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EXISTENCE AND ASYMPTOTICS OF NONLINEAR HELMHOLTZ EIGENFUNCTIONS*

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Abstract. We prove the existence and asymptotic expansion of a large class of solutions to nonlinear Helmholtz equations of the form $(\Delta - \lambda^2)u = N[u]$, where $\Delta = -\sum_j \partial_j^2$ is the Laplacian on $\mathbb{R}^n$, λ is a positive real number, and $N[u]$ is a nonlinear operator depending polynomially on u and its derivatives of order up to order two. Nonlinear Helmholtz eigenfunctions with $N[u] = \pm|u|^{p-1}u$ were first considered by Gutiérrez [*Math. Ann.*, 328 (2004), pp. 1–25]. We show that for suitable nonlinearities and for every $f \in H^{k+4}(\mathbb{S}^{n-1})$ of sufficiently small norm, there is a nonlinear Helmholtz function taking the form $u(r,\omega) = r^{-(n-1)/2}(e^{-i\lambda r} f(\omega) + e^{+i\lambda r} b(\omega) + O(r^{-\epsilon}))$, as $r \to \infty$, $\epsilon > 0$, for some $b \in H^k(\mathbb{S}^{n-1})$. Moreover, we prove the result in the general setting of asymptotically conic manifolds. The proof uses an elaboration of anisotropic Sobolev spaces defined by Vasy [*A minicourse on microlocal analysis for wave propagation*, in Asymptotic Analysis in General Relativity, London Math. Soc. Lecture Note Ser. 443, Cambridge University Press, Cambridge, 2018, pp. 219–374], between which the Helmholtz operator $\Delta - \lambda^2$ acts invertibly. These spaces have a variable spatial weight $\mathsf{l}_\pm$, varying in phase space and distinguishing between the two "radial sets" corresponding to incoming oscillations, $e^{-i\lambda r}$, and outgoing oscillations, $e^{+i\lambda r}$. Our spaces have, in addition, module regularity with respect to two different "test modules" and have algebra (or pointwise multiplication) properties that allow us to treat nonlinearities $N[u]$ of the form specified above.

Key words. nonlinear eigenfunctions, nonlinear Helmholtz equation, incoming boundary data, asymptotic expansions, anisotropic Sobolev spaces, module regularity

AMS subject classifications. 35P25, 35P30, 81Uxx

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1. Introduction.

In this article we prove the existence and asymptotic expansion of a large class of solutions to nonlinear Helmholtz equations of the form

$$(\Delta - \lambda^2)u = N[u], \tag{1.1}$$

where $\Delta = -\sum_j \partial_j^2$ is the Laplacian on $\mathbb{R}^n$ with the sign convention that it is positive as an operator, λ is a positive real number, and $N[u]$ is a nonlinear operator that is a polynomial in u, $\overline{u}$ and their derivatives of order up to two. Such equations are of interest in part because, for certain nonlinearities $N[u]$, they furnish standing waves for nonlinear evolution equations, that is, solutions that are time-harmonic. Indeed this is the case whenever $N[e^{i\theta}u] = e^{i\theta}N[u]$ for all $\theta \in \mathbb{R}$. For example, if $N[u] = \alpha|u|^{2q}u$, then $\Psi(z,t) = u(z)e^{i\lambda^2 t}$ solves the nonlinear Schrödinger equation

$$-i\partial_t \Psi = \Delta\Psi - \alpha|\Psi|^{2q}\Psi, \tag{1.2}$$

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while if $N[u] = |\nabla u|^2 u$, then $v(z,t) = u(z)e^{i\lambda t}$ solves the nonlinear wave equation

$$(\partial_t^2 + \Delta)v = |\nabla v|^2 v. \tag{1.3}$$

In this article, we will study the existence and asymptotic behavior of "small" solutions to (1.1). Moreover, we shall do this not just for the standard Laplacian on $\mathbb{R}^n$ but for potential and/or metric perturbations of this Laplacian and even more generally for the Laplacian on asymptotically conic manifolds. However, in this introduction we shall mostly discuss the flat Euclidean case, as our results are new even in this setting.

Since the linearization of this equation at $u = 0$ is just the standard Helmholtz equation,

$$(\Delta - \lambda^2)u = 0, \tag{1.4}$$

it is intuitively clear that nonlinear eigenfunctions which are small in an appropriate sense should behave similarly to linear Helmholtz eigenfunctions. The structure of these is well known. The space of Helmholtz eigenfunctions of polynomial growth is parametrized by distributions on the "sphere at infinity," $\mathbb{S}^{n-1}$. Given $f \in C^\infty(\mathbb{S}^{n-1})$, there is a unique Helmholtz eigenfunction satisfying (in standard polar coordinates, $r = |z|$, $\omega = z/|z|$)

$$(\Delta - \lambda^2)u_0 = 0, \quad u_0 = r^{-(n-1)/2}\Big(e^{-i\lambda r} f(\omega) + e^{+i\lambda r} b_0(\omega) + O(r^{-1})\Big), \text{ as } \ r \to \infty, \tag{1.5}$$

where $b_0 \in C^\infty(\mathbb{S}^{n-1})$ is determined by f. (In fact, in the simple case of the flat Laplacian, $b_0(\omega) = i^{(n-1)} f(-\omega)$, but in the presence of metric or potential perturbations, b_0 is not so explicit and is indeed related to the scattering matrix of the perturbed operator.) We call f the "incoming data" or "incoming radiation pattern" for the eigenfunction u, while b_0 is referred to as the "outgoing data" or "outgoing radiation pattern." It is an arbitrary choice whether to parametrize eigenfunctions by their incoming or their outgoing data; each determines the other.

1.1. Main results. Our main result, at least as it applies to the flat Laplacian on $\mathbb{R}^n$, is that small nonlinear eigenfunctions can be parametrized in a similar way. We state our result first for the equation (with a reminder that $\Delta \geq 0$ is the *positive* Laplacian)

$$(\Delta - \lambda^2)u = \alpha u^{q_1} \overline{u}^{q_2}, \quad \lambda > 0, \quad \alpha \in \mathbb{C}. \tag{1.6}$$

THEOREM 1.1 (main theorem, Euclidean case). *Let $n \geq 2$, $q_1, q_2 \in \mathbb{N}_0$, $p = q_1 + q_2$, and assume that*

$$(p-1)\frac{n-1}{2} > 2. \tag{1.7}$$

Let k be an integer greater than $(n-1)/2$. There exist $\epsilon, \epsilon' > 0$ sufficiently small, such that for every $f \in H^{k+4}(\mathbb{S}^{n-1})$ with $\|f\|_{H^{k+4}(\mathbb{S}^{n-1})} < \epsilon$, there is a solution u to (1.6)*, satisfying*

$$u = r^{-(n-1)/2}\Big(e^{-i\lambda r} f(\omega) + e^{+i\lambda r} b(\omega) + O(r^{-\epsilon'})\Big), \text{ as } r \to \infty, \tag{1.8}$$

for some $b \in H^k(\mathbb{S}^{n-1})$. The $O(r^{-\epsilon'})$ remainder can be taken either in the space $H^k(\mathbb{S}^{n-1})$ or, using Sobolev embeddings, in a pointwise sense.

Moreover, uniqueness holds in the following sense. Fix a C^∞ *function* $\chi(r)$ *equal to zero for* r *small and* 1 *for* r *large, and let* $\ell = -1/2 - \delta$ *for any* δ *satisfying* $0 < \delta \le (4p)^{-1}$. *Let* $u_- = \chi(r) r^{-(n-1)/2} e^{-i\lambda r} f(\omega)$. *Then given* f *with* $\|f\|_{H^{k+4}(\mathbb{S}^{n-1})}$ *sufficiently small, there is exactly one nonlinear eigenfunction* u *of the form* (1.8) *with the property that* $u - u_-$ *has small norm in the Hilbert space* $H_+^{2,\ell;1,k+1}$ *defined in* (2.34).

Remark 1.2. The solution is a scattering type solution, not an L^2 solution. From the Pohozaev identity, it is known that the sign of the right-hand side (RHS) of (1.6) plays an important role in the existence of finite energy solutions to (1.6), while the sign of the RHS of (1.6) plays no role in Theorem 1.1.

Remark 1.3. The nonlinearity above can be of the form $\pm|u|^{p-1}u$ if p is odd, but not if p is even, since the nonlinear term is required to be polynomial.

Remark 1.4. As mentioned above, $\Psi(z,t) = u(z)e^{i\lambda^2 t}$ is a global-in-time solution which solves (1.2) but it is time-periodic without any decay. This is quite different from the classical finite-energy solution to (1.2).

Our proof of Theorem 1.1 principally makes use of the asymptotically conic structure of $\mathbb{R}^n$ near infinity; in particular it uses neither the translation symmetries of $\mathbb{R}^n$ nor exact formulae for resolvent kernels. The more general version of our main result is valid in the setting of asymptotically conic manifolds. To prepare for the definition of such spaces, let us recall that, given a compact Riemannian manifold $(\boldsymbol{N}, g_{\boldsymbol{N}})$, the metric cone over $\boldsymbol{N}$ is the Riemannian manifold $(0,\infty)_r \times \boldsymbol{N}$ with metric of the form $dr^2 + r^2 g_{\boldsymbol{N}}$.

We define an *asymptotically conic manifold* to be the interior M° of a compact manifold with boundary M, with Riemannian metric g taking a particular form near the boundary. To specify this, let x be a boundary defining function for ∂M (that is, the boundary ∂M is given by $x = 0$, where x vanishes to first order at ∂M and $x > 0$ on M°) and let $y = (y_1, \dots, y_{n-1})$ be local coordinates on ∂M extended to a collar neighborhood $\{x \le c\}$ of the boundary, where $c > 0$ is some small fixed positive number. We assume that the metric g has the property that, near any point on ∂M, there are coordinates $(x, y_1, \dots, y_{n-1})$ as above such that, in this coordinate patch, g takes the form

$$g = \frac{dx^2}{x^4} + \frac{h(x,y,dy)}{x^2}, \tag{1.9}$$

where h is a smooth $(0,2)$-tensor that restricts to a metric on ∂M. This definition is better understood by passing to the variable $r = 1/x$, which goes to infinity at the boundary of M. The metric then takes the form

$$g = dr^2 + r^2 h\left(\frac{1}{r}, y, dy\right). \tag{1.10}$$

If h is independent of x for small x, then this is precisely a conic metric for large r, where $g_{\boldsymbol{N}} = h(0,y,dy)$. More generally, it is asymptotic to this conic metric (smoothness of h in x is equivalent to having an asymptotic expansion in powers of $1/r$ as $r \to \infty$). In particular, the metric is always complete, as the boundary is at $r = \infty$, which is an infinite distance from any interior point. Thus, we can think of an asymptotically conic manifold as a complete noncompact Riemannian manifold that is asymptotic, at infinity, to the "large end of a cone," but having no conic singularity (as a true cone usually does at $r = 0$). Such spaces have curvature tending to zero at infinity, and local injectivity radius tending to infinity, so balls of a fixed size are asymptotically Euclidean as their center tends to infinity.

Particular instances of asymptotically conic manifolds include flat Euclidean space, or any compact metric perturbation of the flat metric on Euclidean space. In this case, M is the radial compactification of $\mathbb{R}^n$, given by the union of $\mathbb{R}^n$ with the "sphere at infinity," $\mathbb{S}^{n-1}$. Connected sums of such manifolds are also asymptotically conic. The topology and geodesic dynamics on such manifolds can be intricate. For example, any convex co-compact hyperbolic manifold can have its metric modified near infinity to be asymptotically conic; while this is an artificial construction, it provides a very large class of asymptotically conic spaces with complicated topology and hyperbolic trapped set.

THEOREM 1.5 (main theorem, asymptotically conic case). *Let (M°, g) be an asymptotically conic manifold of dimension n, and let V be a conormal short range potential, that is, a smooth potential on M° satisfying estimates near infinity of the form*

$$\left|(rD_r)^k D_y^\alpha V(r,y)\right| \leq C\langle r\rangle^{-\gamma} \quad \text{for all } k \geq 0,\ \alpha \in \mathbb{N}^{n-1} \tag{1.11}$$

for some $\gamma > 1$. Let $H = \Delta_g + V$, where Δ_g is the Laplace–Beltrami operator on (M°, g). Let $N(u, \overline{u}, \nabla u, \nabla\overline{u}, \nabla^{(2)}u, \nabla^{(2)}\overline{u})$ be a sum of monomial terms, each of which has degree not less than p, in u and $\overline{u}$ and their derivatives up to order two, with coefficients smooth on M, and assume that p satisfies (1.7). *Let k be an integer greater than $(n-1)/2$. There exist $\epsilon, \epsilon' > 0$ sufficiently small, such that for every $f \in H^{k+4}(\partial M)$ with $\|f\|_{H^{k+4}(\partial M)} < \epsilon$, there is a function u on M° satisfying*

$$(H - \lambda^2)u = N(u, \overline{u}, \nabla u, \nabla\overline{u}, \nabla^{(2)}u, \nabla^{(2)}\overline{u})$$

with asymptotics

$$u = r^{-(n-1)/2}\Big(e^{-i\lambda r} f(\omega) + e^{+i\lambda r} b(\omega) + O(r^{-\epsilon'})\Big), \text{ as } r \to \infty, \tag{1.12}$$

for some $b \in H^k(\partial M)$. Moreover, uniqueness holds in the same sense as in Theorem 1.1.

Remark 1.6. We first clarify the meaning of a "monomial of degree not less than p in u and $\overline{u}$ and their derivatives up to order two, with coefficients smooth on M." These derivatives are understood to be taken with respect to a frame of vector fields that are uniformly bounded with respect to the metric g. Thus, as $r \to \infty$ we could take ∂_r and $r^{-1}\partial_{y_j}$, for example; these are the natural analogues of the gradient in the Euclidean sense, written with respect to polar coordinates. For example, if $p = 3$, then on Euclidean $\mathbb{R}^n$ the nonlinear term N could take the form

$$|u|^2 u + |\nabla u|^2 u + |\nabla^{(2)} u|^2 u + \frac{\partial^2 u}{\partial z_1^2}\frac{\partial u}{\partial z_2}\overline{u}^2 + u^5.$$

Remark 1.7. The first result along the lines of Theorem 1.1 was obtained by Gutiérrez [11]. Curiously, the set of pairs (n, p) treated in that paper is almost disjoint to ours: it covers the case $n = 3, 4$ and $p = 3$, for example, but higher n and p are excluded, while our method works most easily with large n and p. In fact, in view of the condition (1.7) in our two theorems we can treat $p \geq 6$ when $n = 2$, $p \geq 4$ when $n = 3$, $p \geq 3$ when $n = 4, 5$, and $p \geq 2$ for $n \geq 6$. We discuss previous literature more fully below.

1.2. Strategy of the proof. The basic strategy of our proof of Theorem 1.1 is a fixed point argument which is similar to [11]. Given incoming data f, Gutiérrez formed the linear eigenfunction u_0 and showed that the map

$$\Phi : u \mapsto u_0 + (\Delta - (\lambda + i0)^2)^{-1}\alpha|u|^{p-1}u \tag{1.13}$$

is a contraction map on some Banach space, provided that the norm of u is sufficiently small. Gutiérrez used L^q spaces, for example, L^4, when $p = 3$ and $n = 3, 4$. Given $u \in L^4$, it is clear that the cubic term $|u|^2u$ lies in $L^{4/3}$, while uniform resolvent bounds of Kenig, Ruiz, and Sogge [20] and the restriction estimates of Stein and Tomas [29] are used to show that the outgoing resolvent maps $L^{4/3}$ back to L^4. The fixed point of Φ is a nonlinear eigenfunction, as one sees by applying $\Delta - \lambda^2$ to both sides, and it has the same incoming data as u_0.

In our approach, we use polynomially weighted L^2-based Sobolev spaces, with an anisotropic weight. Vasy [32] has shown how to construct two families of Hilbert spaces between which $\Delta - \lambda^2$ maps *as a bounded invertible operator*:

$$\Delta - \lambda^2 \colon \mathcal{X}^{s,\mathsf{l}_\pm} \longrightarrow \mathcal{Y}^{s-2,\mathsf{l}_\pm+1}. \tag{1.14}$$

In (1.14), the space $\mathcal{Y}^{s,\mathsf{l}_\pm} = H^{s,\mathsf{l}_\pm}$ is a variable order L^2-based Sobolev space. The index $s \in \mathbb{R}$ is a regularity parameter, specifying how many derivatives are locally in L^2, while $\mathsf{l}_\pm$ is a variable spatial weight, which varies "microlocally," i.e., in phase space $T^*\mathbb{R}^n$. The weight l_+ is chosen so that $u \in \mathcal{X}^{s,\mathsf{l}_+}$, localized in frequency close to the incoming radial oscillation $e^{-i\lambda r}$, decays at least as $r^{-(n-1)/2-\delta}$ with $\delta > 0$ fixed but small, while near the outgoing radial oscillation $e^{i\lambda r}$, slower decay, as $r^{-(n-1)/2+\delta}$, is permitted. The weight l_- has the opposite property: the decay must be faster than $r^{-(n-1)/2}$ near the outgoing radial oscillation but can be slower near the incoming radial oscillation.

This means that, for the $+$ sign, the "outgoing" expansion at infinity typical of generalized eigenfunctions is permitted, while the "incoming" expansion is not, while for the $-$ sign, the situation is reversed. This is consistent with the statement that the inverse map to (1.14) is, for the $+$ sign, the outgoing resolvent $(\Delta - (\lambda + i0)^2)^{-1}$, and for the $-$ sign, the incoming resolvent $(\Delta - (\lambda - i0)^2)^{-1}$, meaning that solutions $(\Delta - \lambda^2)u = f \in \mathcal{S}(\mathbb{R}^n)$ with $u \in \mathcal{X}^{s,\mathsf{l}_\pm}$ admit asymptotic expansions of the form

$$u = r^{-(n-1)/2}e^{\pm i\lambda r}\sum_{j=0}^{\infty} r^{-j}v_j, \qquad v_j \in C^\infty(\mathbb{S}^{n-1}).$$

The domain of (1.14) is defined by an a priori regularity condition

$$\mathcal{X}^{s,\mathsf{l}_\pm} := \{u \in H^{s,\mathsf{l}_\pm} : (\Delta - \lambda^2)u \in H^{s-2,\mathsf{l}_\pm+1}\}. \tag{1.15}$$

The exponents $(s-2, \mathsf{l}_\pm + 1)$ reflect the order $(2, 0)$ of the operator P, as well as the ellipticity of P at fiber-infinity and the fact that P is of real principal type at spatial-infinity, leading to a loss of one order of decay in the spatial regularity $\mathsf{l}_\pm$. It is a tautology that $\Delta - \lambda^2$ is a bounded operator from $\mathcal{X}^{s,\mathsf{l}_\pm}$ to $H^{s-2,\mathsf{l}_\pm+1}$. What is *not* obvious is that this is an invertible map, a result due to Vasy [33] with methods going back to Melrose [24], and of which we give a detailed proof below. The inverse operator depends on the choice of sign $\pm$ (the choice giving either the incoming or outgoing resolvent), and as in the work of Gutiérrez, although a choice must be made, the only effect of this choice is to determine whether one prescribes the incoming data f in the main theorems or the outgoing data b.

One thus obtains an inverse mapping $R(\lambda + i0)\colon H^{s-2,\mathsf{l}_+ +1} \longrightarrow H^{s,\mathsf{l}_+}$, but this is not enough to solve nonlinear problems. Given that our nonlinear term is assumed to be polynomial, we need to work with spaces of functions with good algebra (or multiplicative) properties. The spaces $\mathcal{X}^{s,\mathsf{l}_\pm}$ and $\mathcal{Y}^{s,\mathsf{l}_\pm} = H^{s,\mathsf{l}_\pm}$ are not suitable for this purpose, even (surprisingly) for large s. Recall that, for $s > n/2$, $H^{s,0}$, the standard Sobolev space of order s forms an algebra, i.e., $H^{s,0} \cdot H^{s,0} \subset H^{s,0}$. If we include spatial weights, then (at least for constant weights), these combine additively, in the sense that we have for $\ell_1, \ell_2 \in \mathbb{R}$, $H^{s,\ell_1} \cdot H^{s,\ell_2} \subset H^{s,\ell_1+\ell_2}$. However, our weights are typically negative—indeed, they are forced to be so to obtain bijectivity of $\Delta - \lambda^2$—so this will not lead to a mapping Φ on a fixed space, as in (1.13); indeed, the nonlinear operation must *gain* one order of spatial decay to account for the loss of one order in the action of the resolvent inverting (1.14).

To do this, we work with spaces with additional regularity with respect to the differential operators with coefficients that grow linearly at infinity but which annihilate the outgoing oscillation $e^{i\lambda r}$. These are generated by the operators $r(\partial_r - i\lambda)$ and purely angular differential operators ∂_{y_j}. This type of regularity condition is precisely the "module regularity" introduced by the second author together with Melrose and Vasy in [14] and used by Hintz and Vasy [15] to solve a semilinear wave equation. Thus for $s, \ell \in \mathbb{R}, \kappa, k \in \mathbb{N}_0$ we define L^2-based Sobolev spaces $H_+^{s,\ell;\kappa,k}$ in which s is the order of differentiability in the usual sense, i.e., relative to constant coefficient vector fields, ℓ is the decay rate relative to L^2, κ is the order of "module" differentiability just described, and k is the order of differentiability with respect to the smaller module of purely angular derivatives. Crucially, provided $\kappa \geq 1$ and $-3/2 < \ell < -1/2$ we can take the spatial weight ℓ to be *constant* and retain the invertibility of $\Delta - \lambda^2$. Indeed, the module regularity—which is asymmetric with respect to the incoming and outgoing oscillations, $e^{\pm i\lambda r}$—enforces additional vanishing of the incoming oscillations, and we arrive at a refinement of the mapping property (1.14), namely, we obtain in Theorem 2.4 below an invertible map

$$\Delta - \lambda^2\colon \mathcal{X}_+^{s,\ell;\kappa,k} \longrightarrow \mathcal{Y}_+^{s-2,\ell+1;\kappa,k}, \tag{1.16}$$

where $\mathcal{Y}_+^{s,\ell;\kappa,k} = H_+^{s,\ell;\kappa,k}$ and, analogously to (1.15), the $\mathcal{X}_+^{s,\ell;\kappa,k}$ are given by

$$\mathcal{X}_+^{s,\ell;\kappa,k} := \{u \in H_+^{s,\ell;\kappa,k} : (\Delta - \lambda^2)u \in H_+^{s-2,\ell+1;\kappa,k}\}.$$

We continue to denote the inverse map to (1.16) by $R(\lambda + i0)$, as it is just the restriction of the inverse of (1.14) to $\mathcal{Y}_+^{s-2,\ell+1;\kappa,k}$ within an appropriate choice of $\mathcal{Y}^{s-2,\mathsf{l}_+ +1}$.

For $\kappa \geq 1$ and $\kappa + k > n/2$, we are in the "Sobolev algebra" range, and it turns out these spaces satisfy *improved* multiplicative properties in comparison to $H^{s,\mathsf{l}_\pm}$ spaces. For example, we have the following containment, which we prove in section 2.5:

$$\left(H_+^{s,\ell;\kappa,k}\right)^p \subset H_+^{s,p\ell+(p-1)n/2-\kappa;\kappa,k}. \tag{1.17}$$

Thus, in contrast to the additivity of weights for products of standard weighted Sobolev spaces $H^{s,\ell}$, the spatial weight of a p-fold product of distributions in the module regularity space can be *larger* (i.e., more decaying) than ℓ even when ℓ is negative. We require a gain of one in the weight, so that we can apply the inverse map to (1.16). Since we must take $\ell < -1/2$, that requires that $p(-1/2-\delta)+(p-1)n/2-\kappa \geq (-1/2-\delta)+1$ for sufficiently small δ, or equivalently, $-p/2+(p-1)n/2-\kappa > 1/2$.

Clearly, to obtain the biggest possible range of (n, p) we should take κ as small as possible; however, we require κ to be *at least* 1 in order to distinguish between the incoming and outgoing oscillations, and so that we can take ℓ constant (which we require in order to prove (1.17)). See Remark 2.6 and Corollary 3.10. For this reason, we take $\kappa = 1$ below. This leads to condition (1.7). On the other hand, for p sufficiently large with respect to n, there is no need for the small module. In this case, we need $\kappa > n/2$ so that we are still in the Sobolev algebra range for such κ, provided one has $\ell + 1 \leq p\ell + \frac{(p-1)n}{2} - \kappa$ (see (4.7) below) for $\ell < -1/2$, which amounts to replacing (1.7) with the stronger condition

$$(p-1)\frac{n-1}{2} > \kappa + 1 \text{ for } \kappa > \frac{n}{2}.$$

For example, the stronger condition excludes $n = 3$, $p = 4$ as well as n large, $p = 2$ which are allowed by (1.7).

For small incoming data $f \in H^{k+4}(\partial M)$, k sufficiently large, and $\kappa = 1$, we obtain our nonlinear eigenfunction using a contraction map on the space $H_+^{s,\ell;1,k+1}$. However, the nonlinear eigenfunction, or even the linear eigenfunction u_0, does not lie in this space as its incoming oscillations do not have the required decay. To deal with this, we decompose u_0, the linear eigenfunction with incoming data f, into two terms, $u_0 = u_+ + u_-$, where u_- contains the leading incoming oscillation (which is the obstruction to membership in $H_+^{s,\ell;1,k+1}$). Consequently, the term u_+ lies in $H_+^{s,\ell;1,k+1}$ but u_- does not. (Indeed, one can think of u_+ as a sum of purely outgoing terms plus the lower order incoming terms, with additional decay, comprising u_0.) We seek a nonlinear eigenfunction satisfying

$$u = u_0 + (\Delta - (\lambda + i0)^2)^{-1} N[u],$$

where $N[u]$ is the nonlinear term. Notice that, since the resolvent gains us two orders of smoothness, according to (1.16), N can involve derivatives of u up to order 2. Subtracting u_- from both sides we have the equivalent equation

$$u - u_- = u_+ + (\Delta - (\lambda + i0)^2)^{-1} N[u],$$

and now defining $w = u - u_-$ we obtain

$$w = u_+ + (\Delta - (\lambda + i0)^2)^{-1} N[u_- + w].$$

Thus, it suffices to show that the map

$$\Phi(w) := u_+ + (\Delta - (\lambda + i0)^2)^{-1} N[u_- + w] \tag{1.18}$$

is a contraction on $H_+^{s,\ell;1,k+1}$ when the norm of w in this space is sufficiently small, which we show provided the norm of f in H^{k+4}, $k > (n-1)/2$, is sufficiently small.

Remark 1.8. In both Theorems 1.1 and 1.5, it is vital to take λ strictly positive. Indeed our entire approach, following Vasy [32, 33], is based on spaces of functions $\mathcal{X}_\pm^{s,\ell_\pm;\kappa,k}$ that distinguish between the incoming oscillations, $e^{-i\lambda r}$, and the outgoing oscillations, $e^{+i\lambda r}$, which would coincide when λ vanishes. The equation $Hu = N[u]$ for $\lambda = 0$ is of a different character and, from a technical point of view, is more closely related to the "b-calculus" used in, say, [32, 8, 15], than the scattering calculus used here (see section 2).

1.3. Previous literature. Standing wave solutions to nonlinear Schrödinger equations have been studied for a long time. The first studies were on finite-energy solutions, where the linearization at $u = 0$ is the operator $\Delta + \lambda^2$ with $\lambda > 0$; this problem is of a different character, as the linearization at $u = 0$ is an invertible operator. See [1, 2, 27] for classical work on this subject on Euclidean space, and [3, 23] for more recent works on hyperbolic and rotationally symmetric manifolds. The more recent literature is vast and we make no attempt to review it.

The first paper to study nonlinear Helmholtz eigenfunctions seems to be [11] by Gutiérrez, already discussed earlier in this introduction. She showed that for the cubic nonlinearity and in dimensions 3 and 4, there are nonlinear eigenfunctions with arbitrary small incoming data $f \in L^2(\mathbb{S}^{n-1})$. We note in passing that the restriction and uniform Sobolev estimates of [10, 9] allow one to extend Gutiérrez's method to all asymptotically conic manifolds.

The result of [11] is a perturbative result from the zero solution, as is ours here. On the other hand, Evequoz and Weth [5] used mountain pass techniques to find nonperturbative solutions far from the zero solution. These approaches have been extended in various ways in [22, 21]. In [4] the topology of the zero level sets of bounded real solutions to $(\Delta - 1)u + u^3 = 0$ are studied.

In the microlocal analysis literature, the underlying theory of real principal type propagation in the setting of "scattering" pseudodifferential operators was developed by Melrose in [24]. The scattering calculus itself appeared earlier (at least on Euclidean space) in work of Hörmander and Parenti (see, for example, [28]). A Fredholm theory for nonelliptic operators was developed by Vasy [32] on anisotropic Sobolev spaces (a precursor is Faure and Sjöstrand [6]). This is elaborated and explained in detail in his lecture notes [33]. His method applies to operators that are of real principal type, except for manifolds of radial points which have a particular structure. The first author with Haber and Vasy [8] used this Fredholm framework to study the Feynman propagator on asymptotically Minkowski spaces and showed that the semilinear wave equation with polynomial nonlinearity is solvable for small data, using a setup very similar to that considered here. This latter result is an extension of a more fundamentally microlocal setting of a previous result of Hintz and Vasy [15]. Indeed, the latter two authors have developed a robust microlocal analysis framework that they use to study quasilinear wave equations in various noncompact settings (see in particular [16, 17, 13]). In a recent series of papers [34, 30, 31], Vasy considers "second-microlocal" regularity for the Helmholtz operators, both at a fixed finite energy and near zero energy, which is very similar to our module regularity here. He proves mapping properties for the resolvent that overlap with our result on the invertibility of the Helmholtz operator on spaces with module regularity in Theorem 2.4 below.

1.4. Outline of this paper. In section 2 we review the theory of pseudodifferential operators with variable order and define anisotropic Sobolev spaces. We discuss the geometry of the bicharacteristic flow of $\Delta - \lambda^2$ at spatial infinity and define the radial sets. We also discuss module regularity and define the corresponding spaces of functions. Finally, we consider algebra properties of these spaces with sufficient module regularity.

In section 3 we prove the invertibility of $\Delta - \lambda^2$ acting between spaces as in (1.16). The proof of this is at least implicitly contained in works of Vasy, particularly his lecture notes [33], but it is not explicitly written out for this operator. Since, in addition, this is quite recently developed technology and not standard, we have decided to give at least some of the details to make the paper more self-contained.

In section 4 we prove the main theorems, using the technical preparation of the previous two sections.

1.5. Future directions. Theorem 1.5 shows that there is a well-defined "nonlinear scattering matrix" with domain $H^{k+4}(\mathbb{S}^{n-1})$, mapping into $H^k(\mathbb{S}^{n-1})$. In the linear case, the scattering matrix preserves the Sobolev order; in fact, it is a Fourier integral operator of order zero, associated to the antipodal map on the sphere, or in the more general context of an asymptotically conic manifold, geodesic flow at time on the boundary π [25]. One of the referees asked whether the nonlinear scattering matrix likewise preserves Sobolev regularity. This question is beyond the scope of the current article but is an ongoing research project by some of the authors. More generally, we think that this paper provides an effective set of tools for investigation of the microlocal properties of the nonlinear scattering matrix.

Another natural question is whether Theorems 1.1 and 1.5 tell us anything about closely related nonlinear evolution equations, such as the nonlinear Schrödinger, wave, or Klein–Gordon equations with nonlinearity of the same structure. As far as we know, the results in this paper do not directly imply any results about such equations. However, the *methods* can certainly be applied to such equations, and this is another ongoing research project of the authors.

A third direction would be to attempt to include fractional values of κ. In principle, the multiplication result (1.17) or Lemma 2.4 only requires $\kappa > 1/2$, provided one can make sense of a fractional power of a module. It seems possible that the calculus of operators from [34, 31] could allow one to do this, by allowing the symbols of operators in the module to have singularities at the zero section of the scattering cotangent bundle at spatial infinity. Were this possible, it would further extend the range of (n, p) that could be treated.

2. Scattering calculus. In this section, we discuss the technical tools that we need for the proof of the main theorems. We begin by discussing the pseudodifferential operators—the scattering calculus—used in the proof, on $\mathbb{R}^n$, and extend this in the following subsection to asymptotically conic manifolds. We refer to [33] and [24] for more detailed treatment of the scattering calculus.

2.1. The scattering calculus on $\mathbb{R}^n$. Throughout this paper, we denote Euclidean coordinates on $\mathbb{R}^n$ by $z = (z_1, \dots, z_n)$ and their dual coordinates by $\zeta = (\zeta_1, \dots, \zeta_n)$. We use the Japanese bracket $\langle z \rangle$ to denote $(1 + |z|^2)^{1/2}$. The Fourier transform, with Hörmander's normalization, will be denoted $\mathcal{F}$, with inverse $\mathcal{F}^{-1}$:

$$\mathcal{F}f(\zeta) = \int e^{-iz\cdot\zeta} f(z)\, dz, \quad \mathcal{F}^{-1}\tilde{f}(z) = (2\pi)^{-n} \int e^{iz\cdot\zeta} \tilde{f}(\zeta)\, d\zeta. \tag{2.1}$$

We denote $-i\partial/\partial z_j$ by D_{z_j} and use multi-index notation D_z^α, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ for higher order derivatives, in the standard way.

Pseudodifferential operators on $\mathbb{R}^n$ are defined via their symbols, which are functions on $T^*\mathbb{R}^n$. For sufficiently decaying symbols, say, $a(z, \zeta) \in \mathcal{S}(T^*\mathbb{R}^n)$, the corresponding pseudodifferential operator (defined by left quantization) is the operator with kernel

$$\mathrm{Op}(a)(z, z') := (2\pi)^{-n} \int e^{i(z-z')\cdot\zeta} a(z, \zeta)\, d\zeta. \tag{2.2}$$

This definition is extended to a larger class of symbols by integration by parts. The scattering calculus is obtained by letting a lie in a (scattering) symbol class

$S^{s,\ell}(T^*\mathbb{R}^n)$. For fixed real numbers s and ℓ this symbol class is defined by the estimates

$$\forall\alpha,\beta\in\mathbb{N}^n,\ \exists C_{\alpha,\beta}<\infty \text{ such that } \left|D_z^\alpha D_\zeta^\beta a(z,\zeta)\right|\le C_{\alpha,\beta}\langle z\rangle^{\ell-|\alpha|}\langle\zeta\rangle^{s-|\beta|}. \tag{2.3}$$

This is a rather restrictive class of symbols in which z and ζ are treated symmetrically: differentiation in ζ leads to decay in ζ and differentiation in z leads to decay in z. It is in the Hörmander class of symbols [19, sect. 18.4] relative to the slowly varying metric

$$\frac{dz^2}{\langle z\rangle^2}+\frac{d\zeta^2}{\langle\zeta\rangle^2}.$$

The class of pseudodifferential operators of order (s,ℓ) is by definition the class of operators obtained from symbols $a\in S^{s,\ell}(T^*\mathbb{R}^n)$ as above and is denoted $\Psi_{\rm sc}^{s,\ell}(\mathbb{R}^n)$. These pseudodifferential operators form a bifiltered algebra; concretely, the composition of an operator in $\Psi_{\rm sc}^{s_1,\ell_1}(\mathbb{R}^n)$ with an operator in $\Psi_{\rm sc}^{s_2,\ell_2}(\mathbb{R}^n)$ is an operator in $\Psi_{\rm sc}^{s_1+s_2,\ell_1+\ell_2}(\mathbb{R}^n)$. The symbol of the composition $\mathrm{Op}(a)\circ\mathrm{Op}(b)$ is given by

$$c(z,\zeta)=e^{iD_y\cdot D_\eta}a(z,\eta)b(y,\zeta)\Big|_{y=z,\eta=\zeta}$$

and has an asymptotic expansion

$$c(z,\zeta)\sim\sum_\alpha i^{|\alpha|}D_\zeta^\alpha a(z,\zeta)D_z^\alpha b(z,\zeta)/\alpha! \tag{2.4}$$

Given this formula, and the decay of derivatives from (2.3), it is clear that the *principal symbol*, which for $a\in S^{s,\ell}(T^*\mathbb{R}^n)$ is its equivalence class in

$$S^{s,\ell}(T^*\mathbb{R}^n)/S^{s-1,\ell-1}(T^*\mathbb{R}^n),$$

is multiplicative under composition. Notice that, unlike in the usual pseudodifferential calculus, here the principal symbol is well defined up to symbols decaying (a full integer order) faster in z, as well as decaying (a full integer order) faster in ζ. That means that the principal symbol is, in effect, completely well-defined at infinity for all finite frequencies ζ and not just asymptotically as $|\zeta|\to\infty$, at least in the case of classical symbols (discussed below).

We elaborate on this point. It is convenient in the scattering calculus to view symbols on the compactification of $T^*\mathbb{R}^n$. We have already mentioned in the introduction the radial compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$. This is obtained via the diffeomorphism $\varphi\colon\mathbb{R}^n\to\mathbb{B}^n$ from $\mathbb{R}^n$ to its unit ball, given by

$$z\mapsto\varphi(z)=\frac{z}{1+\langle z\rangle}\in\mathbb{B}^n.$$

The closure of the image of this map is obviously the closed unit ball $\overline{\mathbb{B}^n}$, and the map realizes $\mathbb{R}^n$ as the interior of this compact manifold with boundary. In keeping with standard notation we write $\overline{\mathbb{R}^n}\simeq\overline{\mathbb{B}^n}$, where the notation indicates that we keep in mind the identification between points in $\mathbb{R}^n$ with points in the ball $\mathbb{B}^n$. We similarly radially compactify the fiber copy of $\mathbb{R}^n$. Thus, we may understand the behavior of symbols by pulling them back via $\varphi^{-1}\times\varphi^{-1}$ to $\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$. This is particularly helpful for classical symbols, which by definition take the form $\langle z\rangle^\ell\langle\zeta\rangle^sC^\infty(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$ (such functions automatically satisfy the symbol estimates (2.15)). The class of such symbols is denoted $S^{s,\ell}_{\rm cl}(T^*\mathbb{R}^n)$. In particular, for classical symbols of order $(0,0)$, the symbol is continuous up to the boundary of $\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n}$, and the principal symbol can be viewed as the boundary value of this symbol. Notice that this has two "components," one a function at fiber-infinity, that is, on $\overline{\mathbb{R}^n}\times\mathbb{S}^{n-1}$, and one at "spatial infinity," that is, at $\mathbb{S}^{n-1}\times\overline{\mathbb{R}^n}$. More generally, for an operator A with classical symbol a of order $(s,0)$ (such as our Helmholtz operator $\Delta_g-\lambda^2$), the principal symbol is conveniently viewed as the combination of a fiber component, $\sigma_{\text{fiber},s,0}(A)(z,\zeta)$, which is homogeneous in ζ of degree s (and is hence determined by ζ restricted to any sphere), and a base (or spatial) component, $\sigma_{\text{base},s,0}(A)(\omega,\zeta)$, where ω is the limiting value of $z/|z|$ on the sphere at infinity, and $\zeta\in\mathbb{R}^n$. These have an obvious compatibility relation at the "corner," where $|z|$ and $|\zeta|$ are both infinite. In particular, for the Helmholtz operator, the principal symbol is given by

$$\sigma_{\text{fiber},2,0}(\Delta-\lambda^2)(z,\zeta)=|\zeta|_g^2:=\sum_{i,j}g_{ij}\zeta_i\zeta_j,$$
$$\sigma_{\text{base},2,0}(\Delta-\lambda^2)(\omega,\zeta)=|\zeta|_g^2-\lambda^2.$$

It is important to understand that the base symbol need not be homogeneous in the fiber variable and indeed is not homogeneous for the Helmholtz operator.

Suppose $A\in\Psi^{s,\ell}_{\rm sc}(\mathbb{R}^n)$ has a classical symbol. The elliptic set of A, $\text{Ell}_{s,\ell}(A)=\text{Ell}(A)$, is the open subset of $\partial(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$ consisting of those points near which the principal symbol is at least as big as $c\langle z\rangle^\ell\langle\zeta\rangle^s$ for some $c>0$. Its complement in $\partial(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$ is called the characteristic variety, $\Sigma_{s,\ell}(A)=\Sigma(A)$. The Helmholtz operator $\Delta_g-\lambda^2$ is elliptic at fiber-infinity, and thus the characteristic variety is contained in the component at spatial infinity and is given by

$$\Sigma(\Delta_g-\lambda^2)=\{(\omega,\zeta)\in\mathbb{S}^{n-1}\times\mathbb{R}^n\mid|\zeta|_g=\lambda\}. \tag{2.5}$$

We also define the *operator wavefront set* or *microlocal support*, $\text{WF}'(A)$ of A, to be the complement of the set of points $\mathsf{q}\in\partial(\overline{\mathbb{R}^n}\times\overline{\mathbb{R}^n})$ such that, in a neighborhood U of q, the full symbol $a(z,\zeta)$ satisfies (2.3) for all $s,\ell\in\mathbb{R}$. Thus, intuitively speaking, A is microlocally of order $-\infty$ in both the fiber and base senses away from $\text{WF}'(A)$.

Returning to the composition formula (2.4), it is straightforward to see from this that the commutator of two pseudodifferential operators $A\in\Psi^{s_1,\ell_1}_{\rm sc}(\mathbb{R}^n)$ and $B\in\Psi^{s_2,\ell_2}_{\rm sc}(\mathbb{R}^n)$ is an operator $[A,B]$ in $\Psi^{s_1+s_2-1,\ell_1+\ell_2-1}_{\rm sc}(\mathbb{R}^n)$, with principal symbol given by the Poisson bracket of the symbols a and b of these operators:

$$\sigma_{\rm pr}([A,B])=\{a,b\}\mod S^{s_1+s_2-2,\ell_1+\ell_2-2}(T^*\mathbb{R}^n). \tag{2.6}$$

We also recall that the Poisson bracket is given in terms of the Hamilton vector fields by

$$\{a,b\}=H_a(b)=-H_b(a),\quad H_a=\sum_j\Big(\frac{\partial a}{\partial\zeta_j}\frac{\partial}{\partial z_j}-\frac{\partial a}{\partial z_j}\frac{\partial}{\partial\zeta_j}\Big). \tag{2.7}$$

This is conceptually important for us in relation to the Fredholm estimates in section 3. In the elliptic region, these Fredholm estimates are easy to obtain, but in a neighborhood of the characteristic variety of $\Delta_g-\lambda^2$, they are obtained from positive commutator estimates, that is, from operators whose commutator with $\Delta_g-\lambda^2$ has positive principal symbol microlocally. Equation (2.7) shows that this amounts to finding symbols b such that $H_p(a)$ is positive, where $p=|\zeta|_g^2-\lambda^2$ is the symbol of the Helmholtz operator. This then motivates considering the properties of the Hamilton vector field of p, and its flow lines (known as bicharacteristics), within the characteristic variety $\Sigma(\Delta_g - \lambda^2)$.

The Hamiltonian vector field H_p for $p = |\zeta|^2 - \lambda^2$, the symbol of the Euclidean Helmholtz operator, is given by

$$\dot{z} = 2\zeta, \quad \dot{\zeta} = 0.$$

We would like to view this on the compactification $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ and investigate its behavior in a neighborhood of $\Sigma(\Delta - \lambda^2)$. To do this, we use polar coordinates, (r, ω), as before, and then choose arbitrary local coordinates y on a patch of the sphere $\mathbb{S}^{n-1}$. We also write $x = r^{-1}$, which serves as a boundary defining function for spatial infinity. Let (ν, η) be the dual coordinates to (r, y). In these coordinates, the full symbol p of $\Delta - \lambda^2$ takes the form

$$p(r, y, \nu, \eta) = \nu^2 - i(n-1)r^{-1}\nu + r^{-2} b_k \eta_k + r^{-2} h^{jk} \eta_j \eta_k - \lambda^2,$$

where $h^{jk} = h^{jk}(y)$ is the dual metric corresponding to the standard round metric $h = h_{jk}(y)$, on $\mathbb{S}^{n-1}$, $b_k = D_{y_j} h^{jk} + h^{jk} D_{y_j} \log(\sqrt{|\det h|})$, and we use the summation convention. We now make the change of variables to $\mu_j = r^{-1}\eta_j$ as these quantities have uniformly bounded length as $r \to \infty$. In terms of (μ, ν) the full symbol is

$$p(r, y, \nu, \mu) = \nu^2 - i(n-1)r^{-1}\nu + r^{-1} b_k \mu_k + h^{jk} \mu_j \mu_k - \lambda^2,$$

and we see that the principal symbol at spatial infinity is

$$\nu^2 + h^{jk}\mu_j\mu_k - \lambda^2 = \nu^2 + |\mu|_y^2 - \lambda^2, \tag{2.8}$$

where $|\mu|_y^2 := h^{jk}\mu_j\mu_k$ is the metric function on $T^*\mathbb{S}^{n-1}$. Thus the characteristic set Σ satisfies

$$\Sigma = \{x = 0, \ \nu^2 + |\mu|_y^2 = \lambda^2\}. \tag{2.9}$$

In the canonical coordinates $(r, y; \nu, \eta)$ we easily compute the Hamilton vector field of the principal symbol:

$$\begin{aligned} \dot{r} &= 2\nu, \quad \dot{y}_l = 2r^{-2}h^{lk}\eta_k, \\ \dot{\nu} &= 2r^{-3}h^{jk}\eta_j\eta_k, \quad \dot{\eta}_l = -r^{-2}\frac{\partial h^{jk}}{\partial y_l}\eta_j\eta_k. \end{aligned} \tag{2.10}$$

Changing to the variable μ, and writing $x = r^{-1}$, the equations become

$$\begin{aligned} \dot{x} &= -2\nu x^2, \quad \dot{y}_j = 2xh^{jk}\mu_k, \\ \dot{\nu} &= 2xh^{jk}\mu_j\mu_k, \quad \dot{\mu}_l = -2x\nu\mu_l - x\frac{\partial h^{jk}}{\partial y_l}\mu_j\mu_k. \end{aligned} \tag{2.11}$$

It is clear that this vector field vanishes to first order as $x \to 0$. Dividing by x we obtain a rescaled Hamilton vector field that we denote by H_p, taking the form

$$\begin{aligned} \dot{x} &= -2\nu x, \quad \dot{y}^j = 2h^{jk}\mu_k, \\ \dot{\nu} &= 2h^{jk}\mu_j\mu_k, \quad \dot{\mu}_l = -2\nu\mu_l - \frac{\partial h^{jk}}{\partial y_l}\mu_j\mu_k. \end{aligned} \tag{2.12}$$

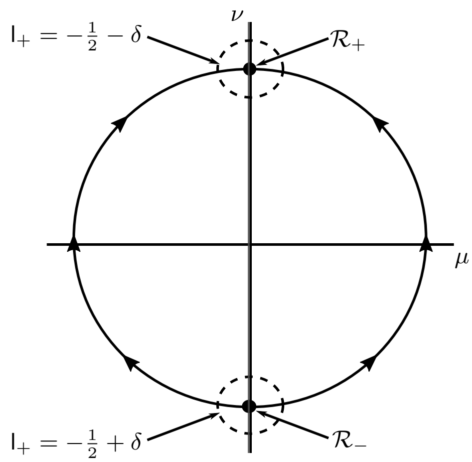

FIG. 1. *The bicharacteristic flow in the characteristic set* $\Sigma(P) = \{x = 0,\, \nu^2 + |\mu|_y^2 = \lambda^2\}$ *with* $P = \Delta - \lambda^2$.

In the coordinates (x, y, ν, μ) this is a smooth vector field on the compactification. See Figure 1. We can write it using derivative notation as follows:

$$\mathsf{H}_p = -2\nu(x\partial_x + R_\mu) + 2|\mu|_y^2\partial_\nu + H_{\mathbb{S}^{n-1}}, \tag{2.13}$$

where $H_{\mathbb{S}^{n-1}}$ is the Hamilton vector field of the round metric h on $T^*\mathbb{S}^{n-1}$ and $R_\mu = \mu \cdot \partial_\mu$ is the radial vector field on the fibers of $T^*\mathbb{S}^{n-1}$. In these coordinates, and on Σ, we have $\mathsf{H}_p = -2\nu R_\mu + 2|\mu|_y^2\partial_\nu + H_{\mathbb{S}^{n-1}}$. We can check directly that $\mathsf{H}_p(\nu^2 + |\mu|_y^2) = 0$ and that H_p vanishes precisely on the two "radial sets"

$$\mathcal{R}_\pm := \{|\mu|_y = 0 = x, \nu = \pm\lambda\}. \tag{2.14}$$

Remark 2.1. Notice that the incoming radial set $\mathcal{R}_-$ is a source, and the outgoing radial set $\mathcal{R}_+$ a sink, for the rescaled Hamilton vector field H_p. Note, also, that the coefficient of $x\partial_x$ in H_p is $\pm\lambda$ at $\mathcal{R}_\pm$, hence always nonzero. This nonvanishing has the important consequence that we can find operators with positive commutators at $\mathcal{R}_\pm$, despite H_p vanishing there. In this sense the radial sets are "nondegenerate."

Up to this point, we have taken the spatial weight ℓ to be constant. To consider variable order spaces, we allow the spatial weight to itself be a classical symbol l of order $(0,0)$ (variable weights will always be written in sans-serif). Choosing an arbitrary small positive number $\boldsymbol{\delta}$, we define the symbol class $S^{s,\mathsf{l}}_{\boldsymbol{\delta}}(T^*\mathbb{R}^n)$ by the estimates

$$\forall \alpha, \beta \in \mathbb{N}^n,\ \exists C_{\alpha,\beta} < \infty \text{ such that } \left|D_z^\alpha D_\zeta^\beta a(z,\zeta)\right| \le C_{\alpha,\beta}\langle z\rangle^{\mathsf{l}-(1-\boldsymbol{\delta})|\alpha| + \boldsymbol{\delta}|\beta|}\langle\zeta\rangle^{s-|\beta|}. \tag{2.15}$$

These are symbol estimates of type $(1 - \boldsymbol{\delta}, \boldsymbol{\delta})$ in the z variable (in the sense of Hörmander), which are slightly "worse" than the standard estimates of type $(1, 0)$. The reason for including this small loss is that the "classical" symbols corresponding to a variable order take the form $\langle z\rangle^{\mathsf{l}}\langle\zeta\rangle^s$ times a C^∞ function on $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$, and these symbols incur logarithmic losses when differentating l.

Changing to these symbol classes makes essentially no difference since we can take $\boldsymbol{\delta}$ arbitrarily small, while the pseudodifferential calculus, as is well known, works with inessential changes provided $\boldsymbol{\delta} < 1/2$. The only differences are that the principal symbol takes values in $S^{s,\ell}(T^*\mathbb{R}^n)/S^{s-1+\boldsymbol{\delta},\ell-1+\boldsymbol{\delta}}(T^*\mathbb{R}^n)$ instead of

$$S^{s,\ell}(T^*\mathbb{R}^n)/S^{s-1,\ell-1}(T^*\mathbb{R}^n),$$

and the commutator $[A,B]$ above will have order $\Psi_{\mathrm{sc}}^{s_1+s_2-1+\boldsymbol{\delta},\ell_1+\ell_2-1+\boldsymbol{\delta}}(\mathbb{R}^n)$ instead of $\Psi_{\mathrm{sc}}^{s_1+s_2-1,\ell_1+\ell_2-1}(\mathbb{R}^n)$.

2.2. The scattering calculus on asymptotically conic manifolds. We now work in the setting of asymptotically conic manifolds. Thus, let M be a compact manifold with boundary, and M° its interior. Let x be a boundary defining function for M (meaning $\partial M = \{x = 0\}$, x vanishes simply at ∂M, and $x > 0$ on M°), and y coordinates on a patch O of ∂M, extended to a collar neighborhood $\{x < c\}$ of ∂M, where $c > 0$ is fixed and small. Given a scattering metric g on M°, we call (x, y) an "adapted coordinate system" near a boundary point $(0, y_0)$ with $y_0 \in O$ provided that g takes the form (1.9) in this coordinate system on the patch O. This condition determines a metric h on the boundary ∂M, such that g is asymptotic to the conic metric $dr^2 + r^2 h$ as $r \to \infty$, where $r := 1/x$, as is clear from the equivalent expression (1.10).

We now define scattering pseudodifferential operators of order (s, ℓ) on M°. We do this by mimicking the behavior of scattering symbols of order (s, ℓ) on $\mathbb{R}^n$. To do this, we choose a diffeomorphism χ from a small open set $O \subset \partial M$ to an open set $O' \subset \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is viewed as the set of vectors of unit length in $\mathbb{R}^n$. We then consider the diffeomorphism

$$(x, y) \mapsto \frac{\chi(y)}{x} = r\chi(y) \in \mathbb{R}^n. \tag{2.16}$$

One can check that the norm of the derivative of this map is uniformly bounded, where we measure with respect to the metric g on M° and with respect to the Euclidean metric on $\mathbb{R}^n$. We now define suitable cotangent variables that are uniformly bounded (with respect to the dual metric g^*). Let (ν, η) be the dual variables to coordinates (r, y), and define $\mu = r^{-1}\eta = x\eta$ as we did in the previous section. Then ν is the symbol of $D_r = -x^2 D_x$ and μ_j is the symbol of $r^{-1}D_{y_j} = xD_{y_j}$; since $(x^2\partial_x, x\partial_{y_j})$ clearly form a uniformly bounded, uniformly nondegenerate frame of functions with respect to the metric g, these are uniformly bounded and uniformly nondegenerate linear coordinates on the cotangent bundle, with respect to the dual metric g^*. We define scattering symbols of order (s, ℓ) on T^*M° to be functions a satisfying usual symbolic estimates of order s away from ∂M, and near the boundary satisfies

$$\left|(xD_x)^j D_y^\alpha D_\nu^k D_\mu^\beta a(x, y, \nu, \mu)\right| \le C_{j,k,\alpha,\beta} x^{-\ell} \langle(\nu, \mu)\rangle^{s-k-|\beta|} \tag{2.17}$$

for all $\alpha, \beta \in \mathbb{N}^{n-1}$. We will denote the class of such symbols by $S^{s,\ell}({}^{\mathrm{sc}}T^*M)$. Scattering pseudodifferential operators $A \in \Psi_{\mathrm{sc}}^{s,\ell}(M)$ are defined as follows: using the local diffeomorphism (2.16), the symbol is mapped (using the induced map on the cotangent bundle) to a symbol of order (s, ℓ) on $T^*\mathbb{R}^n$; we then quantize to a pseudodifferential operator when (z, z') are in the range of this diffeomorphism and pull back to M° by the same map (2.16). By covering M° with a finite number of coordinate charts and using a partition of unity, we get a globally defined operator. This quantization procedure (that is, the precise way of relating symbols a and opeartors A) depends on the choice of charts, partition of unity, etc, but all choices lead to the same operator modulo an operator in $\Psi_{\mathrm{sc}}^{s-1,\ell-1}(M)$ so this is of no importance. To complete

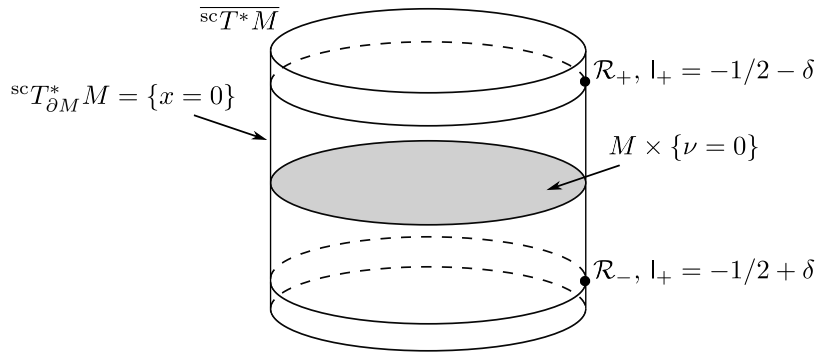

FIG. 2. *The radial sets in the compactified cotangent bundle. Only the ν-direction of each fiber is depicted. In the case $M^\circ = \mathbb{R}^n$, ${}^{sc}T^*_{\partial M}M = \partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$.*

the picture, we include in $\Psi^{s,\ell}_{\mathrm{sc}}(M)$ all kernels $K(z,z')$ that are smooth and rapidly decreasing, with all derivatives, as the distance between z and z' tends to infinity in M°. This definition is equivalent to the definition of the scattering pseudodifferential operators defined in [24] using the "scattering double space."

We see in (2.17) that there are two types of vector fields (in terms of their behavior near the boundary) that play a role in M. First, there are the *b-vector fields*, which by definition are smooth vector fields that at the boundary are tangent to M. These are generated over $C^\infty(M)$ by $x\partial_x$ and ∂_{y_j} near the boundary and govern the regularity of scattering symbols in the spatial coordinates (x, y) (this is called *conormal regularity* in the microlocal literature). Second, there are the *scattering vector fields*, which are just x times b-vector fields, so generated by $x^2\partial_x$ and $x\partial_{y_j}$. These have the property of generating, over $C^\infty(M)$, all smooth vector fields on M that are uniformly bounded with respect to g. Scattering differential operators of order $(s, 0)$, $s \in \mathbb{N}$, are precisely differential operators of order s that, near the boundary, can be written in terms of scattering vector fields with $C^\infty(M)$-coefficients. In the case that (M°, g) is flat Euclidean space, we can take the constant coefficient vector fields ∂_{z_j} as generators of the scattering vector fields. Both will play a role in our analysis; the s parameter in our pseudodifferential calculus is regularity with respect to scattering vector fields, while b-vector fields define module regularity (measured by the κ and k parameters, as discussed in the introduction).

Similarly to the Euclidean case, we can compactify the cotangent bundle T^*M° in a way that mimics the compactification $\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}$ above. We have (by assumption) a compactification M of M°. In the interior of M°, we can compactify each cotangent fiber radially. It only remains to say how the fibers are compactified in the limit as we approach the boundary. Following Melrose [24], we work on the *scattering cotangent bundle*, denoted ${}^{\mathrm{sc}}T^*M$ over M, which over the interior is naturally isomorphic to the usual cotangent bundle and has the property that, near ∂M, using adapted coordinate system (x, y), the corresponding coordinates (ν, μ) (as defined above) are linear coordinates on the fibers of this bundle that remain valid uniformly up to the boundary ∂M. Compactifying each fiber radially gives us a compactification, denoted $\overline{{}^{\mathrm{sc}}T^*M}$, analogous to the cylinder in Figure 2. This is a manifold with corners of codimension two. Clearly x is a boundary defining function at spatial infinity. Let ρ denote a boundary defining function for fiber-infinity—we may take $\rho = \langle(\nu, \mu)\rangle^{-1}$ when x is small. We will call (x, y, ν, μ) adapted coordinates on the scattering cotangent bundle over the neighborhood $\{x < c, y \in O\}$ of $(0, y_0) \in \partial M$.

We then can consider the subspace of operators A with "classical" symbols a of order (s,ℓ) that take the form $x^{-\ell}\rho^{-s}$ times a smooth function on $\overline{{}^{\mathrm{sc}}T^*M}$. The class of such symbols will be denoted $S^{s,\ell}_{\mathrm{cl}}({}^{\mathrm{sc}}T^*M)$. For such operators, the principal symbol can be defined similarly to the classical case on $\mathbb{R}^n$. Supposing for simplicity that $\ell = 0$, we have a symbol at fiber-infinity, $\sigma_{\mathrm{fiber},s,0}(A)$, which is a function on ${}^{\mathrm{sc}}T^*M$ homogeneous of degree s on each fiber, and a symbol at spatial infinity, $\sigma_{\mathrm{base},s,0}(A)$, which is the symbol a restricted to $x = 0$, a function on the scattering cotangent bundle restricted to ∂M (which we denote ${}^{\mathrm{sc}}T^*_{\partial M}M$). These functions are well-defined, that is, they depend only on A, not on the particular quantization. The two symbols have the compatibility condition that $\sigma_{\mathrm{base},s,0}(A)$ is asymptotically homogeneous of degree s, that is, $\rho^s\sigma_{\mathrm{base},s,0}(A)$ has a limit at $\rho = 0$ and agrees at the corner, $x = \rho = 0$, with the limiting value of $\rho^s\sigma_{\mathrm{fiber},s,0}(A)$.

Now let P denote the operator $\Delta_g + V - \lambda^2$ on M°, where Δ_g is the (positive) Laplacian with respect to g and V is a short-range conormal potential as in Theorem 1.5. (The positive real number λ will be fixed throughout.) Then the principal symbols of P, in adapted coordinates (x, y, ν, μ) near the boundary, are

$$\sigma_{\mathrm{fiber},2,0}(P) = \nu^2 + h^{jk}(y)\mu_j\mu_k \tag{2.18}$$

and

$$\sigma_{\mathrm{base},2,0}(P) = \nu^2 + h^{jk}(y)\mu_j\mu_k - \lambda^2. \tag{2.19}$$

This looks very similar to the form of the base symbol of the flat Laplacian on $\mathbb{R}^n$—compare with (2.8). It follows that the characteristic variety is given by (2.9), just as in the flat case. Moreover, exactly the same computation can be made as in the flat case to deduce that the Hamilton vector field of p, the symbol of P, takes the form

$$\mathsf{H}_p = -2\nu(x\partial_x + R_\mu) + 2|\mu|^2_y\partial_\nu + H_{\mathbb{S}^{n-1}} + xW, \tag{2.20}$$

where W is a b-vector field (that is, tangent to $x = 0$). That is, the Hamilton vector field takes the same form as (2.13), up to a error xW. In particular, the radial sets, where the Hamilton vector field vanishes, take the same form (2.14) as in the Euclidean case. *This means that the microlocal analysis of the operator P is no more complicated than that of the flat Laplacian on* $\mathbb{R}^n$ and means that, from this point of view, Theorem 1.5 is a very natural generalization of Theorem 1.1.

Our last topic to discuss is variable order scattering pseudodifferential operators. This is completely analogous to the case of variable order operators on $\mathbb{R}^n$. We allow the spatial weight to be a classical symbol l of order $(0,0)$ on $\overline{{}^{\mathrm{sc}}T^*M}$ and allow a $\boldsymbol{\delta}$ loss in the symbol estimates. Thus, (2.17) is replaced by

$$\left|(xD_x)^j D^\alpha_y D^k_\nu D^\beta_\mu a(x,y,\nu,\mu)\right| \le C_{j,k,\alpha,\beta}\, x^{-\mathsf{l}-\boldsymbol{\delta}(j+k+|\alpha|+|\beta|)}\langle(\nu,\mu)\rangle^{s-k-|\beta|} \tag{2.21}$$

and the rest of the theory proceeds as in the Euclidean case.

2.3. Sobolev spaces of variable order. We begin with the Euclidean case. The Sobolev spaces $H^{s,\ell}(\mathbb{R}^n)$, for $s, \ell \in \mathbb{R}$, are the usual weighted Sobolev spaces defined by

$$H^{s,\ell}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \mid \langle D\rangle^s\langle z\rangle^\ell f \in L^2(\mathbb{R}^n)\}.$$

Here $\langle D\rangle^s$ is the Fourier multiplier, given by $\mathcal{F}^{-1}\langle\zeta\rangle^s\mathcal{F}$. These can be equivalently defined by the condition that $f \in H^{s,\ell}(\mathbb{R}^n)$ if and only if $Af \in L^2$ for all $A \in \Psi^{s,\ell}_{\mathrm{sc}}(\mathbb{R}^n)$;

it is enough to require this for just one A that is "totally elliptic," that is, both its symbol at fiber-infinity and at spatial infinity are everywhere elliptic, or equivalently, $\mathrm{Ell}(A) = \partial(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$. Then the space of Schwartz functions, $\mathcal{S}(\mathbb{R}^n)$, and the space of tempered distributions, $\mathcal{S}'(\mathbb{R}^n)$, can be characterized as the intersection, resp., union, of Sobolev spaces $H^{s,\ell}(\mathbb{R}^n)$ over all s and ℓ.

We use a similar characterization to define the Sobolev spaces for variable orders: if l is a variable order as as defined above, we say that $f \in H^{s,\mathsf{l}}(\mathbb{R}^n)$ if $Af \in L^2$ for all $A \in \Psi^{s,\mathsf{l}}_{\mathrm{sc}}(\mathbb{R}^n)$; again, it is enough to require this for one totally elliptic operator. If in addition, A is invertible, with $A^{-1} \in \Psi^{-s,-\mathsf{l}}_{\mathrm{sc}}(\mathbb{R}^n)$, then the norm of f in $H^{s,\mathsf{l}}(\mathbb{R}^n)$ can be taken to be $\|Af\|_{L^2}$.

The Sobolev spaces for an asymptotically conic manifold are defined analogously: we say $f \in H^{s,\mathsf{l}}(M^\circ)$ if $Af \in L^2$ for all $A \in \Psi^{s,\mathsf{l}}_{\mathrm{sc}}(M)$. Pseudodifferential operators of variable order act on Sobolev spaces with variable order in the expected way: if $A \in \Psi^{s,\mathsf{l}}_{\mathrm{sc}}(M)$, then A is a bounded linear map from $H^{s',\mathsf{l}'}(M^\circ)$ to $H^{s'-s,\mathsf{l}'-\mathsf{l}}(M^\circ)$. Also, the dual space of $H^{s,\mathsf{l}}(M^\circ)$ is $H^{-s,-\mathsf{l}}(M^\circ)$. The duality between these two spaces can be realized by choosing any invertible $A \in \Psi^{s,\mathsf{l}}_{\mathrm{sc}}(M)$. Then for $u \in H^{s,\mathsf{l}}(M^\circ)$, $v \in H^{-s,-\mathsf{l}}(M^\circ)$ we define

$$(u, v) \coloneqq \langle A^{-1}u, A^* v\rangle_{L^2}.$$

It is easy to check that this pairing is independent of the particular invertible operator $A \in \Psi^{s,\mathsf{l}}_{\mathrm{sc}}(M)$.

We shall denote the intersection, resp., union, of all the Sobolev spaces $H^{s,\ell}(M^\circ)$ by $\mathcal{S}(M^\circ)$, resp., $\mathcal{S}'(M^\circ)$; these spaces are sometimes denoted $\dot{C}^\infty(M)$, resp., $C^{-\infty}(M)$ in other works.

We now define the spaces $\mathcal{X}^{s,\mathsf{l}_+}$ and $\mathcal{Y}^{s,\mathsf{l}_+}$. We have already discussed in the introduction that $\mathcal{Y}^{s,\mathsf{l}_+}$ is precisely the variable order Sobolev space H^{s,l_+} as defined above, for a variable spatial weight l_+ with specific properties on $\Sigma(P)$. First, we require that, for some small $\delta > 0$,

$$\begin{gathered} \mathsf{l}_+ \text{ takes values in } [-1/2-\delta, -1/2+\delta] \\ \text{and is equal to } -1/2 \mp \delta \text{ in a neighborhood of } \mathcal{R}_\pm. \end{gathered} \tag{2.22}$$

This ensures elements of $\mathcal{Y}^{s,\mathsf{l}_+}_+$ are permitted to have outgoing oscillations of the form $r^{-(n-1)/2}e^{i\lambda r}$ but not incoming oscillations of the form $r^{-(n-1)/2}e^{-i\lambda r}$. Second, we require that

$$\mathsf{l}_+ \text{ is nonincreasing along the Hamilton flow of } P \text{ within } \Sigma(P). \tag{2.23}$$

Since bicharacteristics within $\Sigma(P)$ start at $\mathcal{R}_-$ and end at $\mathcal{R}_+$, these two conditions are compatible. We also define

$$\mathsf{l}_- = -1 - \mathsf{l}_+. \tag{2.24}$$

This automatically means that l_- has analogous properties to l_+ with the incoming and outgoing radial sets swapped. In particular, we have

$$\begin{gathered} \mathsf{l}_- \text{ takes values in } [-1/2-\delta, -1/2+\delta], \\ \text{is equal to } -1/2 \pm \delta \text{ in a neighborhood of } \mathcal{R}_\pm, \\ \text{and is nondecreasing along the Hamilton flow of } P \text{ within } \Sigma(P). \end{gathered} \tag{2.25}$$

Remark 2.2. Condition (2.23) is imposed so that regularity of approximate solutions of $Pu = 0$ can be propagated from the incoming radial set $\mathcal{R}_-$ toward the outgoing radial set $\mathcal{R}_+(\lambda)$, as we show in the following section.

We then define the spaces $\mathcal{X}^{s,\mathsf{l}_\pm}$ by

$$\mathcal{X}^{s,\mathsf{l}_\pm} := \{u \in H^{s,\mathsf{l}_\pm} \mid Pu \in H^{s-2,\mathsf{l}_\pm+1}\} \tag{2.26}$$

with norm

$$\|u\|^2_{\mathcal{X}^{s,\mathsf{l}_\pm}} = \|u\|^2_{H^{s,\mathsf{l}_\pm}} + \|Pu\|^2_{H^{s-2,\mathsf{l}_\pm+1}}. \tag{2.27}$$

2.4. Test modules and Sobolev spaces with module regularity. We next introduce the "test modules" with respect to which we will assume further differentiability. A test module $\mathcal{M}$, as defined in [14], is a subspace of $\Psi^{1,1}_{\mathrm{sc}}(\mathbb{R}^n)$, or $\Psi^{1,1}_{\mathrm{sc}}(M)$ in the general case, that is closed under commutators, contains the identity, and is a module over $\Psi^{0,0}_{\mathrm{sc}}$. (Here we adapt the definition of [14] slightly to allow order 1 in the fiber as well as the spatial slot, as is convenient here.) We shall also work only with finitely generated modules $\mathcal{M}$, which have the form

$$\mathcal{M} = \left\{ \sum_{j=0}^{N} C_j A_j \mid C_j \in \Psi^{0,0}_{\mathrm{sc}}(M) \right\},$$

for some fixed finite set $A_0 = \mathrm{Id}, A_1, \dots A_N \subset \Psi^{1,1}_{\mathrm{sc}}(M)$, the *generators* of the module, which should be closed under taking commutators in the sense that

$$[A_j, A_k] = \sum_{l=0}^{N} E_{jkl} A_l \quad E_{jkl} \in \Psi^{0,0}_{\mathrm{sc}}(M).$$

The most important modules for us will be the modules $\mathcal{M}_\pm$ defined by the characteristic condition

$$\mathcal{M}_\pm := \mathcal{M}_{\pm\lambda} = \{A \in \Psi^{1,1}_{\mathrm{sc}} \colon \sigma_{\mathrm{base},1,1}(A) \text{ vanishes at } \mathcal{R}_\pm\}, \tag{2.28}$$

where $\mathcal{R}_\pm$ are the radial sets in (2.14). Analytically, the significance of these two modules is that the generators of the module $\mathcal{M}_+$, resp., $\mathcal{M}_-$, annihilate the corresponding radial oscillation, $e^{+i\lambda r}$, resp., $e^{-i\lambda r}$ (as discussed in section 1.2). They are examples of what will become (for us) a useful general class of modules $\mathcal{M}_\gamma$ also defined by a characteristic condition given in terms of a real parameter γ. Recalling the coordinates (x, y, ν, μ), defined near the boundary $\partial M = \{x = 0\}$ with respect to local coordinates y on an open set $O \subset \partial M$, with (x, y) adapted coordinates on M, let

$$\mathcal{R}_\gamma = \{x = 0,\ |\mu|_y = 0,\ \nu = \gamma\}.$$

We then define

$$\mathcal{M}_\gamma := \{A \in \Psi^{1,1}_{\mathrm{sc}} : \sigma_{\mathrm{base},1,1}(A) \text{ vanishes at } \mathcal{R}_\gamma\}. \tag{2.29}$$

Writing $\mathrm{Diff}^{1,1} \subset \Psi^{1,1}_{\mathrm{sc}}$ for the subspace of *differential* operators, one can choose a generating set for these modules containing three types of operators,

(i) $A_N \in \mathrm{Diff}^{1,1}(M)$ and

$$A_N = r(D_r - \gamma) = -xD_x - \frac{\gamma}{x} \text{ on } x < c,$$

where $\{x < c\}$ is our fixed collar neighborhood of ∂M,

(ii) $A_j \in \mathrm{Diff}^{1,1}(M)$, $j = 1, \ldots, N_1$, which are *purely angular* in the sense that they are in the $C^\infty(M)$ linear span of the D_{y_i} for some coordinates (y_i) on ∂M and adapted coordinates (x, y) on M, and last

(iii) $A'_k \in \mathrm{Diff}^1(M^\circ)$, $k = 1, \ldots, N_2$, which are supported in $\{x > c/2\}$, so in particular $A'_k \in \Psi^{1,-\infty}_{\mathrm{sc}}$,

and using these one has

$$\mathcal{M}_\gamma := \left\{ C_0 + \sum_{j=1}^{N_1} C_j A_j + \sum_{k=1}^{N_2} C'_k A'_k + C_N A_N \mid C_0, C_j, C_k, C_N \in \Psi^{0,0}_{\mathrm{sc}}(M) \right\}. \tag{2.30}$$

Here $N_1 + N_2 + 1 = N$, i.e., there are $N + 1$ total generators including $A_0 = \mathrm{Id}$. (To be overly concrete, one can cover the boundary with m total coordinate charts O_q with coordinates $y_i^{(q)}, i = 1, \ldots, n-1, q = 1, \ldots m$, with the $(y_i^{(q)})$ coordinates on O_q, and let $A_j = \chi_q \partial_{y_j^{(q)}}$ for the χ_q a partition of unity subordinate to the O_q, and then choose the A'_k to be any finite family of vector fields for which, for all $\mathsf{q} \in T^*M^\circ$ with $x(\mathsf{q}) > c$, there is an A'_k with $\sigma(A'_k)(\mathsf{q}) \neq 0$.) In the case $\gamma = 0$, these generators form a basis (over $C^\infty(M)$) of the b-vector fields, that is, all vector fields tangent to the boundary of M. (In the case of $\mathbb{R}^n$, this includes all constant coefficient vector fields times a factor r.)

We note in particular the A_j and A'_k are all elements of $\mathcal{M}_\gamma$. (This is not a requirement for test modules in other contexts, e.g., [12].) The operators A_N in (i) and A_j in (ii) taken together have the feature that for any point $\mathsf{q} \in \Sigma_{2,0}(P) \setminus \mathcal{R}_\gamma$ there is an element $A = \sum C A_N + \sum_{j=1}^{N_1} C_j A_j$ in the module that is elliptic at q, i.e., such that $\sigma_{1,1}(A)(\mathsf{q}) \neq 0$. Indeed, using adapted coordinates (x, y, ν, μ) for one of our coordinate charts O_q and writing $\mathsf{q} = (0, y, \nu, \mu)$, $\nu^2 + |\mu|^2 = \gamma^2$, if $\mu \neq 0$, then we can choose a vector field $\mathcal{V} := \chi c_i D_{y_i}$ with $\sigma_{\partial M}(\mathcal{V})(\mathsf{q}) \neq 0$, where this is the standard symbol of a vector field on a closed manifold, χ is supported in O_q, and $c_i \in \mathbb{R}$. Then $\sigma_{\mathrm{base},1,1}(\mathcal{V}) = \sigma_{\mathrm{base},1,0}(r^{-1}\mathcal{V}) = \chi \sum c_i \mu_i$, and thus

$$\sigma_{\mathrm{base},1,1}(\mathcal{V}) = \chi c_i \mu_i,$$

in particular $\sigma_{\mathrm{base},1,1}(\mathcal{V})(q) \neq 0$. Similarly

$$\sigma_{\mathrm{base},1,1}(A_N) = \nu - \gamma,$$

so A_N is elliptic on the whole of $\mathcal{R}_{\gamma'}$ for $\gamma' \neq \gamma$.

For any $\gamma \in \mathbb{R}$, we can then define the weighted Sobolev spaces $H^{s,\ell;\kappa}_{\mathcal{M}_\gamma}$ with module regularity of order $\kappa \in \mathbb{N}$ with respect to $\mathcal{M}_\gamma$. These consist of functions in $H^{s,\ell}$ that remain in this space under the application of any κ elements in $\mathcal{M}_\gamma$. For $\gamma = \pm\lambda$, these spaces will be denoted $H^{s,\ell;\kappa}_{\pm}$ for brevity. A distribution u will lie in $H^{s,\ell;\kappa}_{\mathcal{M}_\gamma}$ if $\chi u \in H^{s+\kappa}$, where $\chi \in C^\infty(M^\circ)$ is supported in $\{x > c/2\}$ and identically one on $\{x \geq c\}$, and if, for any adapted coordinate system (x, y),

$$(r(D_r - \gamma))^j D_y^\beta u \in H^{s,\ell} \text{ whenever } j + |\beta| \leq \kappa.$$

To impart the structure of a Hilbert space to $H^{s,\ell;\kappa}_{\mathcal{M}_\gamma}$ we use the generators $\mathsf{A}_j, j = 0, \ldots, N$ of $\mathcal{M}_\gamma$, where the A_j run over all the A_j and A'_k in the definition of $\mathcal{M}_\gamma$ above. Then, using standard multi-index notation $\mathsf{A}^\alpha = \mathsf{A}_0^{\alpha_0} \cdots \mathsf{A}_N^{\alpha_N}$, $\alpha \in \mathbb{N}^{N+1}$, we define

$$
\begin{aligned}
&H^{s,\ell;\kappa}_{\mathcal{M}_\gamma} := \big\{u \in H^{s,\ell} : \mathsf{A}^\alpha u \in H^{s,\ell} \text{ whenever } |\alpha| \le \kappa\big\},\\
&\|u\|^2_{H^{s,\ell;\kappa}_{\mathcal{M}_\gamma}} := \sum_{|\alpha|\le\kappa} \|\mathsf{A}^\alpha u\|^2_{H^{s,\ell}}.
\end{aligned} \tag{2.31}
$$

In particular, when $\gamma = \pm\lambda$, we define

$$
H^{s,\ell;\kappa}_{\pm} := H^{s,\ell;\kappa}_{\mathcal{M}_{\pm\lambda}}. \tag{2.32}
$$

As well as the modules $\mathcal{M}_\gamma$, we shall need to consider the smaller module $\mathcal{N} \subset \mathcal{M}_\gamma$ (for any γ) generated only by the purely angular and purely interior derivatives, i.e., in the notation preceding (2.30), only the generators $A_0 = \mathrm{Id}$, the purely angular A_j for $j = 1, \dots, N_1$ and the interior A'_k, $k = 1, \dots, N_2$,

$$
\mathcal{N} = \left\{ C_0 + \sum_{j=1}^{N_1} C_j A_j + \sum_{k=1}^{N_2} C'_k A'_k \mid C_j, C'_k \in \Psi^{0,0}_{\mathrm{sc}}(M) \right\}. \tag{2.33}
$$

In direct analogy with $H^{s,\ell;\kappa}_{\mathcal{M}_\gamma}$, writing the generators of $\mathcal{M}_\gamma$ as A_j, $j = 0, \dots, N$ and those of $\mathcal{N}$ as B_k, $k = 0, \dots, N-1$, we define $u \in H^{s,\ell;\kappa,k}_{\mathcal{M}_\gamma}$ if an only if $u|_{x>c} \in H^{s+\kappa+k}$, and for any adapted coordinate system (x, y) we have

$$
\begin{aligned}
&H^{s,\ell;\kappa,k}_{\mathcal{M}_\gamma} := \{u \in H^{s,\ell} : \mathsf{A}^\alpha \mathsf{B}^\beta u \in H^{s,\ell},\ |\alpha| \le \kappa, |\beta| \le k\},\\
&\|u\|^2_{H^{s,\ell;\kappa,k}_{\mathcal{M}_\gamma}} := \sum_{|\alpha|\le\kappa, |\beta|\le k} \|\mathsf{A}^\alpha \mathsf{B}^\beta u\|^2_{H^{s,\ell}}.
\end{aligned} \tag{2.34}
$$

In particular, for $\gamma = \pm\lambda$, we put

$$
H^{s,\ell;\kappa,k}_{\pm} := H^{s,\ell;\kappa,k}_{\mathcal{M}_{\pm\lambda}}. \tag{2.35}
$$

Notice that we have the simple relation between these spaces.

LEMMA 2.1. *Let $\gamma, \gamma' \in \mathbb{R}$.*

$$
H^{s,\ell;\kappa,k}_{\mathcal{M}_{\gamma'}} = e^{i(\gamma'-\gamma)r} H^{s,\ell;\kappa,k}_{\mathcal{M}_\gamma}. \tag{2.36}
$$

Proof. This follows directly from the relation $(D_r - \gamma') e^{i(\gamma'-\gamma)r} u = e^{i(\gamma'-\gamma)r}(D_r - \gamma)u$. ∎

We also note without proof the simple mapping property of scattering pseudodifferential operators on these spaces.

LEMMA 2.2. *Let $A \in \Psi^{m,\ell'}_{\mathrm{sc}}(M)$. Then A is a bounded operator*

$$
A : H^{s,\ell;\kappa,k}_{\pm} \to H^{s-m,\ell-\ell';\kappa,k}_{\pm}. \tag{2.37}
$$

The modules $\mathcal{M}_\pm$ and $\mathcal{N}$ enjoy important positivity properties at the radial sets, which we describe now. Returning to the general situation for a moment, let $\mathcal{M}$ be any test module, generated by $A_0 = \mathrm{Id}, A_1, \dots, A_N$, and let P be the Helmholtz operator on $\mathbb{R}^n$ or the generalized Helmholtz operator on M°.

Definition 2.3. *We say $\mathcal{M}$ is P-*positive *at the subset $S \subset \Sigma(P)$ if there exist $C_{jk} \in \Psi^{1,0}_{sc}(M)$ and $C'_j \in \Psi^{0,1}_{sc}(M)$ with*

$$ir[A_j, P] = \sum_{k=0}^{N} C_{jk} A_k + C'_j P \tag{2.38}$$

with

$$\sigma_{1,0}(C_{jk}) = 0 \text{ on } S \ \forall 0 < k \neq j \tag{2.39}$$

and

$$\operatorname{Re} \sigma_{1,0}(C_{jj}) \geq 0 \text{ on } S. \tag{2.40}$$

*Similarly we say that $\mathcal{M}$ is P-*negative *at S if* (2.40) *holds with the reversed inequality. We say that $\mathcal{M}$ is P-*critical *at S if it is both P-positive and P-negative, that is, if* (2.40) *is replaced by the stronger condition*

$$\operatorname{Re} \sigma_{1,0}(C_{jj}) = 0 \text{ on } S. \tag{2.41}$$

Notice that there is no sign condition on the symbol of the operator C'_j in either case.

Lemma 2.3. *The module $\mathcal{M}_+$ is P-positive at $\mathcal{R}_+ \cup \mathcal{R}_-$, while $\mathcal{M}_-$ is P-negative at $\mathcal{R}_+ \cup \mathcal{R}_-$. The module $\mathcal{N}$ is P-critical at $\mathcal{R}_+ \cup \mathcal{R}_-$.*

Proof. The operator P takes the form near the boundary

$$D_r^2 - i(n-1)r^{-1}D_r + r^{-2}Q + r^{-2}\tilde{Q} + V - \lambda^2, \tag{2.42}$$

where Q is a differential operator of order 2 in the tangential derivatives, ∂_{y_j} with coefficients smooth on M, and $\tilde{Q} \in \mathrm{Diff}^{2,0}_{\mathrm{sc}}$, i.e., $r^{-2}\tilde{Q}$ is a scattering differential operator which vanishes to second order. Consider first the commutators with generators of $\mathcal{N}$. The commutator of Δ_g with D_{y_j} leads to an operator of the form r^{-2} times a tangential differential operator of order 2, depending smoothly on r^{-1}. Multiplying by r we obtain an operator of the form r^{-1} times a tangential differential operator of order 2. This is a sum of terms of the form $C_{jk}A_k$ for $1 \leq k \leq n-1$, where $A_k = D_{y_k}$; C_{jk} is equal to $\sum_l b_l r^{-1} D_{y_l}$, where $b_l \in C^\infty(M)$; in this case we take $C'_j = 0$. Since the symbol of $r^{-1}D_{y_l}$ vanishes at $\mathcal{R}_\pm$, this satisfies the conditions of the lemma and shows the P-criticality of $\mathcal{N}$. The commutators of D_{y_j} with V are acceptable using (1.11).

It remains to consider the commutator of $r(D_r \pm \lambda)$ with P. The argument is similar for each sign, so we just consider the case of $r(D_r - \lambda)$, which, along with the module $\mathcal{N}$, generates $\mathcal{M}_+$. We find that

$$\begin{aligned}
ir[r(D_r - \lambda), P] &= ir\big[r(D_r - \lambda), D_r^2 - i(n-1)r^{-1}D_r + r^{-2}Q + r^{-2}\tilde{Q} + V\big] \\
&= ir\big[rD_r, D_r^2 - i(n-1)r^{-1}D_r + r^{-2}Q + r^{-2}\tilde{Q}\big] \\
&\quad + ir\lambda\big[D_r^2 - i(n-1)r^{-1}D_r + r^{-2}Q + r^{-2}\tilde{Q}, r\big] + r^2\partial_r V \\
&= r\Big(-2(D_r^2 - i(n-1)r^{-1}D_r + r^{-2}Q) + r^{-1}Q'\Big) + r\lambda(2D_r + r^{-1}Q'') + r^2\partial_r V \\
&= -2rP + 2\lambda r(D_r - \lambda) + r^{-1}(Q' + Q'') + r^2\partial_r V.
\end{aligned} \tag{2.43}$$

Here, Q' and Q'' are both scattering differential operators of order $(2, 0)$. Notice that the principal terms of P, aside from the λ^2 term, are all homogeneous of order -2 in r

to leading order, which gives rise to the factor of -2 in front of P after commutation with $r\partial_r$. Examining the last line of (2.43), we see that this takes the required form (2.38) since

- the diagonal term C_{jj} in (2.40) is $2\lambda > 0$,
- the remainder terms $r^{-1}(Q' + Q'') + r^2\partial_r V$ are of the form $C_{j0}A_0$, where $A_0 = \mathrm{Id}$ and C_{j0} has negative boundary order (using (1.11)), and hence satisfy (2.39). ∎

By analogy with the spaces $\mathcal{X}^{s,\mathsf{l}_+}$ and $\mathcal{Y}^{s,\mathsf{l}_+}$, we define

$$\mathcal{Y}_+^{s,\ell;\kappa,k} = H_+^{s,\ell;\kappa,k} \tag{2.44}$$

and

$$\mathcal{X}_+^{s,\ell;\kappa,k} := \{u \in H_+^{s,\ell;\kappa,k} \mid Pu \in H_+^{s-2,\ell+1;\kappa,k}\} \tag{2.45}$$

with norm

$$\|u\|^2_{\mathcal{X}_+^{s,\ell;\kappa,k}} = \|u\|^2_{H_+^{s,\ell;\kappa,k}} + \|Pu\|^2_{H_+^{s-2,\ell+1;\kappa,k}}. \tag{2.46}$$

The main technical result of this paper is the following mapping property of the Helmholtz operator P on these spaces with module regularity.

THEOREM 2.4. *Let $s, \ell \in \mathbb{R}$, and assume $\ell \in (-3/2, -1/2)$. For any natural numbers $\kappa \geq 1$, $k \geq 0$, the map*

$$P\colon \mathcal{X}_\pm^{s,\ell;\kappa,k} \longrightarrow \mathcal{Y}_\pm^{s-2,\ell+1;\kappa,k} \tag{2.47}$$

is an isomorphism of Hilbert spaces. In particular the inverse map, i.e., the outgoing (+), resp., incoming (−), resolvent is bounded as a map

$$R(\lambda \pm i0)\colon H_\pm^{s-2,\ell+1;\kappa,k} \longrightarrow H_\pm^{s,\ell;\kappa,k}. \tag{2.48}$$

We defer the proof to section 3.

2.5. Multiplicative properties of weighted Sobolev spaces with module regularity. We prove multiplicative properties of weighted Sobolev spaces on $\mathbb{R}^n$, or more generally on asymptotically conic manifolds, with additional module regularity. We use the module $\mathcal{M}_0$ generated by b-vector fields, as that gives us the best multiplicative properties, and deduce more general multiplicative properties as a corollary.

LEMMA 2.4. *Let $\ell, \ell' \in \mathbb{R}$, $s, \kappa, k \in \mathbb{N}_0$. If $\kappa \geq 1$ and $k \geq (n-1)/2$, multiplication on $C_c^\infty(M)$ extends to a bounded bilinear map*

$$H_{\mathcal{M}_0}^{s,\ell;\kappa,k} \cdot H_{\mathcal{M}_0}^{s,\ell';\kappa,k} \longrightarrow H_{\mathcal{M}_0}^{s,\ell+\ell'+n/2;\kappa,k}. \tag{2.49}$$

Before the proof, we make some remarks concerning spaces of distributions on $\mathbb{R}^n$ whose regularity in an L^2-based Sobolev sense is a given order, κ, with some additional order k of regularity only in certain directions. Write $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$, $z = (z', z'')$, where $z' \in \mathbb{R}^d$ and $z'' \in \mathbb{R}^{n-d}$. Define

$$\mathcal{Y}_d^{\kappa,k}(\mathbb{R}^d \times \mathbb{R}^{n-d}) = \{u : \langle\zeta\rangle^\kappa \langle\zeta''\rangle^k \hat{u} \in L^2\} \tag{2.50}$$

with $\zeta = (\zeta', \zeta'')$. Thus distributions in $\mathcal{Y}_d^{\kappa,k}$ have κ total derivatives in L^2 and an additional k derivatives in z'' in L^2. For our purposes, below we extend this definition to the case where one factor is a closed manifold $\boldsymbol{N}$ of dimension $n-d$; for $\kappa, k \in \mathbb{N}_0$, and $d\mu_N$ a measure on $\boldsymbol{N}$,

$$\mathcal{Y}_d^{\kappa,k}(\mathbb{R}_z^d \times \boldsymbol{N}) = \{u \in L^2(\mathbb{R}_z^d \times \boldsymbol{N}, dz\, d\mu_N) : D_z^\alpha A u \in L^2(\mathbb{R}_z^d \times \boldsymbol{N}, dz\, d\mu_N)$$
$$\forall |\alpha| \le \kappa,\ A \in \mathrm{Diff}^{k+\kappa-|\alpha|}(\boldsymbol{N})\},$$

and below we will take $d = 1$ and $\boldsymbol{N} = \partial M$.

We will use the following lemma, which is proven in [15, Lemma 4.4].

LEMMA 2.5. *Let $\kappa, k \in \mathbb{R}$. If $\kappa > d/2$ and $k \ge (n-d)/2$, then $\mathcal{Y}_d^{\kappa,k}(\mathbb{R}_z^d \times \boldsymbol{N})$ is an algebra.*

Proof of Lemma 2.4. First, we suppose $\ell = \ell' = s = 0$. Let $u, v \in C_c^\infty(M^\circ)$, and let $\chi = \chi(x) \in C^\infty(M)$ be a cutoff function, identically one near the boundary, and supported in the collar neighborhood $\{x < c\}$. We decompose the product uv as

$$uv = (1-\chi^2)uv + \chi^2 uv = (1-\chi^2)uv + (\chi u)(\chi v)$$

and bound the $H_{\mathcal{M}_0}^{0,n/2;\kappa,k}$ norm of both pieces.

Since $\kappa + k > n/2$ and $1-\chi^2$ has compact support, the standard algebra property for $H^{\kappa+k}(M^\circ)$, along with the equivalence of the $H^{\kappa+k}$ and $H_{\mathcal{M}_0}^{0,\tilde{\ell};\kappa,k}$ norms (any $\tilde{\ell} \in \mathbb{R}$) on a fixed compact subset of M°, yields

$$\|(1-\chi^2)uv\|_{H_{\mathcal{M}_0}^{0,n/2;\kappa,k}} \le C\|u\|_{H_{\mathcal{M}_0}^{0,0;\kappa,k}} \|v\|_{H_{\mathcal{M}_0}^{0,0;\kappa,k}}.$$

On the other hand, $u_1 := \chi u, v_1 := \chi v$ are supported in the collar neighborhood of ∂M, so can be viewed as belonging to $C_c^\infty([0,c) \times \partial M)$. The functions

$$\tilde{u}(t,y) = u_1(e^t, y), \quad \tilde{v}(t,y) = v_1(e^t, y)$$

are then defined for $(t,y) \in \mathbb{R} \times \partial M$. Taking into account that $xD_x = D_t$ if $x = e^t$, we see that

$$\|x^{-n/2}(xD_x)^j D_y^\beta w\|_{L^2([0,c)_x \times \partial M;\, x^{-1-n} dx d\mu)} = \|D_t^j D_y^\beta \tilde{w}\|_{L^2(\mathbb{R}_t \times \partial M; dt d\mu)} \tag{2.51}$$

for all $w \in C_c^\infty([0,c) \times \partial M)$. Using Lemma 2.5 and (2.51), a short calculation shows

$$\|u_1 v_1\|_{H_{\mathcal{M}_0}^{0,n/2;\kappa,k}} \le C\|u_1\|_{H_{\mathcal{M}_0}^{0,0;\kappa,k}} \|v_1\|_{H_{\mathcal{M}_0}^{0,0;\kappa,k}},$$

completing the proof of Lemma 2.4 for the case $\ell = \ell' = s = 0$.

One then proves the general case of $\ell, \ell' \in \mathbb{R}$, $s \in \mathbb{N}$ using $H_{\mathcal{M}_0}^{0,\tilde{\ell};\kappa,k} = x^{\tilde{\ell}} H_{\mathcal{M}_0}^{0,0;\kappa,k}$ (any $\tilde{\ell} \in \mathbb{R}$) and the Leibniz rule. ∎

The following corollary follows from Lemmas 2.4 and 2.1.

COROLLARY 2.5. *Let $s \in \mathbb{N}$ and let $\gamma_1, \gamma_2, \ldots, \gamma_{p+1}$ be real parameters. Then provided $\kappa \ge 1$ and $k \ge (n-1)/2$, pointwise multiplication of functions in $C_c^\infty(M)$ induces a bounded multilinear map*

$$H_{\mathcal{M}_{\gamma_1}}^{s,\ell_1;\kappa,k} \cdot H_{\mathcal{M}_{\gamma_2}}^{s,\ell_2;\kappa,k} \cdots H_{\mathcal{M}_{\gamma_p}}^{s,\ell_p;\kappa,k} \longrightarrow H_{\mathcal{M}_{\gamma_{p+1}}}^{s,\ell;\kappa,k}, \tag{2.52}$$

where

$$\ell = \ell_1 + \ell_2 + \cdots + \ell_p + \frac{(p-1)n}{2} - \kappa. \tag{2.53}$$

Proof. When all $\gamma_j = 0$, the result follows from applying Lemma 2.4 $p-1$ times, and indeed this gives a better result without the loss of κ on the right side of (2.53). In general, we use Lemma 2.1 to write an element of u_j of $H^{s,\ell_j;\kappa,k}_{\mathcal{M}_{\gamma_j}}$ as $e^{i\gamma_j r}$ times an element w_j of $H^{s,\ell_j;\kappa,k}_{\mathcal{M}_0}$. The product of the u_j is then the product of the w_j, which lies in $H^{s,\ell+\kappa;\kappa,k}_{\mathcal{M}_0}$, times the exponential factor $e^{i\gamma' r}$, where $\gamma' = \gamma_1 + \cdots + \gamma_p$. Using Lemma 2.1 once more, we see that the product of the u_j is in fact an element of $H^{s,\ell+\kappa;\kappa,k}_{\mathcal{M}_{\gamma_{p+1}}}$ times the exponential $e^{i(\gamma' - \gamma_{p+1})r}$. Finally, multiplication by an exponential in r leads to a loss of κ in the spatial weight since we incur a factor of r each time the operator $r(D_r - \gamma)$ is applied to the exponential factor. ∎

Remark 2.6. It is because of this loss of κ in the spatial weight in (2.53) that we work below with spaces where κ is as small as possible, namely, $\kappa = 1$.

3. Proof of Theorem 2.4.

The organization of this section is as follows. We first prove the invertibility of P acting between variable order spaces, as in (1.14). We then use this to prove the invertibility on the spaces with extra module regularity, as in Theorem 2.4. The proof of invertibility is achieved by first proving that the map in question is Fredholm and then establishing the triviality of the kernel and cokernel. The Fredholm property is established by patching together microlocal estimates of various sorts. In the elliptic region, we use a very standard elliptic estimate; on the characteristic variety, we use a standard positive commutator estimate, where the Hamilton vector field is nonvanishing, and radial-point estimates originating with Melrose at the radial sets, where the Hamilton vector field vanishes.

THEOREM 3.1 (see [33, Prop. 5.28]). *Let* $\mathsf{l}_\pm \in S^{0,0}_{\mathrm{cl}}({}^{\mathrm{sc}}T^*M)$ *(that is, let them be classical scattering symbols of order* $(0,0)$ *on* M*, see section* 2*) satisfy conditions* (2.22)*,* (2.23)*, and* (2.24)*. Let* $s \in \mathbb{R}$*. Then the map* (1.14) *is invertible.*

As just mentioned, the strategy is first to prove that the map (1.14) is Fredholm. We will prove the following.

LEMMA 3.1. *Suppose* $\mathsf{l}_\pm \in S^{0,0}({}^{\mathrm{sc}}T^*M)$ *satisfying the conditions in Theorem* 3.1*, let* $s \in \mathbb{R}$ *be arbitrary, and let* M, N *be such that* $M < \min\{s, 2-s\}$ *and* $N < \min\{\mathsf{l}_+, \mathsf{l}_-\} = 1/2 - \delta$*. Then there is a* $C > 0$ *so that for all* $u \in \mathcal{X}^{s,\mathsf{l}_\pm}$*,*

$$\|u\|_{s,\mathsf{l}_\pm} \le C \left(\|Pu\|_{s-2,\mathsf{l}_\pm+1} + \|u\|_{M,N} \right). \tag{3.1}$$

Moreover, estimates (3.1) *for both signs* $\pm$ *imply that the map* (1.14) *(for either sign) is Fredholm.*

To motivate the microlocal estimates below, we briefly explain why the estimates (3.1) imply the Fredholm statement. For definiteness, we consider only the + sign, so we are considering

$$P \colon \mathcal{X}^{s,\mathsf{l}_+} \longrightarrow \mathcal{Y}^{s-2,\mathsf{l}_++1}. \tag{3.2}$$

The estimate in (3.1) implies that the kernel (3.2) is finite dimensional by a standard argument. Indeed, on the kernel of P we have $\|u\|_{s,\mathsf{l}_+} \le C \|u\|_{M,N}$, and the containment $H^{s,\mathsf{l}_+} \subset H^{M,N}$ is compact, proving that the kernel must be finite dimensional.

To show that the range is closed, consider a sequence u_j in $\mathcal{X}^{s,\mathsf{l}_+}$ in the subspace orthogonal to the kernel of P, for which Pu_j converges to some $f \in H^{s-2,\mathsf{l}_++1}_+$. Then we apply (3.1) to u_j and observe that the $\|u_j\|_{M,N}$ norms must be uniformly bounded, for if not, then one can pass to a subsequence where $\|u_j\|_{M,N}$ tends to infinity, rescale the u_j to $\hat{u}_j$ that $\|\hat{u}_j\|_{M,N}$ is fixed to be 1, and show, using the compact embedding $H^{s,\mathsf{l}_+} \subset H^{M,N}$ again, that a subsequence of the $\hat{u}_j$ converges to a limit v such that $Pv = 0$. This implies that $v = 0$ since the u_j were chosen orthogonal to the kernel of P, and this is a contradiction since it would imply convergence of the subsequence to zero also in the weaker norm $H^{M,N}$, where the norm was fixed to be 1.

Having thus observed that the $\|u_j\|_{M,N}$ quantities are uniformly bounded, it follows from (3.1) that the $\|u_j\|_{s,\mathsf{l}_+}$ quantities are uniformly bounded. Using the compactness of the inclusion $H^{s,\mathsf{l}_+} \subset H^{M,N}$ once again we can extract a subsequence convergent in the $H^{M,N}$ norm, and applying (3.1) we obtain convergence in the H^{s,l_+} norm (since the Pu_j are converging in H_+^{s-2,l_++1}). Thus we obtain a limit u for this subsequence, and hence Pu_j converges to Pu. So $f = Pu$ is in the range, proving the closedness of the range.

The last step is finite dimensionality of the cokernel. The cokernel can be identified with those $v \in (H^{s-2,\mathsf{l}_++1})^* = H^{2-s,-1-\mathsf{l}_+}$ with $Pv = 0$, using the formal self-adjointness of the operator P. Now recall from (2.24) that $-1 - \mathsf{l}_+$ is precisely l_-. So it suffices to prove that the kernel of P acting on H^{2-s,l_-} is finite dimensional. But this follows from the estimate (3.1) for the opposite sign $-$, exactly as above. This completes the proof that the estimate (3.1), for both signs, implies the Fredholm property of the map (1.14) for the + sign (and the argument for the $-$ works in an exactly similar manner).

3.1. Microlocal estimates. In this section we review the specific microlocal estimates that we shall use to prove the estimate (3.1), which we shall refer to as the "Fredholm estimate."

The first type of estimate is an elliptic estimate that applies on the elliptic set of P. This is very familiar from the theory of elliptic pseudodifferential operators. The only novelty is that it applies here to the full elliptic set in the sense of the scattering calculus, and thus, everywhere on the boundary of $\overline{{}^{\mathrm{sc}}T^*M}$ away from $\Sigma(P)$.

PROPOSITION 3.2 (microlocal elliptic regularity [33, Cor. 5.5]). *Let $u \in \mathcal{S}'$ and let $Q_1, G_1 \in \Psi_{\mathrm{sc}}^{0,0}$ be such that $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(G_1) \cap \mathrm{Ell}(P)$. Assume $G_1 Pu \in H^{s-2,\mathsf{l}}$. Then $Q_1 u \in H^{s,\mathsf{l}}$, and for all $M, N \in \mathbb{R}$, there is a constant $C > 0$ such that if $u \in H^{M,N}$, then*

$$\|Q_1 u\|_{s,\mathsf{l}} \leq C \left(\|G_1 Pu\|_{s-2,\mathsf{l}} + \|u\|_{M,N} \right). \tag{3.3}$$

Thus, on the elliptic set, P, as an operator of order $(2, 0)$, acts microlocally with no loss of derivatives or spatial order; u is in $H^{s,\mathsf{l}}$ microlocally on the elliptic set $\mathrm{Ell}(P)$ if and only if Pu is in $H^{s-2,\mathsf{l}}$ microlocally.

On the characteristic set $\Sigma(P)$, wherever the Hamilton vector field H_p is nonvanishing, we have propagation of singularities (as it is conventionally called—though it is more accurately called "propagation of regularity"). In the following proposition we specialize to the variable order l_+ defined in section 2.3, although only the condition (2.23) is necessary for the following result. Propagation of singularities goes back to Hörmander's paper [18] and was first used at spatial infinity in the scattering calculus by Melrose [24], who viewed it as a microlocal version of the Mourre estimate [26]. The version we state is from [33, Thm. 5.4].

PROPOSITION 3.3 (propagation of singularities/regularity estimate). *Let $u \in \mathcal{S}'$ and let $Q_2, Q_2', G_2 \in \Psi_{\mathrm{sc}}^{0,0}$. Assume $\mathrm{WF}'(Q_2) \subseteq \mathrm{Ell}(G_2)$. Moreover, assume that*

$$\begin{gathered}\text{for every } \alpha \in \mathrm{WF}'(Q_2) \cap \Sigma(P), \text{ there is a point } \alpha' \in \mathrm{Ell}(Q_2') \\ \text{and a forward bicharacteristic segment } \gamma \text{ from } \alpha' \text{ to } \alpha \text{ such that } \gamma \subseteq \mathrm{Ell}(G_2).\end{gathered} \tag{3.4}$$

If $Q_2'u \in H^{s,\mathsf{l}_+}$ and $G_2Pu \in H^{s-2,\mathsf{l}_++1}$, then $Q_2u \in H^{s,\mathsf{l}_+}$, and for all M, N there is $C > 0$ such that if $u \in H^{M,N}$, then

$$\|Q_2u\|_{s,\mathsf{l}_+} \le C\left(\|Q_2'u\|_{s,\mathsf{l}_+} + \|G_2Pu\|_{s-2,\mathsf{l}_++1} + \|u\|_{M,N}\right). \tag{3.5}$$

That is, if u is in $H^{s,\mathsf{l}}$ microlocally near a point $\alpha' \in \Sigma(P)$, if α is another point on the bicharacteristic γ through q, and if Pu is sufficiently regular (namely, in $H^{s-2,\mathsf{l}+1}$) along γ between α' and α, then the regularity "propagates" to α, in the sense that u is in $H^{s,\mathsf{l}}$ at α, *provided*, in the case of a variable order l, that l is nonincreasing between α' and α in the direction of bicharacteristic flow. (If a variable weight is nondecreasing in the direction of bicharacteristic flow, as is the case with l_-, then regularity propagates in the opposite direction.)

Neither of these estimates gives any information at the radial sets, which are the locations within $\Sigma(P)$ where the Hamilton vector field vanishes. At these sets, we have the following radial point estimates, which come in two versions, one below and one above the spatial regularity level $-1/2$ that is critical for the behavior of solutions of $Pu = 0$. We only state these for constant spatial weight, which suffices as we have assumed that $\mathsf{l}_\pm$ are constant in a neighborhood of the radial sets. We also have stated this proposition with l_+ in mind, and thus the below threshold result applies at the outgoing radial set $\mathcal{R}_+$, while the above threshold result applies at the incoming radial set $\mathcal{R}_-$; for the other weight l_-, the roles of the incoming and outgoing radial sets switch. It will help in understanding the statements below to recall that $\mathcal{R}_+$ is a sink, and $\mathcal{R}_-$ a source, for the bicharacteristic flow, and all bicharacteristics inside $\Sigma(P)$ start at $\mathcal{R}_-$ and end at $\mathcal{R}_+$. Again, these estimates were first made by Melrose in [24, sect. 9].

PROPOSITION 3.4 (see [33, Prop. 5.27]).

(i) *Below threshold regularity radial point estimate: Assume $\ell < -1/2$. Let $Q_3, Q_3', G_3 \in \Psi_{\mathrm{sc}}^{0,0}$. Let U, U' denote two open neighborhoods of $\mathcal{R}_+$ with $U \Subset U' \Subset {}^{\mathrm{sc}}T^*_{\partial M}M$, and assume $U \subset \mathrm{Ell}(Q_3) \subset \mathrm{WF}'(Q_3) \subset \mathrm{Ell}(G_3) \subset U'$. Assume that $\mathrm{WF}'(Q_3')$ is contained in $U' \setminus U$ and that*

$$\begin{gathered}\text{for every } \alpha \in \mathrm{WF}'(Q_3) \cap (\Sigma(P) \setminus \mathcal{R}_+), \text{ there is a point } \alpha' \in \mathrm{Ell}(Q_3') \\ \text{and a forward bicharacteristic segment } \gamma \text{ from } \alpha' \text{ to } \alpha \text{ such that } \gamma \subseteq \mathrm{Ell}(G_3).\end{gathered} \tag{3.6}$$

If $Q_3'u \in H^{s,\ell}$ and $G_3Pu \in H^{s-2,\ell+1}$, then $Q_3u \in H^{s,\ell}$, and for all M, N there is $C > 0$ such that if $u \in H^{M,N}$, then

$$\|Q_3u\|_{s,\ell} \le C\left(\|Q_3'u\|_{s,\ell} + \|G_3Pu\|_{s-2,\ell+1} + \|u\|_{M,N}\right). \tag{3.7}$$

(ii) *Above threshold regularity: Assume $\ell, \ell' > -1/2$ and $s, s' \in \mathbb{R}$. Let $U_- \Subset {}^{sc}T^*M$ be a sufficiently small neighborhood of $\mathcal{R}_-$. Then for all $Q_4, G_4 \in \Psi_{\mathrm{sc}}^{0,0}$ such that*

$$\mathcal{R}_- \subset \mathrm{Ell}(Q_4) \subset \mathrm{WF}'(Q_4) \subset \mathrm{Ell}(G_4) \subset U_-,$$

if $G_4Pu \in H^{s-2,\ell+1}$ and $G_4u \in H^{s',\ell'}$, then $Q_4u \in H^{s,\ell}$. Moreover, for all M, N, there is $C > 0$ so that if $u \in H^{M,N}$, then

$$\|Q_4u\|_{s,\ell} \le C\left(\|G_4u\|_{s',\ell'} + \|G_4Pu\|_{s-2,\ell+1} + \|u\|_{M,N}\right). \tag{3.8}$$

3.2. Fredholm estimate. In this subsection we explain how to piece together the microlocal estimates to produce a global estimate. See also Vasy [33, sect. 5.4.6], where this piecing together of estimates is discussed in terms of wavefront sets.

We first note that if U' and U_- are chosen small enough such that l_+ is constant, equal to $-1/2-\delta$ near U' and $-1/2+\delta$ near U_-, and if we choose r to be equal to these respective values in (3.7) and (3.8), then we can deduce the estimates with the variable weight l_+, as follows:

$$\|Q_3u\|_{s,\mathsf{l}_+} \le C\left(\|Q_3'u\|_{s,\mathsf{l}_+} + \|G_3Pu\|_{s-2,\mathsf{l}_++1} + \|u\|_{M,N}\right) \tag{3.9}$$

and

$$\|Q_4u\|_{s,\mathsf{l}_+} \le C\left(\|G_4u\|_{s',r'} + \|G_4Pu\|_{s-2,\mathsf{l}_++1} + \|u\|_{M,N}\right). \tag{3.10}$$

This is because the "microlocal difference" between, say, the norms $\|Q_3u\|_{s,-1/2-\delta}$ and $\|Q_3u\|_{s,\mathsf{l}_+}$ is disjoint from the microlocal support of Q_3, so the difference can be controlled by $\|u\|_{M,N}$ for arbitrary M and N; exactly the same argument applies to each of the other terms in these two estimates.

We then combine the estimates (3.3), (3.5), (3.9), and (3.10), noting that we may assume that $Q_1 + Q_2 + Q_3 + Q_4 = \mathrm{Id}$. In addition, we may assume that $Q_2' = Q_4$ and $Q_3' = Q_2$, as these satisfy the propagation conditions in (3.4) and (3.6). We add up these estimates, building in a large multiple of the second estimate and an even larger multiple of the fourth. Thus, for a constant C equal to the maximum constant in the four estimates (3.3), (3.5), (3.7), and (3.8), we have

$$\begin{aligned}
\|u\|_{s,\mathsf{l}_+} &\le \|Q_1u\|_{s,\mathsf{l}_+} + K\|Q_2u\|_{s,\mathsf{l}_+} + \|Q_3u\|_{s,\mathsf{l}_+} + K^2\|Q_4u\|_{s,\mathsf{l}_+} \\
&\le C\Big(\|G_1Pu\|_{s-2,\mathsf{l}_+} + \|u\|_{M,N} + K\Big(\|Q_4u\|_{s,\mathsf{l}_+} + \|G_2Pu\|_{s-2,\mathsf{l}_++1} + \|u\|_{M,N}\Big) \\
&\qquad + \|Q_2u\|_{s,\mathsf{l}_+} + \|G_3Pu\|_{s-2,\mathsf{l}_++1} + \|u\|_{M,N} \\
&\qquad + K^2\Big(\|G_4u\|_{s',\ell'} + \|G_4Pu\|_{s-2,\mathsf{l}_++1} + \|u\|_{M,N}\Big)\Big).
\end{aligned} \tag{3.11}$$

Next, we estimate the Q_2 and Q_4 terms on the RHS as just done, by using (3.5), resp. (3.10). Noting that estimating the Q_2 term produces an additional Q_4 term, we again estimate using (3.10). So, finally, we obtain, for a new larger constant C (noting that s' is arbitrary),

$$\|u\|_{s,\mathsf{l}_+} \le C\left(\|Pu\|_{s-2,\mathsf{r}_++1} + \|u\|_{M,N} + \|G_4u\|_{s',r'}\right). \tag{3.12}$$

Now, to handle the G_4 term, choose $\ell' = -1/2+\delta/2$ and $M < s' < s$. By Sobolev space interpolation, for appropriate $\eta \in (0,1)$,

$$\|G_4u\|_{s',-1/2+\delta/2} \le \|G_4u\|_{s,-1/2+\delta}^{1-\eta}\|G_4u\|_{M,N}^{\eta}.$$

We can replace the norm $\|G_4u\|_{s,-1/2+\delta}$ with $\|G_4u\|_{s',\mathsf{l}_+}$ since $\mathsf{l}_+ = -1/2+\delta$ on the microlocal support of G_4 (which is contained in U_-). Then, by Young's inequality,

$$\|G_4u\|_{s,\mathsf{l}_+}^{1-\eta}\|G_4u\|_{M,N}^{\eta} \le \epsilon\|u\|_{s,\mathsf{l}_+} + C(\epsilon)\|u\|_{M,N}$$

for arbitrary $\epsilon > 0$ provided $C(\epsilon)$ is sufficiently large. The $\epsilon \|u\|_{s,\mathsf{l}_+}$ term can be absorbed into the RHS of (3.12) and the other term is a multiple of $\|u\|_{M,N}$. This yields the Fredholm estimate (3.1).

3.3. Invertibility on variable order spaces. We now prove Theorem 3.1 using the result of Lemma 3.1. Given that the map (1.14) is Fredholm, it only remains to show that the kernel and cokernel are both trivial. In fact, due to the formal self-adjointness of the operator P, this amounts to showing that if $Pu = 0$, and either $u \in H^{s,\mathsf{l}_+}$ or H^{s,l_-}, then $u = 0$. As the argument is essentially the same in both cases, we only consider the case that $u \in H^{s,\mathsf{l}_+}$.

So, assume that $u \in H^{s,\mathsf{l}_+}$ and $Pu = 0$. Then, u is in $H^{s,-1/2+\delta}$ microlocally in a neighborhood of $\mathcal{R}_-$, i.e., u is above threshold near $\mathcal{R}_-$. Therefore, we can apply Proposition 3.4, part (ii), and deduce that u is in $H^{s,L}$ for arbitrarily large L microlocally near $\mathcal{R}_-$. The propagation theorem, Proposition 3.3, then shows that u is in $H^{s,L}$ for arbitrary L everywhere on Σ except, possibly, at $\mathcal{R}_+$. The elliptic estimate implies that, in fact, u is microlocally trivial except possibly at $\mathcal{R}_+$, in the sense that if A is such that $\mathrm{WF}'(A)$ is disjoint from $\mathcal{R}_+$, then $Au \in \mathcal{S}$, i.e., $Au \in H^{S,L}$ for all S and L.

We can thus apply [24, Prop. 12], which tells us that if u has a wavefront set contained in $\mathcal{R}_+$, and $Pu \in \mathcal{S}$, then u has the form[1]

$$u = r^{-(n-1)/2} e^{i\lambda r} \sum_{j=0}^{\infty} r^{-j} v_j(y), \quad r \to \infty, \text{ where } v_j \in C^\infty(\partial M).$$

On the other hand, the "boundary pairing" lemma [24, Prop. 13] shows that the leading coefficient v_0 in the expansion of u satisfies

$$-2i\lambda \int_{\partial M} |v_0|^2 = 2 \operatorname{Re} \int_M u\overline{Pu}.$$

Since the RHS is zero, $v_0 \equiv 0$, and thus $u \in H^{\infty,-1/2+\delta}(M)$ for δ small enough. Thus, *u is above threshold decay at both radial sets.* It is a key point that this is impossible, unless u vanishes identically. To see this, we apply Proposition 3.4, part (ii) (at the outgoing rather than incoming radial set), with ℓ arbitrary. Since $Pu = 0$, this implies that u is also microlocally trivial at $\mathcal{R}_+$, i.e., u itself is Schwartz. Finally, we can apply a unique continuation theorem, e.g., [19, Thm. 17.2.8], or alternatively [7], to deduce that $u \equiv 0$. This completes the proof of Theorem 3.1.

Remark 3.5. The fact that P^{-1} on these variable order spaces is equal to the action of the outgoing resolvent is shown in [24, sect. 11] or [30].

3.4. Module regularity. The next step is to adapt the argument above to the module regularity spaces $H_+^{s,\ell;\kappa,k}$ instead of variable order spaces H^{s,l_+}. We shall prove each of the microlocal estimates above in the module regularity setting.

PROPOSITION 3.6 (microlocal elliptic regularity—module version). *Let $u \in \mathcal{S}'$ and let $Q_1, G_1 \in \Psi_{\mathrm{sc}}^{0,0}$ be such that $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(G_1) \cap \mathrm{Ell}(P)$. Assume $G_1 Pu \in H_+^{s-2,\ell;\kappa,k}$. Then $Q_1 u \in H_+^{s,\ell;\kappa,k}$, and for all $M, N \in \mathbb{R}$, there is a constant $C > 0$ such that if $u \in H^{M,N}$, then*

$$\|Q_1 u\|_{s,\ell;\kappa,k} \le C \left(\|G_1 Pu\|_{s-2,\ell;\kappa,k} + \|u\|_{M,N} \right). \tag{3.13}$$

[1]The fact that there is a leading term in this expansion—which is all we need here—follows from the argument in section 4 of the present paper, which is modeled on the proof of [24, Prop. 12].

Proof. We prove this by induction on (κ, k). For $(\kappa, k) = (0, 0)$ this is just Proposition 3.2. Now assume, for a given (κ, k) that the result is true for all $(\kappa', k') < (\kappa, k)$ in the sense that $\kappa' \leq \kappa$, $k' \leq k$ and $(\kappa', k') \neq (\kappa, k)$. Then, for generators $A_1, \dots, A_m$ of $\mathcal{M}_+$ and $B_1, \dots, B_l$ of $\mathcal{N}$, we have

$$A_1 \cdots A_m B_1 \cdots B_l Q_1 u = Q_1 A_1 \cdots A_m B_1 \cdots B_l u + \sum_{j=1}^{m} A_1 \cdots [A_j, Q_1] \cdots A_m B_1 \cdots B_l u + \sum_{j=1}^{l} A_1 \cdots A_m B_1 \cdots [B_j, Q_1] \cdots B_l u. \tag{3.14}$$

We can shift the commutator factors to the left of the product modulo double commutator factors, shift the double commutator factors to the left modulo triple commutator factors, and so on. Notice that all of these multiple commutator factors are order $(0, 0)$, with microlocal support no bigger than $\mathrm{WF}'(Q_1)$, hence contained in the elliptic set of G_1. Note first that $G_1 P A_1 \cdots A_m B_1 \cdots B_l u \in H^{s-2,\ell}$. Indeed this can be seen by writing the operator in terms of commutators as

$$G_1 P A_1 \cdots A_m B_1 \cdots B_l = A_1 \cdots A_k B_1 \cdots B_l G_1 P + \sum_{j=1}^{m} A_1 \cdots [G_1 P, A_j] \cdots A_m B_1 \cdots B_l + \sum_{j=1}^{l} A_1 \cdots A_m B_1 \cdots [G_1 P, B_j] \cdots B_l, \tag{3.15}$$

so using that $[G_1 P, A_j], [G_1 P, B_j] \in \Psi^{2,0}$, we can therefore apply Proposition 3.2 to obtain

$$\begin{aligned} \|A_1 \cdots A_m B_1 \cdots B_l Q_1 u\|_{s,\ell} \lesssim{}& \|G_1 P A_1 \cdots A_m B_1 \cdots B_l u\|_{s-2,\ell} \\ &+ \text{commutator terms} + \|u\|_{M,N}, \end{aligned} \tag{3.16}$$

and we can perform a similar process as above, shifting the commutator factors to the left modulo double commutator factors, shifting those to the left modulo triple commutator factors, and so on. Each of these multiple commutator factors are order $(2, 0)$. Substituting into (3.16) we obtain

$$\begin{aligned} \|A_1 \cdots A_m B_1 \cdots B_l Q_1 u\|_{s,\ell} \lesssim{}& \|A_1 \cdots A_m B_1 \cdots B_l G_1 P u\|_{s-2,\ell} \\ &+ \sum \|\tilde{C}_\bullet \prod A_\bullet \prod B_\bullet u\|_{s,\ell} + \sum \|C_\bullet \prod A_\bullet \prod B_\bullet u\|_{s-2,\ell} + \|u\|_{M,N}, \end{aligned} \tag{3.17}$$

where $C_\bullet \in \Psi^{2,0}_{\mathrm{sc}}$ and $\tilde{C}_\bullet \in \Psi^{0,0}_{\mathrm{sc}}$ are multicommutators with wavefront set contained in $\mathrm{WF}'(Q_1)$, and we have fewer than $\kappa + k$ factors of the $A_\bullet$ and the $B_\bullet$ in total. The other terms in (3.16) are estimated similarly. We thus obtain

$$\|A_1 \cdots A_m B_1 \cdots B_l Q_1 u\|_{s,\ell} \lesssim \|G_1 P u\|_{s-2,\ell;\kappa,k} + \sum_{(\kappa',k')<(\kappa,k)} \|G' u\|_{s,\ell;\kappa',k'} + \|u\|_{M,N}, \tag{3.18}$$

where G' is chosen so that $\mathrm{WF}'(Q_1) \subset \mathrm{Ell}(G') \subset \mathrm{WF}'(G') \subset \mathrm{Ell}(G_1)$ and $\mathrm{WF}(I - G') \cap \mathrm{WF}'(Q_1) = \varnothing$. These conditions imply that $G'C = C$ modulo an operator of order $(-\infty, -\infty)$ which contributes to the $\|u\|_{M,N}$ term. We apply the inductive assumption to the term $G'u$, where G' now plays the role of Q_1, and arrive at

$$\|A_1 \cdots A_k B_1 \cdots B_l Q_1 u\|_{s,\ell} \lesssim \|G_1 P u\|_{s-2,\ell;\kappa,k} + \|u\|_{M,N}. \tag{3.19}$$

After summing over all possible choices of the $A_1, \dots, A_m$ and the $B_1, \dots, B_l$ we obtain (3.13). ∎

PROPOSITION 3.7 (propagation of regularity estimate—module version). *Let $u \in \mathcal{S}'$ and let $Q_2, Q_2', G_2 \in \Psi_{\mathrm{sc}}^{0,0}$. Assume $\mathrm{WF}'(Q_2) \subseteq \mathrm{Ell}(G_2)\backslash(\mathcal{R}_+ \cup \mathcal{R}_-)$. Moreover, assume that* (3.4) *holds. If $Q_2'u \in H_+^{s,\ell;\kappa,k}$ and $G_2Pu \in H_+^{s-2,\ell+1;\kappa,k}$, then $Q_2u \in H_+^{s,\ell;\kappa,k}$, and for all M, N there is $C > 0$ such that if $u \in H^{M,N}$, then*

$$\|Q_2u\|_{s,\ell;\kappa,k} \leq C\left(\|Q_2'u\|_{s,\ell;\kappa,k} + \|G_2Pu\|_{s-2,\ell+1;\kappa,k} + \|u\|_{M,N}\right). \tag{3.20}$$

Remark 3.8. Here we included the extra assumption that $\mathrm{WF}'(Q_2)$ is disjoint from the radial sets, so that the modules $\mathcal{M}_+$ and $\mathcal{N}$ both become *elliptic* on $\mathrm{WF}'(Q_2)$, in the sense that at each point of $\mathrm{WF}'(Q_2)$ there exists a module element of $\mathcal{N}$ (and hence also $\mathcal{M}_+$) that is elliptic. (It was not necessary to include this assumption in Proposition 3.3, but we could have done so as the proposition gives no information at the radial sets.)

Proof. When the modules are elliptic the proof becomes almost trivial. We note that the module norm $\|\cdot\|_{s,\ell;\kappa,k}$ is equivalent, microlocally on $\mathrm{WF}'(Q_2)$, to the $\|\cdot\|_{s+\kappa+k,\ell+\kappa+k}$ norm. So the proposition is actually equivalent to the previous one, with a shift in orders s and ℓ by $\kappa + k$. ∎

In the next proposition, all operators have microlocal support in a compact region of ${}^{\mathrm{sc}}T^*M$ by assumption, thus disjoint from fiber-infinity. Hence the differential order is irrelevant for both the operators and the spaces. We write it $*$ to emphasize this irrelevance.

PROPOSITION 3.9 (radial point estimates—module version).

(i) *Below threshold regularity radial point estimate: Assume $\ell < -1/2$. Let $Q_3, Q_3', G_3 \in \Psi_{\mathrm{sc}}^{0,0}$. Let U, U' denote two open neighborhoods of $\mathcal{R}_+$ with $U \Subset U' \Subset {}^{\mathrm{sc}}T^*_{\partial M}M$, and assume that $U \subset \mathrm{Ell}(Q_3) \subset \mathrm{WF}'(Q_3) \subset \mathrm{Ell}(G_3) \subset U'$. Assume that $\mathrm{WF}'(Q_3')$ is contained in $U' \setminus U$ and that* (3.6) *holds. If $Q_3'u \in H_+^{*,\ell;\kappa,k}$ and $G_3Pu \in H_+^{*,\ell+1;\kappa,k}$, then $Q_3u \in H_+^{*,\ell;\kappa,k}$, and for all M, N there is $C > 0$ such that if $u \in H^{M,N}$, then*

$$\|Q_3u\|_{*,\ell;\kappa,k} \leq C\left(\|Q_3'u\|_{*,\ell;\kappa,k} + \|G_3Pu\|_{*,\ell+1;\kappa,k} + \|u\|_{M,N}\right). \tag{3.21}$$

(ii) *Above threshold regularity: Assume $\ell, \ell' > -1/2$. Let $U_- \Subset {}^{sc}T^*_{\partial M}M$ be a sufficiently small neighborhood of $\mathcal{R}_-$. Then for all $Q_4, G_4 \in \Psi_{\mathrm{sc}}^{0,0}$ such that*

$$\mathcal{R}_- \subset \mathrm{Ell}(Q_4) \subset \mathrm{WF}'(Q_4) \subset \mathrm{Ell}(G_4) \subset U_-,$$

if $G_4Pu \in H_+^{*,\ell+1;\kappa,k}$ and $G_4u \in H_+^{*,\ell';\kappa,k}$, then $Q_4u \in H_+^{*,\ell;\kappa,k}$. Moreover, for all M, N, there is $C > 0$ so that if $u \in H^{M,N}$, then

$$\|Q_4u\|_{*,\ell;\kappa,k} \leq C\left(\|G_4u\|_{*,\ell';\kappa,k} + \|G_4Pu\|_{*,\ell+1;\kappa,k} + \|u\|_{M,N}\right). \tag{3.22}$$

Moreover, all of the above holds with $H_+^{*,\ell;\kappa,k}, \mathcal{R}_\pm$ replaced by $H_-^{*,\ell;\kappa,k}, \mathcal{R}_\mp$.

Proof. In this case, the argument is more elaborate than the previous two proofs and relies on the construction of a positive commutator. The key fact we use is Lemma 2.3, that is, the P-positivity/criticality of the modules $\mathcal{M}_\pm$ and $\mathcal{N}$. It is very similar to the argument from [14, sect. 6], where test modules were introduced. To avoid a long exposition about test modules and how positive commutator estimates are used to prove module regularity, we will use section 6 of [14] as a basis and only indicate the minor differences that arise in the present case.

We fix a basis $A_0 = \mathrm{Id}, \dots, A_N$ of the module $\mathcal{M}_+$, where $A_N = r(D_r - \lambda)$ and the rest form a basis for $\mathcal{N}$. We use the notation A_α, $\alpha = (\alpha_1, \dots, \alpha_N)$ a multi-index, for the operator

$$A_1^{\alpha_1} \dots A_N^{\alpha_N}$$

and $A_{\alpha,\ell}$ for the operator $r^\ell A_\alpha$. Given (κ, k), we consider A_α, where $|\alpha| = \kappa + k$ and $\alpha_N \le \kappa$, that is, there are at least k module elements in the product $A_{\alpha,\ell}$ lying in the small module $\mathcal{N}$. Such A_α, as α ranges over such all multi-indices of length $\kappa + k$, together with Id form a basis for $\mathcal{M}_+^\kappa \mathcal{N}^k$, the vector space of sums of products of κ elements of $\mathcal{M}_+$ and k elements of $\mathcal{N}$, as a module over $\Psi_{\mathrm{sc}}^{0,0}$.

We prove the estimate by induction on (κ, k), the module orders. For $\kappa = k = 0$ the result is precisely Proposition 3.4. We now assume inductively that the result has been proved for all $(\kappa', k') < (\kappa, k)$, that is, (κ', k') such that $\kappa' \le \kappa$, $k' \le k$ and $(\kappa', k') \neq (\kappa, k)$.[2]

The positive commutator estimate arises from the following operator identity, which is essentially (6.16) in [14]. In the following, Q is arbitrary, but we will choose it to be an operator which is microlocally equal to the identity on $\mathrm{WF}'(Q_3)$, and with $\mathrm{WF}'(Q) \subset \mathrm{Ell}(G_3)$. In the following identity, the C_{jk} are defined by the commutators of P with basis elements A_j, as in (2.38).

$$\begin{aligned} i[A^*_{\alpha,\ell+1/2} Q^* Q A_{\alpha,\ell+1/2}, P] &= A^*_{\alpha,\ell} Q^* \left(C_0 + C_0^* + \sum_{j=1}^N \alpha_j (C_{jj} + C_{jj}^*) \right) Q A_{\alpha,\ell} \\ &\quad + \sum_{|\beta|=k, \beta\neq\alpha} A^*_{\alpha,\ell} Q^* C_{\alpha\beta} Q A_{\beta,\ell} + \sum_{|\beta|=k,\beta\neq\alpha} A^*_{\beta,\ell} Q^* C^*_{\alpha\beta} Q A_{\alpha,\ell} \\ &\quad + A^*_{\alpha,\ell} Q^* E_{\alpha,\ell} + E^*_{\alpha,\ell} Q A_{\alpha,\ell} + A^*_{\alpha,\ell+1/2} i[Q^*Q, P] A_{\alpha,\ell+1/2} \\ &\quad + \sum_{j=1}^N A^*_{\alpha,\ell} Q^* E'_{\alpha,\ell+1} P + P {E'_{\alpha,\ell+1}}^* Q A_{\alpha,\ell}, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} \sigma_{\mathrm{base},0}(C_0)|_{\mathcal{R}_+} &= -\lambda(2\ell+1), \\ \sigma_{\mathrm{base},0}(C_{\alpha\beta})|_{\mathcal{R}_+} &= 0, \ C_{\alpha\beta} \in \Psi_{\mathrm{sc}}^{1,0}(M), \\ E_{\alpha,\ell} = r^\ell E_\alpha, \quad E'_{\alpha,\ell+1} &= r^{\ell+1} E'_\alpha, \quad E_\alpha, E'_\alpha \in \mathcal{M}_+^{\kappa'} \mathcal{N}^{k'}, \quad (\kappa', k') < (\kappa, k). \end{aligned} \tag{3.24}$$

This coincides with [14, eq. (6.16)] except for the last line, which arises from the $C'_j P$ terms in (2.38).[3] Note that in our case, C'_j is only nonzero for $j = N$, that is, for the $r(D_r - \lambda)$ module element, when it is equal to $2r$. The P factor can be commuted to be either at the left or the right of the composition, at the cost of commutator terms which can be absorbed in the second last line. The r factor in C'_N accounts for the increase of growth from r^ℓ to $r^{\ell+1}$ in the E' terms in the last line.

[2]Technically, the inductive estimate has to be for a slightly different operator $\tilde{Q}_3$ in place of Q_3, with slightly larger elliptic set and microlocal support.

[3]Notice that no vanishing condition is imposed on the C'_j term in (2.38), unlike in [14]. The vanishing condition on C'_j is not required; instead we use the extra order of vanishing of Pu, to order $\ell + 1$ instead of ℓ, to estimate the terms involving $C'_j P$ below.

The key point above is that the symbol of C_0, arising from the $-2\nu r\partial_r$ component of minus the Hamilton vector field from (2.20) hitting the r^ℓ factor, has a definite sign near $\mathcal{R}_+$—positive for ℓ less than the threshold exponent $-1/2$. Moreover, the P-positivity of the modules $\mathcal{M}_+$ and $\mathcal{N}$ means that the diagonal operators C_{jj} have nonnegative symbols at $\mathcal{R}_+$—cf. (2.40). Similarly, the off-diagonal terms $C_{\alpha\beta}$ vanish at $\mathcal{R}_+$ due to (2.39). Now, we define a matrix $C' = (C'_{\alpha\beta})$ of operators, as the indices α, β vary over multi-indices of length k, as follows: for $\alpha \neq \beta$,

$$C'_{\alpha\beta} = C_{\alpha\beta} + C^*_{\beta\alpha}$$

and on the diagonal, we define

$$C'_{\alpha\alpha} = C_0 + C_0^* + \sum_{j=1}^N \alpha_j (C_{jj} + C_{jj}^*). \tag{3.25}$$

Thus, due to (3.24), the symbol of C' at $\mathcal{R}_+$ is diagonal with positive entries, and it is therefore positive as an matrix, provided that the microlocal support of Q is sufficiently close to $\mathcal{R}_+$ (which we arrange below). This means that we can write

$$Q^* C' Q = Q^*(B^*B + G)Q,$$

where B is a matrix of operators of order $(*, 0)$ and G a matrix of operators of order $(*, -1)$. For a function u we also write Au for $(QA_{\alpha,\ell}u)$, regarded as a column vector indexed by multi-indices α of length m. Thus, in this compact notation we can write the first two lines on the RHS of (3.23) as $A^*(B^*B+G)A$. We similarly use compact notation for norms and inner products, e.g.,

$$\|Au\|^2 = \langle Au, Au\rangle := \sum_\alpha \langle QA_{\alpha,\ell}u, QA_{\alpha,\ell}u\rangle.$$

We now assume that the neighborhood U' of $\mathcal{R}_+$ is sufficiently small that the symbol of C' is invertible on U'. We choose Q so that Q is equal to the identity microlocally on U and such that the commutator of any pseudodifferential operator with Q has microlocal support contained in $\mathrm{Ell}(Q_3')$. To do this, we take the full symbol of Q to be identically 1 in U, and to be identically zero outside U', such that along each bicharacteristic it transitions from 1 to 0 on the elliptic set of Q_3', a set which is always met before exiting U' due to (3.6) and the condition $\mathrm{Ell}(G_3) \subset U'$.

Now we follow the argument of the proof of [14, Prop. 6.7]. We let u' be an element of $H_+^{*,\ell;\kappa,0}$. From (3.23) we have, in matrix notation,

$$\begin{aligned}\sum_{|\alpha|=k} &\big\langle u', i[A^*_{\alpha,\ell+1/2}Q^*QA_{\alpha,\ell+1/2}, P]u'\big\rangle \\ &= \|BAu'\|^2 + \langle Au', GAu'\rangle + \sum_{|\alpha|=k} \big(\langle QA_{\alpha,\ell}u', E_{\alpha,\ell}u'\rangle + \langle E_{\alpha,\ell}u', QA_{\alpha,\ell}u'\rangle\big) \\ &+ \sum_{|\alpha|=k} \langle A_{\alpha,\ell}u', FA_{\alpha,\ell}u'\rangle + \sum_{|\alpha|=k} \big(\langle QA_{\alpha,\ell}u', E'_{\alpha,\ell+1}Pu'\rangle + \langle E'_{\alpha,\ell+1}Pu', QA_{\alpha,\ell}u'\rangle\big).\end{aligned} \tag{3.26}$$

Here $F = r^{1/2}[Q^*Q, P]r^{1/2} \in \Psi_{\mathrm{sc}}^{*,0}$ has an operator wavefront set contained in $\mathrm{Ell}(Q_3')$. We can therefore write $F = Q_3'F'Q_3' + E$, where F' has order $(0,0)$ and E has order $(-\infty, -\infty)$. Using this, and also employing the elementary inequality $2ab \le \epsilon a^2 + \epsilon^{-1} b^2$, we deduce an estimate for $\|BAu'\|$:

$$\begin{aligned}\|BAu'\|^2 &\le 2 \sum_\alpha \Big| \langle QA_{\alpha,\ell+1/2} u', QA_{\alpha,\ell+1/2} Pu' \rangle \Big| \\ &+ 4\epsilon \|Au'\|^2 + \epsilon^{-1} \Big(\|GAu'\|^2 + \sum_\alpha \big(\|E_{\alpha,\ell} u'\|^2 + \|E'_{\alpha,\ell+1} Pu'\|^2 \big) \Big) \\ &+ \|F'\| \sum_\alpha \|Q'_3 A_{\alpha,\ell} u'\|^2 + C\|u'\|^2_{M,N}.\end{aligned} \tag{3.27}$$

We next estimate the commutator term, that is, the first term on the RHS. (This is not done in [14], since there it was assumed that Pu' is Schwartz.) We can commute a power of $r^{1/2}$ through Q, at the cost of commutator terms of order $(*, -1)$. This gives us

$$\langle QA_{\alpha,\ell+1/2} u', QA_{\alpha,\ell+1/2} Pu' \rangle = \langle QA_{\alpha,\ell} u', QA_{\alpha,\ell+1} Pu' \rangle + \langle E''_{\alpha,\ell} u', QA_{\alpha,\ell+1} Pu' \rangle,$$

where $E''_{\alpha,\ell} = r^\ell E''_\alpha$, $E''_\alpha \in \mathcal{M}_+^{\kappa'} \mathcal{N}^{k'}$ for some $(\kappa', k') < (\kappa, k)$. Therefore

$$2 \Big| \langle QA_{\alpha,\ell+1/2} u', QA_{\alpha,\ell+1/2} Pu' \rangle \Big| \le \epsilon \|QA_{\alpha,\ell} u'\|^2 + \|E''_{\alpha,\ell} u'\|^2 + (1+\epsilon^{-1}) \|QA_{\alpha,\ell+1} Pu'\|^2.$$

Summing over α and combining this with (3.27) we have

$$\begin{aligned}\|BAu'\|^2 &\le 5\epsilon \|Au'\|^2 + \|E''_{\alpha,\ell} u'\|^2 \\ &+ \epsilon^{-1} \Big(\|GAu'\|^2 + \sum_\alpha \big(\|E_{\alpha,\ell} u'\|^2 + 2\|QA_{\alpha,\ell+1} Pu'\|^2 + \|E'_{\alpha,\ell+1} Pu'\|^2 \big) \Big) \\ &+ C \sum_\alpha \|Q'_3 A_{\alpha,\ell} u'\|^2 + C\|u'\|^2_{M,N}.\end{aligned} \tag{3.28}$$

The terms proportional to 5ϵ can be absorbed in the LHS, up to a term of the form $C\|u'\|^2_{M,N}$. In fact, on the microlocal support of Q, B has a microlocal inverse, which we will denote B^{-1} (despite not being an actual inverse of B). So we have $A = B^{-1} BA + E'$, where E' has order $(-\infty, -\infty)$. Then, estimating B^{-1} by its operator norm, we can absorb the $\|Au'\|^2$ terms provided ϵ is small compared to $\|B^{-1}\|$, while the E' term only contributes a multiple of $\|u'\|^2_{M,N}$.

We now notice that GA can be treated as being in $\mathcal{M}_+^{\kappa'} \mathcal{N}^{k'}$ for $(\kappa', k') < (\kappa, k)$ since G has order $(*, -1)$. So this term, as well as the $E_{\alpha,\ell}$ and $E''_{\alpha,\ell}$ terms, can be estimated using the inductive assumption. In exactly the same way, we can commute the Q to the right of the $A_{\alpha,\ell+1}$ factor and then replace it with G_3. Similarly, Q'_3 can be moved to the right of the $A_{\alpha,\ell}$. After these manipulations, we obtain the estimate

$$\|Q_3 u'\|^2_{*,\ell;0,k} \le C \Big(\|G_3 Pu'\|^2_{*,\ell+1;0,k} + \|Q'_3 u'\|^2_{*,\ell;0,k} + \|u'\|^2_{M,N} \Big). \tag{3.29}$$

Now we let $u' = u'(\eta) := (1 + \eta r)^{-1} u$, $u \in H_+^{*,\ell;0,k-1}$ for $\eta > 0$ tending to zero. Then $u' \in H_+^{*,\ell;0,k}$ for each $\eta > 0$, so the above computation is valid. Assuming that $Q'_3 u \in H_+^{*,\ell;0,k}$ and $G_3 Pu \in H_+^{*,\ell+1;0,k}$, then the RHS of (3.29) stays bounded as $\eta \to 0$. Therefore, the LHS also stays bounded, and using the strong convergence of $(1 + \eta r)^{-1}$ to the identity as in [14, Lem. 4.3], we see that we obtain estimate (3.29) also with $u' = u$.

We next turn to the proof of (3.22). This works quite differently in relation to the two modules. At the incoming radial set $\mathcal{R}_-$, the module $\mathcal{M}_+$ is elliptic, while $\mathcal{N}$ is characteristic. The effect of the κth power of the module $\mathcal{M}_+$ is thus just to increase the spatial order ℓ by κ. So, without loss of generality, we may assume that $\kappa = 0$.

We then employ a very similar argument to the one above. Notice that, instead of having $\sigma_{\mathrm{base},0}(C_0) = -\lambda(2\ell+1)$ at the radial set, as above, we now have $\sigma_{\mathrm{base},0}(C_0) = \lambda(2\ell+1)$. On the other hand, now $\ell > -1/2$, so the $2\ell+1$ factor has also switched sign, so this symbol remains positive at the radial set (now $\mathcal{R}_-$). Using the fact that the module $\mathcal{N}$ is P-critical, we find that the matrix C' in this case is again positive definite at the incoming radial set. Then we run the same argument as above, with the following twist: In this case, the F term arising from $[Q^*Q, P]$ has the same sign as C', namely, it is positive, as it arises from minus the (rescaled) Hamilton vector field H_p hitting $\sigma(Q)^2$. Taking $\sigma(Q)$ to be a function only of $|\mu|_h$ at $x = 0$ we see from (2.20) that $-\mathsf{H}_p(\sigma(Q)^2)$ is nonnegative. Since this has the same sign as that of C', to leading order, we can discard this term up to a lower order term. This lower order term requires the presence of $\|G_4 u\|_{*,\ell',k}$ on the RHS of the estimate; at first sight it appears that we could take $\ell' = \ell - 1/2$ (and then iterate to reduce ℓ' as much as we like), but the regularization required to justify the estimate requires that ℓ' is greater than the threshold value of $-1/2$. See the proof of [33, Prop. 5.26], between (5.61) and (5.62), for the details of the standard regularization step.

This proves the proposition for $H_+^{*,\ell;\kappa,k}$. For the space $H_-^{*,\ell;\kappa,k}$, where the index κ now indicates module regularity with respect to $\mathcal{M}_-$, a similar argument applies. Here, it is important that the module $\mathcal{M}_-$ is P-negative at the radial sets. In this case, the below threshold estimate is localized near $\mathcal{R}_-$. In this case the symbol of C_0 in (3.23) switches sign, due to the switch of sign of ν in (2.20). The fact that $\mathcal{M}_-$ is P-negative means that the symbol of the C_{jj} terms also switch sign, and hence the matrix C' is negative definite, rather than positive definite, near $\mathcal{R}_-$. Then the argument proceeds as above. Similar remarks apply to the above threshold case. □

COROLLARY 3.10. *Suppose* $\ell', \ell > -3/2$ *and* $\kappa \geq 1$. *Then, under the same assumptions as in part* (ii) *of Proposition* 3.9, *we have* (3.22).

Proof. This follows since the module $\mathcal{M}_+$ is elliptic at $\mathcal{R}_-$. So when $\kappa \geq 1$, the estimate (3.22) is equivalent to the same estimate with ℓ, ℓ' increased by κ, and κ set to zero. Under the assumption that $\ell', \ell > -3/2$ and $\kappa \geq 1$ this puts us in the range of applicability of part (ii) of Proposition 3.9. □

3.5. Invertibility on module regularity spaces. Our final piece of preparation for the proof of Theorem 2.4 is the following result relating our module regularity spaces to variable order spaces.

LEMMA 3.2. *Assume* $\mathsf{l}_+ \in C^\infty(\overline{{}^{\mathrm{sc}}T^*M})$ *satisfies* (2.22) *and* (2.23). *Then for* $\epsilon < \delta$ *and* $\ell = -1/2 - \epsilon$,

$$\kappa \geq 1 \;\Rightarrow H_+^{s,\ell;\kappa,k} \subset H^{s,\mathsf{l}_+}. \tag{3.30}$$

Proof. Since $H_+^{s,\ell;\kappa,k} \subset H_+^{s,\ell;1,0}$, to show (3.30), it suffices to assume $\kappa = 1$ and $k = 0$. Let U be a small neighborhood of $\mathcal{R}_+$ near which $\mathsf{l}_+ = -1/2 - \delta$, and let $\tilde{U} \Subset U$ be a smaller neighborhood of $\mathcal{R}_+$. For each $\mathsf{q} \in \overline{{}^{\mathrm{sc}}T^*M} \setminus \tilde{U}$, there is an element A_{q} of $\mathcal{M}_+$, elliptic on a neighborhood U_{q} of q. Form a partition of unity subordinate to the cover of $\overline{{}^{\mathrm{sc}}T^*M}$, consisting of U and finitely many of the U_{q}, say, $U_{\mathsf{q}_1}, \dots, U_{\mathsf{q}_m}$. We take $\{Q_{\mathsf{q}_j}\}_{j=1}^m$, $Q \in \Psi_{\mathrm{sc}}^{0,0}$ to be the corresponding left quantizations of the microlocal cutoffs that comprise the partition of unity. Clearly, $Qu \in H^{s,\ell}$, since $u \in H^{s,\ell}$. Thus $Qu \in H^{s,\mathsf{l}_+}$ since, on $\mathrm{WF}'(Q)$, we have $\mathsf{l}_+ = -1/2 - \delta < -1/2 - \epsilon = \ell$.

On the other hand, by the assumption of module regularity, each $A_{\mathsf{q}_j} u \in H^{s,\ell}$. Because A_{q_j} is order $(1,1)$ and $\mathrm{WF}'(Q_{\mathsf{q}_j}) \subset \mathrm{Ell}(A_{q_j})$, microlocal elliptic regularity (Proposition 3.2) asserts that $Q_{\mathsf{q}_j} u \in H^{s+1,\ell+1}$. Then we note that $H^{s+1,\ell+1} \subset H^{s,\mathsf{l}_+}$ since $\ell + 1 = 1/2 - \epsilon \geq -1/2 + \delta = \max \mathsf{l}_+$ for δ sufficiently small. ∎

Remark 3.11. The point of this lemma is that the different behavior at the incoming and outgoing radial sets is enforced by the module regularity instead of by a variable weight function, due to the fact that the module $\mathcal{M}_+$ is elliptic at the incoming radial set $\mathcal{R}_-$ but characteristic at the outgoing radial set $\mathcal{R}_+$.

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. Let $s \in \mathbb{R}$, $\ell \in (-3/2, -1/2)$, $\kappa \geq 1$, and k be given. We first combine the estimates (3.13), (3.20), (3.21), and (3.22) (in the latter case, for $\ell \in (-3/2, -1/2)$, as allowed by Corollary 3.10). This is done exactly as in section 3.2, so we do not repeat the argument. We obtain, for $u \in H^{M,N}$ such that $Pu \in H_+^{s,\ell;\kappa,k}$, that $u \in H_+^{s,\ell;\kappa,k}$ and

$$\|u\|_{s,\ell;\kappa,k} \leq C\Big(\|Pu\|_{s-2,\ell;\kappa,k} + \|u\|_{M,N}\Big). \tag{3.31}$$

We next use Lemma 3.2 to assert that $H_+^{s,\ell;\kappa,k} \subset H^{s,\mathsf{l}_+}$, provided ℓ is sufficiently close to $-1/2$. The proof of Lemma 3.2 may be trivially modified to show $H_+^{s-2,\ell+1;\kappa,k} \subset H^{s-2,\mathsf{l}_+ +1}$. Thus we have the following diagram:

$$\begin{array}{ccc} \mathcal{X}^{s,\mathsf{l}_+} & \xrightarrow{\ P\ } & \mathcal{Y}^{s-2,\mathsf{l}_+ +1} \\ \uparrow & & \uparrow \\ \mathcal{X}_+^{s,\ell;\kappa,k} & & \mathcal{Y}_+^{s-2,\ell+1;\kappa,k} \end{array}. \tag{3.32}$$

Our goal is to show that the restriction of P to $\mathcal{X}_+^{s,\ell;\kappa,k}$ yields a bijection $\mathcal{X}_+^{s,\ell;\kappa,k} \to \mathcal{Y}_+^{s-2,\ell+1;\kappa,k}$. Injectivity follows immediately since the top row is an injective map. To show surjectivity, we suppose $f \in \mathcal{Y}_+^{s-2,\ell+1;\kappa,k} = H_+^{s-2,\ell+1;\kappa,k}$. In, particular $f \in H^{s-2,\mathsf{l}_+ +1}$. So, by surjectivity of the top row there is $u \in \mathcal{X}^{s,\mathsf{l}_+}$ with $Pu = f$. Then, thanks to (3.31), we see that, provided (M, N) are sufficiently small, $u \in H_+^{s,\ell;\kappa,k}$. As a bounded bijection, the map $\mathcal{X}_+^{s,\ell;\kappa,k} \to \mathcal{Y}_+^{s-2,\ell+1;\kappa,k}$ is automatically a Hilbert space isomorphism. ∎

4. Proof of the main theorem.

In this section, we find a nonlinear eigenfunction with prescribed incoming data by finding a fixed point of the map in (1.18), which we shall show is a contraction map on the space $\mathcal{X}_+^{2,\ell;1,k+1}$. Here and below we fix $s = 2$, and let k be any integer strictly larger than $(n-1)/2$. We also put $\ell = -1/2 - \delta$ for some fixed δ with $0 < \delta \leq (4p)^{-1} \leq 1/8$.

4.1. Linear eigenfunction.

Fix $f \in H^{k+4}(\partial M)$. Define

$$u_-(r, y) := \chi(r) r^{-(n-1)/2} e^{-i\lambda r} f(y),$$

where χ is a cutoff function, supported in $r > R$, and identically 1 for $r \geq 2R$. By inspection, we see that $u_- \in H_-^{2,\ell;1,k+1}$. Moreover, it is clear that if $\|f\|_{H^{k+4}(\partial M)}$ is sufficiently small, then $\|u_-\|_{H_-^{2,\ell;1,k+1}}$ is small.

Moreover, by direct calculation via (2.42), we have

$$(Pu_-)(r,y) = \tilde{\chi}(r) r^{-(n+3)/2} e^{-i\lambda r} g(r,y), \quad x = r^{-1}, \tag{4.1}$$

where $\tilde{\chi}$ is another cutoff function, equal to 1 on $\operatorname{supp}\chi$ and supported in $r > R$, and $g(r,y)$ is a smooth function of r^{-1} with values in $H^{k+2}(\partial M)$. The key point is the gain of two powers of r^{-1} as $r \to \infty$. It follows that

$$Pu_- \in H_-^{0,\ell+2;1,k+1}.$$

Now we want to view this as an element of $H_+^{0,\ell+1;1,k+1}$; to accommodate the $\mathcal{M}_+$-module regularity of order $\kappa = 1$, we lose one order of vanishing as discussed at the end of the proof of Corollary 2.5. Thus

$$Pu_- \in H_+^{0,\ell+1;1,k+1} = \mathcal{Y}_+^{0,\ell+1;1,k+1}.$$

Next, we define

$$u_+ := R(\lambda + i0)(Pu_-), \tag{4.2}$$

which belongs to $\mathcal{X}_+^{2,\ell;1,k+1}$ by the mapping property (2.48) of Theorem 2.4. We also put

$$u_0 := u_- + u_+$$

and notice that $Pu_0 = 0$.

4.2. Contraction mapping on $\boldsymbol{\mathcal{X}_+^{2,\ell;1,k+1}}$. We return to the discussion of section 1.2. There, it was explained how finding a nonlinear eigenfunction amounts to finding a fixed point of the map Φ given by

$$\Phi(w) = u_+ + R(\lambda + i0)\big(N[u_- + w]\big). \tag{4.3}$$

Let us check that a fixed point w provides us with a nonlinear eigenfunction $u := u_- + w$. Adding u_- to both sides of (4.3), we obtain

$$\Phi(w)+u_- = w+u_- = u_+ + u_- + R(\lambda+i0)\big(N[u_-+w]\big) = u_0 + R(\lambda+i0)\big(N[u_-+w]\big). \tag{4.4}$$

Thus,

$$u = u_0 + R(\lambda + i0)\big(N[u]\big). \tag{4.5}$$

Now we apply P to both sides. This annihilates u_0 and we find that $Pu = N[u]$, as claimed.

We now show that Φ is a contraction mapping on $\mathcal{X}_+^{2,\ell;1,k+1}$, provided $\|f\|_{H^{k+4}(\partial M)}$ is sufficiently small (and hence $\|u_-\|_{H_+^{2,\ell;1,k+1}}$ is small), and provided that w is small. The first thing to check is that Φ *is* a mapping on this space.

We observed above that $u_+ \in H_+^{2,\ell;1,k+1}$, so it remains to show $R(\lambda+i0)(N[u_- + w])$ belongs to this space, provided w does.

Recall that the nonlinear expression $N[v]$ is a sum of monomial terms, each of which is a product of $\tilde{p} \geq p$ factors of the form Qv or $Q\overline{v}$, where Q is a scattering differential operator of order $(2,0)$. (In the case of $\mathbb{R}^n$ it just means a combination of the usual coordinate partial derivatives multiplied by $C^\infty(M)$ functions; see Remark 1.6.) It follows that $N[u_- + w]$ is a finite sum of products of such factors. We have already seen that u_- lies in $H_-^{2,\ell;1,k+1}$, and w by assumption lies in $H_+^{2,\ell;1,k+1}$. Also, we notice that complex conjugation is an involution between $H_-^{2,\ell;1,k+1}$ and $H_+^{2,\ell;1,k+1}$. Using these facts along with Lemma 2.2, we conclude $N[u_- + w]$ is a sum of products of factors, each of which lies in $H_-^{0,\ell;1,k+1}$ or $H_+^{0,\ell;1,k+1}$. Applying Corollary 2.5, we find that the product lies in $H_+^{0,\ell';1,k+1}$, provided that for all $\tilde{p} \geq p$,

$$\ell' \leq \tilde{p}\ell + \frac{(\tilde{p}-1)n}{2} - 1. \tag{4.6}$$

We would like to know when this product is in $H^{0,\ell+1;1,k}$. This is the case provided that

$$\ell + 1 \leq \tilde{p}\ell + \frac{(\tilde{p}-1)n}{2} - 1 \quad \Longleftrightarrow \quad 2 \leq (\tilde{p}-1)\ell + \frac{(\tilde{p}-1)n}{2}. \tag{4.7}$$

Since $\ell \geq -5/8$, the RHS is increasing in $\tilde{p}$. So it is only necessary to demand (4.7) for $\tilde{p} = p$. Since $\ell = -1/2 - \delta$ and $0 < \delta \leq (4p)^{-1}$, it is straightforward to check that a sufficient condition for (4.7) is

$$2 < (p-1)\frac{n-1}{2}, \tag{4.8}$$

which is precisely condition (1.7). When this holds, we automatically have

$$5/2 \leq (p-1)\frac{n-1}{2} \tag{4.9}$$

since n and p are integers. In fact, in anticipation of Proposition 4.1, we note that we can take $\ell' = 3/4$ in (4.6).

Next, we verify that Φ is a contraction on a small ball in $\mathcal{X}_+^{2,\ell;1,k+1}$, provided the prescribed incoming data f is small in $H^{k+4}(\partial M)$. We have

$$\Phi(w_1) - \Phi(w_2) = R(\lambda + i0)\big(N[u_- + w_1] - N[u_- + w_2]\big). \tag{4.10}$$

Since N is a sum of monomials, each of degree $\tilde{p} \geq p$, the second factor on the RHS is a sum of terms of the form $Q(w_1 - w_2)$ or its complex conjugate, times a monomial of degree at least $p-1$ in various $Q'u_-$, $Q''w_1$, or $Q'''w_2$ or their complex conjugates, where the Q, Q', and Q''' are scattering differential operators of order $(2,0)$.

Let $\eta > 0$ be a small parameter, to be chosen later. By direct calculation, we see that the map

$$f \mapsto u_-, \quad u_- = \chi(r)r^{-(n-1)/2}e^{-i\lambda r}f(y) \tag{4.11}$$

is a bounded map from $H^{k+4}(\partial M)$ to $H_-^{2,\ell;1,k+1}$, so we may assume that the norm of u_- in $H_-^{2,\ell;1,k+1}$ is sufficiently small, say, $\leq \eta$. Supposing w_1 and w_2 are also less than η in $H_+^{2,\ell;1,k+1}$-norm, and combining this with Corollary 2.5 and our argument in the preceding paragraph, we find that $N[u_- + w_1] - N[u_- + w_2]$ is a finite number, say, $c(p)$, of terms, each of which is in $H_+^{0,\ell+1;1,k+1}$, with norm in this space bounded by

$$C\|w_1 - w_2\|_{H_+^{2,\ell;1,k+1}} \Big(\|u_-\|_{H_-^{2,\ell;1,k+1}} + \|w_1\|_{H_+^{2,\ell;1,k+1}} + \|w_2\|_{H_+^{2,\ell;1,k+1}}\Big)^{p-1}.$$

Applying $R(\lambda + i0)$, the inverse of P acting boundedly $\mathcal{Y}_+^{0,\ell+1;1,k+1} \to \mathcal{X}_+^{2,\ell;1,k+1}$, the norm of $\Phi(w_1) - \Phi(w_2)$ in $\mathcal{X}_+^{2,\ell;1,k+1}$ is at most $\|w_1 - w_2\|_{H_+^{2,\ell;1,k+1}}$ times $c(p)C\|R(\lambda + i0)\|(3\eta)^{p-1}$.

It follows that provided η is chosen small enough so that $c(p)C\|R(\lambda+i0)\|(3\eta)^{p-1}$ is strictly less than 1, the map Φ is a contraction on the ball of radius η centered at the origin in $\mathcal{X}_+^{2,\ell;1,k+1}$. By the contraction mapping theorem, we deduce the existence of a fixed point $w \in \mathcal{X}_+^{2,\ell;1,k+1}$. In view of the previous discussion, this furnishes us with a nonlinear eigenfunction $u_- + w$.

4.3. Outgoing boundary data. Continuing the proof of Theorem 1.5, we show that w, the fixed point of Φ given incoming data f, has zero incoming boundary data and well-defined outgoing boundary data.

PROPOSITION 4.1. *Let $k > (n-1)/2$, and let $u = u_- + w$ be the nonlinear eigenfunction constructed above given $f \in H^{k+4}(\partial M)$. Then u has an asymptotic expansion at infinity of the form*

$$u = r^{-(n-1)/2}\Big(e^{-i\lambda r} f(y) + e^{i\lambda r} b(y) + O(r^{-\epsilon'})\Big), \quad r \to \infty, \tag{4.12}$$

for some $\epsilon' > 0$, where $b \in H^k(\partial M)$.

Proof. We know that $Pu = N[u] \in H_+^{0,3/4;1,k+1}$, in view of the discussion below (4.8), and have observed that $Pu_- \in H_+^{0,\ell+2;0,k+1}$.

The proof is therefore completed by the following lemma. ∎

LEMMA 4.1. *Let γ be as in Theorem 1.5. Suppose $1/2 - \gamma < \ell < -1/2$, $k > (n-1)/2$, and that $w \in H_+^{2,\ell;1,k+1}$ satisfies the equation*

$$Pw = F, \quad F \in H_+^{0,\ell';0,k}(M) \tag{4.13}$$

for some $\ell' > 1/2$. Then $\lim_{r\to\infty} r^{(n-1)/2} e^{-i\lambda r} w(r,\cdot)$ exists in $H^k(\partial M)$. Letting $b \in H^k(\partial M)$ denote the limit, we have

$$r^{(n-1)/2} e^{-i\lambda r} w(r,\cdot) - b = O(r^{-\epsilon'}) \text{ in } H^k(\partial M), \quad r \to \infty, \tag{4.14}$$

for $\epsilon' > 0$ sufficiently small.

Remark 4.2. Since $H^k(\partial M)$ embeds into $C(\partial M)$, due to the assumption $k > (n-1)/2$, this also shows that we have the asymptotic (4.14) pointwise in $y \in \partial M$.

Proof. It suffices to decompose $w = w_+ + w_-$, where

$$r^{(n-1)/2} e^{-i\lambda r} w_+(r,\cdot) - b = O(r^{-\epsilon'}) \text{ in } H^k(\partial M), \quad r \to \infty,$$

and

$$r^{(n-1)/2} e^{+i\lambda r} w_-(r,\cdot) = O(r^{-\epsilon'}) \text{ in } H^k(\partial M), \quad r \to \infty.$$

We do this by choosing a pseudodifferential cutoff $B \in \Psi_{\mathrm{sc}}^{0,0}$ such that B is microlocally equal to the identity near a neighborhood U of the outgoing radial set $\mathcal{R}_+$ and microlocally equal to zero outside some slightly larger neighborhood U', i.e., for some open neighborhoods $U \subset U'$ of $\mathcal{R}_+$ we have $\mathrm{WF}'(I-B) \cap U = \varnothing, \mathrm{WF}'(B) \subset U'$. Then we set

$$w_+ = Bw, \quad w_- = (\mathrm{Id} - B)w. \tag{4.15}$$

From (4.13) we get

$$Pw_+ = P(Bw) = BF + [P,B]w. \tag{4.16}$$

We claim that the RHS is in $H_+^{0,1/2+\epsilon;0,k}$ for some $\epsilon > 0$. Certainly this is true for the term BF since $F \in H_+^{0,\ell';0,k}$ and $B \in \Psi_{\mathrm{sc}}^{0,0}$. For the term $[P,B]w$, we claim

$$[P,B] = r^{-2}A \tag{4.17}$$

for some $A \in \mathcal{M}_+$. Since $w \in H_+^{2,\ell;1,k+1}$, this would imply $Aw \in H_+^{2,\ell;0,k+1}$, and so $[P,B]w = r^{-2}Aw \in H_+^{2,\ell+2;0,k+1} \subseteq H_+^{0,1/2+\epsilon;0,k}$ for ϵ sufficiently small. But (4.17) follows immediately for U' sufficiently small since $r^2[P,B]$ has order $(1,1)$ and is microsupported in $U' \setminus U$. Therefore it is characteristic at $\mathcal{R}_+$, which is a sufficient condition for an operator of order $(1,1)$ to belong to $\mathcal{M}_+$.

Write $\tilde{w}_+ = \chi(r) r^{(n-1)/2} e^{-i\lambda r} w_+$, where χ is supported in $r > R$ and identically 1 near $r \geq 2R$. We assume R is large enough so that the support of χ is contained in the collar neighborhood $\{x < c\}$ of the boundary. Our first goal is to show that $\tilde{w}_+(r,y)$ has a limit $b(y)$ in $H^k(\partial M)$ and that $\tilde{w}_+(r,y) - b(y) = O_{H^k(\partial M)}(r^{-\epsilon'})$ as $r \to \infty$. To do this, we write the operator P in the form (2.42):

$$P = D_r^2 - i(n-1)r^{-1}D_r + r^{-2}Q + r^{-2}\tilde{Q} + V - \lambda^2, \tag{4.18}$$

where Q is a second order differential operator involving only tangential D_{y_j} derivatives, and $\tilde{Q}$ is a scattering differential operator of order $(1,0)$. We substitute this expression for P into (4.16) and rearrange to obtain

$$\begin{aligned} &\Big(D_r + \lambda\Big)\Big(D_r - \lambda - \frac{i(n-1)}{2r}\Big) w_+ \\ &\quad = BF + [P,B]w + \frac{i(n-1)}{2r^2}(r(D_r - \lambda))w_+ + \Big(\frac{n-1}{2r^2} - r^{-2}Q - r^{-2}\tilde{Q} - V\Big)w_+. \end{aligned} \tag{4.19}$$

Each term on the right side of (4.19) is in $H_+^{0,1/2+\epsilon;0,k}$ for ϵ small enough. For the V term this follows from Lemma 2.4 and (2.36), since $V \in H_{\mathcal{M}_0}^{s,-n/2+\gamma';\kappa,k}$ for all $\gamma' < \gamma$ and any κ, k. For the remaining terms, this follows from (4.13), our observation that $[P,B]w \in H_+^{2,\ell+2;0,k}$, and that w_+ has scattering regularity of order 2, $\mathcal{M}_+$ module regularity of order 1 and $\mathcal{N}$ module regularity of order $k+1$. Moreover, the operator $D_r + \lambda$ is elliptic everywhere except at the set $\{\nu = -\lambda\}$; in particular, it is elliptic on $\mathrm{WF}'(B)$, provided that U' is taken sufficiently small. Thus we may invert this operator microlocally, obtaining

$$\Big(D_r - \lambda - \frac{i(n-1)}{2r}\Big) w_+ \in H^{0,1/2+\epsilon;0,k}. \tag{4.20}$$

Now, observing that

$$D_r \tilde{w}_+ = \chi(r) r^{(n-1)/2} e^{-i\lambda r}\Big(D_r - \lambda - \frac{i(n-1)}{2r}\Big) w_+ + (D_r\chi) r^{(n-1)/2} e^{-i\lambda r} w_+,$$

we find that

$$D_r \tilde{w}_+ \in H^{0,1/2+\epsilon-(n-1)/2;0,k} \Longleftrightarrow D_r \tilde{w}_+ \in r^{n/2-1-\epsilon} L^2\big(([R,\infty), r^{n-1}dr); H^k(\partial M)\big),$$

where we used the support property of $D_r\chi$ for the inclusion in $H^{0,1/2+\epsilon-(n-1)/2;0,k}$ of the second term. We can express this with respect to the measure dr as follows:

$$D_r \tilde{w}_+ \in r^{-1/2-\epsilon} L^2\big(([R,\infty), dr); H^k(\partial M)\big) \subseteq r^{-\epsilon'} L^1\big(([R,\infty), dr); H^k(\partial M)\big). \tag{4.21}$$

We note that, since $\tilde{w}_+$ is locally H^1 in r with values in $H^k(\partial M)$, it is in fact continuous in r with values $H^k(\partial M)$. By (4.21), we can integrate to infinity to find

$$b(y) = \int_R^\infty \partial_{r'} \tilde{w}_+(r', y)\, dr'$$

is well-defined as an element of $H^k(\partial M)$. Moreover,

$$\tilde{w}_+(r, y) - b(y) = -\int_r^\infty \partial_{r'} \tilde{w}_+(r', y)\, dr' = O_{H^k(\partial M)}(r^{-\epsilon'}).$$

A very similar argument can be applied to the w_- term. We define $\tilde{w}_- = \chi(r) r^{(n-1)/2} e^{i\lambda r} w_-$ and compute as above. However, we switch the sign of λ in (4.19) to obtain

$$\begin{aligned} (4.22)\quad & \Big(D_r - \lambda\Big)\Big(D_r + \lambda - \frac{i(n-1)}{2r}\Big) w_- \\ &= (\mathrm{Id} - B)F - [P, B]w + \frac{i(n-1)}{2r^2}(r(D_r + \lambda)) w_- + \Big(\frac{n-1}{2r^2} - r^{-2}Q - r^{-2}\tilde{Q} - V\Big) w_-. \end{aligned}$$

Note that $w_- = (I - B)w \in H_+^{2,\ell;1,k+1}$ since B has scattering order $0, 0$ and $I - B$ is characteristic on the outgoing radial set. Hence the fourth term on the RHS lies in $H_+^{0,1/2+\epsilon;0,k}$. Moreover, for the third term, $r(D_r + \lambda)w_- = r(D_r + \lambda)(I - B)w \in H^{2,\ell;0,k+1}$ since $r(D_r + \lambda)(I - B) \in \mathcal{M}_+$ (it is characteristic on the outgoing radial set.) Hence the third term on the the right side belongs to $H^{0,1/2+\epsilon;0,k}$, and so as in the analysis of w_+ above we conclude that the whole of the RHS lies in this space. We may assume that $D_r - \lambda$ is elliptic on the microsupport of $\mathrm{Id} - B$, so we may invert it microlocally to obtain

$$(4.23)\qquad \Big(D_r + \lambda - \frac{i(n-1)}{2r}\Big) w_- \in H_+^{0,1/2+\epsilon;0,k}.$$

Now, observing that

$$D_r \tilde{w}_- = r^{(n-1)/2} e^{i\lambda r}\Big(D_r + \lambda - \frac{i(n-1)}{2r}\Big) w_- + (D_r \chi) r^{(n-1)/2} e^{i\lambda r} w_-,$$

we find that

$$D_r \tilde{w}_- \in r^{-1/2-\epsilon} L^2\big(([R, \infty), dr); H^k(\partial M)\big) \subset r^{-\epsilon'} L^1\big(([R, \infty), dr); H^k(\partial M)\big).$$

The rest of the argument can be followed to obtain a limit

$$b_-(y) = \lim_{r \to \infty} \tilde{w}_-(r, y)$$

in $H^k(\partial M)$ with

$$\tilde{w}_-(r, y) - b_-(y) = O_{H^k(\partial M)}(r^{-\epsilon'}).$$

This is only possible if b_- vanishes identically. Indeed, otherwise w_- would fail to be in $H^{2,-1/2}$; however, since $\mathrm{WF}'(\mathrm{Id} - B) \cap \mathcal{R}_+ = \varnothing$ and the module $\mathcal{M}_+$ is elliptic off $\mathcal{R}_+$, $(\mathrm{Id} - B)w \in H^{2,\ell+1}$. This completes the proof of the lemma. ∎

4.4. Uniqueness. To complete the proof of Theorem 1.5, we show that the solution u obtained above is unique in the following sense.

PROPOSITION 4.3. *Suppose that* (1.7) *is satisfied, that the nonlinearity* N *satisfies the conditions of Theorem* 1.5, *and that* f *is sufficiently small in* $H^{k+4}(\partial M)$, *so that the proof above of the existence of a nonlinear eigenfunction* u *with incoming data* f *is valid. Let* u_- *be given in terms of* f *by* (4.11).

Then the solution u *is unique in the following sense. Let* u_1, u_2 *satisfy* $Pu_i = N[u_i]$ *and assume* $u_i - u_- = w_i$ *both lie in* $H_+^{2,\ell;1,k+1}$. *Then there exists* $\eta > 0$ *such that*

$$\|f\|_{H^{k+4}(\partial M)}, \|w_1\|_{H_+^{2,\ell;1,k+1}}, \|w_2\|_{H_+^{2,\ell;1,k+1}} < \eta \implies u_1 = u_2.$$

Proof. Let u_1 and u_2 be nonlinear eigenfunctions as in the proposition. It suffices to show that the corresponding w_i are both fixed points of the map Φ, since a contraction map has only one fixed point.

We first note that the w_i are in $\mathcal{X}_+^{2,\ell;1,k+1}$. Indeed, recall that $u_- \in H_-^{2,\ell;1,k+1}$, and w_i is by assumption in $H_+^{2,\ell;1,k+1}$, so, as we saw above, this means that $N[u_i] = N[u_- + w_i] \in H_+^{0,\ell+1;1,k+1}$. Then, because $Pu_- \in H_+^{0,\ell+1;1,k+1}$, we have $Pw_i = -Pu_- + N[u_i] \in H_+^{0,\ell+1;1,k+1}$. This confirms that $w_i \in \mathcal{X}_+^{2,\ell;1,k+1}$.

Next, from

$$P(u_- + w_i) = N[u_i],$$

we apply $R(\lambda + i0)$ and note that $R(\lambda + i0)Pw_i = w_i$ since $w_i \in \mathcal{X}_+^{2,\ell;1,k+1}$, while by definition $R(\lambda + i0)Pu_- = -u_+$. Therefore,

$$-u_+ + w_i = R(\lambda + i0)N[u_- + w_i]$$

and this rearranges to $\Phi(w_i) = w_i$ for each i. Since Φ has a unique fixed point, the proof is complete. ∎

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