On Harnack inequalities and singularities of admissible metrics in the Yamabe problem

Neil S. Trudinger · Xu-Jia Wang

Received: 12 April 2007 / Accepted: 11 September 2008 / Published online: 2 December 2008
© Springer-Verlag 2008

Abstract In this paper we study the local behaviour of admissible metrics in the k -Yamabe problem on compact Riemannian manifolds (M, g_0) of dimension $n \geq 3$. For $n/2 < k < n$, we prove a sharp Harnack inequality for admissible metrics when (M, g_0) is not conformally equivalent to the unit sphere S^n and that the set of all such metrics is compact. When (M, g_0) is the unit sphere we prove there is a unique admissible metric with singularity. As a consequence we prove an existence theorem for equations of Yamabe type, thereby recovering as a special case, a recent result of Gursky and Viaclovsky on the solvability of the k -Yamabe problem for $k > n/2$.

Mathematics Subject Classification (2000) 53C21 · 58J05

1 Introduction

Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and $[g_0]$ the set of metrics conformal to g_0 . For $g \in [g_0]$ we denote by

$$A_g = \frac{1}{n-2} \left(\text{Ric}_g - \frac{R_g}{2(n-1)} g \right) \quad (1.1)$$

the Schouten tensor and by $\lambda(A_g) = (\lambda_1, \dots, \lambda_n)$ the eigenvalues of A_g with respect to g (so one can also write $\lambda = \lambda(g^{-1}A_g)$), where Ric and R are respectively the Ricci tensor and the scalar curvature. We also denote as usual

This work was supported by the Australian Research Council.

N. S. Trudinger (✉) · X.-J. Wang
Centre for Mathematics and its Applications, Australian National University,
Canberra, ACT 0200, Australia
e-mail: neil.trudinger@maths.anu.edu.au

X.-J. Wang
e-mail: wang@maths.anu.edu.au

$$\sigma_k(\lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \quad (1.2)$$

the k th elementary symmetric polynomial and

$$\Gamma_k = \{\lambda \in \mathbf{R}^n \mid \sigma_j(\lambda) > 0 \text{ for } j = 1, \dots, k\} \quad (1.3)$$

the corresponding open, convex cone in $\mathbf{R}^n$. Denote

$$[g_0]_k = \{g \in [g_0] \mid \lambda(A_g) \in \bar{\Gamma}_k\}. \quad (1.4)$$

Following [3], we call a metric in $[g_0]_k$ k -admissible. In this paper we prove three main theorems pertaining to the cases $k > \frac{n}{2}$.

Theorem A *If $(\mathcal{M}, g_0)$ is not conformally equivalent to the unit sphere S^n and $\frac{n}{2} < k \leq n$, then $[g_0]_k$ is compact in C^0 and satisfies the following Harnack inequality, namely for any $g = \chi g_0 \in [g_0]_k$,*

$$\max_{x, y \in \mathcal{M}} \frac{\chi(x)}{\chi(y)} \leq \exp \left(C |x - y|^{2 - \frac{n}{k}} \right) \quad (1.5)$$

for some fixed constant C depending only on n, k and $(\mathcal{M}, g_0)$, where $|x - y|$ denotes the geodesic distance in the metric g_0 between x and y .

By compactness we mean in the sense of restriction to conformal metrics with fixed volume. Equivalently the compactness can be understood as the set of all conformal factors χ with $\sup \chi = 1$. See also (3.28) below. When the manifold $(\mathcal{M}, g_0)$ is the unit sphere, the compactness is no longer true. In this case $(\mathcal{M}, g_0)$ with one point removed is conformally equivalent to the Euclidean space $\mathbf{R}^n$ so that without loss of generality, it suffices to study conformal metrics on $\mathbf{R}^n$. For our investigation we will allow singular metrics. Accordingly we call a metric $g = \chi g_0$ k -admissible if $\chi : \mathcal{M} \rightarrow (0, \infty]$, χ is lower semi-continuous, $\not\equiv \infty$ and there exists a sequence of k -admissible metrics $g_m = \chi_m g_0$, $\chi_m \in C^2(\mathcal{M})$, such that $\chi_m \rightarrow \chi$ almost everywhere in $\mathcal{M}$. If g is k -admissible, then the function $v = \chi^{(n-2)/4}$ is subharmonic with respect to the operator

$$\square := -\Delta_g + \frac{n-2}{4(n-1)} R_g \quad (1.6)$$

and hence by the weak Harnack inequality [11, 25], the set $\{\chi = \infty\}$ has measure zero. Our next result classifies the possible singularities of k -admissible metrics on $\mathbf{R}^n$.

Theorem B *Let g be k -admissible on $\mathbf{R}^n$ with $\frac{n}{2} < k \leq n$. Then either*

$$g(x) = \frac{C}{|x - x_0|^4} g_0(x) \quad (1.7)$$

for some point $x_0 \in \mathbf{R}^n$ and positive constant C , or the conformal factor χ is Hölder continuous with exponent $\alpha = 2 - \frac{n}{k}$, where g_0 is the standard metric on $\mathbf{R}^n$.

Remark Theorems A and B also extend to metrics where the condition $g \in [g_0]_k$ (namely $\lambda(A_g) \in \Gamma_k$) is replaced by $P_\delta(\lambda) > 0$, for some constant $\delta < 1/(n-2)$, where P_δ is the Pucci operator [11], given by,

$$P_\delta(\lambda) = \min \lambda_i + \delta \sum \lambda_i.$$

By (1.1), the Ricci curvature $\mu = (\mu_1, \dots, \mu_n)$ is given by

$$\mu_i = (n-2)\lambda_i + \sum \lambda_j.$$

Hence the Ricci curvature $\mu_i \geq 0$ for all $1 \leq i \leq n$ if and only if $P_{\frac{1}{n-2}}(\lambda) \geq 0$. We remark that when $\lambda \in \Gamma_k$ and $k \geq \frac{n}{2}$, the Ricci curvature is nonnegative [15]. Indeed, for any $\lambda \in \Gamma_k$, let $f(x) = \frac{1}{2} \sum \lambda_k x_k^2$. By direct computation, $\Delta_p f \geq 0$ for $p \leq 2 + \frac{n(k-1)}{n-k}$ (this is the simplest case, namely $l = 1$, in Lemma 4.2 in [27]). Hence

$$\sum \lambda_i + \frac{n(k-1)}{n-k} \lambda_j \geq 0 \quad (1.8)$$

for every j . Hence $\text{Ric}_g \geq 0$ if $\lambda(A_g) \in \Gamma_{n/2}$, and $\text{Ric}_g > 0$ if $\lambda(A_g) \in \Gamma_k$ for $k > \frac{n}{2}$.

Theorems A and B have various interesting consequences, in particular to the existence of conformal metrics with prescribed k -curvature. This is the problem of finding a conformal metric $g \in [g_0]$ such that

$$\sigma_k(\lambda(A_g)) = f, \quad (1.9)$$

where f is a given positive smooth function on $\mathcal{M}$. When $f \equiv 1$, (1.9) is the k -Yamabe problem, which has been studied by many authors; see [1, 22, 26] for $k = 1$ and [5, 6, 10, 14, 18, 20, 21, 24, 29] for $k \geq 2$. For $k > \frac{n}{2}$, it was recently resolved by Gursky and Viaclovski [18], by proving compactness results and characterizations of singularities for solutions. Our Theorems A and B above show that such results are already valid for admissible metrics, that is those in the conformal class $[g_0]$ in which solutions are sought. As well, Theorem A provides a sharp version of the solution estimates in [18].

By writing $g = v^{4/(n-2)} g_0$, we see that (1.9) is equivalent to the *conformal k -Hessian equation*

$$\sigma_k(\lambda(V)) = \varphi(x, v), \quad (1.10)$$

where

$$V = -\nabla^2 v + \frac{n}{n-2} \frac{\nabla v \otimes \nabla v}{v} - \frac{1}{n-2} \frac{|\nabla v|^2}{v} g_0 + \frac{n-2}{2} v A_{g_0}, \quad (1.11)$$

$\lambda(V)$ denotes the eigenvalues of the matrix V , and $\varphi = f v^{\frac{n+2}{n-2}}$. When $k \geq 2$, (1.10) is a fully nonlinear partial differential equation, which is elliptic if the eigenvalues $\lambda(A_g) \in \Gamma_k$. Therefore to study problem (1.9), we always assume $[g_0]_k \neq \emptyset$. Under this assumption, the k -Yamabe problem has been solved in [24] if $2 \leq k \leq \frac{n}{2}$ and (1.9) is variational. Equation (1.9) is automatically variational when $k = 2$, but when $k \geq 3$, it is variational when the manifold is locally conformally flat. When $\frac{n}{2} < k \leq n$, the existence of solutions to (1.9) was proved in [18] for any smooth, positive function f ; see also [6] for the solvability when $k = 2$ and $n = 4$, and [14, 20] when the manifold is locally conformally flat. For completeness and to illustrate the application of Theorem A, we prove the following extension.

Theorem C *Let $(\mathcal{M}, g_0)$ be a compact n -manifold not conformally equivalent to the unit sphere S^n . Suppose $\frac{n}{2} < k \leq n$ and $[g_0]_k \neq \emptyset$. Then for any smooth, positive function f and any constant $p \neq k$, there exists a positive admissible solution to the equation*

$$\sigma_k(\lambda(V)) = f(x) v^p. \quad (1.12)$$

The solution is unique if $p < k$. When $p = k$, then there exists a unique constant $\theta > 0$ such that

$$\sigma_k(\lambda(V)) = \theta f(x)v^k \quad (1.13)$$

has a solution, which is unique up to a constant multiplication.

We may call the constant θ in (1.13) (with $f \equiv 1$) the *eigenvalue* of the conformal k -Hessian operator in (1.10). As a special case of Theorem C, letting $p = k\frac{n+2}{n-2}$, we obtain the existence of solutions to the k -Yamabe problem (1.9) for $\frac{n}{2} < k \leq n$, as proved in [18]. As indicated above, the full strength of Theorem A is not necessary for these applications and solution estimates as in [18] would suffice. We also include some extensions of Theorem C at the end of Sect. 4.

As in [24] we will use conformal transformations of different forms,

$$g = \chi g_0 = v^{\frac{4}{n-2}} g_0 = u^{-2} g_0 = e^{-2w} g_0 \quad (1.14)$$

so that

$$u = v^{-2/(n-2)} = e^w. \quad (1.15)$$

We say u , v , or w is *conformally k -admissible*, or simply *k -admissible* if no confusion arises, if the metric g is k -admissible. In the smooth case, from the matrix V in (1.11), we see that u , w are k -admissible if the eigenvalues of the matrices

$$U = \left\{ u_{ij} - \frac{|Du|^2}{2u} g_0 + u A_{g_0} \right\}, \quad (1.16)$$

$$W = \left\{ w_{ij} + w_i w_j - \frac{1}{2} |Dw|^2 g_0 + A_{g_0} \right\} \quad (1.17)$$

lie in $\overline{\Gamma}_k$, the closure of Γ_k . Note that if g is the metric given by (1.7), then

$$v = \frac{C}{|x - x_0|^{n-2}} \quad (1.18)$$

is the fundamental solution of the Laplace operator.

The conformal k -Hessian equation is closely related to the k -Hessian equation

$$\sigma_k(\lambda(D^2 u)) = \varphi \quad \text{in } \Omega, \quad (1.19)$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain. For the k -Hessian equation (1.19), it is proved in [27] that when $\frac{n}{2} < k \leq n$, a k -admissible function (relative to (1.19)) is locally Hölder continuous with Hölder exponent $\alpha = 2 - \frac{n}{k}$. The existence of solutions to (1.19) with right hand side $\varphi = f(x)|u|^p$ for some constant $p > 0$ was studied in [9] for $k \leq \frac{n}{2}$ and in [8, 30] for $k = n$. By the Hölder continuity one can extend the results in [8, 30] to the cases $\frac{n}{2} < k \leq n$. The argument in [30] uses a degree theory, which does not require a variational structure. We will employ the same degree argument to prove our Theorem C.

The proof of Theorem A can roughly be divided into two steps. In the first one we study the singularity behavior of admissible functions. Let $g_m = v_m^{\frac{4}{n-2}} g_0$ be a sequence of admissible metrics in $[g_0]_k$ for $k > \frac{n}{2}$. Then $v_m / \inf v_m$ converges in locally uniformly to an admissible function v . Moreover, if x_0 is a singular point of v , we prove that

$$v(x) = \frac{C_0 + o(1)}{|x - x_0|^{n-2}}, \quad (1.20)$$

where C_0 is a positive constant, $|x - x_0|$ denotes the geodesic distance from x to x_0 in the metric g_0 . In the second step we show that if (1.20) holds at all singular points, then by Bishop's volume growth formula, as used in [18], w is smooth away from x_0 , and the manifold $(\mathcal{M}, g_0)$ is conformally equivalent to the unit sphere.

The asymptotic behavior (1.20) has been established for the blow-ups of solutions to the classical Yamabe problem. Also a weak form of (1.20) was proved by Gursky and Viaclovsky [18], namely if $g_m = v_m^{\frac{4}{n-2}} g_0$ is a sequence of solutions to (1.5) for $k > \frac{n}{2}$, then $v_m / \inf v_m$ converges locally uniformly away from finitely many singular points to an admissible function v , and v satisfies

$$\frac{C_1}{|x - x_0|^{n-2}} \leq v(x) \leq \frac{C_2}{|x - x_0|^{n-2}} \quad (1.21)$$

at any singular point x_0 . From (1.21) they proved that $(\mathcal{M}, g_0)$ is conformally equivalent to the unit sphere. Basically our proof in the second step is inspired by that in [18], however our proof is somewhat simpler due to the stronger asymptotic estimate (1.20).

In this paper we also introduced a *new* technique. That is we reduce the singularity analysis of a k -admissible function to its *maximal radial function*. Given an admissible function v , we say $\tilde{v}$ is the maximal radial function of v with center at x_0 if $\tilde{v}$ is a rotationally symmetric with respect to x_0 , $\tilde{v} \leq v$, and $\tilde{v} \geq \varphi$ for any radial function φ satisfying $\varphi \leq v$. Obviously $\tilde{v}$, as a function of $|x - x_0|$, is monotone increasing, namely $\tilde{v}(r) \geq \tilde{v}(r')$ for any $0 < r' < r$. We reduce the proof of (1.20) for v to that for $\tilde{v}$. This new technique also applies to other nonlinear elliptic equations, which will be shown in a subsequent paper [28].

This paper is arranged as follows. In Sect. 2 we first prove Theorem B for radially symmetric, k -admissible functions defined on $\mathbf{R}^n$, then extend it to general k -admissible functions by considering the corresponding minimal radial function for $w = -\frac{2}{n-2} \log v$. In Sect. 3 we prove Theorem A. By studying the corresponding minimal radial function, we show that if an admissible function w is a k -admissible function on a manifold $\mathcal{M}$, then either w is Hölder continuous, or (1.20) holds at some singular point $x_0 \in \mathcal{M}$. We then show that if (1.20) occurs, then w is smooth away from x_0 and the manifold $\mathcal{M}$ is conformally equivalent to the unit sphere. In Sect. 4 we prove Theorem C by a degree argument.

In a subsequent paper [28], we will establish the asymptotic behavior (1.20) for any blow-up functions of solutions to the k -Yamabe problem (for $1 \leq k \leq \frac{n}{2}$) and the problem of prescribing more general symmetric curvature functions. In particular we prove the existence and compactness of solutions to the k -Yamabe problem for $k = \frac{n}{2}$.

2 Proof of Theorem B

2.1 Radial functions

We first demonstrate our idea by considering radially symmetric functions. Let w be a radially symmetric, k -admissible function on $\mathbf{R}^n \setminus \{0\}$. For any given point $x \neq 0$, by a rotation of axes we assume $x = (0, \dots, 0, r)$. Regard w as a function of $r = |x|$, $r \in (0, \infty)$. Then the matrix W in (1.17) is diagonal,

$$W = \text{diag} \left(\frac{1}{r} w' - \frac{1}{2} w'^2, \dots, \frac{1}{r} w' - \frac{1}{2} w'^2, w'' + \frac{1}{2} w'^2 \right).$$

Denote $a = w'' + \frac{1}{2}w'^2$ and $b = \frac{1}{r}w' - \frac{1}{2}w'^2$. We have

$$\begin{aligned}\sigma_k(\lambda(W)) &= b^k C_{n-1}^k + ab^{k-1} C_{n-1}^{k-1} \\ &= C_{n-1}^{k-1} b^{k-1} \left(a + \frac{n-k}{k} b \right).\end{aligned}\quad (2.1)$$

Since $\lambda(W) \in \bar{\Gamma}_k$ and $k > \frac{n}{2}$,

$$b = \frac{w'}{r} - \frac{1}{2}w'^2 \geq 0, \quad (2.2)$$

$$a + \frac{n-k}{k}b = \left(w'' + \frac{w'}{r} \right) - (1-\theta) \left(\frac{w'}{r} - \frac{1}{2}w'^2 \right) \geq 0, \quad (2.3)$$

where $\theta = \frac{n-k}{k} < 1$. It follows that

$$0 \leq w' \leq \frac{2}{r}, \quad (2.4)$$

$$w'' + \frac{w'}{r} \geq 0. \quad (2.5)$$

By sub-harmonic property of w , we have $w'(r) = 0$ for $r \in (0, r_0)$ if $w'(r_0) = 0$. Note that (2.5) can also be written as $(rw')' \geq 0$. Therefore we have

Lemma 2.1 *The function rw' is nonnegative, monotone increasing, and $rw' \leq 2$.*

It follows that w must be locally uniformly bounded from above, namely if $w(r_0) \leq 0$, then $w \leq C$ for some constant C depending only on r_0 . Next we prove

Lemma 2.2 *The function w is either Hölder continuous in $\mathbf{R}^n$ with exponent $\alpha = 2 - \frac{n}{k}$, or*

$$w(r) = 2 \log r + C \quad (2.6)$$

for some constant C .

Proof It suffices to prove that w is either Hölder continuous near $r = 0$ or (2.6) holds. If w is not Hölder continuous near $r = 0$, then $w' > 0$ as $v = e^{\frac{n-2}{2}w}$ is superharmonic with respect to the operator (1.6).

If $rw' \not\equiv 2$, then by Lemma 2.1, $\lim_{r \rightarrow 0} rw' = c_0 < 2$. For any $c_1 \in (c_0, 2)$,

$$w'' + \frac{w'}{r} \geq (1-\theta) \frac{w'}{r} \left(1 - \frac{1}{2}rw' \right) \geq (1-\theta) \left(1 - \frac{c_1}{2} \right) \frac{w'}{r} \quad (2.7)$$

if r is sufficiently small. Hence

$$\frac{w''}{w'} + \frac{\sigma}{r} \geq 0,$$

where $\sigma = 1 - (1-\theta)(1 - \frac{c_1}{2}) < 1$. We obtain

$$\log(w'r^\sigma) \Big|_r^{r_0} \geq 0.$$

Hence

$$w' \leq \frac{C}{r^\sigma}. \quad (2.8)$$

Hence w is bounded and continuous.

To show that w is Hölder continuous with Hölder exponent $\alpha = 2 - \frac{n}{k}$, by Lemma 2.1 it suffices to prove it at $r = 0$. Note that

$$a + \theta b = w'' + \theta \frac{w'}{r} + \frac{1 - \theta}{2} w'^2 \geq 0.$$

Hence

$$\frac{w''}{w'} + \frac{\theta}{r} \geq -\frac{1 - \theta}{2} w'.$$

Taking integration from r to r_0 , we obtain

$$\log(w' r^\theta) \Big|_{r_0}^{r_0} \geq C.$$

Hence

$$w' \leq \frac{C}{r^\theta}, \quad (2.9)$$

so that w is Hölder continuous with exponent $1 - \theta = 2 - \frac{n}{k}$. $\square$

Remark 2.1 The Hölder continuity also follows from [27]. Let $u = e^w$ be as in (1.15). From the matrix U in (1.16), one sees that u is k -admissible with respect to the k -Hessian operator $\sigma_k(\lambda(D^2u))$. To show that u is Hölder continuous with exponent $\alpha = 2 - \frac{n}{k}$, one applies the comparison principle to u and $\varphi(x) = C|x - y|^{2-n/k} + u(y)$ (relative to the k -Hessian equation), where C is chosen such that $u \leq \varphi$ on $\partial B_R(y)$, one sees that $u(x) - u(y) \leq \varphi(x)$ in $B_R(y)$.

It follows that for any constant $c > 0$, $w_c = \max(w, -c)$ is also Hölder continuous with exponent $2 - \frac{n}{k}$. In particular, if w_m converges to w a.e., then w_m converges to w uniformly in $\{w > -c\}$ for any $c > 0$.

2.2 Proof of Theorem B

Let w be a k -admissible function. For any $h \in \mathbf{R}$, denote $\Omega_h = \{w < h\}$. Since w is upper semi-continuous, Ω_h is an open set. For any given point 0, we define a function $\tilde{w}$ of one variable r by

$$\tilde{w}(r) = \inf\{h : \text{dist}(0, \partial\Omega_h) > r\}. \quad (2.10)$$

We call $\tilde{w}$ the *minimal radial function* of w (with respect to 0). It is a function of $|x|$, satisfying $\tilde{w} \geq w$ and $\tilde{w} \leq \varphi$ for any radial function φ satisfying $\varphi \geq w$. Obviously $\tilde{w}$ is monotone increasing, namely $\tilde{w}(r) \geq \tilde{w}(r')$ for any $0 < r' < r$.

Let $x_h \in \partial\Omega_h$ such that $|x_h| = r_h := \text{dist}(0, \partial\Omega_h)$. Assume that $\partial\Omega_h$ and w are smooth at x_h . Rotate the axes such that $x_h = (0, \dots, 0, r_h)$. Then the x_n -axis is the outer normal of $\partial\Omega_h$ at x_h . Hence

$$\begin{aligned} \tilde{w}(r_h) &= w(x_h), \\ \tilde{w}(r_h + t) &\geq w(x_h + t e_n) \end{aligned} \quad (2.11)$$

for t near 0, where $e_n = (0, \dots, 0, 1)$. We obtain

$$\begin{aligned} \tilde{w}'(r_h) &= w_n(x_h) = |Dw|(x_h), \\ \tilde{w}''(r_h) &\geq w_{nn}(x_h) \end{aligned} \quad (2.12)$$

provided $\tilde{w}$ is twice differentiable point at r_h .

Let $\kappa_1, \dots, \kappa_{n-1}$ be the principal curvatures of $\partial\Omega_h$ at x_h . Then, after a rotation of the axes $(x_1, \dots, x_{n-1})$,

$$w_{ij} = |Dw| \kappa_i \delta_{ij} \quad i, j \leq n-1. \quad (2.13)$$

By our choice of x_h , we have

$$\kappa_i \leq \frac{1}{r}, \quad (2.14)$$

where $r = r_h$. Hence the matrix

$$(w_{ij})_{i,j=1}^{n-1} \leq \frac{1}{r} |Dw| I. \quad (2.15)$$

At x_h , the matrix W is given by

$$\begin{aligned} W &= \left\{ w_{ij} + w_i w_j - \frac{1}{2} |Dw|^2 I \right\} \\ &= \begin{pmatrix} w_{11} - \frac{1}{2} |Dw|^2, & 0, & \dots, & w_{1n} \\ 0, & w_{22} - \frac{1}{2} |Dw|^2, & \dots, & w_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ w_{1n}, & w_{2n}, & \dots, & w_{nn} + \frac{1}{2} |Dw|^2 \end{pmatrix} \end{aligned}$$

Let

$$\hat{W} = \text{diag} \left(w_{11} - \frac{1}{2} |Dw|^2, w_{22} - \frac{1}{2} |Dw|^2, \dots, w_{nn} + \frac{1}{2} |Dw|^2 \right) \quad (2.16)$$

be a diagonal matrix. By Lemma 2.3 below, the eigenvalues $\lambda(\hat{W}) \in \bar{\Gamma}_k$.

Lemma 2.3 *Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with eigenvalues $\lambda(A) \in \Gamma_k$. Then $\lambda(\hat{A}) \in \Gamma_k$, where $\hat{A} = \begin{pmatrix} A' & 0 \\ 0 & a_{nn} \end{pmatrix}$, $A' = (a_{ij})_{1 \leq i, j \leq n-1}$.*

In particular if $\lambda(A) \in \Gamma_k$, then the vector $(a_{11}, \dots, a_{nn}) \in \Gamma_k$.

Proof It suffices to prove $\sigma_j(\lambda(\hat{A})) \geq 0$ for all $j = 1, \dots, k$. Let us prove it for $j = k$. Recalling that $\sigma_k(\lambda(A))$ is the sum of all principal $k \times k$ minors, we have

$$\sigma_k(\lambda(A)) = \sigma_k(\lambda(\hat{A})) - \sum_{i < n} \sigma_{k-2}(\lambda(A_{|in})) a_{in}^2, \quad (2.17)$$

where $A_{|ij}$ denotes the matrix obtained by canceling the i th row and j th column of A . Since $\lambda(A) \in \bar{\Gamma}_k$, we have

$$\sigma_{k-2}(\lambda(A_{|in})) = \frac{\partial^2 \sigma_k(\lambda(A))}{\partial a_{ii} \partial a_{nn}} \geq 0. \quad (2.18)$$

Hence $\sigma_k(\lambda(\hat{A})) \geq \sigma_k(\lambda(A)) \geq 0$. □

From (2.15),

$$\hat{W} \leq \text{diag} \left(\frac{1}{r} \tilde{w}' - \frac{1}{2} (\tilde{w}')^2, \dots, \frac{1}{r} \tilde{w}' - \frac{1}{2} (\tilde{w}')^2, \tilde{w}'' + \frac{1}{2} (\tilde{w}')^2 \right). \quad (2.19)$$

Therefore as in Sect. 2.1, we see that $\tilde{w}$ satisfies

$$\frac{\tilde{w}'}{r} - \frac{1}{2}(\tilde{w}')^2 \geq 0 \quad (2.20)$$

$$\left(\tilde{w}'' + \frac{\tilde{w}'}{r}\right) - (1 - \theta) \left(\frac{\tilde{w}'}{r} - \frac{1}{2}(\tilde{w}')^2\right)^2 \geq 0 \quad (2.21)$$

if $\tilde{w}$ is twice differentiable at r .

To proceed further we make a remark.

Remark 2.2 Inequalities (2.20) and (2.21) are exactly (2.2) and (2.3). In the present case, the function $\tilde{w}$ may not be smooth. But by definition, w can be approximated by smooth k -admissible functions. Hence $\tilde{w}$ can be approximated by piecewise smooth functions, which satisfy $\lim_{r \rightarrow r_0^-} \tilde{w}'(r) \leq \lim_{r \rightarrow r_0^+} \tilde{w}'(r)$ by (2.11).

For a piecewise smooth function w with nonsmooth points $r_1 > r_2 > \dots > r_j > \dots$ at which $\lim_{r \rightarrow r_j^-} \tilde{w}'(r) < \lim_{r \rightarrow r_j^+} \tilde{w}'(r)$, we can mollify w at the points r_j to get a smooth function w^* which satisfies $\lim_{r \rightarrow 0} w^*(r) = \lim_{r \rightarrow 0} w(r)$, and also satisfies (2.20) and (2.21). The proof of Lemma 2.2 implies that if $\lim_{r \rightarrow 0} w^*(r) = -\infty$, then $r(w^*)' \equiv 2$; if $\lim_{r \rightarrow 0} w^*(r) > -\infty$, then w^* is Hölder continuous with its Hölder norm independent of the mollification.

We can now prove Theorem B easily. First we consider the case when w is unbounded from below.

Lemma 2.4 *Let w be a k -admissible function which is locally unbounded from below, then there exists a point $x_0 \in \mathbf{R}^n$ and a constant C such that*

$$w(x) \equiv 2 \log |x - x_0| + C. \quad (2.22)$$

Proof If w is locally unbounded from below, the singular set $S = \bigcap_{\{c < 0\}} \{w < c\}$ is not empty. Choose a point $0 \in S$. By (2.20) and (2.21), and from the argument in Sect. 2.1, we must have $\tilde{w}(r) = 2 \log r + C$ for some constant C .

Let $\hat{w} = 2 \log |x| + C$. Then

$$\begin{aligned} \sigma_1(\lambda(W_{\hat{w}})) &= 0, \\ \sigma_1(\lambda(W_w)) &\geq \sigma_k^{1/k}(\lambda(W_w)) \geq 0, \end{aligned}$$

where $W_{\hat{w}}$ is the matrix corresponding to $\hat{w}$, given in (1.17). Note that $\sigma_1(\lambda(W))$ is indeed the Laplace operator in $\mathbf{R}^n$ (in terms of v , by the relation (1.15)). By our definition of $\tilde{w}$, we have $\hat{w} \geq w$. Since $\tilde{w} = 2 \log r + C$, we see that $w - \hat{w}$ attains its local maximum 0 at some interior point. By the maximum principle for the Laplace equation, we conclude that $w \equiv \hat{w}$. $\square$

Next we consider the case when w is bounded from below.

Lemma 2.5 *Let w be a k -admissible function w . Suppose w is bounded from below. Then w is Hölder continuous with exponent $\alpha = 2 - \frac{n}{k}$.*

Proof For any given point x_0 , we may take x_0 as the origin and let $\tilde{w}$ be the minimal radial function of w with respect to x_0 , as defined in (2.10). Then to prove that w is Hölder continuous at x_0 with exponent $\alpha = 2 - \frac{n}{k}$, it suffices to show that $\tilde{w}$ is Hölder continuous with exponent α . But by (2.20), (2.21), the Hölder continuity of $\tilde{w}$ readily follows from the argument in Sect. 2.1, see (2.9). $\square$

The Hölder continuity also follows from Remark 2.1 above.

Note that the function $w = 2 \log |x|$ is k -admissible. By truncating at $w = -K$ (for large K), we see that the set of Hölder continuous k -admissible functions is not compact.

2.3 Applications

First we remark that, by the above proof, Theorem B also holds for k -admissible functions defined on a domain. Here we restate the theorem for the function $v = e^{-\frac{n-2}{2}w}$. Note that by Lemma 2.1, w is locally uniformly bounded, and so v is locally strictly positive when $k > \frac{n}{2}$.

Theorem B' *Let Ω be a domain in $\mathbf{R}^n$. Let v be a k -admissible function in Ω with $\frac{n}{2} < k \leq n$. If v is unbounded from above near some point $x_0 \in \Omega$, then*

$$v(x) = C|x - x_0|^{2-n}. \quad (2.23)$$

Otherwise v is locally Hölder continuous in Ω with exponent $\alpha = 2 - \frac{n}{k}$.

It was proved in [20] that if v is a k -admissible function, so is the function v_ψ in a punctured small ball $B_r(0) \setminus \{0\}$, where

$$v_\psi = |J_\psi|^{\frac{n-2}{2n}} v \circ \psi \quad (2.24)$$

$\psi(x) = \frac{x}{|x|^2}$, and J_ψ is the Jacobian of the mapping ψ . From Theorem B' we have

Corollary 2.5 *Let v be a k -admissible function defined in $\mathbf{R}^n \setminus B_1(0)$ with $\frac{n}{2} < k \leq n$. Then either $v \equiv \text{constant}$ or $|x|^{n-2}v(x)$ converges to a positive constant as $x \rightarrow \infty$.*

Proof We cannot apply Theorem B' directly, as the function v_ψ has a singular point at 0. Denote $w = \frac{-2}{n-2} \log v_\psi$. If $w(x) \rightarrow -\infty$ as $x \rightarrow 0$, the argument in Sect. 2.2 implies that $w = 2 \log |x| + C$. Changing back to v , we obtain $v \equiv \text{constant}$. Otherwise it suffices to show that w is continuous at 0.

Let $w(0) = \lim_{x \rightarrow 0} w(x)$ so that w is upper semi-continuous. If $a := \lim_{x \rightarrow 0} w(x) < w(0)$, by subtracting a constant we assume $w(0) = 0$. Let $x_m \rightarrow 0$ such that $w(x_m) = -\frac{a}{2}$. Let $\tilde{w} = \tilde{w}_{x_m}$ be the minimal radial function of w with respect to x_m , as defined in (2.10). We claim that when m is sufficiently large, the point x_h in (2.11) at $h = 0$ cannot be the origin. In other words, there is a point y_m with $w(y_m) = 0$ such that $|y_m - x_m| < |x_m|$. Indeed, if $x_h = 0$, by the Hölder continuity of w in $B_{r_m}(x_m)$, where $r_m = |x_m|$ (Remark 2.1), we see that $w(x) \leq -\frac{a}{2}$ in $B_{\delta r_m}(x_m)$, for some $\delta > 0$ independent of m . Namely $v > e^{\frac{(n-2)a}{4}} > 1$ in $B_{\delta r_m}(x_m)$, where $v = v_\psi$. Recall that $\lim_{x \rightarrow 0} v(x) \geq 1$, by the mean value inequality for super-harmonic function, we have

$$v(0) \geq \frac{1}{|B_{2r_m}|(0)} \int_{B_{2r_m}(0)} v(x) dx > 0.$$

We reach a contradiction.

When the singular point $x = 0$ is not the extreme point x_h in (2.11), the argument in Sect. 2.2 applies to $\tilde{w} = \tilde{w}_{x_m}$ in $B_1(0)$. In particular $\tilde{w}$ is uniformly Hölder continuous. Hence if $w(0) = 0$ and $w(x_m) \leq -1$, we have $|x_m| \geq c_0 > 0$ for some c_0 independent of m . This is again a contradiction. Hence w is continuous at 0, and so $|x|^{n-2}v(x)$ converges to a positive constant as $x \rightarrow \infty$. $\square$

Theorem B also implies the non-existence of solutions to the Dirichlet problem in general. Let Ω be a bounded domain in $\mathbf{R}^n$ which is not a ball. Then if $k > \frac{n}{2}$, there is no solution to the Dirichlet problem

$$\begin{aligned}\sigma_k(\lambda(V)) &= f \quad \text{in } \Omega, \\ v &= c \quad \text{on } \partial\Omega\end{aligned}\tag{2.25}$$

in general, where c is any positive constant, and f is a positive smooth function. Indeed, let $\{f_m\}$ be a sequence of smooth, positive functions which converges to zero locally uniformly in $\Omega \setminus \{x_0\}$ such that $\sup v_m \rightarrow \infty$, where v_m is the corresponding solution (if the solution v_m does not exist, we are through). Then v_m must converge to the function $v = C|x - x_0|^{2-n}$ by Theorem B. Hence Ω must be a ball centered at x_0 .

For the existence of solutions to the Dirichlet problem, it was proved recently by Guan [12] that for any smooth, bounded domain with smooth boundary data, the Dirichlet problem is classically solvable if there exists a smooth sub-solution with the same boundary trace.

3 Proof of Theorem A

3.1 Hölder continuity

We start with a Hölder continuity property of k -admissible functions.

Lemma 3.1 *Let $(\mathcal{M}, g_0)$ be a compact manifold. Suppose $g = u^{-2}g_0 \in [g_0]_k$ and $k > \frac{n}{2}$. Then u is Hölder continuous with exponent $\alpha = 2 - \frac{n}{k}$.*

$$\frac{u(x) - u(y)}{|x - y|^\alpha} \leq C \int_{\mathcal{M}} u, \tag{3.1}$$

where C is independent of u .

Proof By approximation it suffices to prove (3.1) for smooth functions. For any given point $0 \in \mathcal{M}$, there exists a conformal normal metric [2, 4, 16], still denoted by g_0 , such that in the normal coordinates at 0,

$$\det(g_0)_{ij} \equiv 1 \quad \text{near } 0. \tag{3.2}$$

Let

$$u_0(x) = |x|^{2-\frac{n}{k}}, \tag{3.3}$$

where $|x|$ denotes the geodesic distance from 0. Note that under condition (3.2), the Laplacian Δ on $\mathcal{M}$ is equal to the Euclidean Laplacian when applying to functions of $r = |x|$ alone [19, 23]. Hence

$$\Delta_{g_0} u_0 = \frac{n(k-1)(2k-n)}{k^2} r^{-\frac{n}{k}}. \tag{3.4}$$

Denote by

$$P[u] = \min \lambda_i + \delta \sum_i \lambda_i, \quad \left(\delta = \frac{n-k}{n(k-1)} \right) \tag{3.5}$$

the Pucci minimal operator [11], where $(\lambda_1, \dots, \lambda_n)$ are the eigenvalues of the Hessian matrix $(\nabla_{ij}u_0)$. Obviously we have

$$\min \lambda_i \leq \partial_r^2 u_0 = -\frac{(2k-n)(n-k)}{k^2} r^{-\frac{n}{k}}. \quad (3.6)$$

Therefore u_0 satisfies

$$P[u_0] \leq 0 \quad \text{in } B_{0,r} \setminus \{0\}.$$

where $B_{y,r}$ denotes the geodesic ball with center y and radius r .

On the other hand, since $\lambda(U) \in \bar{\Gamma}_k$, where U is given in (1.16), we have $\lambda(u_{ij} + uA_{g_0}) \in \bar{\Gamma}_k \subset \bar{\Gamma}_1$. Namely $\Delta u + \text{tr}(A_{g_0})u \geq 0$. By the weak Harnack inequality [11] it follows

$$\sup u \leq C \int_{\mathcal{M}} u, \quad (3.7)$$

where C is a constant independent of u . Therefore to prove (3.1) we may assume that $\int_{\mathcal{M}} u = 1$ and u is uniformly bounded.

Let $u_a = u + a|x|^2$. Then $\nabla^2 u_a > \nabla^2 u + aI$ near 0, where I is the unit matrix. Since $\lambda(\nabla^2 u + uA_{g_0}) \in \bar{\Gamma}_k$, we have $\lambda(\nabla^2 u_a) \in \Gamma_k$ when a is suitably large. Taking $l = 1$ in the proof of Lemma 4.2 in [27], one has

$$\lambda_i + \frac{n-k}{n(k-1)} \sum_i \lambda_i \geq 0, \quad (3.8)$$

namely $P[u_a] \geq 0$ near 0. Hence by applying the comparison principle to the functions u_a and u_0 with respect to the operator P , we conclude the Hölder continuity (3.1). $\square$

Remark 3.1 The estimate (3.1) (with exponent $\alpha < 2 - \frac{n}{k}, k > \frac{n}{2}$) also follows from gradient estimates from our reduction to p -Laplacian subsolution in [27]. Since $\lambda(U) \in \Gamma_k$, we have $\lambda(D^2 u + uA_{g_0}) \in \Gamma_k$. By (3.8) it follows that

$$\Delta_p u := \nabla_i (|\nabla u|^{p-2} \nabla_i u) \geq -Cu |\nabla u|^{p-2} \quad (3.9)$$

for $p-2 = \frac{n(k-1)}{n-k}$ and some constant C . From our argument in [27], we obtain $\int_{\mathcal{M}} |\nabla u|^q \leq C$ for any $q < nk/(n-k)$, whence by the Sobolev inequality, we infer (3.1) for $\alpha < 2 - \frac{n}{k}$; (see also [17]).

By the relation $u = e^w$, we have the following

Corollary 3.2 *Let w be a k -admissible function. Suppose $w \leq 0$. Then for any $K > 0$, there exists $C = C_K > 0$, independent of w , such that when $w(y) > -K$,*

$$\frac{w(x) - w(y)}{|x - y|^\alpha} \leq C, \quad (3.10)$$

with the same α as above.

From (3.10), we see that if $w(x) \leq -K - 1$, then $|x - y| \geq C_{K+1}^{1/\alpha}$. Also note that in Corollary 3.2, if we assume that $w \leq 0$ in $B_{y,r}$, then (3.10) holds for $x, y \in B_{y,r/2}$ for some C depending on r .

3.2 Singularity behaviour of k -admissible functions

Suppose w is a k -admissible function. At any given point $0 \in \mathcal{M}$, we choose a conformal normal coordinate near 0 such that (3.2) holds. In the conformal metric, the Ricci curvature vanishes at 0 [19, 23]. Hence

$$|A_{g_0}(x)| \leq Cr \quad \text{near } 0, \quad (3.11)$$

where r is the geodesic distance from 0 . Let $\tilde{w}$ be the minimal radial function of w with respect to 0 , as defined in (2.10). Then the argument thereafter is still valid, except that (2.14) should be replaced by $\kappa_i \leq \frac{1}{r} + C$. Also note that in the conformal normal coordinate, at the point $(0, \dots, 0, r)$ in the exponential map, the metric $g_0 = I + J$, where I is the unit matrix, $|J| = O(r)$ and its n th-row and n th-column vanishes. Hence from (2.19), we have

$$(\tilde{b}, \dots, \tilde{b}, \tilde{a}) \in \bar{\Gamma}_k, \quad (3.12)$$

where

$$\begin{aligned} \tilde{b} &= \left(\frac{1}{r} + C \right) \tilde{w}' - \frac{1 - Cr}{2} (\tilde{w}')^2 + Cr, \\ \tilde{a} &= \tilde{w}'' + \frac{1}{2} (\tilde{w}')^2 + Cr. \end{aligned}$$

Similarly to the argument in Sect. 2.1, we have $\tilde{b} \geq 0$ and $\tilde{a} + \frac{n-k}{k} \tilde{b} \geq 0$. Note that $\tilde{b} \geq 0$ is equivalent to that $\frac{1}{2} (\tilde{w}')^2 \leq \left(\frac{1}{r} + C \right) \tilde{w}' + Cr$. Hence

$$\tilde{a} + \frac{n-k}{k} \tilde{b} \geq \left[\tilde{w}'' + \left(\frac{1}{r} + C \right) \tilde{w}' + Cr \right] - (1 - \theta) \left[\left(\frac{1}{r} + C \right) \tilde{w}' - \frac{1}{2} (\tilde{w}')^2 + Cr \right] \geq 0,$$

where $\theta = \frac{n-k}{k}$ is as in (2.3). It follows that, when $r > 0$ is small,

$$\tilde{w}' \leq \frac{2}{r} + \frac{Cr}{\tilde{w}'} + C, \quad (3.13)$$

$$\tilde{w}'' + \left(\frac{1}{r} + C \right) \tilde{w}' + Cr \geq 0. \quad (3.14)$$

From (3.13),

$$\tilde{w}' \leq \frac{2}{r} + C$$

for a different C . It follows that $\tilde{w}(r) > -\infty$ for any $r > 0$, namely the set $\{w = -\infty\}$ cannot be an open set containing the origin 0 . But this property also follows from the weak Harnack inequality for the operator (1.6), as noted there.

From (3.14), we therefore obtain

$$(r\tilde{w}')' + C \geq 0.$$

It follows that $r\tilde{w}' + Cr$ is increasing. By the compactness of $\mathcal{M}$, a k -admissible function w must be bounded from above.

If $r\tilde{w}' < 2$ near $r = 0$, then similarly to (2.7)–(2.8), $\tilde{w}$ is bounded and Hölder continuous.

If $r\tilde{w}' \rightarrow 2$ as $r \rightarrow 0$, then $r\tilde{w}' + Cr \geq 2$, namely $\tilde{w}' \geq \frac{2}{r} - C$. Hence we obtain

$$\frac{2}{r} + C \geq \tilde{w}' \geq \frac{2}{r} - C. \quad (3.15)$$

We obtain

$$\tilde{w}(r) = 2 \log r + C' + O(r). \quad (3.16)$$

By subtracting a constant we assume that $C' = 0$.

Lemma 3.3 *If $\tilde{w}$ satisfies (3.16), then near 0,*

$$w(x) = 2 \log |x| + o(1). \quad (3.17)$$

Proof We prove (3.17) by a blow-up argument. In a normal coordinate system at 0, let $y = c_m x$ and $w_m(y) = w(x) + 2 \log c_m$, where c_m is any sequence converging to infinity. Let $\tilde{w}_m$ be the radial function corresponding to w_m . Then by (3.16),

$$\tilde{w}_m(r) = 2 \log r + O(c_m^{-1}). \quad (3.18)$$

Hence $\tilde{w}_m \rightarrow 2 \log r$.

For any fixed $r_0 > 0$ small, let $w_m(y_m) = \tilde{w}_m(r_0)$ ($|y_m| = r_0$). We may assume that $y_m \rightarrow y_0$. By the Hölder continuity (Corollary 3.2), we may also assume that in a neighborhood of y_0 , w_m converges uniformly to w_∞ . Then w_∞ is a k -admissible function defined on $\mathbf{R}^n$. The comparison principle argument of Lemma 2.4 implies that $w_\infty \equiv 2 \log r$ in a neighborhood of y_0 . The Hölder continuity in Corollary 3.2 implies that if $w_\infty = 2 \log r$ at some point, w_∞ is well-defined nearby. The comparison principle then implies that $w_\infty \equiv 2 \log r$ near the point. Hence $\tilde{w}_m$ converges uniformly to $w_\infty = 2 \log r$ on $|x| = r_0$, and converges locally uniformly to $w_\infty = 2 \log r$ in $\mathbf{R}^n \setminus \{0\}$. Hence (3.17) is proved. $\square$

Note that (3.17) is equivalent to (1.20), by the relation $v = e^{-\frac{n-2}{2}w}$.

Remark 3.2 From the proof of Lemma 3.3, we see that w has only isolated singularities. For if there is a sequence of singular points $x_m \in \mathcal{M}$ which converges to a point 0, we may choose $c_m = |x_m|^{-1}$ in the above argument. Then the limit function w_∞ has at least two singular points 0 and $x^* = \lim x_m / |x_m|$. To see that x^* is a singular point of the limit function w_∞ , we notice that the constant C' is uniformly bounded from above if w is negative in a neighbourhood of 0, which in turn implies that $\lim_{x \rightarrow x^*} w_\infty(x^*) = -\infty$. But the above argument shows that $w_\infty = 2 \log r$. This is a contradiction.

Similarly one can show that the set of all isolated singular points has no limit points, for if there is a sequence of singular points $y_m \rightarrow 0$, we may for each $m > 1$ choose a conformal normal coordinate with origin at y_m and consider the sequence $x_m = y_{m+1} - y_m$. The above argument also leads to a contradiction.

Next we show that w has at most one singular point.

Lemma 3.4 *Let w be a k -admissible function. Then the singularity set*

$$S_w = \bigcap_{h < 0} \{x \in \mathcal{M} \mid w(x) < h\} \quad (3.19)$$

contains at most one point.

Proof We adapt the corresponding argument in [18]. If S_w is not empty, it consists of finitely many isolated points. Let $g = e^{-2w} g_0$. By Lemma 3.3, $(\mathcal{M} \setminus S_w, g)$ is a complete manifold with finitely many ends. Now fixing a point $y \notin S_w$, we consider the ratio

$$Q(r) = \frac{\text{Vol}(B_{y,r})}{r^n}, \quad (3.20)$$

where $B_{y,r} = B_{y,r}[g]$ is the geodesic ball of $(\mathcal{M}, g)$. By definition, as well as the local Hölder continuity (Lemma 3.1), there is a sequence of smooth k -admissible functions w_m which converges to w locally uniformly. It is easy to verify that for any fixed y and r , $\text{Vol}(B_{y,r}[g_m]) \rightarrow \text{Vol}(B_{y,r}[g])$ as $m \rightarrow \infty$, where $g_m = e^{-2w_m}g_0$. Note that when $k \geq n/2$ the Ricci curvature of $(\mathcal{M}, g_m)$ is positive, as shown in (1.8). Hence by the Bishop Theorem, the ratio $Q_m(r) = \text{Vol}(B_{y,r}[g_m])/r^n$ is decreasing for all m . Sending $m \rightarrow \infty$, we see that Q is non-increasing in r . Hence

$$Q(0) \leq \lim_{r \rightarrow 0} Q(r) \leq \frac{1}{n} \omega_n, \quad (3.21)$$

where ω_n is the area of the unit sphere S^{n-1} .

On the other hand, denote $A_{r_1,r_2} = B_{0,r_2}[g_0] - B_{0,r_1}[g_0]$, where $r_2 > r_1 > 0$ are sufficiently small. We identify A_{r_1,r_2} with the Euclidean annulus $A_{r_1,r_2}^e = \{x \in \mathbf{R}^n \mid r_1 < |x| < r_2\}$ by the exponential map. By the asymptotic behavior (3.17), the volume of A_{r_1,r_2} in the metric $g = e^{-2w}g_0$ is a lower order perturbation of that in the metric $g' = e^{-2w'}g_0$, where $w' = 2 \log |x|$. But in our normal coordinates at 0, by (3.2) the volume of A_{r_1,r_2} in g' is the same as that of A_{r_1,r_2}^e with the metric $g'_e = e^{-2w'}g_e$, where g_e is the standard Euclidean metric. Hence $\text{Vol}_{g'} A_{r_1,r_2} = \frac{1}{n} \omega_n (r_1^{-n} - r_2^{-n})$. Therefore as $r \rightarrow \infty$, each end of the metric g will contribute to the ratio $Q(r)$ a factor $\frac{1}{n} \omega_n$. Therefore we obtain

$$\lim_{r \rightarrow \infty} Q(r) = \frac{m}{n} \omega_n, \quad (3.22)$$

where m is the number of singular points of w . From (3.21) and (3.22) we see that if S_w is not empty, then m must be equal to 1, namely S_w is a single point. $\square$

3.3 Smoothness of k -admissible functions

In this subsection we prove the following smoothness result. The proof is again inspired by the corresponding proof in [18].

Lemma 3.5 *Let w be a k -admissible function w with a singular point 0. Then w is C^∞ smooth away from 0.*

Proof First we prove

$$\sigma_k(\lambda(A_g)) \equiv 0 \quad \text{in } \mathcal{M} \setminus \{0\}, \quad (3.23)$$

where $g = e^{-2w}g_0$. It suffices to prove that for any given point $x_0 \neq 0$ and a sufficiently small $r > 0$ ($r < \frac{1}{4}|x_0|$), (3.23) holds in $B_{x_0,r} = B_{x_0,r}[g_0]$.

By definition, there exists a sequence w_m of smooth k -admissible functions which converges to w in $B_{x_0,2r}$ uniformly. Let φ_m be the admissible solution of the Dirichlet problem [12]

$$\begin{aligned} \sigma_k(\lambda(A_{g_{\varphi_m}})) &= \varepsilon_m && \text{in } B_{x_0,r}, \\ \varphi_m &= w_m && \text{on } \partial B_{x_0,r}, \end{aligned} \quad (3.24)$$

where $g_{\varphi_m} = e^{-2\varphi_m}g_0$, and ε_m is a small positive constant ($\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$) such that $\sigma_k(\lambda(A_{g_{w_m}})) > \varepsilon_m$ ($g_{w_m} = e^{-2w_m}g_0$). By the comparison principle we have $\varphi_m \geq w_m$ in $B_{x_0,r}$. Let $\hat{w}_m = w_m$ in $\mathcal{M} - B_{x_0,r}$ and $\hat{w}_m = \varphi_m$ in $B_{x_0,r}$. Then $\hat{w}_m$ is k -admissible (see Corollary 3.8 below). Let $\hat{w} = \lim_{m \rightarrow \infty} \hat{w}_m$. Then $\hat{w}$ is a k -admissible function with

singularity point 0. Define the metric $\hat{g} = e^{-2\hat{w}}g_0$ and the ratio $\hat{Q}(r) = \frac{\text{Vol}(B_{y,r}[\hat{g}])}{r^n}$. Then from the proof of Lemma 3.4, we also have $\hat{Q} \equiv \frac{1}{n}\omega_n$.

To prove (3.23) it suffices to show that $\hat{w} \equiv w$. Noting that $\hat{w} = w$ in $\mathcal{M} - B_{x_0,r}$ and $\hat{w} \geq w$ in $B_{x_0,r}$, we have $B_{y,r}[\hat{g}] \supset B_{y,r}[g]$ for any $r > 0$ and $y \neq 0$. If there exists a point $y \in B_{x_0,r}$ such that $\hat{w} > w$ at y , then there exists a positive constant $\delta > 0$ such that for any $r > 1$,

$$B_{y,r}[\hat{g}] \supset B_{y,r+\delta}[g].$$

But this is impossible as both the ratios $Q(r)$ and $\hat{Q}(r)$ are constant.

By the interior second order derivative estimate [13], we see that w is $C^{1,1}$ smooth. Next we prove that w is C^∞ smooth away from 0. By the regularity of linear elliptic equations [11], it suffices to prove that $v = e^{-\frac{n-2}{2}w} \in C^{1,1}$ is a strong solution to the uniformly elliptic equation

$$-\Delta_{g_0}v + \frac{n-2}{4(n-1)}R_{g_0}v = 0 \quad \text{in } \mathcal{M} \setminus \{0\}, \quad (3.25)$$

where R_{g_0} is the scalar curvature of $(\mathcal{M}, g_0)$; namely the scalar curvature of $g = e^{-2w}g_0$ vanishes identically.

Equation (3.25) may be verified as in Sect. 7.6 in [18]. Here we provide a proof for completeness. Since $w \in C^{1,1}$, it is twice differentiable almost everywhere. Suppose at a point 0, w is twice differentiable and the scalar curvature $R > 0$. Then with respect to normal coordinates of g at 0, we have the expansion

$$\det g_{ij} = 1 - \frac{1}{3}R_{ij}x_i x_j + o(|x|^2), \quad (3.26)$$

see (5.2) in [19]. Hence

$$\begin{aligned} \text{Vol}(B_{0,r}[g]) &= \int_{B_{0,r}} \sqrt{\det g_{ij}} \\ &= \int_{B_{0,r}} \left[1 - \frac{1}{6}R_{ij}x_i x_j + o(|x|^2) \right] \\ &= \frac{1}{n}\omega_n r^n \left[1 - \frac{R}{6(n+2)}r^2 + o(r^2) \right], \end{aligned} \quad (3.27)$$

where R_{ij} and R are respectively the Ricci curvature and the scalar curvature in g . This is a contradiction when $R > 0$ at 0, as the ratio Q is a constant. Hence the scalar curvature of g vanishes almost everywhere. $\square$

3.4 End of proof of Theorem A

From Sects. 3.2 and 3.3, we see that if $(\mathcal{M}, g_0)$ is a compact manifold and there exists a k -admissible function w with singularity at some point 0, then w has the asymptotic formula (3.17) and w is smooth away from 0. The manifold $\mathcal{M} \setminus \{0\}$ equipped with the metric $g = e^{-2w}g_0$ is a complete manifold with nonnegative Ricci curvature, and satisfies furthermore the volume growth formula $Q(r) \equiv 1$. Hence $(\mathcal{M} \setminus \{0\}, g)$ is isometric to the Euclidean space [7]. Hence $(\mathcal{M}, g_0)$ is conformally equivalent to the unit sphere S^n . See also [18].

To finish the proof of Theorem A, it suffices to prove

Lemma 3.6 *Let $(\mathcal{M}, g_0)$ be a compact manifold. If $(\mathcal{M}, g_0)$ is not conformally equivalent to the unit sphere S^n , then there exists $K > 0$ such that if w is a k -admissible function,*

$$\sup_{\mathcal{M}} w - \inf_{\mathcal{M}} w \leq K, \quad (3.28)$$

$$|w(x) - w(y)| \leq K|x - y|^{2-\frac{n}{k}}. \quad (3.29)$$

Proof If (3.28) is not true, there exists a sequence of k -admissible functions w_m such that $\sup_{\mathcal{M}} w_m = 0$ (by subtracting a constant if necessary) and $\inf_{\mathcal{M}} w_m \rightarrow -\infty$. Suppose that $w_m(0) \rightarrow -\infty$. By Lemma 3.1, $\{e^{w_m}\}$ are uniformly Hölder continuous. Hence we may assume that e^{w_m} converges locally uniformly to e^w in $\mathcal{M}$. The Hölder continuity also implies that $e^w(0) = 0$, namely $\lim_{x \rightarrow 0} w(x) = -\infty$. By a weak Harnack inequality for (1.6), the set $\{w = -\infty\}$ has measure zero and so w is k -admissible. Applying Lemma 3.3 to the limit function w we conclude that w has the asymptotic behavior (3.17) and that 0 is an isolated singularity of w (see Remark 3.2). Therefore by Lemmas 3.4, 0 is the unique singular point of w . By Lemma 3.5, w is C^∞ -smooth away from 0. Hence as above, we see that $(\mathcal{M}, g_0)$ is conformally equivalent to the unit sphere S^n , which is ruled out by our assumption. Hence (3.28) holds. $\square$

The Hölder continuity (3.29) follows from Lemma 3.1. $\square$

3.5 Remarks on the set $[g_0]_k$

In this section we prove some properties of k -admissible functions.

Lemma 3.7 *If w_1, w_2 are smooth and k -admissible, then $w = \max(w_1, w_2)$ is k -admissible.*

Proof It is convenient to consider the function $u = e^w$. By approximation we suppose u_1 and u_2 are smooth and k -admissible functions such that the eigenvalues $\lambda(U)$ lie strictly in the open convex cone Γ_k , where U is the matrix (1.16) with $u = u_1$ and u_2 . Hence when $r > 0$ is sufficiently small, the eigenvalues of the matrix

$$U_r = \left\{ u_{ij} - \frac{|\nabla u|^2}{2u_{x,r}} + uA_{g_0} \right\} \quad (3.30)$$

lie in Γ_k for $u = u_1$ and u_2 , where $u_{x_0,r} = \inf_{B_{x_0,r}} u$.

Let $u = \max(u_1, u_2)$. Since u_1, u_2 are smooth functions, u is twice differentiable almost everywhere. Let $\rho \in C_0^\infty(\mathbf{R}^n)$ be a mollifier. In particular we choose ρ to be a radial, smooth, nonnegative function, supported in the unit ball $B_{0,1}$, with $\int_{B_{0,1}} \rho = 1$. Let

$$u_{[\varepsilon]}(x) = \int_{B_{x,\varepsilon}} \varepsilon^{-n} \rho\left(\frac{|x-y|}{\varepsilon}\right) u(y) \sqrt{\det(g_0)_{ij}} dy \quad (3.31)$$

be the mollification of u , where $B_{x,\varepsilon}$ is the geodesic ball. For each point x , using normal coordinates and the exponential map, we have, by (3.26),

$$\begin{aligned} u_{[\varepsilon]}(x) &= \int_{B_{0,1}} \rho(y) u(x - \varepsilon y) \sqrt{\det(g_0)_{ij}} dy \\ &= \int_{B_{0,1}} \rho(y) u(x - \varepsilon y) \left(1 - \frac{\varepsilon^2}{6} R_{ij}(x) y_i y_j + O(\varepsilon^3) \right) dy, \end{aligned} \quad (3.32)$$

where $B_{0,1}$ is the Euclidean space. If g_0 is a flat metric, we have

$$\nabla u_{[\varepsilon]} = \int_{B_{0,1}} \rho(y) \nabla u(x - \varepsilon y) dy, \quad (3.33)$$

$$\nabla^2 u_{[\varepsilon]} \geq \int_{B_{0,1}} \rho(y) \nabla^2 u(x - \varepsilon y) dy, \quad (3.34)$$

$$\begin{aligned} |\nabla u_{[\varepsilon]}|^2 &= \left[\int_{B_{0,1}} \rho(y) \nabla u(x - \varepsilon y) dy \right]^2 \\ &\leq \int_{B_{0,1}} \rho(y) |\nabla u(x - \varepsilon y)|^2 dy. \end{aligned} \quad (3.35)$$

Hence $u_{[\varepsilon]}$ is k -admissible by (3.30). If g_0 is not flat, by (3.32), an extra term of magnitude $O(\varepsilon^2)$ arises. Letting $\varepsilon > 0$ be sufficiently small and noting that the eigenvalues of U (with respect to u_1 and u_2) lie strictly in the open set Γ_k , we conclude again that $u_{[\varepsilon]}$ is k -admissible. $\square$

Corollary 3.8 *Suppose φ is a smooth k -admissible function on $\mathcal{M}$ with $\sigma_k(\lambda(A_{g_\varphi})) > f$, where $g_\varphi = e^{-2\varphi} g_0 \in [g_0]_k$ and f is a smooth, positive function. Let w be the admissible solution of*

$$\begin{aligned} \sigma_k(\lambda(W)) &= f \quad \text{in } \Omega, \\ w &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \quad (3.36)$$

where W is given in (1.17), and Ω is a smooth domain on $\mathcal{M}$. Extend w to $\mathcal{M}$ by letting $w = \varphi$ on $\mathcal{M} - \Omega$. Then w is k -admissible.

It was proved in [12] that (3.36) admits a solution w , smooth up to the boundary. By the comparison principle we have $w > \varphi$ in Ω and $\partial_\nu(\varphi - w) > 0$ on $\partial\Omega$, where ν is the unit outward normal. Hence we can extend w to a neighbourhood of Ω such that it is k -admissible. Hence Corollary 3.8 follows from Lemma 3.7.

Corollary 3.9 *Consider the Dirichlet problem (3.36). Suppose the set of sub-solutions W_{sub} is not empty. Let*

$$w(x) = \sup\{\varphi(x) \mid \varphi \in W_{\text{sub}}\}. \quad (3.37)$$

If w is bounded from above, then it is a solution to (3.36).

By the interior a priori estimates [13], the proof is standard. Note that in Corollary 3.9, we allow Ω to be the whole manifold $\mathcal{M}$.

4 Proof of Theorem C

We divide the proof into three cases, according to $p < k$, $p = k$, and $p > k$.

Case 1 $p < k$. By (1.15), we can write Eq. (1.12) as

$$\sigma_k(\lambda(W)) = f e^{aw}, \quad (4.1)$$

where f is a constant multiple of that in (1.12),

$$a = \frac{1}{2}(n-2)(k-p). \quad (4.2)$$

For any given k -admissible function w , the functions $w + c$ and $w - c$ are respectively a super and a sub solution of (4.1) provided the constant c is sufficiently large. By the a priori estimates in [13] and the comparison principle, the solution of (4.1) is uniformly bounded. When $a > 0$, the linearized equation of (4.1) is invertible. Hence by the continuity method, there is a unique smooth solution to (4.1).

Case 2 $p = k$. We prove that for any positive smooth function f , there is a unique constant $\theta > 0$ such that the equation

$$\sigma_k(\lambda(W)) = \theta f \quad (4.3)$$

has a solution. For $a > 0$ small, let w_a be the solution of (4.1). Let $c_a = \inf w_a$. We write (4.1) in the form

$$\sigma_k(\lambda(W_a)) = (f e^{ac_a}) e^{a(w_a - c_a)}, \quad (4.4)$$

where W_a is the matrix (1.17) relative to w_a . Assume $g_0 \in [g_0]_k$ so that $\lambda(A_{g_0}) \in \Gamma_k$. Then at the maximum point of w_a ,

$$\sigma_k(\lambda(A_{g_0})) \geq \sigma_k(\lambda(W_a)) \geq f e^{ac_a}.$$

At the minimum point of w_a ,

$$\sigma_k(\lambda(A_{g_0})) \leq \sigma_k(\lambda(W_a)) = f e^{ac_a}.$$

Hence e^{ac_a} is strictly positive and uniformly bounded as $a \rightarrow 0$. By the a priori estimates [13], where the estimates depend only on $\inf(w_a - c_a)$, we see that $w_a - c_a$ is uniformly bounded from above and sub-converges to a solution w_0 of (4.3) with $\theta = \lim_{a \rightarrow 0} e^{ac_a}$. By the maximum principle it is easy to see that if w' is another solution, then necessarily $w' = w_0 + \text{const}$; and furthermore (4.3) has no (k -admissible) solution for different θ .

Case 3 $p > k$. In this case we adopt the degree argument from [30], see the proof of Theorem 5.1 there. Alternatively we can also use the degree argument in Sect. 3 of [30]. We will study the auxiliary problem

$$\sigma_k(\lambda(V)) = t(\delta_t + f v^p), \quad (4.6)$$

where $t \geq 0$ is a parameter and δ_t is a positive constant depending on t , $\delta_t = \delta_0 \leq 1$ when $t \leq 1$ and $\delta_t = 1$ when $t > 2$, and δ_t is smooth and monotone increasing when $1 \leq t \leq 2$.

Claim 1 For any $t_0 > 0$, the solution of (4.6) is uniformly bounded when $t \geq t_0$. Indeed, if there exists a sequence of solutions (t_j, v_j) of (4.6) such that $t_j \geq t_0$ and $\sup v_j \rightarrow \infty$, we have $m_j = \inf v_j \rightarrow \infty$ by (1.5). The function $v'_j = v_j/m_j$ satisfies

$$\begin{aligned} \sigma_k(\lambda(V')) &\geq t_j f m_j^{p-k} (v'_j)^p \\ &\geq t_j f m_j^{p-k} \rightarrow \infty, \end{aligned} \quad (4.7)$$

where V' is the matrix (1.11) relative to v' . From (4.7) and the comparison principle we have $\sup v'_j \rightarrow \infty$. Hence $\inf v'_j \rightarrow \infty$ by (1.5), which contradicts to the definition of v'_j .

Define the mapping T_t so that for any $v_1 \in C^2(\mathcal{M})$, $T_t(v_1)$ is the solution of

$$\sigma_k(\lambda(V)) = t(\delta_t + f v_1^p). \quad (4.8)$$

Then a solution of (4.6) is a fixed point of T_t .

Claim 2 There is a solution of (4.6) when $t > 0$ is small. Indeed, for any smooth, positive function φ^* , denote $\Phi = \{\varphi \in C^2(\mathcal{M}) \mid 0 < \varphi < \varphi^*, \|\varphi\|_{C^2} < R\}$ for some large constant $R > 0$. Then when $\delta_0, t > 0$ are small, $T(\Phi)$ is strictly contained in Φ . Hence the degree $\deg(I - T_t, \Phi, 0)$ is well defined for $t > 0$ small. Extend T_t to $t = 0$ by letting $T_t(v) = 0$ for all v , so that T_t is also continuous at $t = 0$. Hence

$$\deg(I - T_t, \Phi, 0) = \deg(I - T_0, \Phi, 0) = 1. \quad (4.9)$$

Hence T_t has a fixed point in Φ for $t > 0$ small.

Note that when δ_0 is sufficiently small, letting φ^* be a small positive constant, we see that (4.6) has a solution for $t \leq 1$.

Claim 3 Let $t^* = \sup\{t \mid (4.6) \text{ admits a solution}\}$. Then t^* is finite. Indeed, if $t^* = \infty$, there is a sequence $t_j \rightarrow \infty$ such that (4.6) has a solution v_j . We have obviously $m_j = \inf v_j \rightarrow \infty$, which is a contradiction with Claim 1.

Claim 4 Equation (4.6) has a solution at $t = t^*$. Indeed, let $t_j \nearrow t^*$ and v_j be the corresponding solution of (4.6). By claim 1, v_j is uniformly bounded. Hence v_j sub-converges to a solution v^* of (4.6) with $t = t^*$.

Now we choose $\varphi^* = v^*$ and define Φ as above. For any $v_1 \in \Phi$, let v be the solution of (4.8). Since for any $t \in (0, t^*)$, v^* is a super-solution of (4.6). We have $0 < v < v^*$ by the maximum principle. Hence by (4.9), $\deg(I - T_t, \Phi, 0) = 1$ for $t \in [0, t^*)$.

On the other hand, for any given $t_0 > 0$, since the solution of (4.6) is uniformly bounded for $t \geq t_0$, the degree $\deg(I - T_t, B_R, 0)$ is well defined for $t \in (t_0, t^* + 1]$ for sufficiently large R , where $B_R = \{v \in C^2(\mathcal{M}) \mid 0 < v < R, \|v\|_{C^2} < R\}$. But when $t > t^*$, (4.6) has no solution. Hence $\deg(I - T_t, B_R, 0) = 0$. Hence for any $t \geq t_0$, (4.6) has a solution $v \notin \Phi$ with degree -1 .

Let $v = v_{\delta_0} \notin \Phi$ be a solution of (4.6) at $t = 1$. We have $\sup v > \inf v^* > 0$. Let $\delta_0 \rightarrow 0$. Since the solution is uniformly bounded, it converges to a solution of (1.12). This completes the proof. $\square$

From the above argument, we have the following extensions.

Corollary 4.1 Let $(\mathcal{M}, g_0)$ be a compact n -manifold not conformally equivalent to the unit sphere S^n . Suppose $\frac{n}{2} < k \leq n$ and $[g_0]_k \neq \emptyset$. Suppose there exists a constant $c_0 > 0$ such that

$$\varphi(x, t) \geq c_0, \quad (4.10)$$

$$\lim_{t \rightarrow \infty} t^{-k} \varphi(x, t) = \infty. \quad (4.11)$$

Then there exists a constant $t^* > 0$ such that the equation

$$\sigma_k(\lambda(V)) = t\varphi(x, v) \quad (4.12)$$

has at least two solutions for $0 < t < t^*$, one solution at $t = t^*$, and no solution for $t > t^*$.

Corollary 4.2 *Let $(\mathcal{M}, g_0)$ be as in Corollary 4.1, $\frac{n}{2} < k \leq n$. Suppose $\varphi > 0$,*

$$\lim_{t \rightarrow 0} t^{-k} \varphi(x, t) = 0, \quad (4.13)$$

and (4.11) holds. Then there exists a solution to (1.10).

In the above theorems, we can also allow that the right hand side depends on the gradient ∇v . Furthermore, (4.11) and (4.13) can be relaxed to

$$\lim_{t \rightarrow \infty} t^{-k} \varphi(x, t) > \theta, \quad (4.14)$$

$$\lim_{t \rightarrow 0} t^{-k} \varphi(x, t) < \theta, \quad (4.15)$$

where θ is the eigenvalue of (1.13) (with $f \equiv 1$). See [30] for the Monge-Ampère equation.

We remark that when $1 \leq k \leq \frac{n}{2}$, Theorem C holds for $p < k \frac{n+2}{n-2}$. Indeed, when $p \leq k$, the proof of the Cases 1 and 2 above also applies to the cases $1 \leq k \leq \frac{n}{2}$. When $k < p < k \frac{n+2}{n-2}$, by a blow-up argument and the Liouville theorem [20], it is known that the set of solutions to (4.6) is uniformly bounded. Hence by the above degree argument, one also obtains the existence of solutions.

Corollary 4.3 *Let $(\mathcal{M}, g_0)$ be a compact n -manifold with $[g_0]_k \neq \emptyset$, $1 \leq k \leq n$. Then for any smooth, positive function f and any constant $p \neq k$, $p < k \frac{n+2}{n-2}$, there exists a positive solution to the (1.12). The solution is unique if $p < k$. When $p = k$, there exists a unique constant $\theta > 0$ such that (1.13) has a solution. The solution is unique up to a constant multiplication.*

Note that in Corollary 4.3 we allow that $(\mathcal{M}, g_0)$ is the unit sphere.

References

1. Aubin, T.: Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl.* **55**, 269–296 (1976)
2. Aubin, T.: Some nonlinear problems in Riemannian geometry. Springer, Heidelberg (1998)
3. Caffarelli, L.A., Nirenberg, L., Spruck, J.: Dirichlet problem for nonlinear second order elliptic equations III. Functions of the eigenvalues of the Hessian. *Acta Math.* **155**, 261–301 (1985)
4. Cao, J.-G.: The existence of generalized isothermal coordinates for higher-dimensional Riemannian manifolds. *Trans. Am. Math. Soc.* **324**, 901–920 (1991)
5. Chang, A., Gursky, M., Yang, P.: An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature. *Ann. Math.* **155**(2), 709–787 (2002)
6. Chang, A., Gursky, M., Yang, P.: An a priori estimate for a fully nonlinear equation on four-manifolds. *J. Anal. Math.* **87**, 151–186 (2002)
7. Chavel, I.: Riemannian geometry—a modern introduction. Cambridge Univ. Press, London (1993)
8. Chou, K.S. (Tso, K.): On a real Monge-Ampère functional. *Invent. Math.* **101** 425–448 (1990)
9. Chou, K.S., Wang, X.J.: A variational theory of the Hessian equation. *Comm. Pure Appl. Math.* **54**, 1029–1064 (2001)
10. Ge, Y., Wang, G.: On a fully nonlinear Yamabe problem. *Ann. Sci. École Norm. Sup.* **39**(4), 569–598 (2006)
11. Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. Springer, Heidelberg (1983)
12. Guan, B.: Conformal metrics with prescribed curvature functions on manifolds with boundary. *Am. J. Math.* **129**, 915–942 (2007)
13. Guan, P., Wang, G.: Local estimates for a class of fully nonlinear equations arising from conformal geometry. *Int. Math. Res. Not.* (26), 1413–1432 (2003)
14. Guan, P., Wang, G.: A fully nonlinear conformal flow on locally conformally flat manifolds. *J. Reine Angew. Math.* **557**, 219–238 (2003)

15. Guan, P., Viaclovsky, J., Wang, G.: Some properties of the Schouten tensor and applications to conformal geometry. *Trans. Am. Math. Soc.* **355**, 925–933 (2003)
16. Günther, M.: Conformal normal coordinates. *Ann. Global Anal. Geom.* **11**, 173–184 (1993)
17. Gursky, M., Viaclovsky, J.: Convexity and singularities of curvature equations in conformal geometry, *Int. Math. Res. Not.*, Art. ID 96890, pp. 43 (2006)
18. Gursky, M., Viaclovsky, J.: Prescribing symmetric functions of the eigenvalues of the Ricci tensor. *Ann. Math.* **166**(2), 475–531 (2007)
19. Lee, J.M., Parker, T.H.: The Yamabe problem. *Bull. Am. Math. Soc.* **17**, 37–91 (1987)
20. Li, A., Li, Y.Y.: On some conformally invariant fully nonlinear equations. *Comm. Pure Appl. Math.* **56**, 1416–1464 (2003)
21. Li, A., Li, Y.Y.: On some conformally invariant fully nonlinear equations II, Liouville, Harnack, and Yamabe. *Acta Math.* **195**, 117–154 (2005)
22. Schoen, R.: Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differ. Geom.* **20**, 479–495 (1984)
23. Schoen, R., Yau, S.T.: *Lectures on Differential geometry*. International Press (1994)
24. Sheng, W.M., Trudinger, N.S., Wang, X.-J.: The Yamabe problem for higher order curvatures. *J. Differ. Geom.* **77**, 515–553 (2007)
25. Trudinger, N.S.: On Harnack type inequalities and their application to quasilinear elliptic equations. *Comm. Pure Appl. Math.* **20**, 721–747 (1967)
26. Trudinger, N.S.: Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa* **22**(3), 265–274 (1968)
27. Trudinger, N.S., Wang, X.J.: Hessian measures II. *Ann. Math.* **150**(2), 579–604 (1999)
28. Trudinger, N.S., Wang, X.-J.: The intermediate case of the higher order Yamabe problem, (in press)
29. Viaclovsky, J.: Conformal geometry, contact geometry, and the calculus of variations. *Duke Math. J.* **101**(2), 283–316 (2000)
30. Wang, X.-J.: Existence of multiple solutions to the equations of Monge-Ampère type. *J. Differ. Eqs.* **100**, 95–118 (1992)