

Measurement-Induced Boolean Dynamics and Controllability for Quantum Networks

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Abstract

In this paper, we study dynamical quantum networks which evolve according to Schrödinger equations but subject to sequential local or global quantum measurements. A network of qubits forms a composite quantum system whose state undergoes unitary evolution in between periodic measurements, leading to hybrid quantum dynamics with random jumps at discrete time instances along a continuous orbit. The measurements either act on the entire network of qubits, or only a subset of qubits. First of all, we reveal that this type of hybrid quantum dynamics induces probabilistic Boolean recursions representing the measurement outcomes. With global measurements, it is shown that such resulting Boolean recursions define Markov chains whose state-transitions are fully determined by the network Hamiltonian and the measurement observables. Particularly, we establish an explicit and algebraic representation of the underlying recursive random mapping driving such induced Markov chains. Next, with local measurements, the resulting probabilistic Boolean dynamics is shown to be no longer Markovian. The state transition probability at any given time becomes dependent on the entire history of the sample path, for which we establish a recursive way of computing such non-Markovian probability transitions. Finally, we adopt the classical bilinear control model for the continuous Schrödinger evolution, and show how the measurements affect the controllability of the quantum networks.

1 Introduction

Quantum systems admit drastically different behaviors compared to classical systems in terms of state representations, evolutions, and measurements, based on which there holds the promise to develop fundamentally new computing and cryptography infrastructures for our society (Nielsen and Chuang, 2010).

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Quantum states are described by vectors in finite or infinite dimensional Hilbert spaces; isolated quantum systems exhibit closed dynamics described by Schrödinger equations; performing measurements over a quantum system yields random outcomes and creates back action to the system being measured. When interacting with environments, quantum systems admit more complex evolutions which are often approximated by various types of master equations. The study of the evolution and manipulation of quantum states has been one of the central problems in the fields of quantum science and engineering (Altafini and Ticozzi, 2012).

For the control or manipulation of quantum systems, we can carry out feedforward control by directly revising the Hamiltonians in the Schrödinger equations (Brockett, 1972), resulting in bilinear control systems. Celebrated results have been established regarding the controllability of such systems from the perspective of geometric nonlinear control (Jurdjevic and Sussman, 1972; Brockett, 1972; Brockett and Khaneja, 2000; Schirmer, Fu and Solomon, 2001; Albertini and D'Alessandro, 2003; Li and Khaneja, 2009; Tsopelakos, Belabbas and Ghahesifard, in press, 2018). In the presence of external environments, one can also directly engineer the interaction between the quantum system of interest and the environments, e.g., (Schirmer and Wang, 2010; Ticozzi, Schirmer and Wang, 2010). Feedforward can also be carried out by designing a sequence of measurements from different bases (Pechen, Ilin, Shuang and Rabitz, 2006), where the quantum back actions from the measurements are utilized as a control mean.

Feedback control can also be carried out for quantum systems via coherent feedback (James, Nurdin and Petersen, 2008) or measurement feedback (Belavkin, 1999; Blok, Bonato, Markham, Twitchen, Dobrovitski and Hanson, 2014). In coherent feedback, the outputs of a quantum system are fed back to the control of the inherent or interacting Hamiltonians. While in measurement feedback, the measurement outcomes are fed back to the selection of the future measurement bases. Introducing feedback to the control of quantum systems on one hand improves the robustness of the closed-loop system, and on the other hand, the resulting quantum back actions intrinsically perturb the system states subject to the quantum uncertainty principle.

Qubits, the so-called quantum bits, are the simplest quantum states with a two-dimensional state space. Qubits naturally form networks in various forms of interactions: they can interact directly with each other by coupling Hamiltonians in a quantum composite system (Altafini, 2002); implicitly through coupling with local environments (Shi, Dong, Petersen and Johansson, 2016); or through local quantum operations such as measurements and classical communications on the operation outcomes (Perseguers, Lewenstein, Acin and Cirac, 2010). Qubit networks have become canonical models for quantum mechanical states and interactions between particles and fields under the notion of spin networks (Kato and Yamamoto, 2014), and for quantum information processing platforms in computing and communication (Perseguers et al., 2010; Shi, Li, Miao, Dower and James, 2017). The control of qubit networks has been studied in various forms (Albertini and D'Alessandro, 2002; Wang, Pemberton-Ross and Schirmer, 2012; Dirr and Helmke, 2008; Shi et al., 2016; Li, Zhang and Wang, 2017).

In this paper, we study dynamical qubit networks which evolve as a collective isolated quantum system

but subject to sequential local or global measurements. Global measurements are represented by observables applied to all qubits in the network, and local measurements only apply to a subset of qubits and therefore the state information of the remaining qubits becomes hidden. We reveal that this type of hybrid quantum dynamics induces probabilistic Boolean recursions representing the measurement outcomes, defining a quantum-induced probabilistic Boolean network. Boolean networks, introduced by Kauffman in the 1960s (Kauffman, 1969) and then extended to probabilistic Boolean networks (Shmulevich, Dougherty, Kim and Zhang, 2002), have been a classical model for gene regulatory interactions. The behaviors of Boolean dynamics are quite different compared to classical dynamical systems described by differential or difference equations due to their combinatorial natures, and their studies have been focused on the analytical or approximate characterizations to the steady-state orbits and controllability (Tournier and Chaves, 2013; Cheng and Qi, 2009). The contributions of the paper are summarized as follows:

- Under global measurements, the induced Boolean recursions define Markov chains for which we establish a purely algebraic representation of the underlying recursive random mapping. The representation is in the form of random linear systems embedded in a high dimensional real space.
- Under local measurements, the resulting probabilistic Boolean dynamics is no longer Markovian. The transition probability at any given time relies on the entire history of the sample path, for which we establish a recursive computation scheme.
- In view of the classical bilinear model for closed quantum systems, we demonstrate how the measurements affect the controllability of the quantum networks. In particular, we show that practical quantum state controllability is already enough to guarantee almost sure Boolean state controllability.

The remainder of the paper is organized as follows. Section 2 presents a collection of preliminary knowledge and theories which are essential for our discussion. Section 3 presents the qubit network model for the study. Section 4 focuses on the induced Boolean network dynamics from the measurements of the dynamical qubit network. Section 5 then turns to the controllability of such qubit networks under bilinear control. Finally Section 6 concludes the paper with a few remarks.

2 Preliminaries

In this section, we present some preliminary knowledge on quantum system states and measurements, quantum state evolution and bilinear control, probabilistic Boolean networks, and Lie algebra and groups, in order to facilitate a self-contained presentation.

2.1 Quantum States and Projective Measurements

The state space of any isolated quantum system is a complex vector space with inner product, i.e., a Hilbert space $\mathcal{H}_N \simeq \mathbb{C}^N$ for some integer $N \geq 2$. The system state is described by a unit vector in \mathcal{H}_N denoted by $|\varphi\rangle$, where $|\cdot\rangle$ is known as the Dirac notation for vectors representing quantum states. The complex conjugate transpose of $|\varphi\rangle$ is denoted by $\langle\varphi|$. One primary feature that distinguishes quantum systems from classical systems is the state space of composite system consisting of one or more subsystems. The state space of a composite quantum system is the tensor product of the state space of each component system. As a result, the states of a composite quantum system of two subsystems with state space \mathcal{H}_A and \mathcal{H}_B , respectively, are complex linear combinations of $|\varphi_A\rangle \otimes |\varphi_B\rangle$, where $|\varphi_A\rangle \in \mathcal{H}_A$, $|\varphi_B\rangle \in \mathcal{H}_B$.

Let $\mathcal{L}_*(\mathcal{H}_N)$ be the space of linear operators over \mathcal{H}_N . For a quantum system associated with state space \mathcal{H}_N , a projective measurement is described by an observable M , which is a Hermitian operator in $\mathcal{L}_*(\mathcal{H}_N)$. The observable M has a spectral decomposition in the form of

$$M = \sum_{m=0}^{N-1} \lambda_m P_m,$$

where P_m is the projector onto the eigenspace of M with eigenvalue λ_m . The possible outcomes of the measurement correspond to the eigenvalues λ_m , $m = 0, \dots, N-1$ of the observable. Upon measuring the state $|\varphi\rangle$, the probability of getting result λ_m is given by $p(\lambda_m) = \langle\varphi|P_m|\varphi\rangle$. Given that outcome λ_m occurred, the state of the quantum system immediately after the measurement is $\frac{P_m|\varphi\rangle}{\sqrt{p(\lambda_m)}}$.

2.2 Closed Quantum Systems

The time evolution of the state $|\varphi(s)\rangle \in \mathcal{H}_N$ of a closed quantum system is described by a Schrödinger equation:

$$|\dot{\varphi}(s)\rangle = -iH(s)|\varphi(s)\rangle, \quad (1)$$

where $H(s)$ is a Hermitian operator over \mathcal{H}_N known as the Hamiltonian of the system at time s . Hamiltonians relate to physical quantities such as momentum, energy etc. for quantum systems. Here without loss of generality the initial time is assumed to be $s = 0$. For any time instants $s_1, s_2 \in [0, \infty)$, there exists a unique unitary operator $U_{[s_1, s_2]}$ such that

$$|\varphi(s_2)\rangle = U_{[s_1, s_2]}|\varphi(s_1)\rangle \quad (2)$$

along the Schrödinger equation (1).

2.3 Bilinear Model for Quantum Control

Let $\mathcal{O}_*(\mathcal{H}_N)$ be the space of Hermitian operators over \mathcal{H}_N . The basic bilinear model for the control of a quantum system is defined by letting $H(s) = H_0 + \sum_{\ell=1}^p u_\ell(s)H_\ell$ in the Schrödinger equation (1),

where $\mathbf{H}_0 \in \mathcal{O}_*(\mathcal{H}_N)$ is the unperturbed or internal Hamiltonian, and $\mathbf{H}_\ell \in \mathcal{O}_*(\mathcal{H}_N)$, $\ell = 1, \dots, p$ are the controlled Hamiltonians with the $u_\ell(s)$, $\ell = 1, \dots, p$ being control signals as real scalar functions. This leads to

$$\begin{aligned} |\dot{\varphi}(s)\rangle &= -i \left(\mathbf{H}_0 + \sum_{\ell=1}^p u_\ell(s) \mathbf{H}_\ell \right) |\varphi(s)\rangle \\ &:= \left(\mathbf{A} + \sum_{\ell=1}^p u_\ell(s) \mathbf{B}_\ell \right) |\varphi(s)\rangle, \end{aligned} \quad (3)$$

where $\mathbf{A} = -i\mathbf{H}_0$, and $\mathbf{B}_\ell = -i\mathbf{H}_\ell$. The background of this model lies in physical quantum systems for which we can manipulate their Hamiltonians. Let $\mathbf{X}(s)$ be the operator defined for $s \in [0, \infty)$ satisfying

$$|\varphi(s)\rangle = \mathbf{X}(s)|\varphi(0)\rangle \quad (4)$$

for all $s \geq 0$ along the equation (3). It can be shown that the evolution matrix operator $\mathbf{X}(s)$ is described by

$$\dot{\mathbf{X}}(s) = \left(\mathbf{A} + \sum_{\ell=1}^p u_\ell(s) \mathbf{B}_\ell \right) \mathbf{X}(s) \quad (5)$$

starting from $\mathbf{X}(0) = \mathbf{I}_N$.

The following two definitions specify basic controllability questions arising from the bilinear model (3).

Definition 1. *The system (3) is pure state controllable if for every pair quantum states $|\varphi_0\rangle, |\varphi_1\rangle \in \mathcal{H}_N$, there exist $\mu > 0$ and control signals $u_1(s), \dots, u_p(s)$ for $s \in [0, \mu]$ such that the solution of (3) yields $|\varphi(\mu)\rangle = |\varphi_1\rangle$ starting from $|\varphi(0)\rangle = |\varphi_0\rangle$.*

Definition 2. *The system (3) is equivalent state controllable if for every pair quantum states $|\varphi_0\rangle, |\varphi_1\rangle \in \mathcal{H}_N$, there exist $\mu > 0$, control signals $u_1(s), \dots, u_p(s)$ for $s \in [0, \mu]$, and a phase factor ϕ such that the solution of (3) yields $|\varphi(\mu)\rangle = e^{i\phi}|\varphi_1\rangle$ starting from $|\varphi(0)\rangle = |\varphi_0\rangle$.*

Remark 1. *From a physical point of view, the states $e^{i\phi}|\varphi\rangle$ and $|\varphi\rangle$ are the same as the phase factor $e^{i\phi}$ contributes to no observable effect.*

2.4 Probabilistic Boolean Networks

A Boolean network consists of n nodes in $V = \{1, 2, \dots, n\}$ with each node i holding a logical value $x_i(t) \in \{0, 1\}$ at discretized time $t = 0, 1, 2, \dots$. Denote $\mathbf{x}(t) = [x_1(t) \dots x_n(t)]$, and let \mathcal{S} denote the space containing all functions that map $\{0, 1\}^n$ to $\{0, 1\}^n$. The evolution of the network states $\mathbf{x}(t)$ can then be described by the functions in \mathcal{S} . In a probabilistic Boolean network, at each time $t = 0, 1, 2, \dots$, a function f_t is drawn randomly from \mathcal{S} according to some underlying distributions, and the network state evolves according to

$$\mathbf{x}(t+1) = f_t(\mathbf{x}(t)). \quad (6)$$

To be precise, $\Omega = \mathcal{S} \times \mathcal{S} \times \dots$ and $\mathcal{F} = 2^{\mathcal{S}} \times 2^{\mathcal{S}} \times \dots$ are the overall sample space and event algebra \mathcal{F} equipped with probability measure \mathbb{P} , where $\omega = (\omega_0, \omega_1, \dots) \in \Omega$. Let \mathcal{F}_t be the filtration

$$\mathcal{F}_t = \underbrace{2^{\mathcal{S}} \times 2^{\mathcal{S}} \times \dots \times 2^{\mathcal{S}}}_{t+1} \times \{\emptyset, \mathcal{S}\} \times \{\emptyset, \mathcal{S}\} \times \dots. \quad (7)$$

Here by saying f_t is randomly drawn, it means $f_t(\omega) = \omega_t$ and therefore $\sigma(f_t) \in \mathcal{F}_t$.

2.5 Lie Algebra and Lie Group

A Lie algebra $\mathcal{L} \subset \mathcal{L}_*(\mathcal{H}_N)$ is a linear subspace of $\mathcal{L}_*(\mathcal{H}_N)$ which is closed under the Lie bracket operation, i.e., if $A, B \in \mathcal{L}$, then $[A, B] = AB - BA \in \mathcal{L}$. For $\{B_1, \dots, B_p\}$ being a subset of \mathcal{L} , the Lie algebra generated by $\{B_1, \dots, B_p\}$, denoted by $\mathcal{L}\{B_1, \dots, B_p\}$, is the smallest Lie subalgebra in $\mathcal{L}_*(\mathcal{H}_N)$ containing $\{B_1, \dots, B_p\}$. Given a Lie algebra \mathcal{L} , the associated Lie group, denoted by $\{e^{\mathcal{L}}\}_G$ or simply $e^{\mathcal{L}}$, is the one-parameter group $\{\exp(tA) : t \in \mathbb{R}, A \in \mathcal{L}\}$. Here $\exp : \mathcal{L} \rightarrow e^{\mathcal{L}}$ denotes the exponential map, i.e., $\exp(tA) = e^{tA} := \sum_{i=0}^{\infty} \frac{t^i A^i}{i!}$.

The space of skew-Hermitian operators over \mathcal{H}_N forms a Lie algebra, which is denoted by $\mathfrak{u}(N)$. The Lie group associated with $\mathfrak{u}(N)$ is denoted by $U(N)$, which is the space of unitary operators over \mathcal{H}_N . Let $\mathfrak{su}(N)$ denote the Lie algebra containing all traceless skew-Hermitian operators over \mathcal{H}_N , and $\mathfrak{sp}(2N)$ be the Lie algebra containing $\{X \in \mathfrak{su}(2N) : XJ + JX^T = 0\}$ with $J \in \mathcal{L}_*(\mathcal{H}_{2N})$ whose matrix representation can be $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ under certain basis.

Theorem 1. (Albertini and D'Alessandro, 2003) *The pure state controllability and equivalent state controllability are equivalent for the system (3). The system (3) is pure state controllable or equivalent state controllable if and only if $\mathcal{L}\{A, B_1, \dots, B_p\}$ is isomorphic to*

$$\begin{cases} \mathfrak{sp}(N/2) \text{ or } \mathfrak{su}(N), & N \text{ is even,} \\ \mathfrak{su}(N), & N \text{ is odd.} \end{cases} \quad (8)$$

3 The Quantum Network Model

In this section, we present the quantum networks model for our study. We consider a network of qubits subject to bilinear control, which aligns with the spin-network models in the literature. We also consider a sequential measurement process where global or local qubit measurements take place periodically.

3.1 Qubit Networks

Qubit is the simplest quantum system whose state space is a two-dimensional Hilbert space $\mathcal{H} (:= \mathcal{H}_2)$. Let n qubits indexed by $V = \{1, \dots, n\}$ form a network with state space $\mathcal{H}^{\otimes n}$. The (pure) states of the qubit network are then in the space $\mathcal{Q}(2^n) := \{q \in \mathcal{H}_{2^n} : |q|^2 = 1\}$.

Let there be a projective measurement (or an observable) for a single qubit as

$$M = \lambda_0 P_0 + \lambda_1 P_1,$$

where $P_m = |v_m\rangle\langle v_m|$ is the projector onto the eigenspace generated by $|v_m\rangle$ with eigenvalue λ_m , $m \in \{0, 1\}$. For the n -qubit network, we can have either global or local measurements.

Definition 3. (i) We term $M^{\otimes n} = M \otimes \cdots \otimes M$ as a global measurement over the n -qubit network.

(ii) Let $V_* = \{i_1, \dots, i_k\} \subset V$. Then

$$M^{V_*} = I \otimes \cdots \otimes I \otimes \overbrace{M}^{i_1\text{-th}} \otimes I \otimes \cdots \otimes I \otimes \overbrace{M}^{i_k\text{-th}} \otimes I \otimes \cdots \otimes I$$

is defined as a local measurement over V_* .

The global measurement $M^{\otimes n}$ measures the individual qubit states of the entire network, which yields 2^n possible outcomes $[\lambda_{m_1}, \dots, \lambda_{m_n}]$, $m_j \in \{0, 1\}$, $j = 1, \dots, n$. Upon measuring the state $|\varphi\rangle$, the probability of getting result $[\lambda_{m_1}, \dots, \lambda_{m_n}]$ is given by $p([\lambda_{m_1}, \dots, \lambda_{m_n}]) = \langle \varphi | P_{m_1} \otimes \cdots \otimes P_{m_n} | \varphi \rangle$. Given that the outcome $[\lambda_{m_1}, \dots, \lambda_{m_n}]$ occurred, the qubit network state immediately after the measurement is $|\varphi\rangle_P = |v_{m_1}\rangle \otimes \cdots \otimes |v_{m_n}\rangle$. On the other hand, the local measurement M^{V_*} measures the states of the qubits in the set V_* only, which yields 2^k possible outcomes $[\lambda_{m_{i_1}}, \dots, \lambda_{m_{i_k}}]$, $i_j \in \{0, 1\}$, $j = 1, \dots, k$ corresponding to the qubits $\{i_1, \dots, i_k\}$. Upon measuring the state $|\varphi\rangle$, the probability of getting result $[\lambda_{m_{i_1}}, \dots, \lambda_{m_{i_k}}]$ is

$$p([\lambda_{m_{i_1}}, \dots, \lambda_{m_{i_k}}]) = \langle \varphi | I \otimes \cdots \otimes I \otimes P_{m_{i_1}} \otimes I \otimes \cdots \otimes I \otimes P_{m_{i_k}} \otimes I \otimes \cdots \otimes I | \varphi \rangle,$$

where $m_{i_j} \in \{0, 1\}$, $j = 1, \dots, k$. Since the local measurement reveals no information about the nodes in $V \setminus V_*$, we term the qubits in V_* as the measured qubits, and those in $V \setminus V_*$ as the dark qubits. For the ease of presentation and without loss of generality, we assume $V_* = \{1, \dots, k\}$ throughout the remainder of the paper.

3.2 Hybrid Qubit Network Dynamics

Consider the continuous time horizon represented by $s \in [0, \infty)$. Let $|q(s)\rangle$ denote the qubit network state at time s . Let the evolution of $|q(s)\rangle$ be defined by a Schrödinger equation with controlled Hamiltonians in the form of (3), and the network state be measured globally or locally from $s = 0$ periodically with a period T . To be precise, $|q(s)\rangle$ satisfies the following hybrid dynamical equations

$$|\dot{q}(s)\rangle = \left(A + \sum_{\ell=1}^p u_\ell(s) B_\ell \right) |q(s)\rangle, \quad s \in [tT, (t+1)T), \quad (9)$$

$$|q((t+1)T)\rangle = |q((t+1)T)^-\rangle_P, \quad (10)$$

for $t = 0, 1, 2, \dots$, where $|q((t+1)T)^-\rangle$ represents the quantum network state right before $(t+1)T$ along (9) starting from $|q(tT)\rangle$, and $|q((t+1)T)^-\rangle_{\text{p}}$ is the post-measurement state of the network when a measurement is performed at time $s = (t+1)T$. For the ease of presentation, we define quantum states

$$\begin{aligned} |\psi(t)\rangle &= |q((tT)^-\rangle), \\ |\psi(t)\rangle_{\text{p}} &= |q(tT)\rangle \end{aligned}$$

for the pre- and post-measurement network states at the $(t+1)$ -th measurement.

In particular, the control signals $u_{\ell}(s)$, $\ell = 1, \dots, p$ will have feedforward or feedback forms.

Definition 4. (i) *The control signals $u_{\ell}(s)$, $\ell = 1, \dots, p$ are feedforward if their values are determined deterministically at $t = 0^-$ for the entire time horizon $s \geq 0$.*

(ii) *The control signals $u_{\ell}(s)$, $\ell = 1, \dots, p$ are feedback if each $u_{\ell}(s)$ for $s \in [tT, (t+1)T)$ depends on the post-measurement state $|\psi(t')\rangle_{\text{p}}$, $t' = 0, 1, \dots, t$.*

3.3 Problems of Interest

The evolution of the quantum system (9)–(10) defines a quantum hybrid with state resets, analogous to the study of classical hybrid systems with state jumps (Ogura and Martin, 2014). We note that such state evolution represents physical systems that exist in the real world, where sequential measurements are performed for quantum dynamical systems (Blok et al., 2014). The mixture of the continuous-time dynamics and the random state resets leads to intrinsic questions related to the relationship between the quantum state and the measurement outcome evolutions. Furthermore, how the continuous bilinear control (9) will be affected by the sequential measurements is also an interesting point for investigation. In this paper, we focus on the following questions:

Q1: How can we characterize the dynamics of the measurement outcomes from the quantum networks with feedforward control?

Q2: How the sequential measurements with feedback control will influence the controllability properties of the classical bilinear model (9)?

4 Boolean Dynamics from Quantum Measurements

In this section, we focus our attention on the induced Boolean dynamics from the sequential measurements of the qubit networks. We impose the following assumption.

Assumption 1. *The $u_{\ell}(s)$, $\ell = 1, \dots, p$ are feedforward signals. Consequently, there exist a sequence of deterministic U_t , $t = 0, 1, 2, \dots$ such that $|\psi(t+1)\rangle = U_t|\psi(t)\rangle_{\text{p}}$.*

4.1 Induced Probabilistic Boolean Networks

Under the global measurement $M^{\otimes n}$, we can use the Boolean variable $x_i(t) \in \{0, 1\}$ to represent the measurement outcome at qubit i for step t , where $x_i(t) = 0$ corresponds to λ_0 and $x_i(t) = 1$ corresponds to λ_1 . We can further define the n -dimensional random Boolean vector

$$\mathbf{x}(t) = [x_1(t), \dots, x_n(t)] \in \{0, 1\}^n$$

as the outcome of measuring $|\psi(t)\rangle$ under $M^{\otimes n}$ at step t . The recursion of $|\psi(t)\rangle_{\text{p}}$ generates the corresponding recursion of $\mathbf{x}(t)$ for $t = 0, 1, 2, \dots$, resulting in an induced probabilistic Boolean network (PBN). Similarly, subject to local measurement, we can define $\mathbf{x}_k(t) = [x_1(t), \dots, x_k(t)] \in \{0, 1\}^k$ as the outcome of measuring $|\psi(t)\rangle$ by $M^{\vee*}$, where $x_i(t) \in \{0, 1\}$ continues to represent the measurement outcome at qubit i .

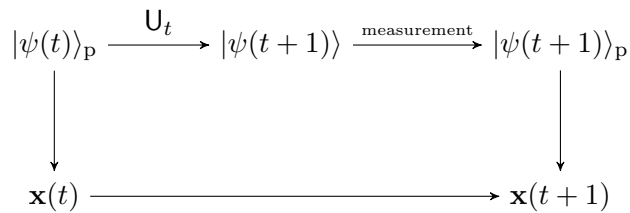


Figure 1: Induced Boolean network dynamics.

We are interested in the interplay between the underlying quantum state evolution and the induced probabilistic Boolean network dynamics.

4.2 Global Measurement: Markovian PBN

4.2.1 Transition Characterizations

We first analyze the behaviors of the induced probabilistic Boolean network dynamics under global qubit network measurements. Let δ_N^i be the i -th column of identity matrix I_N . Denote $\Delta_N = \{\delta_N^i | i = 1, \dots, N\}$, and particularly $\Delta := \Delta_2$ for simplicity. Identify $\{0, 1\} \simeq \Delta$ under which $0 \sim \delta_2^1$ and $1 \sim \delta_2^2$. Let $\mathbf{x} = [x_1, \dots, x_n] \in \{0, 1\}^n$ be associated with

$$\mathbf{x}^\sharp := \delta_2^{x_1+1} \otimes \dots \otimes \delta_2^{x_n+1} = \delta_{2^n}^{\sum_{i=1}^n x_i 2^{n-i} + 1}, \quad (11)$$

where \otimes represents the Kronecker product. In this way, we have identified $\{0, 1\}^n \simeq \Delta_{2^n}$. For the ease of presentation, we also denote $[\mathbf{x}] := \sum_{i=1}^n x_i 2^{n-i} + 1$, and consider \mathbf{x} , $[\mathbf{x}]$, and $\mathbf{x}^\sharp = \delta_{2^n}^{[\mathbf{x}]}$ interchangeable without further mentioning. Recall \mathcal{S} as the set containing all $(2^n)^{2^n}$ Boolean mappings from $\{0, 1\}^n$ to $\{0, 1\}^n$. Each element in \mathcal{S} is indexed by $f_{[\alpha_1, \dots, \alpha_{2^n}]} \in \mathcal{S}$ with $\alpha_i = 1, \dots, 2^n$, $i = 1, \dots, 2^n$, where

$$f_{[\alpha_1, \dots, \alpha_{2^n}]}(s_i) = s_{\alpha_i}, \quad s_i \in \{0, 1\}^n, \quad i = 1, \dots, 2^n. \quad (12)$$

In this way, the matrix $f_{[\alpha_1, \dots, \alpha_{2^n}]} = [\delta_{2^n}^{\alpha_1}, \dots, \delta_{2^n}^{\alpha_{2^n}}]$ serves as a representation of $f_{[\alpha_1, \dots, \alpha_{2^n}]}$ since

$$f_{[\alpha_1, \dots, \alpha_{2^n}]} \delta_{2^n}^i = \delta_{2^n}^{\alpha_i}, \quad i = 1, \dots, 2^n. \quad (13)$$

Recall the observable $M = \lambda_0 P_0 + \lambda_1 P_1$ for one qubit. We choose $\{|0\rangle, |1\rangle\}$ as the standard orthonormal basis of \mathcal{H} , and denote $Q_0 = |0\rangle\langle 0|$, $Q_1 = |1\rangle\langle 1|$. Then there exists a unitary operator $u = |v_0\rangle\langle 0| + |v_1\rangle\langle 1| \in \mathfrak{L}_*(\mathcal{H})$, whose representation under the chosen basis $\{|0\rangle, |1\rangle\}$ is $u \in \mathbb{C}^{2 \times 2}$ which is a unitary matrix, such that $P_0 = u Q_0 u^\dagger$ and $P_1 = u Q_1 u^\dagger$.

Let $\{|0\rangle, |1\rangle\}^{\otimes n}$ be the standard computational basis of the n -qubit network. We denote for $i = 1, \dots, 2^n$ that

$$|b_i\rangle = |b_{i_1} \cdots b_{i_n}\rangle \quad (14)$$

where $|b_{i_1} \cdots b_{i_n}\rangle \in \{|0\rangle, |1\rangle\}^{\otimes n}$ with $b_{i_j} \in \{0, 1\}$, $j = 1, \dots, n$. Now we can sort the elements of $\{|0\rangle, |1\rangle\}^{\otimes n}$ by the value of $|b_i\rangle$ in an ascending order. Let U_t have the representation $U_t \in \mathbb{C}^{2^n \times 2^n}$ under such an ordered basis. Note that $u \otimes \cdots \otimes u$ has its matrix representation as $u \otimes \cdots \otimes u$ under the same sorted basis. Define

$$U_t^M = (u \otimes \cdots \otimes u)^\dagger U_t (u \otimes \cdots \otimes u). \quad (15)$$

For the induced Boolean series $\{\mathbf{x}(t)\}_{t=0}^\infty$, the following result holds, whose proof is omitted as it is a direct verification of quantum measurement postulate.

Proposition 1. *Let Assumption 1 hold. With global measurement, the $\{\mathbf{x}(t)\}_{t=0}^\infty$ form a Markov chain over the state space $\{0, 1\}^n$, whose state transition matrix \mathbf{P}_t at time t is given by*

$$[\mathbf{P}_t]_{i,j} = \mathbb{P}(\mathbf{x}(t+1) | \mathbf{x}(t)) = |[U_t^M]_{j,i}|^2,$$

for $i = \lfloor \mathbf{x}(t) \rfloor, j = \lfloor \mathbf{x}(t+1) \rfloor \in \{1, 2, \dots, 2^n\}$, where $[\cdot]_{i,j}$ stands for the (i, j) -th entry of a matrix. In fact, there holds $\mathbf{P}_t = (U_t^M)^\dagger \circ (U_t^M)^\top$, where \circ stands for the Hadamard product.

The following theorem establishes an algebraic representation of the recursion for $\{\mathbf{x}(t)\}_{t=0}^\infty$.

Theorem 2. *Let Assumption 1 hold. The recursion of $\{\mathbf{x}(t)\}_{t=0}^\infty$ can be represented as a random linear mapping*

$$\mathbf{x}^\sharp(t+1) = F_t \mathbf{x}^\sharp(t), \quad (16)$$

where $\langle F_t \rangle$ is a series of independent random matrices in $\mathbb{R}^{2^n \times 2^n}$. Moreover, the distribution of F_t is described by

$$\mathbb{P}(F_t = f_{[\alpha_1, \dots, \alpha_{2^n}]}) = \prod_{i=1}^{2^n} |[U_t^M]_{\alpha_i, i}|^2.$$

The proofs of Proposition 1 and Theorem 2 are deferred to the Appendix.

Remark 2. Although Theorem 2 provides a way of explicitly representing the evolution of the measurement outcomes, the inherent computational complexity does not get reduced. The dimension of $\mathbf{x}^\sharp(t)$ grows exponentially as the number of qubits grows. However, the state transition F_t is in general a sparse matrix, which might lead to potential computational reduction in the establishment on usage of (16).

Remark 3. Note that Proposition 1 and Theorem 2 hold for general quantum states and unitary evolution U_t . Let M be taken as the standard computational basis. Then from the identity $\mathbf{P}_t = (U_t)^\dagger \circ (U_t)^\top$, the structure of U_t is fully inherited by \mathbf{P}_t . As a result, if U_t is an entangling unitary operator, the same entangling structure will be preserved by the state-transition matrix \mathbf{P}_t . In fact, the correlations between the $x_i(t)$ arise from \mathbf{P}_t , in contrast to the correlation of the qubit states induced by U_t .

4.2.2 Quantum Realization of Classical PBN

From Theorem 2, one can see that the n -qubit network under global sequential measurement $M^{\otimes n}$ always induces a Markovian probabilistic Boolean network. When U_t is time invariant, $\{\mathbf{x}(t)\}_{t=0}^\infty$ is a homogeneous chain. A natural question lies in whether any classic probabilistic Boolean network with a homogeneous transition could be realized by the qubit networks under investigation. This question is related to the unistochastic matrix theory. A matrix $W \in \mathbb{R}^{N \times N}$ is doubly stochastic if it is a square matrix of nonnegative real numbers, each of whose rows and columns sums to 1, i.e., $\sum_i [W]_{i,j} = \sum_j [W]_{i,j} = 1$. A doubly stochastic matrix T is unistochastic if its entries are the squares of the absolute values of the entries from certain unitary matrix, i.e., there exists a unitary matrix U such that $[W]_{i,j} = |[U]_{i,j}|^2$ for $i, j = 1, \dots, N$. It is still an open problem to tell whether a given doubly stochastic matrix is unistochastic or not (Dunkl and Życzkowski, 2009).

Note that instead of using the global measurement $M^{\otimes n}$, we may choose another global measurement as $M_1 \otimes \dots \otimes M_n$, i.e., the observable of qubit i is $M_i = \lambda_{i_0}|v_{i_0}\rangle\langle v_{i_0}| + \lambda_{i_1}|v_{i_1}\rangle\langle v_{i_1}|$, then assume the matrix representation of $u_i = |v_{i_0}\rangle\langle 0| + |v_{i_1}\rangle\langle 1|$ is u_i for qubit i under the basis $\{|0\rangle, |1\rangle\}$. Then we have $U^M = (u_1 \otimes \dots \otimes u_n)^\dagger U (u_1 \otimes \dots \otimes u_n)$, which is still a unitary matrix. As a result, using a more general measurement $M_1 \otimes \dots \otimes M_n$ does not reduce the difficulty of the quantum realization problem.

Alternatively, we can try to solve the quantum realization problem approximately. Given a column stochastic matrix $W \in \mathbb{R}^{N \times N}$, we define

$$\begin{aligned} & \text{minimize} && \sum_{i,j=1,\dots,N} \left| |[U]_{i,j}|^2 - [W]_{i,j} \right|^2 \\ & \text{subject to} && UU^\dagger = I, \quad U \in \mathbb{C}^{N \times N}, \end{aligned}$$

which is a polynomial optimization problem.

In general, this optimization problem may lead to multiple solutions, implying potential ambiguity in identifying the unitary operator U from the state-transition probability matrix of the induced Markov chain. However, whenever such an optimization problem yields exact solutions, or a solution with a sufficiently small gap compared to exact solutions, our quantum network with sequential measurements

becomes a potential resource for the realization of the given Markov chain. For a Markov chain with L states, it suffices to use $\log L$ qubits for the quantum network realization, where the quantum measurements become the intrinsic resource of the randomness.

4.2.3 Examples

We consider a two-qubit network. Let an observable be given for one qubit along standard computational basis $\{|0\rangle, |1\rangle\}$ as $M = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|$. The resulting global network measurement is $M \otimes M$. Then the set of possible outcomes is $\{0, 1\}^2$. The random Boolean mapping $F_t : \mathcal{S} \rightarrow \mathcal{S}$ has $4^4 = 256$ possible realizations.

Example 1. Let the unitary operator acting on the two-qubit network be

$$U_t \equiv U_1 = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes (-i|0\rangle\langle 1| + i|1\rangle\langle 0|).$$

The state transition map of the homogeneous Markov chain induced by U_1 and M is shown in Fig. 2, and F_t has only one realization.

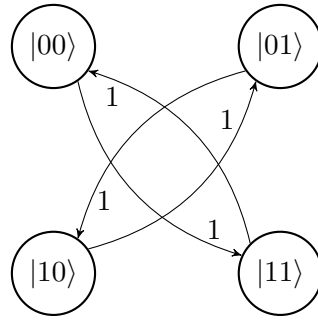


Figure 2: State transition map of the Markov chain induced by U_1 and M .

Example 2. Let the unitary operator be alternatively given as

$$U_t \equiv U_2 = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes \frac{|0\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|}{\sqrt{2}}.$$

The state transition map of the homogeneous Markov chain induced by U_2 and M is shown in Fig. 3. Moreover, F_t has 16 realizations each of which happens with equal probability $1/16$.

Example 3. Let $H = \frac{\pi}{3}(|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|) + \frac{\pi}{6}(-i|0\rangle\langle 1| + i|1\rangle\langle 0|) \otimes (-i|0\rangle\langle 1| + i|1\rangle\langle 0|)$. Then

$$e^{-iH} = \frac{\sqrt{3}}{2}|00\rangle\langle 00| - i\frac{1}{2}|00\rangle\langle 11| - i|01\rangle\langle 10| \\ - i|10\rangle\langle 01| - i\frac{1}{2}|11\rangle\langle 00| + \frac{\sqrt{3}}{2}|11\rangle\langle 11|.$$

is an entangling unitary operator (e.g., Cohen (2011)). Let $U_t \equiv U_3 = e^{-iH}$ for all $t = 0, 1, 2, \dots$. The state transition map of the Markov chain induced by U_3 and M is shown in Fig. 4. Also, the state transition maps for each qubit when the two-qubit network starts from the state $|00\rangle$ are shown in Fig. 5.

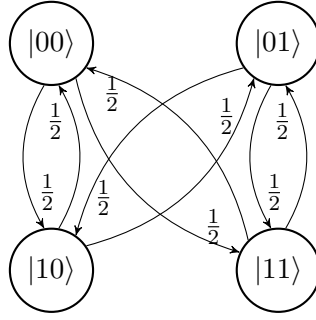


Figure 3: State transition map of the Markov chain induced by U_2 and M .

As we can see, starting from the product state $|00\rangle$ and after the operation of U_3 , the measurement outcomes $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ become statistically correlated. The entangling relationship generated by U_t is then reflected in the state transition of the induced Boolean dynamics.

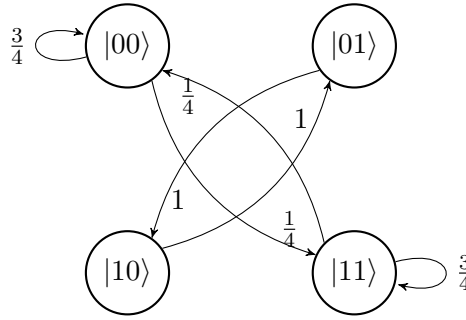


Figure 4: State transition map of the Markov chain induced by U_3 and M .

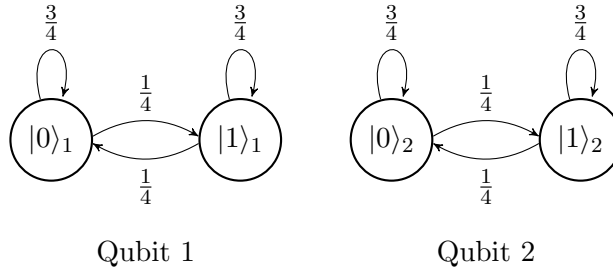


Figure 5: State transition maps of individual qubits starting from $|00\rangle$ in the Markov chain induced by U_3 and M .

Example 4. Consider the following doubly stochastic matrix in $\mathbb{R}^{4 \times 4}$

$$W = \begin{pmatrix} \frac{1}{12} & \frac{1}{6} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{12} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{12} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \frac{1}{12} \end{pmatrix}.$$

Then we can find the following unitary matrix

$$U = \begin{pmatrix} \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ -\frac{1}{\sqrt{6}}i & \frac{1}{2\sqrt{3}}i & -\frac{\sqrt{2}}{2}i & \frac{1}{2}i \\ -\frac{1}{4} - \frac{\sqrt{3}}{4}i & -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}i & \frac{1}{4\sqrt{3}} + \frac{1}{4}i & \frac{1}{2\sqrt{6}} + \frac{\sqrt{2}}{4}i \\ -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}i & \frac{\sqrt{3}}{4} - \frac{1}{4}i & \frac{\sqrt{2}}{4} - \frac{1}{2\sqrt{6}}i & -\frac{1}{4} + \frac{1}{4\sqrt{3}}i \end{pmatrix},$$

such that $U^\dagger \circ U^\top = W$.

Let a Markov chain over a four-state space $\{s_1, s_2, s_3, s_4\}$ with state transition matrix W evolve from initial distribution $p_0 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{4})^\top$. Let $\mathbf{M}^{\otimes 2}$ be the measurement of a qubit network. We encode $s_1 \simeq |00\rangle, s_2 \simeq |01\rangle, s_3 \simeq |10\rangle, s_4 \simeq |11\rangle$. Let the qubit network start from

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle + \frac{1}{2\sqrt{3}}|10\rangle + \frac{1}{2}|11\rangle.$$

We numerically simulate the dynamics of $\mathbf{x}(t)$ for 10^4 rounds and therefore obtain 10^4 independent sample paths of $\mathbf{x}(t)$ with the same initial condition. Then we plot the trajectory of

$$\begin{aligned} \hat{p}(t) &= (\hat{p}_1(t), \hat{p}_2(t), \hat{p}_3(t), \hat{p}_4(t))^\top \\ &:= \left(\mathbb{P}(\mathbf{x}(t) = 00), \mathbb{P}(\mathbf{x}(t) = 01), \right. \\ &\quad \left. \mathbb{P}(\mathbf{x}(t) = 10), \mathbb{P}(\mathbf{x}(t) = 11) \right)^\top \end{aligned}$$

from the experimental data as shown in Fig. 6. Here $\hat{p}_i(t) = \frac{\#\{\mathbf{x}(t) = i\}}{10^4}$, as an unbiased estimate of $p_i(t)$.

We can also define

$$p(t) = (p_1(t), p_2(t), p_3(t), p_4(t))^\top = W^t p_0,$$

which trajectory is displayed in Fig. 7. Since it is homogeneous Markov chain, which will converge to a steady distribution, one can obtain that $\lim_{t \rightarrow \infty} p(t) = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]^\top$. From these two figures, one can easily see that $\hat{p}(t)$ is an excellent estimate of $p(t)$.

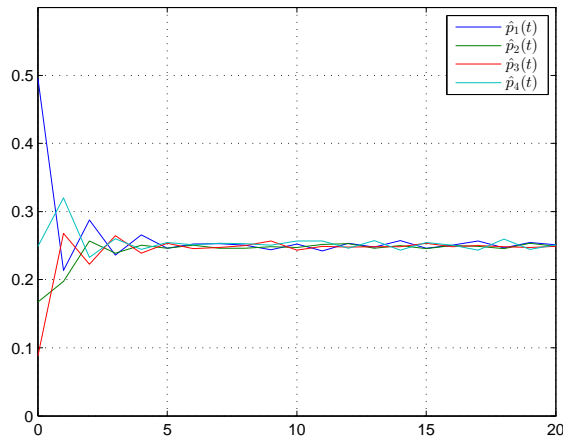


Figure 6: The trajectory of $\hat{p}(t)$ starting from the state $|\psi(0)\rangle$.

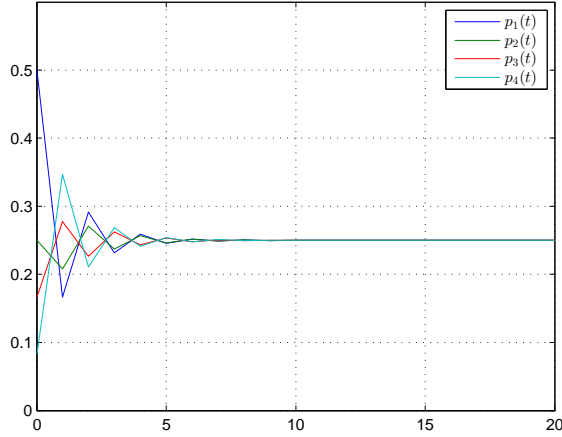


Figure 7: The trajectory of $p(t)$ starting from p_0 .

4.3 Local Measurement: Non-Markovian PBN

We now turn to the local measurement case, where at time t , $M^{V*} = M^{\otimes k} \otimes I^{\otimes(n-k)}$ is performed over $|\psi(t)\rangle$ and produces outcome $\mathbf{x}_k(t) = [x_1(t), \dots, x_k(t)]$. The operators U_t and M collectively determine the dynamics of the quantum states and the resulting Boolean states, while any two different measurement bases M are only subject to a coordinate change. Therefore, without loss of generality, we assume that $M = \lambda_0 P_0 + \lambda_1 P_1 = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1|$.

Given $\mathbf{x}_k(t)$, the post-measurement state $|\psi(t)\rangle_p$ depends on $\mathbf{x}_k(0), \dots, \mathbf{x}_k(t-1)$ due to the local measurement effect as $\mathbf{x}_k(t)$ alone is not enough to determine $|\psi(t)\rangle$. Therefore $\{\mathbf{x}_k(t)\}_{t=0}^{\infty}$ is no longer Markovian. Let $r : \chi_k(0), \dots, \chi_k(t)$ be a path of measurement realization. Define

$$\begin{aligned} \mathcal{P}_r(0) &:= \mathbb{P}(\chi_k(0)) \\ \mathcal{P}_r(1) &:= \mathbb{P}(\chi_k(1) | \chi_k(0)) \\ &\vdots \\ \mathcal{P}_r(t+1) &:= \mathbb{P}(\chi_k(t+1) | \chi_k(t), \dots, \chi_k(0)). \end{aligned}$$

We aim to provide a recursive way of calculating the above transition probabilities. Recall from (14) that $\{|0\rangle, |1\rangle\}^{\otimes n} = \{|b_i\rangle, i = 1, \dots, 2^n\}$ is a sorted basis for $\mathcal{H}^{\otimes n}$. Let

$$|\psi(0)\rangle = \sum_{i=1}^{2^n} a_i |b_i\rangle$$

with $\sum_{i=1}^{2^n} |a_i|^2 = 1$ be the state of the quantum network at time $t = 0$. Let U_t be the matrix representation of U_t under the chosen basis for $t = 0, 1, 2, \dots$. Let P_0, P_1 be defined in (19) as the matrix representations of P_0, P_1 under the standard computational basis, respectively. Recall $[\chi_k(t)] := \sum_{i=1}^k x_i(t) 2^{k-i} + 1$, and $\chi_k^\sharp(t) := \delta_{2^k}^{[\chi_k(t)]}$. Then we have the following theorem.

Theorem 3. *Let Assumption 1 hold and $M = \lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1|$. Let $r : \chi_k(0), \dots, \chi_k(t)$ be a realization of the random measurement outcomes. Then there exist $\beta^r(t) \in \mathbb{C}^{2^{n-k}}$ with $\beta^r(t) = [\beta_1^r(t), \dots, \beta_{2^{n-k}}^r(t)]^\top$ for $t = 0, 1, 2, \dots$, such that $\mathcal{P}_r(t) = \|\beta^r(t)\|^2$ for all $t \geq 0$, where $\beta^r(t)$ satisfies the recursion*

$$\beta^r(t+1) = \left((\chi_k^\#(t+1))^\top \otimes I^{\otimes(n-k)} \right) U_t \left(\chi_k^\#(t) \otimes I^{\otimes(n-k)} \right) \frac{\beta^r(t)}{\|\beta^r(t)\|} \quad (17)$$

with $\beta_i^r(0) = a_{(\lfloor \chi_k(0) \rfloor - 1)2^{n-k} + i}$, $i = 1, \dots, 2^{n-k}$.

The fact that with local measurements the induced Boolean dynamics becomes non-Markovian is indeed quite natural. The dark qubits carry out information that is needed for determining the full state-transition, whose evolution in turn depends on the entire history. Note that to calculate $\mathcal{P}_r(t+1)$ from basic quantum measurement mechanism, one needs to record the entire path history $\chi_k(0), \dots, \chi_k(t+1)$. While the computing process from Theorem 3 is recursive as from $\mathcal{P}_r(t)$ to $\mathcal{P}_r(t+1)$ we only need $\chi_k(t)$, $\chi_k(t+1)$, and $\beta^r(t)$. The proof of Theorem 3 can be found in the Appendix.

The following example is an illustration of the computation for non-Markovian transition probabilities.

Example 5. We consider a three-qubit network. Let a local measurement be $M \otimes M \otimes I$ over qubits 1 and 2. Then the set of possible measurement outcomes is $\{0, 1\}^2$. Let the unitary operator resulting from the continuous evolution be

$$U_t \equiv U = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| - \frac{1}{2}|1\rangle\langle 0| + \frac{\sqrt{3}}{2}|1\rangle\langle 1| \right) \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|). \quad (18)$$

Let the network initial state be given by

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{1}{2\sqrt{3}}|011\rangle + \frac{1}{2}|101\rangle.$$

Let a sample path of $\chi_k(t)$ for $t = 0, 1, 2, 3$ be given by

$$\chi_k(0) = 10, \quad \chi_k(1) = 00, \quad \chi_k(2) = 11, \quad \chi_k(3) = 10.$$

From the quantum state evolution one can directly verify that

$$\begin{aligned} \mathcal{P}_r(0) &= \mathbb{P}(\chi_k(0) = 10) = \frac{1}{4}, \\ \mathcal{P}_r(1) &= \mathbb{P}(\chi_k(1) = 00 | \chi_k(0) = 10) = \frac{3}{4}, \\ \mathcal{P}_r(2) &= \mathbb{P}(\chi_k(2) = 11 | \chi_k(1) = 00, \chi_k(0) = 10) = \frac{1}{4}, \\ \mathcal{P}_r(3) &= \mathbb{P}(\chi_k(3) = 10 | \chi_k(2) = 11, \chi_k(1) = 00, \chi_k(0) = 10) \\ &= \frac{3}{4}. \end{aligned}$$

Alternatively, from the recursion (17) one has

$$\begin{aligned}\beta^x(0) &= \left(0, \frac{1}{2}\right)^\top, \\ \beta^x(1) &= \left(0, \frac{\sqrt{3}}{2}\right)^\top, \\ \beta^x(2) &= \left(0, -\frac{1}{2}\right)^\top, \\ \beta^x(3) &= \left(0, -\frac{\sqrt{3}}{2}\right)^\top.\end{aligned}$$

We can easily verify $\mathcal{P}_r(t) = \|\beta^x(t)\|^2$ for $t = 0, 1, 2, 3$. This validates Theorem 3.

5 Controllability Conditions

The controllability of the quantum states under the bilinear model described by (9) has been well understood (Albertini and D'Alessandro, 2002). However, it is unclear how the random jumping in (10) from the sequential measurements affects the controllability of the quantum states, or how the quantum state controllability determines the controllability of the induced Boolean dynamical states. This section attempts to provide clear answers to these two questions.

5.1 Quantum State Controllability

It is natural to define the quantum network state controllability over the discrete state sequence $|\psi(t)\rangle = |q((tT)^-)\rangle$, $t = 0, 1, 2, \dots$. Note that, the sequence $|\psi(t)\rangle$, $t = 0, 1, 2, \dots$ along the system (9)–(10) defines a random process in its own right as the randomness in the $|\psi(t)\rangle_p$ will be inherited by $|\psi(t+1)\rangle$ for any t . The classical definition of the controllability of bilinear quantum systems therefore needs to be refined to accommodate the existence of the measurements.

We introduce the following definition of controllability for the hybrid bilinear quantum system (9)–(10).

Definition 5. *The quantum network (9)–(10) is quantum state controllable if for any pair of network states $|\psi_0\rangle, |\psi_1\rangle \in \mathcal{H}^{\otimes n}$, there exist an integer $T_0 > 0$, a global measurement $\mathbf{M}^{\otimes n}$, and control signals $u_\ell(s)$, $s \in [0, T_0T]$ that steer the state of the quantum hybrid network from $|\psi(0)\rangle = |\psi_0\rangle$ to $|\psi(T_0)\rangle = |\psi_1\rangle$ with probability one.*

Here steering the state of the quantum network from $|\psi(0)\rangle = |\psi_0\rangle$ to $|\psi(T_0)\rangle = |\psi_1\rangle$ deterministically means the event that $|\psi(T_0)\rangle = |\psi_1\rangle$ conditioned that $|\psi(0)\rangle = |\psi_0\rangle$ is a sure event along (9)–(10). If the control signals $u_\ell(s)$, $\ell = 1, \dots, p$ are feedforward, there exist deterministic unitary operators \mathbf{U}_t for $t = 0, 1, 2, \dots$ such that $\mathbf{U}_t|\psi(t)\rangle_p = |\psi(t+1)\rangle$. Clearly, in this case, it is possible for the sequence $|\psi(t)\rangle$, $t = 0, 1, 2, \dots$ to have degenerate probability distribution taking one possible path, but only for specially selected $|\psi(0)\rangle$, $t = 0, 1, 2, \dots, M$, and $u_\ell(s)$, $\ell = 1, \dots, p$. In particular, for that probabilistically

degenerate path to take place $|\psi(t)\rangle$ must be one of the eigenvectors of the measurement $M^{\otimes n}$. As a result, the above deterministic quantum state controllability can only be achieved by feedback controllers. We present the following result.

Proposition 2. *Let $H_0 = 0$. Fix an arbitrary global measurement $M^{\otimes n}$. Then for any $T > 0$, the quantum network (9)–(10) is quantum state controllable if and only if $\mathcal{L}\{B_1, \dots, B_p\}$ is isomorphic to $\mathfrak{sp}(2^{n-1})$ or $\mathfrak{su}(2^n)$.*

When the network dynamics contains uncontrolled drift item, the analysis becomes more involved and we introduce the following definition.

Definition 6. *The quantum network (9)–(10) is Quantum Equivalent State Controllable if for any pair quantum states $|\psi_0\rangle, |\psi_1\rangle \in \mathcal{H}^{\otimes n}$, there exist an integer $T_0 > 0$, a global measurement $M^{\otimes n}$, control signals $u_\ell(s), s \in [0, T_0T]$, and a phase factor ϕ that steer the state of the quantum network from $|\psi(0)\rangle = |\psi_0\rangle$ to $|\psi(T_0)\rangle = e^{i\phi}|\psi_1\rangle$ deterministically.*

We recall the following definition introduced in Jurdjevic and Sussman (1972):

$$R(I, s) = \left\{ U \in e^{\mathcal{L}\{A, B_1, \dots, B_p\}} : U = X(s) \text{ is the solution of (5) under some controls } u_\ell(\cdot) \right. \\ \left. \text{(or is reachable along (5)) at time } s \text{ from } I \right\}.$$

We also define $\mathbf{R}(I, T) = \bigcup_{0 \leq s \leq T} R(I, s)$.

Proposition 3. *Suppose $A = iH_0^{\otimes n}$ for some $H_0 \in \mathcal{L}_*(\mathcal{H})$. The quantum network (9)–(10) is quantum equivalent state controllable if the following conditions hold:*

- (i) $\mathcal{L}\{A, B_1, \dots, B_p\}$ is isomorphic to $\mathfrak{sp}(2^{n-1})$ or $\mathfrak{su}(2^n)$;
- (ii) T is sufficiently large so that $\mathbf{R}(I, T) = e^{\mathcal{L}\{A, B_1, \dots, B_p\}}$.

5.2 Boolean State Controllability

We can also define the controllability on the induced Boolean network dynamics $\{\mathbf{x}(t)\}_{t=0}^\infty$.

Definition 7. *Let a global network measurement be given as $M^{\otimes n}$. The quantum network (9)–(10) is almost surely Boolean controllable if for any pair $X_0, X_1 \in \{0, 1\}^{\otimes n}$, there exist an integer $T_0 > 0$, and control signals $u_\ell(s), s \in [0, T_0T]$ that steer the state of the random Boolean network from $\mathbf{x}(0) = X_0$ to $\mathbf{x}(T_0) = X_1$ with probability one along the induced Boolean dynamics $\{\mathbf{x}(t)\}_{t=0}^\infty$.*

It is straightforward to verify that Boolean controllability is an inherently relaxed controllability notion. We introduce the following definition of practical controllability of the quantum states concerning whether controllability can be achieved in the approximate sense (Moreau and Aeyels, 2000).

Definition 8. *The bilinear control system (9) is practically controllable with respect to $\delta > 0$ if for any $|\psi_0\rangle$ and $|\psi_f\rangle$ there exist $u_\ell(s) : s \in [0, T], \ell = 1, \dots, p$ such that*

$$|q(0)\rangle = |\psi_0\rangle \implies \left\| |q(T)\rangle - |\psi_f\rangle \right\| < \delta$$

We now present the following result suggesting that practical controllability for the quantum states implies almost sure controllability for the induced Boolean states.

Theorem 4. *Let the bilinear control system (9) be practically controllable with respect to some δ with $\delta < \sqrt{2}$. Then*

(i) *The hybrid qubit network (9)–(10) is almost surely Boolean controllable.*

(ii) *For any $X_0, X_* \in \{0, 1\}^{\otimes n}$, for*

$$T_{\text{hit}} = \inf_{t \geq 0} \{X(t) = X_*\}$$

with $X(0) = X_0$ there holds

$$\max_{u_\ell(s): s \in [0, tT)} \mathbb{P}(T_{\text{hit}} \leq t) > 1 - e^{-t \log\left(\frac{4}{4\delta^2 - \delta^4}\right)}.$$

Theorem 4.(ii) shows that in the presence of practical quantum state controllability, the probability of arriving at any measurement outcome $X_* \in \{0, 1\}^{\otimes n}$ approaches one at an exponential rate. Moreover, the measurement outcome $\mathbf{x}(t) = X_*$ corresponds uniquely to the quantum state $|q(tT)\rangle = |X_*\rangle$. Therefore, this Boolean state controllability also provides a way of realizing verifiable quantum state manipulation by the combination of Bilinear control and sequential measurements. The proofs of Proposition 2, Proposition 3, and Theorem 4 are in the Appendix.

5.3 Further Discussions

The controllability definition of the hybrid bilinear quantum network under local measurement can be similarly introduced. For any initial $|\psi_0\rangle$, after being measured its post-measurement state $|\psi_0\rangle_{\text{p}}$ is in $\{|0\rangle, |1\rangle\}^{\otimes n}$, which is known even when $|\psi_0\rangle$ is unknown. Therefore, an advantage in terms of controllability from global measurement is the fact that the initial quantum state can be uncertain for reaching any target state. However, with local measurements, the initial state $|\psi_0\rangle$ must be fully known in order to establish any post-measurement state initial $|\psi(t)\rangle_{\text{p}}$, which is critical for the design of any feedback controller. This point represents the most significant difference between these two types of measurements for the controllability properties. When the initial state $|\psi_0\rangle$ is known, similar results can be established along the same line of analysis for the controllability of the quantum network with local measurements.

It is certainly of interest to investigate how the graphical network structure influences the controllability of the quantum networks. The network structure can be defined by the drift Hamiltonian H_0 , or controlled Hamiltonians H_ℓ , where edges arise from the qubit interactions encoded in H_0 or H_ℓ . Alternatively, generalized network structures can be defined over the interaction relationship among the 2^n quantum states. Excellent results have been established regarding how such an interaction structure would lead to the Lie-algebra controllability condition (Altafini, 2002; Li et al., 2017; Tsopelakos et al., in press, 2018). We note that such results can be applied to the hybrid network model considered in the current paper as well, since the controllability in the presence of measurements is still closely related to the original bilinear controllability as shown in the results.

6 Conclusions

We have studied dynamical quantum networks subject to sequential local or global measurements leading to probabilistic Boolean recursions which represent the measurement outcomes. With global measurements, such resulting Boolean recursions were shown to be Markovian, while with local measurements, the state transition probability at any given time depends on the entire history of the sample path. Under the bilinear control model for the Schrödinger evolution, we showed that the measurements in general enhance the controllability of the quantum networks. The global or local measurements were assumed to be prescribed in the current framework. It is of interest as a future direction to investigate the co-design of the continuous control signals and the measurements, which may both have local structures, for more robust and efficient methods of manipulating the states of large-scale quantum networks.

Appendix

A. Proof of Theorem 2

From the definition of $f_{[\alpha_1, \dots, \alpha_{2^n}]}$, F_t taking value as $f_{[\alpha_1, \dots, \alpha_{2^n}]}$ is equivalent to obtaining outcomes $\delta_{2^n}^1, \dots, \delta_{2^n}^{2^n}$, respectively, when measuring quantum states independently prepared at $\delta_{2^n}^1, \dots, \delta_{2^n}^{2^n}$. Then the probability of $F_t : \mathbf{x}^\sharp(t) \rightarrow \mathbf{x}^\sharp(t+1)$ taking $f_{[\alpha_1, \dots, \alpha_{2^n}]}$ as the transition matrix is

$$p(f_{[\alpha_1, \dots, \alpha_{2^n}]}) = \prod_{i=1}^{2^n} \mathbb{P} \left(\mathbf{x}^\sharp(t+1) = \delta_{2^n}^{\alpha_i} \mid \mathbf{x}^\sharp(t) = \delta_{2^n}^i \right).$$

To express this probability, we need to figure out each $\mathbb{P} \left(\mathbf{x}^\sharp(t+1) = \delta_{2^n}^{\alpha_i} \mid \mathbf{x}^\sharp(t) = \delta_{2^n}^i \right)$. At time t , if the outcome is $[\lambda_{x_1(t)}, \dots, \lambda_{x_n(t)}] \sim \mathbf{x}^\sharp(t)$, $x_j(t) \in \{0, 1\}, j = 1, \dots, n$ after the network state $|\psi(t)\rangle$ being measured, then the probability of getting outcome $[\lambda_{x_1(t+1)}, \dots, \lambda_{x_n(t+1)}] \sim \mathbf{x}^\sharp(t+1)$ is

$$\begin{aligned} & \mathbb{P} \left(\mathbf{x}^\sharp(t+1) \mid \mathbf{x}^\sharp(t) \right) \\ &= \left| \langle v_{x_1(t+1)} \cdots v_{x_n(t+1)} \mid \mathbf{U}_t \mid v_{x_1(t)} \cdots v_{x_n(t)} \rangle \right|^2 \\ &= \left| \langle x_1(t+1) \cdots x_n(t+1) \mid \underbrace{(\mathbf{u} \otimes \cdots \otimes \mathbf{u})^\dagger}_n \mathbf{U}_t \underbrace{\mathbf{u} \otimes \cdots \otimes \mathbf{u}}_n \mid x_1(t) \cdots x_n(t) \rangle \right|^2 \\ &= \left| [U_t^M]_{[\mathbf{x}(t+1)], [\mathbf{x}(t)]} \right|^2. \end{aligned}$$

Since $[\mathbf{x}(t)], [\mathbf{x}(t+1)] \in \{1, 2, \dots, 2^n\}$, we have

$$\mathbb{P} \left(\mathbf{x}^\sharp(t+1) = \delta_{2^n}^{\alpha_i} \mid \mathbf{x}^\sharp(t) = \delta_{2^n}^i \right) = \left| [U_t^M]_{\alpha_i, i} \right|^2.$$

Thus, the probability of F_t taking $f_{[\alpha_1, \dots, \alpha_{2^n}]}$ is

$$\begin{aligned} p(f_{[\alpha_1, \dots, \alpha_{2^n}]}) &= \prod_{i=1}^{2^n} \mathbb{P} \left(\mathbf{x}^\sharp(t+1) = \delta_{2^n}^{\alpha_i} \mid \mathbf{x}^\sharp(t) = \delta_{2^n}^i \right) \\ &= \prod_{i=1}^{2^n} \left| [U_t^M]_{\alpha_i, i} \right|^2. \end{aligned}$$

This completes the proof.

B. Proof of Theorem 3

We first present the following technical lemma on the tensor product of projector matrices, which can be verified directly.

Lemma 1. *Denote*

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (19)$$

Let $\gamma = [\gamma_1, \dots, \gamma_k]$, where $\gamma_i \in \{0, 1\}$, $i = 1, \dots, k$. Then

$$[P_{\gamma_1} \otimes \dots \otimes P_{\gamma_k}]_{i,j} = \begin{cases} 1, & i = j = \lfloor \gamma \rfloor, \\ 0, & \text{otherwise.} \end{cases}$$

First, if we measure $|\psi(0)\rangle$ and get outcome $\chi_k(0)$, then the probability of getting $\chi_k(0)$ is

$$\begin{aligned} \mathcal{P}_r(0) &= \langle \psi(0) | P_{x_1(0)} \otimes \dots \otimes P_{x_k(0)} \otimes I^{\otimes(n-k)} | \psi(0) \rangle \\ &= \sum_{i=1}^{2^{n-k}} |a_{(\lfloor \chi_k(0) \rfloor - 1)2^{n-k} + i}|^2 \\ &= \|\beta^r(0)\|^2 \end{aligned}$$

with $\beta_i^r(0) = a_{(\lfloor \chi_k(0) \rfloor - 1)2^{n-k} + i}$, $i = 1, \dots, 2^{n-k}$. Moreover, given $\chi_k(0)$ occurred, the vector form of the post-measurement state of $|\psi(0)\rangle$ under the chosen basis is

$$\begin{aligned} |\psi(0)\rangle_p^r &= \frac{P_{x_1(0)} \otimes \dots \otimes P_{x_k(0)} \otimes I^{\otimes(n-k)} |\psi(0)\rangle}{\sqrt{\|\beta^r(0)\|^2}} \\ &= \frac{\sum_{i=1}^{2^{n-k}} \beta_i^r(0) |b_{(\lfloor \chi_k(0) \rfloor - 1)2^{n-k} + i}\rangle}{\|\beta^r(0)\|}, \\ &= |x_1(0) \dots x_k(0)\rangle \otimes \frac{\sum_{i=1}^{2^{n-k}} \beta_i^r(0) |b_i\rangle^{(n-k)}}{\|\beta^r(0)\|}, \end{aligned}$$

where Lemma 1 is used in the second equality, and $\{|b_i\rangle^{(n-k)}, i = 1, \dots, 2^{n-k}\} = \{|0\rangle, |1\rangle\}^{\otimes(n-k)}$.

Next, we compute $\mathcal{P}_r(1)$. Given $\chi_k(0)$, the network state at time 1 is

$$|\psi(1)\rangle^r = U_0 |\psi(0)\rangle_p^r.$$

Subject to $M^{V*} = M^{\otimes k} \otimes I^{\otimes(n-k)}$, the probability of getting outcome $\chi_k(1)$ is

$$\begin{aligned} \mathcal{P}_r(1) &= \langle \psi(1) | P_{x_1(1)} \otimes \dots \otimes P_{x_k(1)} \otimes I^{\otimes(n-k)} | \psi(1) \rangle \\ &= \sum_{i=1}^{2^{n-k}} \left| \frac{\sum_{j=1}^{2^{n-k}} \beta_i^r(0) [U_0]_{(\lfloor \chi_k(1) \rfloor - 1)2^{n-k} + i, (\lfloor \chi_k(0) \rfloor - 1)2^{n-k} + j}}{\|\beta^r(0)\|} \right|^2 \\ &= \|\beta^r(1)\|^2 \end{aligned}$$

with

$$\begin{aligned}\beta_i^r(1) &= \frac{\sum_{j=1}^{2^{n-k}} \beta_i^r(0) [U_0]_{(\lfloor \chi_k(1) \rfloor - 1)2^{n-k} + i, (\lfloor \chi_k(0) \rfloor - 1)2^{n-k} + j}}{\|\beta^r(0)\|} \\ &= \frac{\left((\chi_k^\#(1))^\top \otimes I^{\otimes(n-k)} \right) U_0 \left(\chi_k^\#(0) \otimes I^{\otimes(n-k)} \right) \beta^r(0)}{\|\beta^r(0)\|},\end{aligned}\quad (20)$$

for $i = 1, \dots, 2^{n-k}$. Similarly, given $\chi_k(0)$ and $\chi_k(1)$, the vector form of post-measurement state of $|\psi(1)\rangle^r$ depending on r is

$$\begin{aligned}|\psi(1)\rangle_p^r &= \frac{P_{x_1(1)} \otimes \dots \otimes P_{x_k(1)} \otimes I^{\otimes(n-k)} |\psi(1)\rangle}{\sqrt{\|\beta^r(1)\|^2}} \\ &= \frac{\sum_{i=1}^{2^{n-k}} \beta_i^r(1) |b_{(\lfloor \chi_k(1) \rfloor - 1)2^{n-k} + i}\rangle}{\|\beta^r(1)\|}, \\ &= |x_1(1) \dots x_k(1)\rangle \otimes \frac{\sum_{i=1}^{2^{n-k}} \beta_i^r(1) |b_i\rangle^{(n-k)}}{\|\beta^r(1)\|}.\end{aligned}$$

Finally, the above process can be carried out recursively, so that $\mathcal{P}_r(2)$, $\mathcal{P}_r(3)$, \dots , $\mathcal{P}_r(t+1)$ can be computed from this procedure. The recursion from $\mathcal{P}_r(i)$ to $\mathcal{P}_r(i+1)$, $i \geq 1$ will follow from the same process as $\mathcal{P}_r(0)$ to $\mathcal{P}_r(1)$, and we can establish (17) eventually.

This completes the proof.

C. Proof of Proposition 2

With feedback controllers, it is clear from the Markovian property of $U_t |\psi(t)\rangle_p$ that we can assume $T_0 = 1$ for the definition of the quantum state controllability. After the measurement at $t = 0$, the post-measurement state $|\psi_0\rangle_p$ of any initial state $|\psi_0\rangle$ belongs to $\{|0\rangle, |1\rangle\}^{\otimes n}$ which is a finite set but is still a subset of $\mathcal{Q}(2^n)$. The sufficiency statement is therefore a special case of classical result, e.g., Theorem 5 in Jurdjevic and Sussman (1972).

Now, we prove the necessity continues to hold. Suppose the quantum network is quantum state controllable. Then with $|\psi_0\rangle \in \{|0\rangle, |1\rangle\}^{\otimes n}$ and for any $|\psi_1\rangle \in \mathcal{Q}(2^n)$, there exist control signal $u_\ell(s)$, $s \in [0, T)$, such that $|\psi(0)\rangle = |\psi_0\rangle$ and $|\psi(T)\rangle = |\psi_1\rangle$. Thus there exists $U_{|\psi_1\rangle}$ such that $|\psi_1\rangle = U_{|\psi_1\rangle} |\psi_0\rangle$. By Theorem 5 in Brockett (1972), the solution at $s = T$ of (5) from I at $s = 0$ is $X(T) = U_{|\psi_1\rangle} \in e^{\mathcal{L}\{\mathbf{B}_1, \dots, \mathbf{B}_p\}}$. Denoting $\mathbf{U} = \{U_{|\psi_1\rangle} : |\psi_1\rangle \in \mathcal{Q}(2^n)\}$, we have the following facts:

- (i) $\mathbf{U} \subseteq e^{\mathcal{L}\{\mathbf{B}_1, \dots, \mathbf{B}_p\}}$;
- (ii) $\mathbf{U} |\psi_0\rangle = \mathcal{Q}(2^n)$;
- (iii) $U_* |\psi_0\rangle \in \mathcal{Q}(2^n)$, for any $U_* \in e^{\mathcal{L}\{\mathbf{B}_1, \dots, \mathbf{B}_p\}}$.

Hence $e^{\mathcal{L}\{\mathbf{B}_1, \dots, \mathbf{B}_p\}} |\psi_0\rangle = \mathcal{Q}(2^n)$. Because of the reversibility of the action of elements in the group $e^{\mathcal{L}\{\mathbf{B}_1, \dots, \mathbf{B}_p\}}$, we can further conclude that $e^{\mathcal{L}\{\mathbf{B}_1, \dots, \mathbf{B}_p\}}$ is transitive on $\mathcal{Q}(2^n)$. Invoking Theorem 4 of (Albertini and D'Alessandro, 2003), the desired conclusion holds.

D. Proof of Proposition 3

Let $M = H_0$. For any pair quantum states $|\psi_0\rangle$ and $|\psi_1\rangle$, the post-measurement state of $|\psi_0\rangle$ being measured by $H_0^{\otimes n}$ is $|\psi_0\rangle_p$, which is an eigenstate of $H_0^{\otimes n}$. We let the corresponding eigenvalue of $|\psi_0\rangle_p$ is λ . If $\mathcal{L}\{A, B_1, \dots, B_p\}$ is isomorphic to $\mathfrak{sp}(2^{n-1})$ or $\mathfrak{su}(2^n)$, then $\mathcal{L}\{A, B_1, \dots, B_p\}$ is transitive. From Theorem 6.5 of Jurdjevic and Sussman (1972) with the condition that T is sufficiently large so that $\mathbf{R}(I, T) = e^{\mathcal{L}\{A, B_1, \dots, B_p\}}$, there exists s_* such that we can find a $U \in R(I, s_*)$ with controls $u_\ell^*(\cdot)$ such that $|\psi_1\rangle = U|\psi_0\rangle_p$. Now we set the admissible control as

$$u_\ell(s) = \begin{cases} 0, & s \in [0, T - s_*] \\ u_\ell^*(s), & s \in (T - s_*, T] \end{cases}, \quad \ell = 1, \dots, p.$$

Under this control, the system state $|\varphi(s)\rangle$ will be driven to (1) $e^{-\lambda}|\psi_0\rangle_p$ at time $s = T - s_*$ from $|\psi_0\rangle_p$; (2) $e^{-\lambda}|\psi_1\rangle$ at time $s = T$. This completes the proof.

E. Proof of Theorem 4

(i) Denote the quantum state corresponding to the measurement outcome $X_0, X_* \in \{0, 1\}^{\otimes n}$ as $|X_0\rangle$ and $|X_*\rangle$, respectively. Since bilinear control system (9) is practically controllable with respect to some δ with $\delta < \sqrt{2}$, for any $|\psi(t)\rangle_p = |\mathbf{x}(t)\rangle$, there always exists $u_\ell(s) : s \in [tT, (t+1)T]$ such that

$$\begin{aligned} \left| \langle X_* | \psi(t+1) \rangle \right| &\geq \operatorname{Re}(\langle X_* | \psi(t+1) \rangle) \\ &\geq \frac{2 - \delta^2}{2} \\ &> 0. \end{aligned} \tag{21}$$

As a result, there holds

$$\begin{aligned} \mathbb{P}(\mathbf{x}(t+1) = X_* | \mathbf{x}(t)) &\geq \left| \langle X_* | \psi(t+1) \rangle \right|^2 \\ &\geq \left(1 - \frac{\delta^2}{2}\right)^2 \end{aligned} \tag{22}$$

for all $t \geq 0$. The desired almost sure Boolean controllability follows directly from the Borel-Cantelli Lemma (cf. Theorem 2.3.6, (Durrett, 2005)).

(ii) In view of (22) and according to the definition of T_{hit} , there holds

$$\begin{aligned} \max_{u_\ell(s): s \in [0, tT]} \mathbb{P}(T_{\text{hit}} \geq t) &\leq \left(1 - \left(1 - \frac{\delta^2}{2}\right)^2\right)^t \\ &= \left(\delta^2 - \frac{\delta^4}{4}\right)^t. \end{aligned}$$

This immediately implies that

$$\begin{aligned} \max_{u_\ell(s): s \in [0, tT]} \mathbb{P}(T_{\text{hit}} \leq t) &\geq 1 - \left(\delta^2 - \frac{\delta^4}{4}\right)^t \\ &= 1 - e^{-t \log\left(\frac{4}{4\delta^2 - \delta^4}\right)}. \end{aligned}$$

This proves the desired theorem.

References

- Albertini, F. and D'Alessandro, D. (2002). The Lie algebra structure and controllability of spin systems, *Linear Alg. Applicat.* **350**: 213–235.
- Albertini, F. and D'Alessandro, D. (2003). Notions of controllability for bilinear multilevel quantum systems, *IEEE Trans. Automatic Control* **48**(8): 1399–1403.
- Altafini, C. (2002). Controllability of quantum mechanical systems by root space decomposition of $\mathfrak{su}(n)$, *J. Mathematical Physics* **43**(5): 2051–2062.
- Altafini, C. and Ticozzi, F. (2012). Modeling and control of quantum systems: an introduction, *IEEE Trans. Automatic Control* **57**(8): 1898–1917.
- Belavkin, V. P. (1999). Optimal measurement and control in quantum dynamical systems, *Rep. Math. Phys.* **43**(3): 405–425. Preprint No. 411, Inst. of Phys., Nicolaus Copernicus University, Torun, February 1979.
- Blok, M. S., Bonato, C., Markham, M. L., Twitchen, D. J., Dobrovitski, V. V. and Hanson, R. (2014). Manipulating a qubit through the backaction of sequential partial measurements and real-time feedback, *Nature Physics* **10**: 189.
- Brockett, R. W. (1972). System theory on group manifolds and coset spaces, *SIAM J. Control* **10**(2): 265–284.
- Brockett, R. W. and Khaneja, N. (2000). On the stochastic control of quantum ensembles, *System Theory: Modeling, Analysis, and Control*, Kluwer Academic Publisher, Boston, pp. 75–96.
- Cheng, D. and Qi, H. (2009). Controllability and observability of Boolean control networks, *Automatica* **45**: 1659–1667.
- Cohen, S. M. (2011). All maximally entangling unitary operators, *Physical Review A* **84**: 052308.
- Dirr, G. and Helmke, U. (2008). Lie theory for quantum control, *GAMM-Mitt* **31**(1): 59–93.
- Dunkl, C. and Życzkowski, K. (2009). Volume of the set of unistochastic matrices of order 3 and the mean Jarlskog invariant, *J. Mathematical Physics* **50**(12): 123521.
- Durrett, R. (2005). *Probability: Theory and Examples*, Pacific Grove, CA, USA: Brooks/Cole.
- James, M. R., Nurdin, H. I. and Petersen, I. R. (2008). h_∞ control of linear quantum stochastic systems, *IEEE Trans. Automatic Control* **53**(8): 1787–1803.

- Jurdjevic, V. and Sussman, H. J. (1972). Control systems on Lie groups, *J. Differential Equations* **12**: 313–329.
- Kato, Y. and Yamamoto, N. (2014). Structure identification and state initialization of spin networks with limited access, *New J. Phys.* **16**: 023024.
- Kauffman, S. A. (1969). Metabolic stability and epigenesis in randomly constructed genetic nets, *J. Theoretical Biology* **22**: 437–467.
- Li, J. S. and Khaneja, N. (2009). Ensemble control of bloch equations, *IEEE Transactions on Automatic Control* **54**: 528–536.
- Li, J. S., Zhang, W. and Wang, L. (2017). Computing controllability of systems on $\mathfrak{so}(n)$ over graphs, *IEEE Conference on Decision and Control* pp. 5511–5516.
- Moreau, L. and Aeyels, D. (2000). Practical stability and stabilization, *IEEE Trans. Automatic Control* **45**(8): 1554–1558.
- Nielsen, M. A. and Chuang, I. L. (2010). *Quantum Computation and Quantum Information*, Cambridge University Press.
- Ogura, M. and Martin, C. F. (2014). Stability analysis of positive semi-markovian jump linear systems with state resets, *SIAM Journal on Control and Optimization* **52**: 1809–1831.
- Pechen, A., Ilin, N., Shuang, F. and Rabitz, H. (2006). Quantum control by von Neumann measurements, *Phys. Rev. A* **74**: 052102.
- Perseguers, S., Lewenstein, M., Acin, A. and Cirac, J. (2010). Quantum random networks, *Nature Physics* **6**: 539.
- Schirmer, S. G., Fu, H. and Solomon, A. I. (2001). Complete controllability of quantum systems, *Phys. Rev. A* **63**: 063410.
- Schirmer, S. G. and Wang, X. (2010). Stabilizing open quantum systems by markovian reservoir engineering, *Physical Review A* **81**(6): 062306.
- Shi, G., Dong, D., Petersen, I. R. and Johansson, K. H. (2016). Reaching a quantum consensus: Master equations that generate symmetrization and synchronization, *IEEE Trans. Automatic Control* **61**: 374–387.
- Shi, G., Li, B., Miao, Z., Dower, P. M. and James, M. R. (2017). Reaching agreement in quantum hybrid networks, *Scientific Reports* **7**: 5989.
- Shmulevich, I., Dougherty, E. R., Kim, S. and Zhang, W. (2002). Probabilistic Boolean networks: a rule-based uncertainty model for gene regulatory networks, *Bioinformatics* **2**: 261–274.

- Ticozzi, F., Schirmer, S. and Wang, X. (2010). Stabilizing quantum states by constructive design of open quantum dynamics, *IEEE Transactions on Automatic Control* **55**(12): 2901–2905.
- Tournier, L. and Chaves, M. (2013). Interconnection of asynchronous Boolean networks, asymptotic and transient dynamics, *Automatica* **49**(4): 884–893.
- Tsopelakos, A., Belabbas, M. A. and Gharesifard, B. (in press, 2018). Classification of the structurally controllable zero-patterns for driftless bilinear control systems, *IEEE Transactions on Control of Network Systems* .
- Wang, X., Pemberton-Ross, P. and Schirmer, S. G. (2012). Symmetry and subspace controllability for spin networks with a single-node control, *IEEE Trans. Automatic Control* **57**(8): 1945–1956.