The eight-vertex model and Painlevé VI

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Abstract

In this paper we establish a connection of Picard-type elliptic solutions of the Painlevé VI equation with the special solutions of the non-stationary Lamé equation. The latter appeared in the study of the ground-state properties of Baxter's solvable eight-vertex lattice model at a particular point, $\eta = \pi/3$, of the disordered regime.

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1. Introduction

The Painlevé transcendents have numerous remarkable applications in the theory of integrable models of statistical mechanics and quantum field theory (see, for instance, [1–3]). We mention, in particular, the calculation of the ‘supersymmetric index’ and related problems of dilute polymers on a cylinder which lead to Painlevé III [4, 5]. These problems are connected with the finite volume massive sine-Gordon model with $N = 2$ supersymmetry. The lattice analogue of this continuous quantum field theory corresponds to a special case of Baxter's famous solvable eight-vertex lattice model [6]. In this paper we continue our study [7] of this special model on a finite lattice and unravel its deep connections with Painlevé VI theory.

We consider the eight-vertex model on a square lattice with an odd number, $N = 2n + 1$, of columns and periodic boundary conditions. The eigenvalues of the row-to-row transfer matrix of the model, $T(u)$, satisfy the TQ equation [6]

$$T(u)Q(u) = \phi(u - \eta)Q(u + 2\eta) + \phi(u + \eta)Q(u - 2\eta), \quad (1)$$

where u is the spectral parameter,

$$\phi(u) = \vartheta_1^N(u | q), \quad q = e^{i\pi\tau}, \quad \text{Im } \tau > 0, \quad (2)$$

and $\vartheta_1(u | q)$ is the standard theta-function with the periods π and $\pi\tau$ (we follow the notation of [8]). Here we consider a special case $\eta = \pi/3$, where the ground-state eigenvalue is known [9, 10] to have a very simple form for all (odd) N

$$T(u) = \phi(u), \quad \eta = \frac{\pi}{3}. \quad (3)$$

Equation (1) with this eigenvalue, $T(u)$, has two different solutions [11, 12], $Q_{\pm}(u) \equiv Q_{\pm}(u, q, n)$, which are entire functions of the variable u and obey the following periodicity conditions [6, 13]:¹

$$\begin{aligned} Q_{\pm}(u + \pi) &= \pm(-1)^n Q_{\pm}(u), \\ Q_{\pm}(u + \pi\tau) &= q^{-N/2} e^{-iNu} Q_{\mp}(u), \\ Q_{\pm}(-u) &= Q_{\pm}(u). \end{aligned} \quad (4)$$

The above requirements uniquely determine $Q_{\pm}(u)$ to within a common u -independent normalization factor. It is convenient to rewrite the functional equation (1) for $Q_{\pm}(u)$ in the form

$$\Psi_{\pm}(u) + \Psi_{\pm}\left(u + \frac{2\pi}{3}\right) + \Psi_{\pm}\left(u + \frac{4\pi}{3}\right) = 0, \quad (5)$$

where

$$\Psi_{\pm}(u) \equiv \Psi_{\pm}(u, q, n) = \frac{\vartheta_1^{2n+1}(u | q)}{\vartheta_1^n(3u | q^3)} Q_{\pm}(u, q, n) \quad (6)$$

are the meromorphic functions of the variable u for any fixed values of q and n . As shown in [7], these functions satisfy the non-stationary Lamé equation

$$6q \frac{\partial}{\partial q} \Psi(u, q, n) = \left\{ -\frac{\partial^2}{\partial u^2} + 9n(n+1)\wp(3u | q^3) + c(q, n) \right\} \Psi(u, q, n), \quad (7)$$

where the elliptic Weierstrass $\wp$ -function, $\wp(v | e^{i\pi\epsilon})$, has the periods π and $\pi\epsilon$ [8]. The constant $c(q, n)$ appearing in (7) is totally controlled by the (u -independent) normalization of $Q_{\pm}(u)$. Note that this equation (in fact, a more general equation, usually called the non-autonomous Lamé equation) arose previously [14–17] in different contexts. We will not explore these connections here.

Let us quote other relevant results of [7]. Define the combinations

$$Q_1(u) = (Q_+(u) + Q_-(u))/2, \quad Q_2(u) = (Q_+(u) - Q_-(u))/2 \quad (8)$$

such that

$$Q_{1,2}(u + \pi) = (-1)^n Q_{2,1}(u). \quad (9)$$

Bearing in mind this simple relation we will only consider $Q_1(u)$, writing it as $Q_1^{(n)}(u)$ to indicate the n -dependence. Obviously the linear relations (5) and (7) remain unaffected if $Q_{\pm}(u)$ in (6) is replaced by $Q_1^{(n)}(u)$. The partial differential equation (7) has, of course, many solutions. Here we are only interested in the very special solutions, relevant to our original problem of the eight-vertex model (by definition the functions $Q_1^{(n)}(u)$ are entire quasi-periodic functions of u with the periods π and $2\pi\tau$). Introduce new variables γ and x , instead of q and u ,

$$\gamma \equiv \gamma(q) = -\left[\frac{\vartheta_1(\pi/3 | q^{1/2})}{\vartheta_2(\pi/3 | q^{1/2})} \right]^2, \quad x = \gamma \left[\frac{\vartheta_3(u/2 | q^{1/2})}{\vartheta_4(u/2 | q^{1/2})} \right]^2, \quad (10)$$

and new functions $\mathcal{P}_n(x, z)$ instead of $Q_1^{(n)}(u)$,

$$Q_1^{(n)}(u) = \mathcal{N}(q, n) \vartheta_3(u/2 | q^{1/2}) \vartheta_4^{2n}(u/2 | q^{1/2}) \mathcal{P}_n(x, z), \quad z = \gamma^{-2}, \quad (11)$$

where $\mathcal{N}(q, n)$ is an arbitrary normalization factor (which remains at our disposal). The properties of the functions $\mathcal{P}_n(x, z)$ corresponding to the required solutions of (5) and (7) can be summarized by the following

¹ The factor $(-1)^n$ in (4) and (9) reflects our convention for labelling the eigenvalues for different n .

Conjecture 1.

(a) The functions $\mathcal{P}_n(x, z)$ are polynomials in x, z of the degree n in x ,

$$\mathcal{P}_n(x, z) = \sum_{k=0}^n r_k^{(n)}(z) x^k, \quad (12)$$

while $r_i^{(n)}(z)$, $i = 0, \dots, n$, are polynomials in z of the degree

$$\deg[r_k^{(n)}(z)] \leq \lfloor n(n-1)/4 + k/2 \rfloor \quad (13)$$

with positive integer coefficients. The normalization of $\mathcal{P}_n(x, z)$ is fixed by the requirement $r_n^{(n)}(0) = 1$ and $\lfloor x \rfloor$ denotes the largest integer not exceeding x .

(b) The coefficients of the lowest and highest powers in x , corresponding to $k = 0$ and $k = n$ in (12), read

$$r_0^{(n)}(z) = \tau_n(z, -1/3), \quad r_n^{(n)}(z) = \tau_{n+1}(z, 1/6), \quad (14)$$

where the functions $\tau_n(z, \xi)$ (for each fixed value of the their second argument ξ) are determined by the recurrence relation

$$\begin{aligned} & 2z(z-1)(9z-1)^2 [\log \tau_n(z)]'_z + 2(3z-1)^2(9z-1) [\log \tau_n(z)]'_z \\ & + 8 \left[2n - 4\xi - \frac{1}{3} \right]^2 \frac{\tau_{n+1}(z)\tau_{n-1}(z)}{\tau_n^2(z)} - [12(3n-6\xi-1)(n-2\xi) \\ & + (9z-1)(n-1)(5n-12\xi)] = 0, \end{aligned} \quad (15)$$

with the initial condition

$$\tau_0(z, \xi) = 1, \quad \tau_1(z, \xi) = -4\xi + 5/3. \quad (16)$$

The functions $\tau_n(z, \xi)$ are polynomials in z for all $n = 0, 1, 2, \dots, \infty$.

The conjecture has been verified by an explicit calculation [7] of $\mathcal{P}_n(x, z)$ for all $n \leq 50$.

The polynomials $\mathcal{P}_n(x, z)$ can be effectively calculated using the algebraic form (see [7]) of equation (7). The first few of them read²

$$\mathcal{P}_0(x, z) = 1, \quad \mathcal{P}_1(x, z) = x + 3, \quad \mathcal{P}_2(x, z) = x^2(1+z) + 5x(1+3z) + 10, \quad (17)$$

$$\mathcal{P}_3(x, z) = x^3(1+3z+4z^2) + 7x^2(1+5z+18z^2) + 7x(3+19z+18z^2) + 35+21z. \quad (18)$$

As explained in [7], equation (7) leads to a descending recurrence relation for the coefficients in (12), in the sense that each coefficient $r_k^{(n)}(z)$ with $k < n$ can be recursively calculated in terms of $r_m^{(n)}(z)$, with $m = k+1, \dots, n$ and, therefore, can eventually be expressed through the coefficient $r_n^{(n)}(z)$ of the leading power of x . The conditions that this procedure truncates (and thus defines a polynomial, but not an infinite series in negative powers of x) completely determine the starting leading coefficient as a function of z . The above conjecture implies that these truncation conditions are equivalent to the particular case ($\xi = 1/6$) of recurrence relations (15), (16). A similar statement applies to the (smallest power in x) coefficient $r_0^{(n)}$ of $\mathcal{P}_n(x, z)$. The results quoted above are due to [7] (except for the expression (14) for $r_0^{(n)}$, which is new).

In this paper we show that the recurrence relation (15) exactly coincides with that for the tau-functions associated with special elliptic solutions of the Painlevé VI equation, revealing hitherto unknown connections of the eight-vertex model and the non-stationary Lamé equation (7) to Painlevé VI theory.

² The higher polynomials with $n \leq 50$ are available in electronic form upon request to the authors.

Before concluding this introduction, let us briefly mention some problems related to the special eight-vertex model considered here. In the trigonometric limit ($q \rightarrow 0$) this model reduces to the special six-vertex model (with the parameter $\Delta = -1/2$), which is closely connected with various interesting combinatorial problems [18–21], particularly, with the theory of alternating sign matrices. The fact that the polynomials $\mathcal{P}_n(x, z)$ have *positive integer coefficients* makes it plausible to suggest that these coefficients have some (yet unknown) combinatorial interpretation.

In the scaling limit

$$n \rightarrow \infty, \quad q \rightarrow 0, \quad t = 8nq^{3/2} = \text{fixed}, \quad (19)$$

functions (6) essentially reduce to the ground-state eigenvalues $\mathcal{Q}_{\pm}(\theta) \equiv \mathcal{Q}_{\pm}(\theta, t)$ of the **Q**-operators [11, 22] of the restricted massive sine-Gordon model (at the so-called, supersymmetric point) on a cylinder of the spatial circumference R , where $t = MR$ and M is the soliton mass and the variable θ is defined as $u = \pi\tau/2 - i\theta/3$. With a suitable t -dependent normalization of $\mathcal{Q}_{\pm}(\theta)$, equation (7) then reduces to the ‘non-stationary Mathieu equation’ [7],

$$t \frac{\partial}{\partial t} \mathcal{Q}_{\pm}(\theta, t) = \left\{ \frac{\partial^2}{\partial \theta^2} - \frac{1}{8} t^2 (\cosh 2\theta - 1) \right\} \mathcal{Q}_{\pm}(\theta, t). \quad (20)$$

This equation determines the asymptotic behaviour of $\mathcal{Q}_{\pm}(\theta)$ at large θ

$$\log \mathcal{Q}_{\pm}(\theta) = -\frac{1}{4} t e^{\theta} + \log \mathcal{D}_{\pm}(t) + 2 (\partial_t \log \mathcal{D}_{\pm}(t) - t/8) e^{-\theta} + O(e^{-2\theta}), \quad \theta \rightarrow +\infty, \quad (21)$$

where $\mathcal{D}_{\pm}(t)$ are the Fredholm determinants, which previously appeared in connection with the ‘supersymmetric index’ and the dilute polymers on a cylinder [4, 5, 23, 24]. Note, in particular, that the quantity

$$F(t) = \frac{d}{dt} U(t), \quad U(t) = \log \frac{\mathcal{D}_+(t)}{\mathcal{D}_-(t)}, \quad (22)$$

describes the free energy of a single incontractible polymer loop and satisfies the Painlevé III equation [23]

$$\frac{1}{t} \frac{d}{dt} t \frac{d}{dt} U(t) = \frac{1}{2} \sinh 2U(t). \quad (23)$$

2. Painlevé VI equation

The Painlevé $\mathbf{P}_{\text{VI}}(\alpha, \beta, \gamma, \delta)$ is the following second-order differential equation [25, 26],

$$q''(t) = \frac{1}{2} \left(\frac{1}{q(t)} + \frac{1}{q(t)-1} + \frac{1}{q(t)-t} \right) q'(t)^2 - \left(\frac{1}{t} + \frac{1}{(t-1)} + \frac{1}{q(t)-t} \right) q'(t) + \frac{q(t)(q(t)-1)(q(t)-t)}{t^2(t-1)^2} \left[\alpha + \beta \frac{t}{q(t)^2} + \gamma \frac{t-1}{(q(t)-1)^2} + \delta \frac{t(t-1)}{(q(t)-t)^2} \right], \quad (24)$$

where the following parameterizations of four constants [27] are chosen

$$\alpha = \frac{1}{2} \kappa_{\infty}^2, \quad \beta = -\frac{1}{2} \kappa_0^2, \quad \gamma = \frac{1}{2} \kappa_1^2, \quad \delta = \frac{1}{2} (1 - \theta^2), \quad (25)$$

$$\kappa_0 = b_1 + b_2, \quad \kappa_1 = b_1 - b_2, \quad \kappa_{\infty} = b_3 - b_4, \quad \theta = b_3 + b_4 + 1. \quad (26)$$

This equation is equivalent to the Hamiltonian system $H_{\text{VI}}(t; q, p)$ described by the equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad (27)$$

with the Hamiltonian function

$$H_{\text{VI}}(t; q, p) = \frac{1}{t(t-1)}[q(q-1)(q-t)p^2 - \{\kappa_0(q-1)(q-t) + \kappa_1 q(q-t) + (\theta-1)q(q-1)\}p + \kappa(q-t)], \quad (28)$$

where $q \equiv q(t)$, $p \equiv p(t)$ and

$$\kappa = \frac{1}{4}(\kappa_0 + \kappa_1 + \theta - 1)^2 - \frac{1}{4}\kappa_\infty^2. \quad (29)$$

One can introduce an auxiliary Hamiltonian $h(t)$,

$$h(t) = t(t-1)H(t) + e_2(b_1, b_3, b_4)t - \frac{1}{2}e_2(b_1, b_2, b_3, b_4), \quad (30)$$

where $e_i(x_1, \dots, x_n)$ is the i th elementary symmetric function in n variables.

Okamoto [27] showed that, for each pair $\{q(t), p(t)\}$ satisfying (27), the function $h(t)$ solves the $\mathbf{E}_{\text{VI}}$ equation,

$$h'(t)[t(1-t)h''(t)]^2 + [h'(t)[2h(t) - (2t-1)h'(t)] + b_1b_2b_3b_4]^2 = \prod_{k=1}^4 (h'(t) + b_k^2), \quad (31)$$

and $q(t)$ solves $\mathbf{P}_{\text{VI}}(\alpha, \beta, \gamma, \delta)$, (24).

Conversely, for each solution $h(t)$ of (31), such that $\frac{d^2}{dt^2}h(t) \neq 0$, there exists a solution $\{q(t), p(t)\}$ of (27), where $q(t)$ solves (24). An explicit correspondence between the three sets $\{q(t), q'(t)\}$, $\{q(t), p(t)\}$ and $\{h(t), h'(t), h''(t)\}$ is given by birational transformations, which can be found in [27].

The $\mathbf{E}_{\text{VI}}$ equation is a particular case of a more general class of the second-order second-degree equation

$$y''(t)^2 = F(t, y(t), y'(t)), \quad (32)$$

where F is rational in $y(t)$, $y'(t)$, locally analytic in t , with the property that the only movable singularities of $y(t)$ are poles. Such equations were classified in [28], where equation (31) is referred to the ‘SD-I type’. Originally, we have obtained equations of the form (32) for asymptotics of the polynomials $\mathcal{P}_n(x, z)$ and only then reduced them to (31) and (24).

The group of Backlund transformations of $\mathbf{P}_{\text{VI}}$ is isomorphic to the affine Weyl group of the type D_4 . It contains the following transformations of parameters (only 5 of them are independent)

$$w_1 : b_1 \leftrightarrow b_2, \quad w_2 : b_2 \leftrightarrow b_3, \quad w_3 : b_3 \leftrightarrow b_4, \quad w_4 : b_3 \rightarrow -b_3, b_4 \rightarrow -b_4, \quad (33)$$

$$x^1 : \kappa_0 \leftrightarrow \kappa_1, \quad x^2 : \kappa_0 \leftrightarrow \kappa_\infty, \quad x^3 : \kappa_0 \leftrightarrow \theta, \quad (34)$$

and the parallel transformation

$$l_3 : \mathbf{b} \equiv (b_1, b_2, b_3, b_4) \rightarrow \mathbf{b}^+ \equiv (b_1, b_2, b_3 + 1, b_4). \quad (35)$$

Here we shall use only two transformations: x^2 and l_3 . The canonical transformation x^2 corresponds to a simple change of variables [27],

$$x_\star^2 : (q, p, H, t) \rightarrow (q^{-1}, \epsilon q - q^2 p, -H/t^2, 1/t), \quad \epsilon = \frac{1}{2}(\kappa_0 + \kappa_1 + \theta - 1 + \kappa_\infty). \quad (36)$$

The birational canonical transformation,

$$\{q, p\} = \{q(\mathbf{b}), p(\mathbf{b})\} \rightarrow \{q^+, p^+\} = \{q(\mathbf{b}^+), p(\mathbf{b}^+)\}, \quad (37)$$

corresponding to l_3 , can be obtained from the observation that [27]

$$h^+(t, q^+, p^+) = h(t, q, p) - q(q-1)p + (b_1 + b_4)q - \frac{1}{2}(b_1 + b_2 + b_4). \quad (38)$$

Here we shall only give the expression for q in terms of $\{q^+, p^+\}$:

$$\begin{aligned} q \{ & p_+^2 q_+ (q_+ - 1)(q_+ - t) + p_+ [2(1 + b_1 + b_3)q_+ (1 - q_+) + (b_1 + b_2)(q_+ - t) + 2b_1(t - 1)q_+] \\ & + (1 + b_1 + b_3)[(1 + b_1 + b_3)(q_+ - 1) + t(1 + b_3 - b_1) + b_1 - b_2] \} \\ & - t[p_+(q_+ - 1) - 1 - b_1 - b_3][p_+ q_+(q_+ - 1) - (1 + b_1 + b_3)q_+ + b_1 + b_2] = 0. \end{aligned} \quad (39)$$

Later we will need the transformation $x^2 l_3 x^2$

$$x^2 l_3 x^2 : \{\mathbf{b}\} = \{b_1, b_2, b_3, b_4\} \rightarrow \{\tilde{\mathbf{b}}\} = \{b_1 + \frac{1}{2}, b_2 + \frac{1}{2}, b_3 + \frac{1}{2}, b_4 + \frac{1}{2}\}. \quad (40)$$

Combining (36)–(39) one obtains

$$\begin{aligned} q = \{ & \tilde{p}^2(\tilde{q} - t)(\tilde{q} - 1) + \tilde{p}(2 + b_1 + b_2 + b_3 + b_4 + (b_1 - b_2)t - (2 + 2b_1 + b_3 + b_4)\tilde{q}) \\ & + (1 + b_1 + b_3)(1 + b_1 + b_4) \} / \{ (1 + b_1 + b_3 + \tilde{p}(1 - \tilde{q}))(1 + b_1 + b_4 + \tilde{p}(1 - \tilde{q})) \}, \end{aligned} \quad (41)$$

where $q = q(\mathbf{b})$ and $\tilde{q} = q(\tilde{\mathbf{b}})$, $\tilde{p} = p(\tilde{\mathbf{b}})$.

3. Special elliptic solutions

It goes back to Picard [29] that if parameters (26) satisfy

$$b_1 = b_2 = 0, \quad b_3 = b_4 = -1/2 \quad (42)$$

then a general solution of $\mathbf{P}_{VI}$ reads (see also [30])

$$q(t) = \wp(c_1 \omega_1 + c_2 \omega_2; \omega_1, \omega_2) + \frac{t+1}{3}, \quad (43)$$

where $\wp(u; \omega_1, \omega_2)$ is the Weierstrass elliptic function with half-periods $\omega_{1,2}$, $c_{1,2}$ are arbitrary constants and $\omega_{1,2}(t)$ are two linearly independent solutions of the hypergeometric equations

$$t(1-t)\omega''(t) + (1-2t)\omega'(t) - \frac{1}{4}\omega(t) = 0. \quad (44)$$

It is convenient to choose

$$\omega_1(t) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right) = \mathbf{K}(t^{1/2}), \quad \omega_2(t) = i\frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-t\right) = i\mathbf{K}'(t^{1/2}), \quad (45)$$

where $\mathbf{K}$ and $\mathbf{K}'$ are the complete elliptic integrals of the modulus $k = t^{1/2}$.

Using expressions for invariants of the Weierstrass function e_1, e_2, e_3 ,

$$e_1 = 1 - \frac{t+1}{3}, \quad e_2 = t - \frac{t+1}{3}, \quad e_3 = -\frac{t+1}{3}, \quad (46)$$

we can rewrite Picard's solution of $\mathbf{P}_{VI}$ as

$$q(t) = \text{ns}^2(c_1 \mathbf{K} + c_2 i \mathbf{K}', k), \quad k = t^{1/2}. \quad (47)$$

Let us choose

$$c_1 = 1, \quad c_2 = \frac{1}{3} \quad (48)$$

and denote corresponding solution of (24) as $q_1(t)$. Using addition theorems for elliptic functions it is easy to show that $q_1(t)$ satisfies the algebraic equation

$$q_1^4(t) - 4tq_1^3(t) + 6tq_1^2(t) - 4tq_1(t) + t^2 = 0. \quad (49)$$

In fact, this algebraic solution of $\mathbf{P}_{VI}$ is very special. It solves (24) not only for Picard's choice of parameters (42) but also for

$$b_1 = \xi - \frac{1}{6}, \quad b_2 = 0, \quad b_3 = \xi - \frac{2}{3}, \quad b_4 = 2\xi - \frac{5}{6}, \quad (50)$$

where ξ is an arbitrary parameter. This happens because $\mathbf{P}_{VI}$, (24), splits up into two different equations which are both satisfied by (49).

Using the above formulae we can easily find expressions for $p_1(t)$ and $h_1(t)$ corresponding to (49):

$$\begin{aligned} p_1(t) &= \frac{(1 - 3\xi)(3t - 2tq_1(t) - q_1^2(t))}{6t(q_1(t) - 1)^2}, \\ h_1(t) &= \frac{1 - 2t}{72} + \frac{(1 - 2\xi)(1 - 3\xi)}{4} \frac{(t + q_1(t) - 2tq_1(t))}{q_1(t) - t}. \end{aligned} \quad (51)$$

Now we shall assume that this solution corresponds to the case $n = 1$ and apply Backlund transformation $x^2(l_3)^{1-n}x^2$ to obtain a series of solutions $\{q_n(t), p_n(t), h_n(t)\}$ with parameters

$$b_1 = \frac{1}{3} + \xi - \frac{n}{2}, \quad b_2 = \frac{1}{2} - \frac{n}{2}, \quad b_3 = -\frac{1}{6} + \xi - \frac{n}{2}, \quad b_4 = -\frac{1}{3} + 2\xi - \frac{n}{2}. \quad (52)$$

At this stage we are ready to establish a connection with a parameterization from the previous sections. Let us assume that the elliptic nome q in (2), (6), (7) and (10) is given by

$$q = \exp \left\{ i\pi \frac{2 \mathbf{K}'(k)}{3 \mathbf{K}(k)} \right\}, \quad k = t^{1/2}, \quad (53)$$

where $\mathbf{K}(k)$ and $\mathbf{K}'(k)$ are defined by (45). Using Landen transformation for elliptic functions it is easy to obtain the following rational parameterization for $z = 1/\gamma(q)^2$ defined in (10), (11), and for $t, q_1(t)$ in (49),

$$z = \frac{1}{\gamma^2(q)} = \frac{1 + \bar{\gamma}}{(3 - \bar{\gamma})\bar{\gamma}}, \quad t = \frac{(1 - \bar{\gamma})(3 + \bar{\gamma})^3}{(1 + \bar{\gamma})(3 - \bar{\gamma})^3}, \quad q_1(t) = \frac{(1 - \bar{\gamma})(3 + \bar{\gamma})}{(1 + \bar{\gamma})(3 - \bar{\gamma})}, \quad (54)$$

in terms of a new parameter $\bar{\gamma} \equiv \gamma(q^{1/2})$, defined by (10) with q replaced by $q^{1/2}$. Note that such parameterization of Picard's solutions of $\mathbf{P}_{VI}$ with the above choice (48) of the parameters c_1 and c_2 has already appeared in [30, 31].

From these formulae one can get an explicit connection of variables t and z :

$$t = \frac{(z - 1)(1 - 9z)^3}{32z} \left[1 + \frac{27z^2 - 18z - 1}{\sqrt{(1 - z)(1 - 9z)^3}} \right]. \quad (55)$$

Now we can construct a sequence of τ -functions associated with a series of auxiliary Hamiltonians $h_n(t)$. It appears that corresponding τ -functions are polynomials in variable z .

First, let us introduce a sequence of functions $\sigma_n(z)$ considering them as functions of z

$$\begin{aligned} \sigma_n(z) &= \frac{1}{tz} \sqrt{\frac{1 - 9z}{1 - z}} \left\{ h_n(t) + \frac{1}{72} (2t - 1) + (n - 1)^2 \left[\frac{t - 1}{4} + \frac{1 - 9z}{8} \sqrt{(1 - t)z} \right] \right. \\ &\quad \left. + (n - 1) \left(\xi - \frac{5}{12} \right) \left[1 - t + \frac{t(1 - 3z)}{\sqrt{(1 - z)(1 - 9z)}} \right] \right. \\ &\quad \left. + \left(\xi - \frac{1}{2} \right) \left(\xi - \frac{1}{3} \right) \left[\frac{3}{2} - \sqrt{\frac{1 - t}{z}} \right] \right\}. \end{aligned} \quad (56)$$

Comparing it with (51) and using (54) it is easy to see that

$$\sigma_1(z) = 0. \quad (57)$$

Then using Backlund transformations $x^2 l_3^{-1} x^2$ and $x^2 l_3 x^2$ it is not difficult to show that

$$\sigma_i(z) = 0, \quad i = 0, 1, 2. \quad (58)$$

The easiest way to do that is to calculate $h_i(t)$, $i = 0, 2$, in terms of $h_1(t)$ (51), substitute into (56) and use a rational parameterization (54).

Now let us introduce a family of τ -functions $\tau_n(z, \xi)$ via

$$\sigma_n(z) = \frac{d}{dz} [\log \tau_n(z, \xi)] \quad (59)$$

and fix a normalization for $n = 0, 1, 2$ as

$$\tau_0(z, \xi) = 1, \quad \tau_1(z, \xi) = -4\xi + 5/3, \quad \tau_2(z, \xi) = 3(2\xi - 1)(3\xi - 1). \quad (60)$$

Using Okamoto's Toda-recursion relations for τ -functions for $\mathbf{P}_{VI}$, generated via successive applications of parallel transformation l_3 [27], one can show that the recurrence relation for $\tau_n(z, \xi)$ exactly coincides with (15). Thus, we showed that the leading coefficient and the constant term of $\mathcal{P}_n(x, z)$ (considered as polynomials in x) can be expressed in terms τ -functions for special solutions of $\mathbf{P}_{VI}$.

At the moment we do not have a complete proof of the polynomiality of $\tau_n(z)$. Note that this property takes place provided that two successive τ -functions $\tau_n(z)$ and $\tau_{n+1}(z)$ do not have a non-trivial common divisor (which is a polynomial in z).

One of the challenging problems is to find a determinant representation for $\tau_n(z, \xi)$ similar to the known other polynomial solutions of the Painlevé equations. It could help to clarify the structure of $\mathcal{P}_n(x, z)$ and possibly to establish a connection with problems of combinatorics.

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