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The number of transversals in a Latin square

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Abstract A Latin square of order n is an $n \times n$ array of n symbols, in which each symbol occurs exactly once in each row and column. A transversal is a set of n entries, one selected from each row and each column of a Latin square of order n such that no two entries contain the same symbol. Define $T(n)$ to be the maximum number of transversals over all Latin squares of order n. We show that $b^n \leq T(n) \leq c^n \sqrt{n}\, n!$ for $n \geq 5$, where $b \approx 1.719$ and $c \approx 0.614$. A corollary of this result is an upper bound on the number of placements of n non-attacking queens on an $n \times n$ toroidal chess board. Some divisibility properties of the number of transversals in Latin squares based on finite groups are established. We also provide data from a computer enumeration of transversals in all Latin squares of order at most 9, all groups of order at most 23 and all possible turn-squares of order 14.

Keywords Transversal · Latin square · n-queens · Turn-square · Cayley table

AMS Classification 05B15

1 Introduction

A *Latin square* of order n is an $n \times n$ array of n symbols, in which each symbol occurs exactly once in each row and column. We can think of a Latin square as a set of n^2 *entries* of the form (row, column, symbol). A set of n entries, one selected from each row and each column

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of a Latin square of order n such that no two entries contain the same symbol, is called a *transversal*. Transversals play a crucial role in the important concept of orthogonality for Latin squares (see, e.g. [9]). Despite this, a number of basic questions about their properties remain unresolved.

To date, literature regarding transversals has been largely concerned with the question of existence. This question is still far from being resolved even if strong additional assumptions are made about the structure of the square. Of particular note is the case of Latin squares based on finite groups, which we discuss further in Section 3.

The number of transversals provides a useful invariant for squares of small orders where this number can be computed in reasonable time (see, e.g., Killgrove et al. [15]). Also Brown and Parker used the number of transversals as a heuristic during their extended search for a triple of mutually orthogonal Latin squares of order 10 (see [4] and the references cited therein), although to possess an orthogonal mate a square of order n need not have more than n transversals [28]. One of the few general results in the area stems from a conjecture made by Ryser [21] in 1967 stating that for every Latin square of order n, the number of transversals is congruent to n mod 2. In [2], Balasubramanian proved:

Theorem 1 *In any Latin square of even order the number of transversals is even.*

Despite this, it has been noted in [1] and [5] (and other places) that there are many known counterexamples of odd order to Ryser's conjecture. Hence the conjecture has now been weakened to the following:

Conjecture 2 (Ryser) *Each Latin square of odd order has at least one transversal.*

The results of counting transversals in all Latin squares of order 9 are presented in Section 4 and verify Conjecture 2 for $n = 9$. This means that Conjecture 2 has now been verified for $n \leq 9$ using computer enumeration; however, this method becomes intractable for $n > 9$.

The other conjecture of interest in this paper is one posed by Vardi [26].

Conjecture 3 (Vardi) *Let t_n denote the number of transversals in a cyclic Latin square of order n. Then there exist two real constants c_1 and c_2 such that*

$$c_1^n n! \leq t_n \leq c_2^n n!$$

where $0 < c_1 < c_2 < 1$ and $n \geq 3$ is odd.

Vardi makes this conjecture while considering a variation on the toroidal n-queens problem. The toroidal n-queens problem is that of determining in how many different ways n non-attacking queens can be placed on a toroidal $n \times n$ chessboard. Vardi considered the same problem using semiqueens in place of queens, where a semiqueen is a piece which moves like a toroidal queen but cannot travel on negative diagonals. The solution to Vardi's problem provides an upper bound on the toroidal n-queens problem. The problem can be translated into one concerning Latin squares by noting that every configuration of n non-attacking semi-queens on a toroidal $n \times n$ chessboard corresponds to a transversal in a cyclic Latin square L of order n, where $L(i, j) \equiv i - j \bmod n$. Note that the toroidal n-queens problem is equivalent to counting transversals in L such that the corresponding positions in L' contain a transversal, where $L'_{ij} = i + j \bmod n$.

Let $T(n)$ be the maximum number of transversals over all Latin squares of order n. In Section 2, we derive an upper bound on $T(n)$ and in doing so prove the upper bound given in Vardi's Conjecture. A simple exponential lower bound on $T(n)$ is also given.

In Section 3, we examine the important special case of transversals in Latin squares based on finite groups. We prove some results in the same spirit as Theorem 1 and also report on computer enumerations for groups of order at most 23. In Section 4, we report further computer enumerations, this time for arbitrary Latin squares of order at most 9 and for the so called turn-squares of order 14.

Before proceeding, we require a few more definitions. For $0 \leq k \leq n$, a set of k entries, each selected from different rows and columns of a Latin square such that no two entries contain the same symbol, is called a *partial transversal* of *length* k. It has been conjectured by Brualdi ([9, p.103]) that every Latin square of order n possesses a partial transversal of length $n-1$. A claimed proof of this by Derienko [11] appears to contain a fatal error [5]. The best reliable result to date states that there must be a partial transversal of length at least $n - O(\log^2 n)$. This was shown by Shor [24], and the implicit constant in the 'big O' was marginally improved by Fu and Lin [12]. It has also been shown by Cameron and Wanless [5] that every Latin square possesses a set of n entries from different rows and columns in which no symbol appears more than twice.

For each Latin square there are $3! = 6$ *conjugate* squares obtained by uniformly permuting the coordinates in each entry. These conjugates can be labelled by a permutation giving the new order of the coordinates, relative to the original order, which we denote by (123). Hence, the (123)-conjugate is the square itself and the (213)-conjugate is its transpose. A Latin square is said to be *symmetric* if its (123) and (213)-conjugates are equal and *semi-symmetric* if its (123), (231) and (312)-conjugates are all equal.

We have now defined the core concepts which are used in our arguments. However, we shall occasionally make remarks which assume the reader is familiar with other basic terminology of Latin squares. In particular, any reader not familiar with the terms isotopic, autotopy, orthogonal mate, MOLS, Latin subsquare, or main class is referred to [6, 7] for definitions of these concepts.

2 Bounds on $T(n)$

We begin by proving two technical lemmas which are required in the proof of Theorem 6.

Lemma 4 *Let $\mathcal{T}$ be a rooted tree, and let $N(v)$ be the number of children of vertex v in $\mathcal{T}$. Define $p_0(\mathcal{T}) = 1$. For each level $l > 0$ of $\mathcal{T}$, let $p_l(\mathcal{T}) = \max \prod_{i=0}^{l-1} N(v_i)$, where the maximum is taken over all paths $v_0, v_1, \ldots, v_{l-1}$ of $\mathcal{T}$, v_i is a vertex at level i of $\mathcal{T}$ and v_0 is the root of the tree. When no such path exists, we take $p_l(\mathcal{T}) = 0$.*

Then, if $V_l(\mathcal{T})$ is the number of vertices of $\mathcal{T}$ at level l,

$$V_l(\mathcal{T}) \leq p_l(\mathcal{T}).$$

Proof We prove this lemma using induction on l. The lemma is trivially true when $l = 0$, and we assume it holds when $l \leq m$, for $m \geq 0$. Now let $l = m+1$ and $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{N(v_0)}$ be the distinct subtrees obtained by deleting the root of $\mathcal{T}$. It follows that

$$p_l(\mathcal{T}) = \max_{1 \leq j \leq N(v_0)} N(v_0)\, p_{l-1}(\mathcal{T}_j) \geq \sum_{j=1}^{N(v_0)} p_{l-1}(\mathcal{T}_j)$$

and $V_l(\mathcal{T}) = \sum_{j=1}^{N(v_0)} V_{l-1}(\mathcal{T}_j)$. Hence by the induction hypothesis, we conclude that $p_l(\mathcal{T}) \geq V_l(\mathcal{T})$. □

Define $I_n = \{0, 1, \ldots, n\}$ and consider a function $f : I_n \to \mathbb{R}$. Let $H(f)$ be the convex hull of the point set $\{(0, f(0)), (1, f(1)), \ldots, (n, f(n))\}$. We define the *concave envelope* of f to be the function $c : I_n \to \mathbb{R}$ given by $c(i) = \max\{y : (i, y) \in H(f)\}$. Note that c is the least concave function such that $c(i) \geq f(i)$ for all $i \in I_n$, where we mean that c is concave in the usual sense that $c(a-1) + c(a+1) \leq 2c(a)$ for all $a \in \{1, 2, \ldots, n-1\}$.

Lemma 5 *Let $b : I_n \to \mathbb{R}$ be a function such that $b(i) < b(n)$ for all $0 \leq i \leq n-1$ and let*

$$F = \{f : I_n \to \mathbb{R} \mid \\ f(0) = b(0), f(n) = b(n) \text{ and } f(i+1) \geq f(i) \geq b(i) \text{ for } 0 \leq i \leq n-1\}.$$

Define the function $g : F \to \mathbb{R}$ as $g(f) = \prod_{i=0}^{n-1}(f(i+1) - f(i))$ and let $c : I_n \to \mathbb{R}$ be the concave envelope of b. Note that $c \in F$. Then if $f \in F$ and $f \neq c$, we have $g(f) < g(c)$.

Proof We note that since F is a compact set and g is continuous, there exists a function $f \in F$ for which $g(f)$ achieves its maximum.

If we suppose that f is not concave, we can choose consecutive integer points $p, p+1, p+2 \in I_n$ such that f is strictly convex at $p+1$. Let $d = \frac{1}{2}(f(p+2) - f(p))$, then it follows that $f(p+2) = f(p) + 2d$ and $f(p+1) = f(p) + d - \epsilon$ for some $0 < \epsilon < d$.

Let $\lambda = \prod_{i=0}^{p-1}(f(i+1) - f(i)) \prod_{i=p+2}^{n-1}(f(i+1) - f(i))$ and note that $\lambda > 0$. Hence

$$\begin{aligned} g(f) &= \lambda(f(p+1) - f(p))(f(p+2) - f(p+1)) \\ &= \lambda(d^2 - \epsilon^2). \end{aligned} \tag{1}$$

Since letting $\epsilon = 0$ yields a larger product and $f(p) + d > f(p+1) \geq b(p+1)$, this contradicts our choice of $f \in F$ such that $g(f)$ attains its maximum value. Hence f must be concave.

Now suppose that f is concave and $f \neq c$. Then there exist consecutive integer points $p, p+1, p+2 \in I_n$ such that f is strictly concave at $p+1$ and $f(p+1) > c(p+1)$. We can apply a similar argument to that given above by noting that in this instance $f(p+1) = f(p) + d + \epsilon$ for some $\epsilon > 0$ where $d = \frac{1}{2}(f(p+2) - f(p))$ as before. Additionally, $c(p+1) = f(p) + d + \delta$ for some $0 \leq \delta < \epsilon$. Considering $g(f)$ as it is in (1), letting $\epsilon = \delta$ will yield a larger product, and as $f(p+1) > c(p+1) \geq b(p+1)$ this again contradicts our choice of f. Therefore, f is the concave envelope of b and hence $f = c$. □

We are now ready to derive an upper bound on $T(n)$.

Theorem 6 *Let $T(n)$ be the maximum number of transversals in a Latin square of order n. Then*

$$T(n) \leq \frac{27}{35}\left(\frac{(2\beta - 3\alpha)(n+1) + 3\alpha^2 - \beta^2}{3(\beta - \alpha)}\right)^{\beta-\alpha} \prod_{k=0}^{\alpha-1}(n - 2k) \prod_{k=\beta}^{n-1} \tfrac{1}{3}(2n - 2k + 1)$$

for $n \geq 13$, where $\alpha = \lfloor \alpha_0 \rfloor$ and $\beta = \lceil \beta_0 \rceil$, with $\beta_0 = \sqrt{3}\,\alpha_0 = \frac{1}{4}(1 + \sqrt{3})(n+1)$.

Proof In order to calculate an upper bound on the number of transversals in a Latin square L of order n, we analyse a tree that contains every transversal in L exactly once.

Let r_i and c_i be the set of entries comprising the ith row and ith column of L, respectively, and let s_i be the set of the n entries of the square containing the ith symbol, where $1 \leq i \leq n$. Let $\{\gamma_1, \ldots, \gamma_k\}$ be the set of entries in a partial transversal of length k, where $0 \leq k < n$. Define $E(\{\gamma_1, \ldots, \gamma_k\})$ to be the first member of the list $r_1, \ldots, r_n, c_1, \ldots, c_n, s_1, \ldots, s_n$ such that

Rule 1: $E(\{\gamma_1, \dots, \gamma_k\}) \cap \{\gamma_1, \dots, \gamma_k\} = \emptyset$, and
Rule 2: of the sets which satisfy Rule 1, $E(\{\gamma_1, \dots, \gamma_k\})$ contains the fewest entries which have no row, column or symbol in common with any of $\{\gamma_1, \dots, \gamma_k\}$.

For example, consider the square

$$\begin{array}{cccccccc}
\mathbf{1} & 2 & 4 & 5 & 3 & 7 & 8 & 6 \\
2 & 1 & \mathbf{5} & 6 & 7 & 8 & 4 & 3 \\
5 & 8 & 3 & 1 & 2 & 4 & 6 & 7 \\
3 & \mathbf{7} & 6 & 8 & 1 & 2 & 5 & 4 \\
4 & 3 & 1 & 7 & 6 & 5 & 2 & \boxed{\textcircled{8}} \\
8 & 4 & 7 & 3 & 5 & 6 & 1 & \textcircled{2} \\
6 & 5 & 8 & 2 & 4 & 3 & 7 & 1 \\
7 & 6 & 2 & 4 & \boxed{8} & 1 & 3 & 5
\end{array}$$

for $k = 3$. The entries in the partial transversal γ_1, γ_2 and γ_3 are shown in **bold**, and both c_8 and s_8 satisfy Rules 1 and 2. The entries in c_8, which have no row, column or symbol in common with entries in the partial transversal are enclosed by a ◯, and those in s_8, by a □. Then $E(\{\gamma_1, \gamma_2, \gamma_3\}) = c_8$ because c_8 appears earlier in the list $r_1, \dots, r_8, c_1, \dots, c_8, s_1, \dots, s_8$ than s_8.

We now use the above rules to describe a rooted tree $\mathcal{T}$ of height at most n, in which each node at level k is a partial transversal of length k and the root is the empty partial transversal. For $k < n$, the children of the node $\{\gamma_1, \dots, \gamma_k\}$ are the partial transversals $\{\gamma_1, \dots, \gamma_{k+1}\}$ such that $\gamma_{k+1} \in E(\{\gamma_1, \dots, \gamma_k\})$. Nodes at level n (if any) are transversals and have no children. Let $N(\{\gamma_1, \dots, \gamma_k\})$ denote the number of children of the node $\{\gamma_1, \dots, \gamma_k\}$. In the example given above, $N(\{\gamma_1, \gamma_2, \gamma_3\}) = 2$.

To prove that $\mathcal{T}$ is a tree we let $\Gamma_k = \{\gamma_1, \dots, \gamma_k\}$ be an arbitrary partial transversal. We will prove by induction that for each i $(0 \le i \le k)$ there is at most one $\Gamma_i \subseteq \Gamma_k$ such that $|\Gamma_i| = i$ and $\Gamma_i \in \mathcal{T}$. This is clearly true for $i = 0$. Given $\Gamma_i \in \mathcal{T}$ $(i < k)$, define $E(\Gamma_i)$ as in Rules 1 and 2. Then at most one $\gamma \in \Gamma_k \backslash \Gamma_i$ belongs to $E(\Gamma_i)$ since no two elements of $\Gamma_k \backslash \Gamma_i$ share the same row, column or symbol. In the case that $k = n$, $\Gamma_k \backslash \Gamma_i$ includes entries in every row, column and symbol not used by Γ_i and so $\gamma \in (\Gamma_k \backslash \Gamma_i) \cap E(\Gamma_i)$ exists. This shows that every transversal appears in $\mathcal{T}$ exactly once. However, not all partial transversals in L are nodes of the tree.

From Lemma 4, we deduce that

$$T(n) \le \max \prod_{k=0}^{n-1} N(\{\gamma_1, \dots, \gamma_k\}), \tag{2}$$

where the maximum is taken over all transversals $\{\gamma_1, \dots, \gamma_n\}$ and $\gamma_1, \gamma_2, \dots, \gamma_n$ is the order in which the entries are added, which determines the path from the root of $\mathcal{T}$ to the node $\{\gamma_1, \dots, \gamma_n\}$. When we consider this product for a particular transversal, we let $N(\{\gamma_1, \dots, \gamma_k\}) = N(k)$.

Let A_k denote the k^2 entries of L which lie at the intersection of the k rows and k columns used by $\{\gamma_1, \dots, \gamma_k\}$. For convenience, we will describe the case where A_k is a block in the upper left corner of L and where $E(\{\gamma_1, \dots, \gamma_k\})$ is row r_{k+1}. All other possibilities give the same calculation. With A_k in the upper left-hand corner, the remainder of L naturally partitions into blocks B_k, C_k and D_k as illustrated below.

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$

Let S_k denote the set of symbols in the entries $\{\gamma_1, \ldots, \gamma_k\}$, and let $\tau(k)$ be the number of entries, including $\{\gamma_1, \ldots, \gamma_k\}$, that lie in A_k and contain symbols from S_k. We obtain a bound for $\tau(k+1)$ by noting that there are $n-k-N(k)$ symbols from S_k in that part of r_{k+1} which lies in D_k, so there must be $k-(n-k-N(k)) = N(k)-n+2k$ symbols from S_k in the $(k+1)$th row of A_{k+1}. By Rule 2, there are also at least $N(k)-n+2k$ symbols from S_k in the same column of A_{k+1} as γ_{k+1} and at least $N(k)-n+2k$ entries with the same symbol as γ_{k+1} in A_k. By definition, these contributions do not overlap, and we have yet to count γ_{k+1} itself. It follows that A_{k+1} inherits at least $3(N(k)-n+2k)+1$ entries containing symbols in S_{k+1} from its $(k+1)$th row and column, and the entries in A_k containing the same symbol as γ_{k+1}. Hence

$$\tau(k+1) \geq \tau(k) + 1 + 3(N(k) - n + 2k).$$

Rearranging gives

$$N(k) \leq n - 2k + \tfrac{1}{3}(\tau(k+1) - \tau(k) - 1). \tag{3}$$

In order to better analyse and eventually maximise $\prod_{k=0}^{n-1} N(k)$, we can rewrite the upper bound on $N(k)$ given in (3) as

$$N(k) \leq \sigma(k+1) - \sigma(k), \tag{4}$$

where $\sigma(k) = kn - k^2 + \frac{1}{3}(\tau(k) + 2k)$.

Next we calculate a lower bound on $\tau(k)$. There are k^2 entries containing the k symbols from S_k in blocks A_k and B_k, and by definition at least k of these are in A_k. This is a lower bound on $\tau(k)$. However, B_k can accommodate at most $k(n-k)$ entries containing symbols from S_k, so $\tau(k)$ must be at least $k^2 - k(n-k)$. Hence

$$\tau(k) \geq \max\{k, k(2k-n)\}. \tag{5}$$

It is possible to improve the lower bound on $\tau(n-2)$ slightly. We do this by noting that whenever our partial transversal is completable to a transversal there will be exactly two entries in D_{n-2}, one in each of the two rows, which contain symbols not in S_{n-2}. This leaves $(n-2)^2 - 2(n-3) = n^2 - 6n + 10$ entries containing symbols from S_{n-2} in A_{n-2}, which is greater than $n^2 - 6n + 8$, the lower bound on $\tau(n-2)$ given in (5).

The bound on $\tau(k)$ given in (5) together with the value of $\tau(n-2)$ implies a lower bound on $\sigma(k)$. In order to derive this bound, we start by using the values in (5) to obtain functions $\sigma_1(x)$ and $\sigma_2(x)$ for real $x \in [0, n]$,

$$\sigma_1(x) = xn - x^2 + x,$$
$$\sigma_2(x) = \frac{x}{3}(2n + 2 - x).$$

We also note that $\sigma(n-2) = \frac{1}{3}(n^2 + 2n - 6)$ when we use the improved lower bound on $\tau(n-2)$. The common tangent to $\sigma_1(x)$ and $\sigma_2(x)$ touches the curves at α_0 and β_0, respectively, where $\beta_0 = \sqrt{3}\,\alpha_0 = \frac{1}{4}(1+\sqrt{3})(n+1)$. We choose $\alpha, \beta \in \mathbb{Z}$ to be $\alpha = \lfloor \alpha_0 \rfloor$ and $\beta = \lceil \beta_0 \rceil$ where $\beta \leq n-2$, and define $b \colon I_n \to \mathbb{R}$

$$b(i) = \begin{cases} \sigma_1(i), & \text{for } 0 \leq i \leq \alpha, \\ 0, & \text{for } \alpha < i < \beta, \\ \sigma_2(i), & \text{for } \beta \leq i \leq n, \quad i \neq n-2, \\ \frac{1}{3}(n^2 + 2n - 6), & \text{for } i = n-2. \end{cases} \tag{6}$$

Then $\sigma(i) \geq b(i)$ for all integers $i \in I_n$.

Next we define a function

$$c(i) = \begin{cases} \dfrac{(\beta - i)b(\alpha) + (i - \alpha)b(\beta)}{\beta - \alpha}, & \text{for } \alpha < i < \beta, \\ b(i), & \text{otherwise,} \end{cases} \tag{7}$$

where $((\beta - i)b(\alpha) + (i - \alpha)b(\beta))/(\beta - \alpha)$ is the linear function which agrees with σ_1 at α and σ_2 at β (see Fig. 1).

It is routine but somewhat tedious to check that c is the concave envelope of b given that $n \geq 13$. Then by Lemma 5, $\prod_{k=0}^{n-1}(\sigma(k+1) - \sigma(k))$ is maximised when $\sigma = c$. So by (4)

$$\max \prod_{k=0}^{n-1} N(k) \leq \prod_{k=0}^{n-1} (c(k+1) - c(k))$$

and it follows from (2) and (7) that

$$\begin{aligned} T(n) &\leq 3\left(\frac{(2\beta-3\alpha)(n+1)+3\alpha^2-\beta^2}{3(\beta-\alpha)}\right)^{\beta-\alpha} \prod_{k=0}^{\alpha-1} (n - 2k) \prod_{\substack{k=\beta \\ k\neq n-2 \\ k\neq n-3}}^{n-1} \tfrac{1}{3}(2n - 2k + 1) \\ &= \tfrac{27}{35}\left(\frac{(2\beta-3\alpha)(n+1)+3\alpha^2-\beta^2}{3(\beta-\alpha)}\right)^{\beta-\alpha} \prod_{k=0}^{\alpha-1} (n - 2k) \prod_{k=\beta}^{n-1} \tfrac{1}{3}(2n - 2k + 1). \end{aligned}$$

□

For $n < 13$ the argument in Theorem 6 needs slight modification because the function c defined by (7) is not concave. For these values, the concave envelope of b is

$$c(i) = \begin{cases} \dfrac{(n - i - 2)\,b(\alpha) + (i - \alpha)\,b(n-2)}{n - \alpha - 2}, & \text{for } \alpha < i < n - 2, \\ b(i), & \text{otherwise.} \end{cases}$$

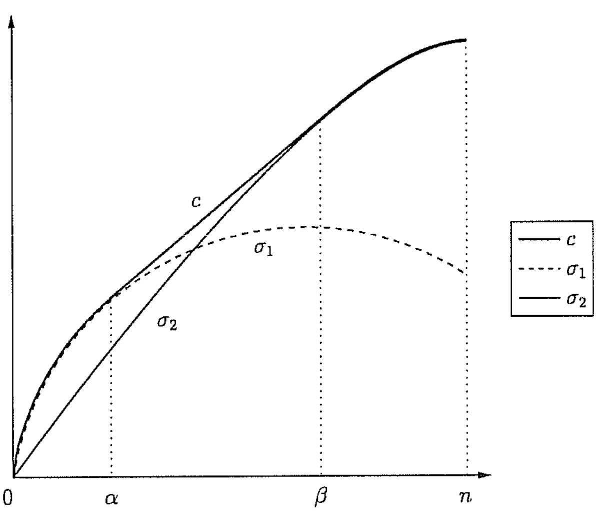

Fig. 1 A graph of the functions σ_1, σ_2 and c

This gives

$$T(1) \le 1,\ T(2) = 0,\ T(3) \le 3,\ T(4) \le 8,\ T(5) \le 25,\ T(6) \le 96,\ T(7) \le 420,$$
$$T(8) \le 2106,\ T(9) \le 12304,\ T(10) \le 75000,\ T(11) \le 528647\ T(12) \le 3965268. \tag{8}$$

We will now prove the upper bound given in Conjecture 3, and in doing so obtain an exponential improvement on the trivial bound $n!$. This requires the following lemma, which can be proved using elementary calculus.

Lemma 7 *If* $\theta(x)$ *is defined by* $1 + x = e^{x-\theta(x)x^2}$ *for* $x > -1$ *with* $\theta(0) = 1/2$, *then* $|\theta(x) - \frac{1}{2}| \le \frac{1}{25}$ *whenever* $|x| \le \frac{1}{10}$.

Theorem 8 *Let* $T(n)$ *be the maximum number of transversals in a Latin square of order* n, *then*

$$T(n) \le c^n \sqrt{n}\, n!$$

for $n \ge 5$ *where* $c = \sqrt{\frac{3-\sqrt{3}}{6}}\, e^{\sqrt{3}/6} \approx 0.61354.$

Proof We can confirm the theorem holds for $n = 5, 6$ by comparing it to experimental values (see Table 2), and by using the bounds on $T(n)$ given in (8) for $n = 7, \ldots, 12$. For $n \ge 13$ we have Theorem 6, which can be restated as

$$T(n) \le \frac{2^{n+\alpha-\beta+1} 3^{\alpha-n+3} (\frac{n}{2})!\, (n - \beta + \frac{1}{2})!\, \left((2\beta - 3\alpha)(n+1) + 3\alpha^2 - \beta^2\right)^{\beta-\alpha}}{35 \sqrt{\pi} (\frac{n}{2} - \alpha)!\, (\beta - \alpha)^{\beta-\alpha}}, \tag{9}$$

where $\alpha = \lfloor \alpha_0 \rfloor$ and $\beta = \lceil \beta_0 \rceil$, with $\beta_0 = \sqrt{3}\alpha_0 = \frac{1}{4}(1 + \sqrt{3})(n + 1)$.

By direct computation of (9) we can verify that the theorem holds for $13 \le n < 85$. For $n \ge 85$ we approximate the bound given in (9) using the following form of Stirling's formula:

$$n! = \sqrt{2\pi n}\, n^n e^{-n} \left(1 + \frac{\phi(n)}{n}\right), \tag{10}$$

where $\frac{1}{12} \le \phi(n) \le \frac{1}{11}$ and $n \ge 1$, including non-integer n.

We define $\epsilon_\alpha, \epsilon_\beta \in [0, 1]$ by $\alpha = \alpha_0 - \epsilon_\alpha$ and $\beta = \beta_0 + \epsilon_\beta$, and apply (10) to (9) to obtain

$$T(n) \le f(n)\, c^n \sqrt{n}\, n!,$$

where

$$c = \sqrt{\frac{3 - \sqrt{3}}{6}}\, e^{\sqrt{3}/6},$$

$$f(n) = \frac{1}{35}\sqrt{\frac{3^7(3-\sqrt{3})}{2\pi}}\, e^{(\sqrt{3}-3)/6+\epsilon_\alpha+\epsilon_\beta}\, x(n)\, y(n),$$

$$\begin{aligned} x(n) = {} & \left(1 - \frac{2\epsilon_\beta(3+\sqrt{3})+\sqrt{3}}{3n}\right)^{n-\beta+1} \\ & \times\left(1 + \frac{3+\sqrt{3}-12\epsilon_\alpha}{n(3-\sqrt{3})-3-\sqrt{3}+12\epsilon_\alpha}\right)^{n/2-\alpha+1/2} \\ & \times\left(1 + \frac{1}{n} - \frac{12(\epsilon_\beta^2-3\epsilon_\alpha^2)}{n(3-\sqrt{3})(\sqrt{3}n+\sqrt{3}+6(\epsilon_\beta+\epsilon_\alpha))}\right)^{\beta-\alpha}, \end{aligned}$$

$$y(n) = \frac{\left(1+\frac{2\phi_1}{n}\right)\left(1+\frac{\phi_2}{n-\beta+1/2}\right)}{\left(1+\frac{\phi_3}{n/2-\alpha}\right)\left(1+\frac{\phi_4}{n}\right)}$$

with $\frac{1}{12} \le \phi_i \le \frac{1}{11}$ for $i = 1, \ldots, 4$ as given in (10).

Applying Lemma 7 to each term of $x(n)$, yields

$$x(n) \le e^{1/2-\epsilon_\beta-\epsilon_\alpha+w(n)/n},$$

where $w(n)$ is a rational function of $\epsilon_\alpha, \epsilon_\beta, \theta_1, \theta_2, \theta_3$ and n, and $\frac{46}{100} \le \theta_i \le \frac{54}{100}$ for $i = 1, \ldots, 3$ whenever $n \ge 85$. Bounding $\epsilon_\alpha, \epsilon_\beta, \theta_1, \theta_2, \theta_3$ and n in the various terms of $w(n)$, we easily obtain $w(n) \le 18$. It follows that

$$\frac{w(n)}{n} \le \frac{18}{85}$$

for $n \ge 85$. Hence $e^{1/2-\epsilon_\beta-\epsilon_\alpha+w(n)/n} \le e^{121/170-\epsilon_\beta-\epsilon_\alpha}$.

We can bound $y(n)$ by noting that

$$y(n) = \frac{\left(1+\frac{2\phi_1}{n}\right)\left(1+\frac{\phi_2}{n-\beta+1/2}\right)}{\left(1+\frac{\phi_3}{n/2-\alpha}\right)\left(1+\frac{\phi_4}{n}\right)} \le \left(1+\frac{2\phi_1}{n}\right)\left(1+\frac{\phi_2}{n-\beta+1/2}\right)$$

for all n. For $n \ge 85$ we have

$$y(n) \le \frac{1003}{1000} \times \frac{1004}{1000} \le \frac{101}{100}.$$

Therefore, when $n \ge 85$,

$$f(n) \le \frac{101}{3500}\sqrt{\frac{3^7(3-\sqrt{3})}{2\pi}}\, e^{\sqrt{3}/6+18/85} \le 1. \qquad \square$$

As a corollary of this result, we can infer that the upper bound in Conjecture 3 is true with $c_2 = 3/4$ (and the value $c_2 = 0.614$ works for large n). This is also an upper bound for the number of solutions to the toroidal n-queens problem.

The following theorem provides an exponential lower bound on the maximum number of transversals in a Latin square of order n.

Theorem 9 $T(n) \geq 15^{n/5}$ *for all* $n \geq 5$.

Proof The examples in Section 4 confirm the result in the range $5 \leq n \leq 19$ (with equality when $n = 5$). So assume that $n \geq 20$ and let $s_1 = s_2 = s_3 = \lfloor \frac{1}{4}n \rfloor$ and $s_4 = n - 3s_1$. By a theorem of Heinrich [13], there exists a Latin square L of order n containing four Latin subsquares S_1, S_2, S_3 and S_4 of respective orders s_1, s_2, s_3 and s_4 such that each row, column and symbol of L is used by exactly one of the S_i. In particular, if we choose one transversal in each of the S_i then their union will be a transversal in L. By replacing, if necessary, S_i with another subsquare on the same symbols and in the same rows and columns, we may assume that S_i has $T(s_i)$ transversals for each $i = 1, \ldots, 4$. Then, by induction on n,

$$T(n) \geq T(s_1)T(s_2)T(s_3)T(s_4) \geq (15^{1/5})^{s_1+s_2+s_3+s_4} = 15^{n/5}. \qquad \square$$

Finding a lower bound of the form given in Conjecture 3 is still an open problem.

Note that [16] establishes exponential lower bounds on the toroidal n-queens problem and hence exponential lower bounds on $T(n)$. However, these bounds only apply when n is a prime such that $(n-1)/2$ is not a prime or when n is divisible by a prime congruent to 1 mod 4. In both cases Theorem 9 provides a better lower bound on $T(n)$.

3 Transversals of finite groups

In this section, we investigate the number of transversals in a finite group G, by which we mean transversals in the (unbordered) Cayley table L_G of G. Consider the following five propositions:

(i) L_G has a transversal.
(ii) L_G can be decomposed into disjoint transversals.
(iii) There exists a Latin square orthogonal to L_G.
(iv) There is some ordering of the elements of G, say $a_1, a_2, \ldots, a_n$, such that $a_1\, a_2 \cdots a_n = \varepsilon$, where ε denotes the identity element of G.
(v) The Sylow 2-subgroups of G are trivial or non-cyclic.

The following relationships are known.

Theorem 10 (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v).

Moreover, it is conjectured that all five statements are equivalent. Hall and Paige, who made this conjecture, showed that it is true for all soluble groups, symmetric groups and alternating groups (see [9, 27] for more details).

An immediate corollary of the proof that (i)$\Leftrightarrow$(ii) is that for any G the number of transversals through a given entry of L_G is independent of the entry chosen and hence that the total number of transversals in G is divisible by the order of G (see Theorem 3.5 of [10]). We also have the following simple results, in the spirit of Theorem 1:

Theorem 11 *The number of transversals in any symmetric Latin square of order* n *is congruent to* n *modulo* 2.

Proof Let S be a symmetric Latin square of odd order n and let T denote the set of transversals in S. The main diagonal of S must be a transversal, as can be seen by noting that each symbol occurs an even number of times in off diagonal positions (in [9, p.31] this argument is credited to Sade). Consider the action μ induced on T by transposition. That is, μ maps each entry (i, j, S_{ij}) of a transversal to (j, i, S_{ji}), and this produces a new transversal since $S_{ij} = S_{ji}$. Note that μ is an involution. Now, if $i \neq j$ then no transversal may include both the entries S_{ij} and S_{ji} since they contain the same symbol. Hence, the only fixed point of μ is the main diagonal of S. Thus μ has an odd number of fixed points so $|T|$ must be odd.

The case when n is even can be resolved by a similar argument to the above or by appealing to Theorem 1. □

Corollary 12 *Let G be a group of order n. If G is abelian or n is even then the number of transversals in G is congruent to n modulo 2.*

Corollary 12 cannot be generalised to non-abelian groups of odd order. We will see in Section 4 that the non-abelian group of order 21 has an even number of transversals.

Theorem 13 *If G is a group of order $n \not\equiv 1 \bmod 3$ then the number of transversals in G is divisible by 3.*

Proof If $n \equiv 0 \bmod 3$ then the result is a trivial consequence of the number of transversals being divisible by n. So we assume that $n \equiv 2 \bmod 3$.

Define a Latin square $C = C(G)$ of order n by $C_{ab} = a^{-1}b^{-1}$, where the rows, columns and symbols of C are indexed by the elements of G. It is routine to check that C is semi-symmetric and isotopic to L_G.

Let ε be the identity element of G and let T be the set of transversals in C containing the entry $(\varepsilon, \varepsilon, \varepsilon)$. Consider the action μ on T induced by (231)-conjugation. Note that since C_n is semisymmetric (231)-conjugation maps transversals to transversals. Also the entry $(\varepsilon, \varepsilon, \varepsilon)$ is the only fixed point of μ since ε is the unique solution of $x = x^{-2}$ in G, given that $n \not\equiv 0 \bmod 3$. All other entries have an orbit of length 3 under μ, so if $n \equiv 2 \bmod 3$ then μ can have no fixed points in T. Thus every element of T has an orbit of length 3, so $|T| \equiv 0 \bmod 3$. The number of transversals in C is $n|T|$. □

We will see below that the cyclic groups of small orders $n \equiv 1 \bmod 3$ have a number of transversals which is not a multiple of three. Also Theorem 13 does not generalise to non-group-based semi-symmetric squares. For example, here is a semi-symmetric square of order 8 which has 16 transversals.

1	2	4	5	3	7	8	6
2	1	5	6	7	8	4	3
5	8	3	1	2	4	6	7
3	7	6	8	1	2	5	4
4	3	1	7	6	5	2	8
8	4	7	3	5	6	1	2
6	5	8	2	4	3	7	1
7	6	2	4	8	1	3	5

Let $z_n = t_n/n$ denote the number of transversals through any given entry of the cyclic square of order n. Since $z_n = 0$ for all even n by Theorem 10, we shall assume for the remainder of this section that n is odd.

The initial values of z_n are known from [14, 22] and Y. P. Shieh (2006, Private Correspondence). They are $z_1 = z_3 = 1$, $z_5 = 3$, $z_7 = 19$, $z_9 = 225$, $z_{11} = 3441$, $z_{13} = 79259$, $z_{15} = 2424195$, $z_{17} = 94471089$, $z_{19} = 4613520889$, $z_{21} = 275148653115$, $z_{23} = 19686730313955$ and $z_{25} = 1664382756757625$. Interestingly, if we take these numbers modulo 8, we find that this sequence begins 1,1,3,3,1,1,3,3,1,1,3,3,1. We know from

Table 1 Transversals in groups of order $n \leq 23$

n	Number of transversals in groups of order n
3	3
4	0, 8
5	15
7	133
8	0, 384, 384, 384, 384
9	2025, 2241
11	37851
12	0, 198144, 76032, 46080, 0
13	1030367
15	36362925
16	0, 235765760, 237010944, 238190592, 244744192, 125599744, 121143296, 123371520, 123895808, 122191872, 121733120, 62881792, 62619648, 62357504
17	1606008513
19	87656896891
20	0, 697292390400, 140866560000, 0, 0
21	5778121715415, 826814671200
23	452794797220965

Theorem 11 that z_n is always odd for odd n, but we will leave as an open question whether there is any deeper pattern modulo 4 or 8. We also know from Theorem 13 that z_n is divisible by 3 when $n \equiv 2 \bmod 3$. The initial terms of $\{z_n \bmod 3\}$ are 1,1,0,1,0,0,2,0,0,1,0,0,2.

We now discuss the number of transversals in groups of small order. When n is congruent to 2 mod 4 we know there can be no transversals by Theorem 10. For each other order $n \leq 23$ the number of transversals in each group is given in Table 1. The groups are ordered according to the catalogue of Thomas and Wood [25]. Bedford and Whitaker [3] offer an explanation for why all the non-cyclic groups of order 8 have 384 transversals. The groups of order 4, 9 and 16 with the most transversals are the elementary abelian groups of those orders. For order 12 the group with the most transversals is $\mathbb{Z}_2 \oplus \mathbb{Z}_6$, and the "best" groups for orders 20 and 21 are $\mathbb{Z}_2 \oplus \mathbb{Z}_{10}$ and $\mathbb{Z}_{21}$, respectively. The numbers of transversals in abelian groups of order at most 16 and cyclic groups of order at most 21 were obtained by Shieh et al. [23]. The remaining values in Table 1 were computed by Shieh [22]. Shieh obtained one different value (for $n = 16$) but agrees that it was incorrect. We did not check the values for cyclic groups of order 21 to 25.

By Corollary 12, we know that in each case covered by Table 1 (except the non-abelian group of order 21), the number of transversals must have the same parity as the order of the square. It is remarkable, though that the groups of even order have a number of transversals which is divisible by a high power of 2. Indeed, we now know that any 2-group of order $n \leq 16$ has a number of transversals which is divisible by 2^{n-1}. It would be very interesting to know if this is true for general n.

4 Latin squares of small order

By exhaustive computation, the transversals were counted in one representative of each of the 115618721533 isotopy classes of Latin square of order 9. Together with information about the autotopy group of each representative, this allows us to calculate statistics for all squares of that order. The squares were generated using the program described in [18]. The counting of transversals was performed twice with code written independently by the first and third authors. Each computation took an aggregate of roughly 2 GHz years.

We found that the expected number of transversals in a random Latin square of order 9 is 4283420654079/20005839187, which is approximately 214. There are only three main classes of order 9 which possess more than 1,000 transversals. They are $\mathbb{Z}_3 \times \mathbb{Z}_3$, which has 2241, $\mathbb{Z}_9$ which has 2025 transversals and third square, L, which has 1620 transversals. Like the two group-based squares, L is composed of nine disjoint order three subsquares and has no intercalates (2×2 Latin subsquares).

The minimum number of transversals for any Latin square of order 9 is 68. In particular, this confirms Conjecture 2 for $n = 9$. The following semi-symmetric square is a representative of the unique main class with the minimum number of transversals. It has 24 intercalates and four subsquares of order 3. The latter are shown in **bold**.

2	**1**	**3**	6	7	8	9	5	4
1	**3**	**2**	5	4	9	**6**	**7**	**8**
3	**2**	**1**	4	9	5	**7**	**8**	**6**
9	5	4	3	2	1	**8**	**6**	**7**
8	4	6	2	5	7	1	9	3
4	**7**	**9**	**8**	3	6	5	1	2
5	**8**	**7**	**9**	6	**2**	**3**	**4**	1
6	**9**	**8**	**7**	1	**4**	**2**	**3**	5
7	6	5	1	8	**3**	**4**	**2**	9

As a by-product of our computation we also counted the intercalates in each Latin square of order 9. We found that the expected number of intercalates in a random Latin square of order 9 is 360629073747/20005839187 (which is approximately 18). This is corroborated by Mckay and Wanless [19], where the same number was calculated by an entirely different method. We also found that the maximum number of intercalates was 72, which is achieved by the following semi-symmetric square (which has 801 transversals).

1	2	3	5	6	4	8	9	7
2	1	4	7	8	9	3	6	5
3	7	1	2	9	8	4	5	6
6	3	7	4	1	5	2	8	9
4	9	8	6	7	1	5	2	3
5	8	9	1	4	7	6	3	2
9	4	2	3	5	6	7	1	8
7	5	6	8	3	2	9	4	1
8	6	5	9	2	3	1	7	4

Table 2 lists the minimum and maximum number of transversals over all Latin squares of order n for $n \leq 9$, and the mean and standard deviation to two decimal places. The columns on either side of $T(n)$ provide the upper and lower bounds given by (8) and Theorem 9, respectively.

In Table 3 we list, for $10 \leq n \leq 21$, the bounds from (8) and Theorems 6 and 9, together with our best guess at the value of $T(n)$. When $n \not\equiv 2 \bmod 4$ we have used the group with the highest number of transversals (see Table 1). It is commonly suspected that $T(10)$ is achieved by one of Parker's turn-squares, which has 5,504 transversals and 12265168 orthogonal mates (see [4, 17]). None of the several billion squares encountered in [18], which included every square with a non-trivial symmetry, had more than 5,504 transversals.

Table 2 Transversals in Latin squares of order $n \leq 9$

n	Min	Mean	Std dev	Lower bound	$T(n)$	Upper bound
2	0	0	0	–	0	0
3	3	3	0	–	3	3
4	0	2	3.46	–	8	8
5	3	4.29	3.71	15	15	25
6	0	6.86	5.19	26	32	96
7	3	20.41	6.00	45	133	420
8	0	61.05	8.66	77	384	2106
9	68	214.11	15.79	131	2241	12304

Table 3 Estimates of $T(n)$ for $10 \leq n \leq 21$

n	Lower bound	$T(n)$ guess	Upper bound
10	225	5504	75000
11	387	37851	528647
12	665	198144	3965268
13	1143	1030367	32837805
14	1964	3477504	300019037
15	3375	36362925	2762962210
16	5801	244744192	28218998328
17	9971	1606008513	300502249052
18	17137	6434611200	3410036886841
19	29455	87656896891	41327486367018
20	50625	697292390400	512073756609248
21	87013	5778121715415	6803898881738477

A turn-square is obtained by starting with the Cayley table of a group (typically a group of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_m$ for some m) and "turning" some of the intercalates (that is, replacing them by the other possible intercalate on the same symbols). A turn-square based on a group $\mathbb{Z}_2 \oplus H$ can be specified by giving the Cayley table for H and marking the entries which will be "turned". For example, the turn-squares defined by

$$\begin{array}{ccc} \mathbf{0} & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathbf{0} & \mathbf{1} & 2 & 3 & 4 \\ \mathbf{1} & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{array}$$

achieve $T(6) = 32$ and 5,504 transversals, respectively. The entries to be turned have been marked in **bold**.

In Table 4, we give data on the number of transversals in all possible turn-squares formed by turning intercalates in the Cayley table of $\mathbb{Z}_{14}$. These turn-squares were classified into main classes and grouped according to their *turn number*, which we define to be the fewest number of intercalates which can be turned to reach that main class, starting from $\mathbb{Z}_{14}$. For each possible turn number t, Table 4 gives the number of main classes with turn number t then the minimum and maximum number of transversals over the set of main classes with turn number t. We found that no such square had more than 3477504 transversals. The following square is from the unique main class achieving this number.

Table 4 Transversals in turn-squares of order 14

Turns	Main classes	Min transversals	Max transversals
0	1	0	0
1	1	1695744	1695744
2	3	2260992	2826240
3	19	2479104	3477504
4	147	2703360	3375104
5	1127	2779136	3216384
6	7721	2652160	3205120
7	41735	2758656	3207168
8	160144	2776064	3145728
9	296010	2801664	3062784
10	105295	2904064	3035136
11	1325	2930688	3035136

0	**1**	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

For order 18 it is hard to test a large number of turn-squares and there are two abelian groups which make promising candidates for the initial square. Thus our "guess" for the value of $T(18)$ is quite likely to be incorrect, but at least it should be of roughly the right order. The given value is achieved by starting with $\mathbb{Z}_2 \oplus \mathbb{Z}_9$ and turning three intercalates in the same pattern as for $n = 10$ and $n = 14$.

Parker's original motivation for studying turn-squares with many transversals was his search for a triple of MOLS of order 10 (the latest evidence [18] suggests that such a triple is highly unlikely to exist). As mentioned above, the turn-square of order 10 with the most transversals has numerous orthogonal mates. However, for order 14 (and possibly for order 18) the turn-square with the most transversals has no orthogonal mates. This can be deduced from a Theorem due to Mann (see Theorem 12.3.2 in [9]), which implies for odd q that a turn-square has no mate if it was formed from $\mathbb{Z}_2 \oplus \mathbb{Z}_q$ by turning no more than $(q-1)/2$ intercalates.

Mann's theorem implies that for $n = 14$ the 24 main classes of turn-squares with turn number at most 3 have no orthogonal mates. With the aid of a randomised hill-climbing algorithm we established that the 613504 main classes with turn number at least 4 all have mates. No pair of MOLS that we found during this search possessed more than 26 common transversals or more than six disjoint common transversals. In particular, we did not find a pair of MOLS that could be extended to a triple.

5 Concluding remarks

Many questions remain in addition to Conjectures 2 and 3. For a given n, which square of order n achieves the most transversals? Is it an abelian group table (and if so, which one?) when $n \not\equiv 2 \bmod 4$ and a turn-square otherwise? Do 2-groups of order n have a number of transversals which is divisible by 2^{n-1}? Is there a pattern to z_n mod 8 (see Section 3)?

It seems likely that neither Theorem 8 nor Theorem 9 is near the true value of $T(n)$, leaving room for much further improvement.

Note added in proof

We have found that upper and lower bounds on $T(n)$ and estimates of its growth rate have been studied in [6, 7, 8, 16]. These results prove the upper bound in Vardi's conjecture (with a worse constant than ours), but only apply to cyclic Latin squares.

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