}^{ij}$ if $-1 < r < 0$. $\qquad\square$

Corollary 2.4. *A homogeneous degree one monotone increasing function f on Γ_+ is in $\mathcal{C}_n$ if and only if it is concave and the function f_{-1} defined in Theorem 2.3 is concave.*

Proof. Concavity of f_{-1} is equivalent to the inequality (2.1) (with $r = 1$), hence stronger than concavity of f^*. $\qquad\square$

Corollary 2.5. *The power-means $H_r = \left(\frac{1}{n}\sum_i x_i^r\right)^{\frac{1}{r}}$ are in $\mathcal{S}_n$ for $|r| \leq 1$.*

Proof. By Corollary 2.2 we can take $f = H$ in Theorem 2.3 to obtain the result for $r \neq 0$. The result holds also for $r = 0$ since H_r converges locally uniformly to $H_0 = (x_1 \ldots x_n)^{1/n}$ as $r \to 0$. $\qquad\square$

A commonly defined class involves the elementary symmetric functions of principal curvatures:

$$S_k(x_1,\ldots,x_n) = \frac{1}{\binom{n}{k}}\sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1}\ldots x_{i_k}, \quad k = 1,\ldots,n.$$

Theorem 2.6. *For $0 \leq k < n$, $\frac{S_{k+1}}{S_k} \in \mathcal{S}_n$.*

Proof. The concavity is proved in [4, Theorem 15.16], and the inverse-concavity follows since $f_{-1} = \frac{S_{n-k}}{S_{n-k-1}}$ is of the same kind. $\qquad\square$

To complete our discussion and give a satisfyingly large class of examples, we note the following:

Theorem 2.7. *If $f_1, \ldots, f_k$ are in $\mathcal{C}_n$, and $\varphi \in \mathcal{C}_k$, then the function f defined by*

$$f(x_1, \ldots, x_n) = \varphi(f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n))$$

is in $\mathcal{C}_n$. If the f_i are in $\mathcal{S}_n$, then so is f.

Proof. It suffices to show that f is concave under these conditions, because

$$f_{-1}(x_1, \ldots, x_n) = \varphi_{-1}((f_1)_{-1}(x_1, \ldots, x_n), \ldots, (f_k)_{-1}(x_1, \ldots, x_n)),$$

and by Corollary 2.4, $\varphi_{-1} \in \mathcal{C}_k$ and $(f_i)_{-1} \in \mathcal{C}_n$ for each i, so f_{-1} must also be concave. The derivatives of f are as follows:

$$\dot{f}^i = \sum_p \dot{\varphi}^p f_p^i;$$

$$\ddot{f}^{ij} = \sum_{p,q} \ddot{\varphi}^{pq} \dot{f}_p^i \dot{f}_q^j + \sum_p \dot{\varphi}^p \ddot{f}_p^{ij},$$

so for any $\xi = (\xi_1, \ldots, \xi_n)$, writing $\eta_p = \sum_i \dot{f}_p^i \xi_i$ for $p = 1, \ldots, k$,

$$\ddot{f}(\xi, \xi) = \ddot{\varphi}(\eta, \eta) + \sum_k \dot{\varphi}^p \ddot{f}_p(\xi, \xi).$$

This is non-positive since $\ddot{\varphi} \leq 0$, $\ddot{f}_p \leq 0$, and $\dot{\varphi}^p > 0$. $\qquad\square$

In particular, the class $\mathcal{S}_n$ is closed under multiplication by positive scalars, addition, and taking weighted geometric means. Therefore the following examples involving elementary symmetric functions are in $\mathcal{S}_n$:

- $S_k^{\frac{1}{k}}$, since this is the geometric mean of $\frac{S_{j+1}}{S_j}$ for $j = 0, \ldots, k-1$;
- $\left(\frac{S_k}{S_l}\right)^{\frac{1}{k-l}}$ for $n \geq k > l \geq 0$, since this is the geometric mean of $\frac{S_{j+1}}{S_j}$ for $j = l, \ldots, k-1$;
- $S_n^{\alpha_n} S_{n-1}^{\alpha_{n-1}-\alpha_n} \ldots S_2^{\alpha_2-\alpha_3} S_1^{\alpha_1-\alpha_2}$, if $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$ (the general form of a weighted geometric mean of $\{\frac{S_{j+1}}{S_j}, \; j = 0, \ldots, n-1\}$);
- positive linear combinations of any of the above examples.

All of the above examples can be used in Theorem 1.1. Note that of these, relatively few are covered by the previously known results: Of the functions $S_k^{1/k}$, only $k = 1$ and $k = n$ were known; for $f = (S_k/S_l)^{1/(k-l)}$, $k > l$, only those with $k = n$ or $k = 1$ were known; and for the more general combinations $S_n^{\alpha_n} S_{n-1}^{\alpha_{n-1}-\alpha_n} \ldots S_2^{\alpha_2-\alpha_3} S_1^{\alpha_1-\alpha_2}$ with $\alpha_i \geq 0$ and $\sum \alpha_i = 1$, only those with $\alpha_n > 0$ were known (except $\alpha_1 = 1$).

Note also that the previously known results allowed speeds given by the power means H_r for $r \geq 1$ (convex case) or $r \leq 0$ (concave and zero on the boundary of the positive cone). The new result therefore extends this to cover all of the remaining values of r.

3. A MAXIMUM PRINCIPLE FOR TENSORS

This section refines the following well-known result from [8]:

Theorem 3.1 ([8], Theorem 9.1). *Let M be a compact manifold with a (possibly time-dependent) Riemannian metric g and Levi-Civita connection ∇. Let S_{ij} be a smooth symmetric tensor field satisfying*

$$\frac{\partial}{\partial t} S_{ij} = \Delta S_{ij} + u^k \nabla_k S_{ij} + N_{ij}$$

on some time interval $[0, T]$, where u is smooth, and $N(p,t)(v,v) \geq 0$ whenever v is a null eigenvector of $S(p,t)$. If S_{ij} is positive definite everywhere at time $t = 0$, then it remains so on $0 \leq t \leq T$.

This generalises easily to the case where the manifold has boundary, and the tensor field satisfies a more general evolution equation of the form

$$\frac{\partial}{\partial t} S_{ij} = a^{kl} \nabla_k \nabla_l S_{ij} + u^k \nabla_k S_{ij} + N_{ij}$$

where a^{kl} is smooth and positive definite at each point and time. The result also remains true for connections other than the Levi-Civita connection. The new result makes particular use of the latter observation:

Theorem 3.2. *Let S_{ij} be a smooth time-varying symmetric tensor field on a compact manifold M (possibly with boundary), satisfying*

$$\frac{\partial}{\partial t} S_{ij} = a^{kl} \nabla_k \nabla_l S_{ij} + u^k \nabla_k S_{ij} + N_{ij}$$

where a^{kl} and u are smooth, ∇ is a (possibly time-dependent) smooth symmetric connection, and a^{kl} is positive definite everywhere. Suppose that

$$N_{ij} v^i v^j + \sup_\Gamma 2 a^{kl} \left(2\Gamma_k^p \nabla_l S_{ip} v^i - \Gamma_k^p \Gamma_l^q S_{pq} \right\} \geq 0$$

whenever $S_{ij} \geq 0$ and $S_{ij} v^j = 0$. If S_{ij} is positive definite everywhere on M at time $t = 0$ and on ∂M for $0 \leq t \leq T$, then it is positive on $M \times [0, T]$.

The result amounts to the observation that in a parabolic equation of this kind for a tensor field (in contrast to the scalar case), the leading elliptic term can be squeezed to yield a non-trivial extra term. In many situations this extra term is not useful, but in equations like (1.3) the 'reaction' term N_{ij} is quadratic in the derivatives of S, and the extra term given by Theorem 3.2 is also of this kind if Γ is taken to be linear in the derivatives of S. The extra term that results is crucial in the proof of Theorem 4.1.

Proof. The tensor inequality $S_{ij} \geq 0$ is equivalent to an inequality for a function on the tangent bundle of M:

$$Z(p, v) = S(p)(v, v) \geq 0$$

for all $p \in M$ and v in $T_p M$. Let p be a point where $S(p)$ has a null eigenvector v. Choose coordinates $x^1, \ldots, x^n$ for M near p such that the

connection coefficients of ∇ vanish at p. Then any vector in $T_q M$ for q near p has the form $\sum_{i=1}^{n} \dot{x}^i \partial_i$, so TM is described locally by the $2n$ coordinates $x^1, \ldots, x^n$ and $\dot{x}^1, \ldots, \dot{x}^n$. The coordinates can be chosen so that $v = \partial_1$.

At (p, v) the first derivatives of Z must vanish, so that

$$(3.1) \qquad 0 = \frac{\partial Z}{\partial x^k} = \left(\frac{\partial}{\partial x^k} S_{ij} \right) \dot{x}^i \dot{x}^j = \frac{\partial}{\partial x^k} S_{11},$$

and

$$(3.2) \qquad 0 = \frac{\partial Z}{\partial \dot{x}^k} = 2 S_{ij} \dot{x}^i \delta_k^j = 2 S_{k1},$$

for $k = 1, \ldots, n$. Equation (3.2) implies that v is a null eigenvector of $S(p)$.

Now consider the second order conditions implied by minimality: The second derivatives form a $(2n) \times (2n)$ matrix which is non-negative at (p, v). The entries in this matrix are as follows:

$$\frac{\partial^2 Z}{\partial x^k \partial x^l} = \frac{\partial^2}{\partial x^k \partial x^l} S_{11};$$

$$\frac{\partial^2 Z}{\partial x^k \partial \dot{x}^q} = 2 \frac{\partial}{\partial x^k} S_{1q};$$

$$\frac{\partial^2 Z}{\partial \dot{x}^p \partial \dot{x}^q} = 2 S_{pq}.$$

For any Γ this implies the inequality

$$0 \le a^{kl} \left(\frac{\partial^2 Z}{\partial x^k \partial x^l} - \Gamma_k^p \frac{\partial^2 Z}{\partial \dot{x}^p \partial x^l} - \Gamma_l^q \frac{\partial^2 Z}{\partial x^k \partial \dot{x}^q} + \Gamma_k^p \Gamma_l^q \frac{\partial^2 Z}{\partial \dot{x}^p \partial \dot{x}^q} \right)$$

$$= a^{kl} \left(\frac{\partial^2 S_{11}}{\partial x^k \partial x^l} - 2 \Gamma_k^p \frac{\partial S_{1p}}{\partial x^l} - 2 \Gamma_l^q \frac{\partial S_{1q}}{\partial x^k} + 2 \Gamma_k^p \Gamma_l^q S_{pq} \right)$$

$$(3.3) \qquad = a^{kl} \left(\frac{\partial^2 S_{11}}{\partial x^k \partial x^l} - 4 \Gamma_k^p \frac{\partial S_{1p}}{\partial x^l} + 2 \Gamma_k^p \Gamma_l^q S_{pq} \right).$$

In the coordinates chosen above, the coordinate derivatives at p relate to the covariant derivatives as follows:

$$(3.4) \qquad \nabla_k \nabla_l S_{11} = \frac{\partial^2 S_{11}}{\partial x^l \partial x^l} - 2 S(\nabla_k \nabla_l \partial_1, \partial_1) = \frac{\partial^2 S_{11}}{\partial x^k \partial x^l},$$

in view of the first order condition (3.2).

The function Z satisfies the scalar evolution equation

$$\frac{\partial Z}{\partial t} = a^{kl} \nabla_k \nabla_l S_{ij} \dot{x}^i \dot{x}^j + u^k \nabla_k S_{ij} \dot{x}^i \dot{x}^j + N_{ij} \dot{x}^i \dot{x}^j.$$

At the minimum point (p, v), the identities (3.1) and (3.4), the vanishing of the connection coefficients, and the inequality (3.3) imply the following:

$$\frac{\partial Z}{\partial t} = a^{kl} \frac{\partial^2 S_{11}}{\partial x^k \partial x^l} + N_{11}$$

$$\ge 2 a^{kl} \left(2 \Gamma_k^p \nabla_l S_{1p} - \Gamma_k^p \Gamma_l^q S_{pq} \right) + N_{11}.$$

The assumption of the theorem implies that the supremum of the right-hand side over all choices of Γ is non-negative, so the left-hand side is non-negative. By the maximum principle (see the argument of Hamilton in [9]) the inequality $Z \geq 0$ is preserved. $\qquad\square$

To illustrate the application of Theorem 3.2 I include the following result on preserving convexity for solutions of fully nonlinear parabolic equations. A result of this kind was first shown to me by Gerhard Huisken (in the context of preserving convexity for evolving hypersurfaces), who proved it by considering the evolution equation for the inverse of the second derivative matrix. It can also be proved by considering the equation satisfied by the second derivatives of the Legendre transform of the solution. The belief that the conclusion should also follow directly from the evolution equation for D^2u led to Theorem 3.2.

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $u : \Omega \times [0, T] \to \mathbb{R}$ be a solution of a fully nonlinear equation of the form*

$$\frac{\partial u}{\partial t} = F(D^2 u)$$

where F is a C^2 function defined on the cone Γ_+ of positive definite symmetric matrices, which is monotone increasing (that is, $F(A + B) \geq F(A)$ whenever B is a positive definite matrix), and such that the function

$$F^*(A) = -F(A^{-1})$$

is concave on Γ_+. If $D^2 u \geq \varepsilon I$ (for some $\varepsilon > 0$) everywhere on Ω for $t = 0$, and on $\partial\Omega$ for $0 \leq t \leq T$, then $D^2 u \geq \varepsilon I$ everywhere on Ω for $0 \leq t \leq T$.

Proof. $D^2 u$ evolves as follows (denoting derivatives by subscripts):

$$\frac{\partial}{\partial t} u_{ij} = \dot{F}^{kl} D_k D_l u_{ij} + \ddot{F}^{kl,pq} u_{ikl} u_{jpq}.$$

The result is obvious for convex F, but not for the weaker condition of the Theorem. Theorem 3.2 with $S_{ij} = u_{ij} - \varepsilon \delta_{ij}$ and $\Lambda^p_k = v^j D_j S_{kq} \mathfrak{r}^{qp}$ (where $\mathfrak{r}$ is the inverse matrix of $D^2 u$) gives

$$\sup_\Gamma \left(4a^{kl} \Gamma^p_k D_l S_{ip} v^i - 2a^{kl} \Gamma^p_k \Gamma^q_l S_{pq} \right) = 4a^{kl} \Gamma^p_k D_l S_{ip} v^i - 2a^{kl} \Gamma^p_k \Gamma^q_l S_{pq}$$

$$= 2v^i v^j \dot{F}^{kl} u_{ikq} \mathfrak{r}^{qp} u_{jpl}$$

$$+ 2\varepsilon v^i v^j \dot{F}^{kl} u_{ika} \mathfrak{r}^{ap} \delta_{pq} \mathfrak{r}^{qb} u_{jlb}$$

$$\geq 2v^i v^j \dot{F}^{kl} u_{ikq} \mathfrak{r}^{qp} u_{jlp}.$$

The result then follows from Theorem 3.2 and the following Lemma. $\qquad\square$

Lemma 3.4. *If F^* is concave, then*

$$(3.5) \qquad \left(\ddot{F}^{kl,pq} + 2\dot{F}^{kp} \mathfrak{r}^{lq} \right) X_{kl} X_{pq} \geq 0$$

for every symmetric matrix X.

Proof. Take the identity $F(A) = -F^*(B)$ with $B = A^{-1}$, and differentiate:

$$\dot{F}^{kl}(A) = \frac{\partial F^*}{\partial B^{ab}}(B)B^{ak}B^{bl}.$$

Further differentiation gives

$$\ddot{F}^{kl,pq} = -\frac{\partial^2 F^*}{\partial B^{ab}\partial B^{cd}}B^{ak}B^{bl}B^{cp}B^{dq} - \frac{\partial F^*}{\partial B^{ab}}\left(B^{ak}B^{bp}B^{ql} + B^{ap}B^{kq}B^{bl}\right)$$

$$\geq -\frac{\partial F^*}{\partial B^{ab}}\left(B^{ak}B^{bp}B^{ql} + B^{aq}B^{kp}B^{bl}\right)$$

$$= -\dot{F}^{kp}(A^{-1})^{ql} - \dot{F}^{ql}(A^{-1})^{kp}.$$

This proves the Lemma. $\square$

Theorem 3.3 extends (with trivial modifications) to equations of the form $\frac{\partial u}{\partial t} = F(D^2 u, Du, u)$ if the same concavity condition holds in the first argument, and F is also convex in the last entry.

4. The pinching estimate

The main result of this paper gives conditions under which an equation will preserve uniform positivity of the second derivatives, in the sense that $D^2 u \geq \varepsilon \Delta u\, \mathrm{I}$ for some $\varepsilon \in (0, 1/n)$. This is a non-trivial extension of Theorem 3.3, and requires considerably more work to prove. The result is stated to allow easy application in a variety of different situations. In the model case Equation (1.3) implies that the tensor $S_{ij} = D_i D_j u - \varepsilon \Delta u\, \delta_{ij}$ evolves according to

$$(4.1) \qquad \frac{\partial S_{ij}}{\partial t} = \dot{F}^{kl}D_k D_l S_{ij} + \ddot{F}^{kl,pq}u_{ikl}u_{jpq} - \varepsilon\delta_{ij}\delta^{ab}\ddot{F}^{kl,pq}u_{akl}u_{bpq}.$$

The import of the inequality stated in the Theorem below should be understood in view of Equation (4.1) and Theorem 3.2. In this context the tensor T_{ijk} which appears there represents u_{ijk}.

Theorem 4.1. *Let f be a smooth symmetric, monotone, concave and inverse-concave function on Γ_+. Let $F(A) = f(\lambda(A))$, where λ is the map which takes a symmetric matrix to its eigenvalues. Let A be a symmetric positive definite matrix and v an eigenvector of A corresponding to the smallest eigenvalue of A, and let $\varepsilon = \frac{A_{ij}v^i v^j}{\mathrm{Tr}\,A|v|^2} \in (0, 1/n)$. If T is a totally symmetric 3-tensor with $T_{ijk}v^i v^j = \varepsilon\delta^{ij}T_{ijk}$ for $k = 1, \ldots, n$, then*

$$v^i v^j \ddot{F}^{kl,pq}(A)T_{ikl}T_{jpq} - \varepsilon|v|^2\delta^{ab}\ddot{F}^{kl,pq}(A)T_{akl}T_{bpq}$$

$$+ 2\sup_\Gamma \dot{F}^{kl}(A)\left(2\Gamma_k^p\left(T_{lpi}v^i - \varepsilon\delta^{ab}T_{lab}v_p\right) - \Gamma_k^p\Gamma_l^q(A_{pq} - \varepsilon\mathrm{Tr}A\delta_{pq})\right) \geq 0.$$

5. Differentiating eigenvalues and eigenvectors

This section prepares for the proof of Theorem 4.1 by establishing results about derivatives of eigenvalues and eigenvectors of symmetric matrices, and of functions of symmetric matrices defined in terms of their eigenvalues.

Theorem 5.1. *Let f be a C^2 symmetric function defined on a symmetric region Ω in $\mathbb{R}^n$. Let $\tilde{\Omega} = \{A \in \mathrm{Sym}(n) : \lambda(A) \in \Omega\}$, and define $F : \tilde{\Omega} \to \mathbb{R}$ by $F(A) = f(\lambda(A))$. Then at any diagonal $A \in \tilde{\Omega}$ with distinct eigenvalues, the second derivative of F in direction $B \in \mathrm{Sym}(n)$ is given by*

$$\ddot{F}(B, B) = \sum_{k,l} \ddot{f}^{kl} B_{kk} B_{ll} + 2 \sum_{k<l} \frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} B_{kl}^2.$$

This result appeared without detailed proof in [1, Equation 2.23]. A proof appeared later in [7], involving somewhat laborious checking of several cases. For this reason I include here an argument which may be more illuminating.

Proof. Let $Z : \mathrm{Sym}(n) \times \mathbb{R}^n \times O(n) \to \mathrm{Sym}(n)$ be the smooth map given by

$$Z(A, \lambda, M) = M^t A M - \mathrm{diag}(\lambda).$$

This vanishes if and only if the columns of M are eigenvectors of A, with eigenvalues λ. The derivative of Z at a point (A, λ, M) with $Z = 0$ in a direction (A', λ', M') is as follows (noting that $M' = M\Lambda$ with $\Lambda_{ij} + \Lambda_{ji} = 0$):

$$DZ(A', \lambda', M')_{ij} = M_{ik}A'_{kl}M_{lj} + \Lambda_{ki}M_{lk}A_{lp}M_{pj} + M_{ki}A_{kl}M_{lp}\Lambda_{pj} - \lambda'_i \delta_{ij}$$
$$= M_{ik}A'_{kl}M_{lj} + (\lambda_j \Lambda_{ji} + \lambda_i \Lambda_{ij}) - \lambda'_i \delta_{ij}$$
$$\text{(5.1)} \qquad = M_{ik}A'_{kl}M_{lj} + (\lambda_i - \lambda_j)\Lambda_{ij} - \lambda'_i \delta_{ij}.$$

The restriction of this to the last two components has no kernel: If it vanishes, the diagonal parts imply $\lambda' = 0$, and the off-diagonal parts imply $\Lambda = 0$ since $\lambda_i - \lambda_j \neq 0$. Therefore this is an isomorphism, and the implicit function theorem gives that the zero set of Z is locally of the form $\{\lambda = \lambda(A), M = M(A)\}$ where λ and M are analytic functions of A. If A is diagonal (so $M = I$) the first derivatives of λ and M can be read off:

$$\lambda'_i = A'_{ii} \qquad \text{and} \qquad M'_{ij} = \Lambda_{ij} = -\frac{A'_{ij}}{\lambda_i - \lambda_j}.$$

Equation (5.1) holds everywhere on $\{Z = 0\}$, so differentiating this along $(A(t), \lambda(A(t)), M(A(t))$ with $A'' = 0$ and $M(0) = I$ gives

$$0 = M'_{ki}A'_{kj} + A'_{ik}M'_{kj} + (\lambda'_i - \lambda'_j)\Lambda_{ij} + (\lambda_i - \lambda_j)\Lambda'_{ij} - \lambda''_i \delta_{ij}.$$

The second derivative of λ_i can be read off from the (i, i) component:

$$\lambda''_i = \sum_k \left(\Lambda_{ki}A'_{ki} + A'_{ik}\Lambda_{ki} \right) = -2 \sum_{k \neq i} \frac{(A'_{ik})^2}{\lambda_k - \lambda_i}.$$

The first and second derivatives of F at a diagonal matrix A with distinct eigenvalues can now be computed directly:

$$(5.2) \qquad F' = \sum_k \dot{f}^k \lambda'_k = \sum_k \dot{f}^k A'_{kk} = \sum_{k,l} \dot{f}^k \delta_{kl} A'_{kl},$$

so $\dot{F}^{kl} = \dot{f}^k \delta_{kl}$. Also,

$$\begin{aligned}
F'' &= \frac{d}{dt}\left(\sum_k \dot{f}^k \lambda'_k\right) \\
&= \sum_{k,l} \ddot{f}^{kl} \lambda'_k \lambda'_l + \sum_k \dot{f}^k \lambda''_k \\
&= \sum_{k,l} \ddot{f}^{kl} A'_{kk} A'_{ll} - 2\sum_{k\neq l} \frac{\dot{f}^k}{\lambda_l - \lambda_k}\left(A'_{kl}\right)^2 \\
&= \sum_{k,l} \ddot{f}^{kl} A'_{kk} A'_{ll} + 2\sum_{k<l} \frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l}\left(A'_{kl}\right)^2.
\end{aligned}$$

This completes the proof. $\qquad\square$

Corollary 5.2. *F is concave at A if and only if f is concave at $\lambda(A)$ and*

$$\frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} \leq 0 \qquad \text{for all } k \neq l.$$

Corollary 5.3. *For Ω convex, F is concave on $\tilde{\Omega}$ if f is concave on Ω.*

Proof. See [1, Lemma 2.2]. $\qquad\square$

Corollary 5.4. *F^* is concave at A if and only if*

$$\left(\ddot{f}^{kl} + 2\frac{\dot{f}^k}{\lambda_k}\delta_{kl}\right) \geq 0 \qquad \text{and} \qquad \frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} + \frac{\dot{f}^k}{\lambda_l} + \frac{\dot{f}^l}{\lambda_k} \geq 0, \quad k \neq l.$$

Proof. This follows immediately from the inequality (3.5) and Theorem 5.1. $\qquad\square$

Corollary 5.5. *If $\Omega^{-1} = \{(x_1^{-1}, \ldots, x_n^{-1}) : (x_1, \ldots, x_n) \in \Omega\}$ is convex, then F^* is concave if and only if f^* is concave.*

In particular, corollaries 5.3 and 5.5 apply for functions defined on Γ_+.

6. Proof of the estimate

This section contains the proof of the main result, Theorem 4.1. Note that the Theorem does not refer at all to a partial differential equation or its solution, but only to a pointwise inequality for the first and second

derivatives of a function defined on the positive cone. If F is C^2, then $\ddot{F}$ and $\dot{F}$ are continuous, so for fixed Γ, v and T the quantity we wish to estimate,

$$v^i v^j \ddot{F}^{kl,pq}(A) T_{ikl} T_{jpq} - \varepsilon |v|^2 \delta^{ab} \ddot{F}^{kl,pq}(A) T_{akl} T_{bpq}$$
$$+ 2\dot{F}^{kl}(A) \left(2\Gamma_k^p \left(T_{lpi} v^i - \varepsilon T_{laa} v_p\right) - \Gamma_k^p \Gamma_l^q (A_{pq} - \varepsilon \mathrm{Tr} A \delta_{pq})\right)$$

is continuous in A. It follows that the supremum over Γ is semi-continuous in A. We will take advantage of this by working only with symmetric matrices A for which all of the eigenvalues are distinct. This is possible since for any positive definite $A \in \mathrm{Sym}(n)$ with $A_{ij} \geq \varepsilon \mathrm{Tr} A \delta_{ij}$ and $A_{ij} v^i v^j = \varepsilon \mathrm{Tr} A |v|^2$ for some $v \neq 0$, there is a sequence $\{A^{(k)}\}_{k \geq 0}$ approaching A, satisfying $A_{ij}^{(k)} \geq \varepsilon \mathrm{Tr} A^{(k)} \delta_{ij}$ and $A_{ij}^{(k)} v^i v^j = \varepsilon \mathrm{Tr} A^{(k)} |v|^2$, and with each $A^{(k)}$ having distinct eigenvalues. Hence it suffices to establish the result in the case where all of the eigenvalues are distinct.

In this case there is an orthonormal basis $e_1, \ldots, e_n$ consisting of eigenfunctions of A, with eigenvalues in increasing order. In this basis, $v = e_1$ and $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and $\lambda_1 = \varepsilon(\lambda_1 + \cdots + \lambda_n)$. Also $\dot{F} = \mathrm{diag}(\dot{f}^1, \ldots, \dot{f}^n)$ by Equation (5.2).

The problem is simplified by the observation that the supremum over Γ can be computed exactly in this case: We can write

$$2\dot{F}^{kl} \left(2\Gamma_k^p \left(T_{lpi} v^i - \varepsilon T_{laa} v_p\right) - \Gamma_k^p \Gamma_l^q (A_{pq} - \varepsilon \mathrm{Tr} A \delta_{pq})\right)$$
$$= 2 \sum_{k=1}^n \sum_{p=2}^n \dot{f}^k \left(\Gamma_k^p T_{kp1} - (\Gamma_k^p)^2 (\lambda_p - \lambda_1)\right)$$
$$= 2 \sum_{k \geq 1, p \geq 2} \left(\frac{\dot{f}^k}{\lambda_p - \lambda_1} T_{kp1}^2 - \dot{f}^k (\lambda_p - \lambda_1) \left(\Gamma_k^p - \frac{T_{kp1}}{\lambda_p - \lambda_1}\right)^2\right)$$

It follows that the supremum is attained by the choice $\Gamma_k^p = \frac{T_{kp1}}{\lambda_p - \lambda_1}$.

The required inequality becomes the following:

$$Q = \sum_{k,l} \ddot{f}^{kl} T_{1kk} T_{1ll} - \varepsilon \sum_{j,k,l} \ddot{f}^{kl} T_{jkk} T_{jll} + 2 \sum_k \sum_{l>1} \frac{\dot{f}^k}{\lambda_l - \lambda_1} T_{1kl}^2$$
$$+ 2 \sum_{k<l} \frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} T_{1kl}^2 - 2\varepsilon \sum_j \sum_{k<l} \frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} T_{jkl}^2.$$
$$\geq 0.$$

We use the identities

$$T_{k11} = \frac{\varepsilon}{1 - \varepsilon} \sum_{j>1} T_{kjj}$$

for $k = 1, \ldots, n$, to eliminate terms involving T_{k11}. This together with the total symmetry of T implies that, as a bilinear form on the space of all

possible T, Q has a block-diagonal form, as follows:

$$Q = \sum_{k=1}^{n} Q_k + \sum_{1 \leq j < k < l} Q_{jkl}$$

where Q_k involves only T_{kii} for $i \geq 2$, and Q_{jkl} involves only T_{jkl}. Precisely, these are as follows:

$$Q_1 = (1 - \varepsilon) \sum_{k,l>1} \left(\dot{f}^{kl} + \frac{\varepsilon}{1 - \varepsilon}(\ddot{f}^{k1} + \ddot{f}^{1l}) + \left(\frac{\varepsilon}{1 - \varepsilon}\right)^2 \ddot{f}^{11} \right) T_{1kk} T_{1ll}$$

$$+ 2 \sum_{k>1} \frac{(1 - \varepsilon)\dot{f}^k + \varepsilon \dot{f}^1}{\lambda_k - \lambda_1} T_{1kk}^2;$$

$$Q_k = -\varepsilon \sum_{i,j>1} \left(\ddot{f}^{ij} + \frac{\varepsilon}{1 - \varepsilon}(\ddot{f}^{i1} + \ddot{f}^{1j}) + \left(\frac{\varepsilon}{1 - \varepsilon}\right)^2 \ddot{f}^{11} \right) T_{kii} T_{kjj}$$

$$+ 2\frac{(1 - \varepsilon)\dot{f}^k + \varepsilon \dot{f}^1}{\lambda_k - \lambda_1} \left(\frac{\varepsilon}{1 - \varepsilon} \sum_{i>1} T_{kii} \right)^2 - 2\varepsilon \sum_{j \neq k,1} \frac{\dot{f}^k - \dot{f}^j}{\lambda_k - \lambda_j} T_{kjj}^2;$$

$$Q_{1kl} = 2\left((1 - \varepsilon)\frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} + \frac{\dot{f}^l}{\lambda_k - \lambda_1} + \frac{\dot{f}^k}{\lambda_l - \lambda_1} - \varepsilon\frac{\dot{f}^k - \dot{f}^1}{\lambda_k - \lambda_1} - \varepsilon\frac{\dot{f}^l - \dot{f}^1}{\lambda_l - \lambda_1} \right) T_{1kl}^2;$$

$$Q_{jkl} = -2\varepsilon \left(\frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} + \frac{\dot{f}^k - \dot{f}^j}{\lambda_k - \lambda_j} + \frac{\dot{f}^l - \dot{f}^j}{\lambda_l - \lambda_j} \right) T_{jkl}^2.$$

We require each of these to be non-negative.

For $k > l > j > 1$, $Q_{jkl} \geq 0$ by concavity (see Corollary 5.2).

The same is true for Q_k: The matrix in the first bracket is $\ddot{f}(\xi, \xi)$, where

$$\xi = \sum_{k>2} T_{1kk} \left(e_k + \frac{\varepsilon}{1 - \varepsilon} e_1 \right).$$

The concavity of f therefore implies that this term is non-positive. The last term is also of the right sign by Corollary 5.2, and the remaining term is manifestly non-negative.

The non-negativity of Q_{1kl} follows from the concavity of both f and f^*:

$$Q_{1kl} = 2(1 - \varepsilon) \left(\frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} + \frac{\dot{f}^k}{\lambda_l - \lambda_1} + \frac{\dot{f}^l}{\lambda_k - \lambda_1} \right)$$

$$+ 2\varepsilon \left(\frac{\dot{f}^k}{\lambda_l - \lambda_1} + \frac{\dot{f}^l}{\lambda_k - \lambda_1} - \frac{\dot{f}^k - \dot{f}^1}{\lambda_k - \lambda_1} - \frac{\dot{f}^l - \dot{f}^1}{\lambda_l - \lambda_1} \right).$$

The first bracket is non-negative by the second inequality of Corollary 5.4. The first two terms in the second bracket are manifestly non-negative, while the other two are non-negative by Corollary 5.2.

Finally, non-negativity of Q_1 follows from concavity of f^* in an indirect way: Consider the function ϕ of $(n-1)$ variables $x_2, \ldots, x_n$ given by $\phi(x_2, \ldots, x_n) = f\left(\frac{\varepsilon}{1-\varepsilon}(x_2 + \cdots + x_n), x_2, \ldots, x_n\right)$. Then

$$Q_1 = (1-\varepsilon) \sum_{k,l>1} \left(\ddot{\phi}^{kl} + 2\frac{\dot{\phi}^k}{\lambda_k - \lambda_1}\delta_{kl}\right) T_{1kk}T_{1ll}$$

$$\geq (1-\varepsilon) \sum_{k,l>1} \left(\ddot{\phi}^{kl} + 2\frac{\dot{\phi}^k}{\lambda_k}\delta_{kl}\right) T_{1kk}T_{1ll}.$$

The first inequality in Corollary 5.4 then implies $Q_1 \geq 0$ provided ϕ^* is concave. To establish concavity of ϕ^*, note

$$\phi^*(x_2, \ldots, x_n) = -\phi\left(\frac{\varepsilon}{1-\varepsilon}(x_2^{-1} + \cdots + x_n^{-1}), x_2^{-1}, \ldots, x_n^{-1}\right)$$

$$= f^*(\psi(x_2, \ldots, x_n), x_2, \ldots, x_n),$$

where $\psi(x_2, \ldots, x_n) = \frac{1-\varepsilon}{\varepsilon}\left(x_2^{-1} + \cdots + x_n^{-1}\right)^{-1}$ is a multiple of the harmonic mean of $x_2, \ldots, x_n$, hence a concave function of $x_2, \ldots, x_n$ (Corollary 2.5). Also, f^* is concave, and increasing in each argument. Therefore

$$\alpha\phi^*(x) + (1-\alpha)\phi^*(y) = \alpha f^*(\psi(x), x) + (1-\alpha)f^*(\psi(y), y)$$

$$\leq f^*(\alpha\psi(x) + (1-\alpha)\psi(y), \alpha x + (1-\alpha)y)$$

$$\leq f^*(\psi(\alpha x + (1-\alpha)y), \alpha x + (1-\alpha)y)$$

$$= \phi^*(\alpha x + (1-\alpha)y)$$

for any $\alpha \in (0,1)$ and x, y in the positive cone of $\mathbb{R}^{n-1}$. Here the first inequality follows from the concavity of f^*, and the second follows since ψ is concave (so $\alpha\psi(x) + (1-\alpha)\psi(y) \leq \psi(\alpha x + (1-\alpha)y)$) and f^* is increasing in the first argument. Therefore ϕ^* is concave, and the proof is complete.

7. Application to evolving hypersurfaces

In this section Theorem 4.1 is applied to prove Theorem 1.1 on evolving hypersurfaces. As mentioned before, the only case not proved elsewhere is case 4, where both the speed f and its dual f^* are concave functions on Γ_+.

The only new ingredient in the proof is the application of Theorem 4.1 to prove that the smallest eigenvalue of $\frac{h_{ij}}{H}$ over M_t is non-decreasing in t, where h_{ij} is the second fundamental form and H is the mean curvature.

First note that in any local coordinates for M the rate of change of the metric tensor $g_{ij} = g(\partial_i, \partial_j)$ under Equation (1.4) is given by

$$\frac{\partial}{\partial t}g_{ij} = -2Fh_{ij}.$$

The evolution of h_{ij} is as follows (see [1, Lemma 3.13]):

$$\frac{\partial}{\partial t}h_{ij} = \dot{F}^{kl}\nabla_k\nabla_l h_{ij} + \ddot{F}^{kl,pq}\nabla_i h_{kl}\nabla_j h_{pq} + h_{ij}\dot{F}^{kl}h_{kp}g^{pq}h_{ql} - 2Fh_{ik}g^{kl}h_{lj}.$$

Suppose $S_{ij} = h_{ij} - \varepsilon H g_{ij}$ is non-negative. Then note

$$(7.1) \qquad \frac{\partial}{\partial t} S_{ij} = \dot{F}^{kl} \nabla_k \nabla_l S_{ij} + \ddot{F}^{kl,pq} T_{ikl} T_{jmn} - \varepsilon g_{ij} g^{rs} \ddot{F}^{kl,pq} T_{rkl} T_{spq}$$
$$+ S_{ij} \dot{F}^{kl} h_{kp} g^{pq} h_{ql} - 2F h_{ik} g^{kl} S_{lj}$$

where $T_{ijk} = \nabla_i h_{jk}$ is totally symmetric by the Codazzi identity, and satisfies $0 = \nabla_k S_{11} = T_{k11} - \varepsilon \sum_j T_{kjj}$ at a point where S_{ij} has e_1 as a null eigenvector. Then Theorem 4.1 gives

$$0 \leq \ddot{F}^{kl,pq} T_{1kl} T_{1mn} - \varepsilon g^{rs} \ddot{F}^{kl,pq} T_{rkl} T_{spq}$$
$$+ 2 \sup_{\Gamma} \dot{F}^{kl} \left(2\Gamma_k^p \left(T_{lp1} - \varepsilon T_{ljj} \delta_{p1} \right) - \Gamma_k^p \Gamma_l^q S_{pq} \right)$$

The terms in the second line of Equation (7.1) vanish at a null eigenvector, so Theorem 3.2 implies that S_{ij} remains non-negative.

The remainder of the proof of Theorem 1.1 is the same as in [1].

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