

On the maximum principle for linear parabolic equations

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Received: 20 September 2007 / Accepted: 21 September 2007 / Published online: 11 October 2007
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Abstract We prove extensions of our previous estimates for linear elliptic equations with inhomogeneous terms in L^p spaces, $p \leq n$ to linear parabolic equations with inhomogeneous terms in L^p , $p \leq n + 1$. As with the elliptic case, our results depend on restrictions on parabolicity determined by certain subcones of the positive cone. They also extend the maximum principle of Krylov for the case $p = n + 1$, corresponding to the usual parabolicity.

Keywords Maximum principles · Linear parabolic equations · Parabolic Hessian equation · Hessian integrals

1 Introduction

In our previous paper [8], we considered extensions of the maximum principles of Aleksandrov, Bakel' man and Pucci, (see [2]) for linear elliptic operators to lower L^p exponents under restricted ellipticity hypotheses. In the present paper we consider analogous results for linear parabolic operators, thereby providing corresponding extensions to the Krylov maximum principle. Specifically, we consider linear parabolic operators of the form

$$Pu := D_t u - a^{ij}(x, t) D_{ij} u, \quad (1)$$

in bounded domains \mathcal{D} in space-time $\mathbb{R}^{n-1}(x, t)$, where the matrix function $\mathcal{A} = [a^{ij}] : \mathcal{D} \rightarrow S^n$ is positive in \mathcal{D} . Here $D_t u$ and $D^2 u = [D_{ij} u]$ denote respectively the time derivative and the spatial Hessian of an appropriately smooth function $u : \mathcal{D} \rightarrow \mathbb{R}$. As in [8],

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our restrictions on parabolicity are expressed in terms of the subcones of the positive cone, determined by the elementary symmetric functions S_k , $k = 1, \dots, n$, given by

$$S_k(\lambda) = \sum \lambda_{i_1}, \dots, \lambda_{i_k} \tag{2}$$

for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, where the summation is taken over all increasing k tuples $(i_1, \dots, i_k) \subset \{1, \dots, n\}$. The cone Γ_k is then defined by

$$\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid S_j(\lambda) > 0, \quad j = 1, \dots, k \} \tag{3}$$

and we denote by Γ_k^* the dual cone, given by

$$\Gamma_k^* = \{ \lambda \in \mathbb{R}^n \mid \lambda \cdot \mu \geq 0, \quad \forall \mu \in \Gamma_k \} \tag{4}$$

Corresponding dual functions ρ_k, ρ_k^* are defined by

$$\begin{aligned} \rho_k(\lambda) &= \left\{ \frac{S_k(\lambda)}{\binom{n}{k}} \right\}^{1/k} \\ \rho_k^*(\lambda) &= \inf \left\{ \frac{\lambda \cdot \mu}{n} \mid \mu \in \Gamma_k, \rho_k(\mu) \geq 1 \right\}. \end{aligned} \tag{5}$$

For matrices $\mathcal{A} \in \mathcal{S}^n$, we write $\mathcal{A} \in \Gamma_k (\Gamma_k^*)$ if the eigenvalues of \mathcal{A} , $\lambda = \lambda(\mathcal{A}) \in \Gamma_k (\Gamma_k^*)$, $\rho_k(\mathcal{A}) = \rho_k(\lambda)$ and $\rho_k^*(\mathcal{A}) = \rho_k^*(\lambda)$. Our main theorem may then be formulated as follows.

Theorem 1 *Let $u \in C^{2,1}(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$ satisfy the differential inequality*

$$Pu \leq f \tag{6}$$

in \mathcal{D} for some coefficient matrix $\mathcal{A} \in \Gamma_k^$, $n/2 < k \leq n$, at each point of \mathcal{D} , with $\rho_k^*(\mathcal{A}) > 0$ in \mathcal{D} . Suppose also $u \leq 0$ on $\partial_p \mathcal{D}$, the parabolic boundary of \mathcal{D} . Then we have the estimate*

$$\sup_{\mathcal{D}} u \leq C \left\| \frac{f}{\rho_k^*(\mathcal{A})^{k/(k+1)}} \right\|_{L^{k+1}(\mathcal{D})}, \tag{7}$$

where C is a constant depending on n, k and \mathcal{D} .

Note that the parabolic boundary $\partial_p \mathcal{D}$ is defined by

$$\partial_p \mathcal{D} = \{ (x, t) \in \partial \mathcal{D} \mid (x, t') \notin \mathcal{D}, \quad \text{for any } t' < t \}. \tag{8}$$

When $k = n$, there is no restriction on \mathcal{A} and we obtain the Krylov maximum principle [3].

As with the elliptic case [8], we will actually derive a sharp form of the estimate (7), using the Hessian measure theory of Trudinger and Wang [14], (see also [15, 16]). Moreover the domain \mathcal{D} in the right hand side of (7) may be replaced by an appropriate upper contact set as in the elliptic case. We postpone the cases, $k \leq n/2$, to a later treatment as there we cannot use the same method.

2 Parabolic Hessian equations

For $k = 1, \dots, n$, we define the operator P_k on $C^{2,1}(\mathcal{D})$ by

$$P_k[u] = - (D_t u) S_k(D^2 u). \tag{9}$$

Suppose now that $u \in C^2(\mathcal{D})$ satisfies the differential inequality (7). From Proposition 2.1 in [8], we have

$$\begin{aligned}
 P_k[-u] &= (D_t u) S_k(-D^2 u) \\
 &\leq \binom{n}{k} \frac{1}{(n\rho_k^*(\mathcal{A}))^k} D_t u \left(-a^{ij} D_{ij} u\right)^k \\
 &\leq \frac{k}{k+1} \binom{n}{k} \left\{ (n\rho_k^*(\mathcal{A}))^{\frac{-k}{k+1}} f \right\}^{k+1}
 \end{aligned}
 \tag{10}$$

provided $-D^2 u \in \overline{\Gamma}_k^-$, $D_t u \geq 0$, $\mathcal{A} \in \Gamma_k^*$. Consequently, replacing u by $-u$, we have the differential inequality,

$$P_k[u] \leq \psi, \tag{11}$$

where

$$\psi = \frac{k}{k+1} \binom{n}{k} \left[\frac{1}{n\rho_k^*(\mathcal{A})} \right]^k f^{k+1},$$

holding on the subset of \mathcal{D} where $D^2 u \in \overline{\Gamma}_k^-$ and $D_t u \leq 0$, that is where the function u is spatially k -convex and non-increasing in time. As with the elliptic case [8], the estimate is thus reduced to existence and estimation of solution of Hessian equations with inhomogeneous terms in L^p spaces. For the parabolic Hessian equation the relevant existence theorem is due to Reyne [12], (with a thorough treatment given in the monograph [9]).

For our purpose here it is enough to formulate it in cylinders $Q_T = \Omega \times (0, T]$, where Ω is a smooth domain in \mathbb{R}^n , which is assumed to be uniformly $(k - 1)$ -convex, that is the principal curvatures $(\kappa_1, \dots, \kappa_{n-1})$ of the boundary $\partial\Omega$, lie in the cone Γ_{k-1} in \mathbb{R}^{n-1} . The function $u \in C^{2,1}(\mathcal{D})$ is admissible with respect to the operator (9), or simply k -admissible, if $D_t u \leq 0$ in \mathcal{D} and $D^2 u(x, t) \in \overline{\Gamma}_k^-$ for each $(x, t) \in \mathcal{D}$.

Theorem 2 *Let $\psi_0 \in C^{2,1}(\overline{Q_T})$, $\inf \psi_0 > 0$ and Ω be uniformly $(k - 1)$ -convex with $\partial\Omega \in C^4$. Then there exists a unique k -admissible solution $u \in C^{2,1}(\overline{Q_T})$ of the initial boundary value problem,*

$$\begin{aligned}
 P_k[u] &= \psi_0, \quad \text{in } Q_T \\
 u &= 0 \quad \text{on } \partial_p Q_T.
 \end{aligned}
 \tag{12}$$

Theorem 2 extends to parabolic equations of the form (2.4), the corresponding elliptic result due to Caffarelli et al. [1], used in [8]. To apply Theorem 2 to the proof of Theorem 1.1, we let $Q_0 = \Omega_0 \times [0, T]$ be a cylinder containing \mathcal{D} with Ω_0 uniformly $(k - 1)$ -convex, with boundary $\partial\Omega_0 \in C^4$, and set

$$\psi' = P_k[u] \chi_{\mathcal{D}_k}, \tag{13}$$

where $\mathcal{D}_k = \mathcal{D}_k^-$ is the lower k -contact set of u in \mathcal{D} given by

$$\begin{aligned}
 \mathcal{D}_k^- &= \{(x, t) \mid \exists k\text{-admissible } v \in C^{2,1}(\mathcal{D}) \text{ satisfying} \\
 &\quad v \leq u \text{ in } \mathcal{D}, \quad v(x, t) = u(x, t)\}.
 \end{aligned}
 \tag{14}$$

Clearly for any $(x, t) \in \mathcal{D}$, $D^2 u(x, t) \in \overline{\Gamma}_k^-$ and $D_t u(x, t) \leq 0$. For $\psi_0 \in C^{2,1}(\overline{Q_0})$, satisfying $\psi_0 > 0$, in Q_0 , $\psi' < \psi_0$ in \mathcal{D} , we define $u_0 \in C^{2,1}(\overline{Q_0})$ to be the unique k -admissible solution of (12). Analogously, to the elliptic case, we have $u \geq u_0$ in \mathcal{D} . To see this, we suppose there exists a point $(x_0, t_0) \in \overline{\mathcal{D}} - \partial_p \mathcal{D}$ such that

$$u_0(x_0, t_0) - u(x_0, t_0) = \sup_{\mathcal{D}} (u_0 - u) > 0.$$

Since u_0 is k -admissible, we must have $(x_0, t_0) \in \mathcal{D}_k^-$. But then $D^2u_0(x_0, t_0) \leq D^2u(x_0, t_0)$, $D_tu_0(x_0, t_0) \leq D_tu(x_0, t_0)$ implies $P_k[u_0] \leq P_k[u]$ at (x_0, t_0) which contradicts (13). Consequently, Theorem 1 will follow from corresponding estimates for k -admissible solutions of (12) with $\psi_0 \in L^1$, which we establish in the next section.

3 Estimates

In the previous section, we reduced the proof of Theorem 1, to L^∞ estimates for parabolic Hessian operators of the form (9). As with the case $k = n$, due to Krylov [3], we accomplish this through the Hessian integrals $I_k[u]$, introduced in [14, 18]. For Ω a domain in \mathbb{R}^n and $u \in C^2(\Omega)$, we define the Hessian integral $I_k[u]$, $k = 1, \dots, n$, by

$$I_k[u] = - \int_{\Omega} u S_k(D^2u). \tag{15}$$

By integration by parts, we have for $u = 0$ on $\partial\Omega$,

$$I_k[u] = \int_{\Omega} S_k^{ij} D_i u D_j u, \tag{16}$$

where, for any matrix $r \in S^n$,

$$S_k^{ij} = \frac{\partial}{\partial r_{ij}} S_k(r).$$

Accordingly, if u is k -convex in Ω , that is $D^2u \in \overline{\Gamma}_k$, then $[S_k^{ij}] \geq 0$ and hence $I_k[u] \geq 0$. To connect the Hessian integral with the operator (9), we let $u \in C^{2,1}(Q_T)$ and calculate

$$\begin{aligned} \frac{d}{dt} I_k[u(\cdot, t)] &= - \int_{\Omega} u_t S_k(D^2u) - \int_{\Omega} u S_k^{ij} D_{ij} u \\ &= \int_{\Omega} P_k[u] - \int_{\Omega} u_t S_k^{ij} D_{ij} u \\ &= (k + 1) \int_{\Omega} P_k[u] \end{aligned} \tag{17}$$

provided $u(\cdot, t) = 0$ on $\partial\Omega$. Here as for (19), we are using the key divergence free property

$$D_i(S_k^{ij}(D^2u)) = 0, \quad j = 1, \dots, n. \tag{18}$$

Consequently, we have the identity,

$$I_k[u(\cdot, t)] = (k + 1) \int_0^t \int_{\Omega} P_k[u] \tag{19}$$

for any $u \in C^{2,1}(Q_T)$ vanishing continuously on ∂Q_T . To proceed further, we need the monotonicity formula for Hessian integrals proved in [14], namely if $u \leq v$ in Ω , $u = v = 0$ on $\partial\Omega$ and $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are k -convex in Ω , then

$$I_k[v] \leq I_k[u]. \tag{20}$$

From the weak continuity of Hessian integrals, also proved in [14], the monotonicity formula (20) extends to general k -convex functions $u, v \in C^0(\overline{\Omega})$, where

$$I_k[u] = - \int_{\Omega} u \, d\mu_k[u] \tag{21}$$

and $\mu_k[u]$ denotes the k -Hessian measure of u . As in [8], we will make a specific choice for v in terms of the Green’s function for the operator, $F_k[u] = S_k(D^2u)$. Indeed if $\partial\Omega$ is uniformly $(k - 1)$ -convex, $k > n/2$, there exists a unique k -convex function $G_y \in C^{0,2-n/k}(\bar{\Omega})$, for any point $y \in \Omega$, such that

$$\begin{aligned} \mu_k[G_y] &= \delta_y, \\ G_y &= 0 \text{ on } \partial\Omega \end{aligned} \tag{22}$$

where δ_y denotes the Dirac delta measure at y . Combining (19) and (20) with the specific choice

$$v = \left[\frac{u(y)}{G_y(y)} \right] G_y \geq u, \tag{23}$$

we thus obtain, for k -admissible $u \in C^{2,1}(Q_T)$,

$$\begin{aligned} [-u(y, t)]^{k+1} &\leq [-G_y(y)]^k I_k[u] \\ &\leq (k + 1) \left| G_y(y) \right|^k \int_0^t \int_{\Omega} P_k[u]. \end{aligned} \tag{24}$$

Applying the estimate (24) to (12), we get the a priori estimate

$$-u(y, t) \leq \left\{ (k + 1) \left| G_y(y) \right|^k \right\}^{\frac{1}{k+1}} \left\{ \int_{Q_T} \psi_0 \right\}^{\frac{1}{k+1}} \tag{25}$$

for all $(y, t) \in Q_T$, and by letting ψ_0 approximate ψ' , we obtain (7), by virtue of (10).

Moreover, defining the upper contact set $\mathcal{D}_k^+ = \mathcal{D}_k^-(-u)$ and taking account of (13), we may write (7) more precisely as

$$\begin{aligned} \sup_{\mathcal{D}} u &\leq \frac{1}{n} \left[nk \binom{n}{k} \right]^{\frac{1}{n+1}} \sup_{y \in \Omega} \left| G_y(y) \right|^{\frac{k}{k+1}} \left\| \frac{f}{\rho^*(\mathcal{A})^{k/(k+1)}} \right\|_{L^k(\mathcal{D}_k^+)} \\ &\leq \left\{ \frac{k(\text{diam } \Omega)^{2k-n}}{[n(n - n/k)]^k \omega_n} \right\}^{\frac{1}{k+1}} \left\| \frac{f}{\rho^*(\mathcal{A})^{k/(k+1)}} \right\|_{L^k(\mathcal{D}_k^+)}, \end{aligned} \tag{26}$$

from formula (17) in [8]. The estimates (21), (26) constitute parabolic analogues of the precise estimates (21) and (22) in [8]. Because of the presence of the upper contact set, the estimate (26) also extends refinements of Krylov’s estimate for $k = n$, obtained in [10–12, 17]. As in [8], we may also express the dependence on the Green’s function G in (26) in terms of the k -Hessian measure of the solution of the homogeneous equation in the punctured domain. We also note that discrete versions of the case $k = n$ are given in [5–7].

Finally we remark that from Theorem 1 we may derive corresponding extensions of local estimates for linear parabolic equations due to Krylov and Safonov [4], (see also [11–13]), as done for the elliptic case in [8].

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