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Transverse stability of strongly nonlinear ion acoustic modes

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Abstract

The transverse stability of strongly nonlinear ion-acoustic structures, described by the general Sagdeev potential, is investigated. It is shown, that the multi-scale stability analysis by [Sov. Phys. Dokl. 15 (1970) 539] for solitary waves, described by a weakly dispersive and weakly nonlinear equation, can be generalized for strongly nonlinear systems. The stability of plasma ion sheath formed at the plasma-boundary interface and the double-layer formed in a two-component plasma has been specifically studied and dispersion relations for transverse ion oscillations have been obtained.

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1. Introduction

Ion-acoustic solitons, double layers, and collisionless plasma sheaths formed at the plasma-boundary interfaces represent strongly nonlinear structures associated with ion-acoustic modes. All of these can be viewed as particular solutions to a more general nonlinear ion-sound wave problem. While ion-acoustic solitons and sheaths occur in a single species electron–ion plasma, double layers require for their existence plasmas several distinct plasma components [2,22–24]. Nonlinear ion-sound waves have attracted a great deal of interest in the last decades

due to their importance in various space and laboratory plasmas [5,6,15,23]. Ion-acoustic shock waves, solitons, and double layers are believed to be important mechanisms of particle acceleration in space plasmas [3,18]. Plasma sheath phenomena are also of great interest for space and technological plasmas [2,16,24].

Nonlinear ion-sound waves have been studied by a number of authors. One of the important issues is the problem of stability of strongly nonlinear structures, since only nonlinearly stable formations can be dynamically relevant. The question of stability with respect to the transverse perturbations is of particular interest, e.g., the stability of the wave-front of a one-dimensional shock with respect to transverse oscillations. Transverse instabilities may result in the wave break-down and self-focusing [8]. It may also indicate that two- or three-dimensional nonlinear structures are more relevant [2].

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One of the very first studies of the two-dimensional stability of nonlinear waves was done by [7] for a special case of a solitary wave described by the now famous Kadomtsev–Petviashvili equation. This approach was further developed by [1,8] and applied to the stability analysis of periodic waves [11,12].

Most of the previous studies of the transverse stability of nonlinear waves consider weakly nonlinear, weakly dispersive situations that are described by various generalizations of the Korteweg–de Vries equation (such as Kadomtsev–Petviashvili or Zakharov–Kuznetsov equations [8,10]). On other hand, a number of nonlinear structures such as double layers and sheaths are strongly nonlinear and, in general, are not described by these equations. In this Letter, we generalize the procedure of [7] for a generic case of the so-called Sagdeev potential [17]). Solitary waves, sheaths, and double layers are particular solutions corresponding to various branches of such a potential. We analyze the transverse stability of the nonlinear plasma sheath and a double layer at the plasma-wall boundary and obtain the dispersion equations for transverse long-wavelength and low-frequency waves.

2. Ion acoustic waves in fluid approximation

We study ion dynamics in the simplest fluid form, neglecting ion thermal and kinetic effects as well as electron inertia and collisional effects. Cold collisionless ions are described by the following continuity and momentum conservation equations,

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{u} = 0, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{e}{M} \nabla \varphi = 0. \quad (2)$$

Electrons are also considered collisionless and having finite temperature such that $v_{te} \gg \omega/k$. The electron thermal velocity is considered to be much larger than any directed electron flow velocity, $v_{te} \gg u$. Thus, the electron momentum equation reads:

$$e \nabla \varphi - \frac{1}{n} \nabla (n T_e) = 0. \quad (3)$$

The simple Boltzmann relation for electrons, that relates density and potential, can be derived,

$$n = n_0 \exp\left(\frac{e\varphi}{T_e}\right). \quad (4)$$

The description of the ion dynamics is then completed by Poisson's equation

$$\Delta \varphi = 4\pi e (n_0 \exp(\varphi) - n). \quad (5)$$

To lighten notation, we use the following normalizations in the remaining sections. The potential is taken in units T_e/e , distances in $\lambda_{De} = \sqrt{T_e/4\pi e^2 n_0}$, velocity in units of sound velocity $c_s = \sqrt{T_e/M_i}$ and time in units of the ion plasma oscillations period $\omega_p = \sqrt{4\pi e^2 n_0/M_i}$.

3. Stationary solutions

Our basic structures of interest are strongly nonlinear, stationary solutions of Eqs. (1)–(5) which are obtained by setting all time derivatives to zero. Density, flow and potential of the equilibrium system are denoted by capital letters. Furthermore, a plane geometry is considered; all inhomogeneities are in z -direction. The equations describing the stationary state are:

$$\frac{\partial NU}{\partial z} = 0, \quad (6)$$

$$\frac{\partial}{\partial z} \left(\frac{U^2}{2} + \Phi \right) = 0, \quad (7)$$

$$\frac{\partial^2 \Phi}{\partial z^2} - \exp(\Phi) + N = 0. \quad (8)$$

These equations are written in the form typical for the sheath solution at plasma boundary $z = 0$. Then, the boundary conditions are defined by quasineutrality condition in the bulk plasma at $z = \infty$. At $z = \infty$ the potential vanishes, $\Phi = 0$, the normalized density is 1 and the normalized flow velocity is a which is the Mach number of the system and is a free parameter in this study. For a sheath to exist the requirement is $a > 1$ (Bohm criteria). For a simple discussion see, for example, the book of Chen [4]. We solve for U and N ,

$$U = \sqrt{a^2 - 2\Phi}, \quad N = 1 / \sqrt{1 - \frac{2\Phi}{a^2}}. \quad (9)$$

Poisson's equation can be integrated once using these boundary conditions to yield:

$$\frac{1}{2} \Phi_z^2 = [\exp \Phi + a(U - a)], \quad (10)$$

where $\Phi_z = \partial \Phi / \partial z$.

Fig. 1. Sagdeev potential for single ion and single electron species. The same analysis as for soliton regime (right half) can be applied to left half of Sagdeev potential.

In analogy to classical mechanics where the equation

$$\frac{1}{2} \left(\frac{dx}{dt} \right)^2 - V(x) = 0, \quad (11)$$

describes a particle moving in a potential V , one can think of Φ as of a coordinate of the particle moving in a general potential $\Psi(\Phi)$ and z is playing the role of time. The function $\Psi(\Phi)$ is defined by the r.h.s. of Eq. (10) and usually called Sagdeev potential [17]. The form of the Sagdeev potential allows to make qualitative statements about the solutions.

Fig. 1 shows the Sagdeev potential of Eq. (10). We can see that for $\Phi > 0$ bound “orbits” exist. If the potential starts at zero and moves to the right, it will reach a maximum at some Φ_m where Ψ crosses zero and Φ will be reflected and return to zero and stay there. This corresponds to the soliton solution. For the sheath plasma problem that we consider in this chapter the potential Φ is always negative. It starts at zero in the bulk and stops at some negative value when it reaches the boundary. This negative value can be a fixed boundary potential or the floating potential which depends on the electron temperature.

4. Perturbative solutions to sheath oscillations

For the transverse stability analysis, we seek a perturbative solution to the full set of equations. The wave vector k is in the direction perpendicular to z (in the x, y -plane). The stationary functions U, N, Φ are independent of x, y and t , the perturbed quantities are proportional to $\exp(-i\omega t + i\mathbf{k} \cdot \mathbf{x})$. Similar to standard

analysis, we restrict ourselves to long-wavelengths, low-frequency perturbations, so that the following ordering applies $k^2 \sim \omega^2 \ll \omega$ [7]. In this ordering, the wavelength of transverse perturbations is much larger than the characteristic length scale of the basic equilibrium. Therefore the perturbation theory approach is possible. All parameters are expanded to third order:

$$u = u_1 + u_2 + u_3, \quad (12)$$

$$n = n_1 + n_2 + n_3, \quad (13)$$

$$\phi = \phi_1 + \phi_2 + \phi_3. \quad (14)$$

The dispersion equation to the $O(k^2)$ order is obtained as a solubility condition for the third order equation, which is the highest order considered in this Letter. In the soliton stability problem, higher order corrections can be found by the methods of the inverse spectral theory [9]. The latter approach is not possible for strongly nonlinear Eqs. (1)–(5). The dynamical equations for the perturbations are written as follows:

$$-i\omega n + \frac{\partial}{\partial z}(Un + Nu) = 0, \quad (15)$$

$$-i\omega u + \frac{\partial}{\partial z}(Uu + \phi) = 0, \quad (16)$$

$$\frac{\partial^2 \phi}{\partial z^2} - \exp(\Phi)\phi + n - k^2\phi = 0. \quad (17)$$

In the following, the analytic solutions, that we find for the first two orders are derived. First order:

$$\frac{\partial}{\partial z}(Un_1 + Nu_1) = 0, \quad (18)$$

$$\frac{\partial}{\partial z}(Uu_1 + \phi_1) = 0, \quad (19)$$

$$\frac{\partial^2 \phi_1}{\partial z^2} - \exp(\Phi)\phi_1 + n_1 = 0. \quad (20)$$

This system can be solved exactly:

$$u_1 = -\frac{\phi_1}{U}, \quad n_1 = \frac{N}{U^2}\phi_1, \quad (21)$$

$$\frac{\partial \phi_1}{\partial z^2} - \left(\exp(\Phi) - \frac{N}{U^2} \right) \phi_1 = 0, \quad (22)$$

with the solutions:

$$\phi_1 = \Phi_z, \quad u_1 = -\frac{\Phi_z}{U}, \quad n_1 = \frac{N}{U^2}\Phi_z. \quad (23)$$

The second order equations take the form:

$$\begin{aligned} \frac{\partial}{\partial z}(Un_2 + Nu_2) \\ = i\omega \frac{N}{U^2} \Phi_z = i\omega \frac{\partial}{\partial z} \frac{1}{\sqrt{1 - \frac{2\Phi}{a^2}}} = i\omega \frac{\partial N}{\partial z}, \end{aligned} \quad (24)$$

$$\frac{\partial}{\partial z}(Uu_2 + \phi_2) = -i\omega \frac{\Phi_z}{U} = i\omega \frac{\partial U}{\partial z}, \quad (25)$$

$$\frac{\partial^2 \phi_2}{\partial z^2} - \exp(\Phi) \phi_2 + n_2 = 0. \quad (26)$$

Solving this system leads to:

$$\begin{aligned} u_2 &= -\frac{\phi_2}{U} + i\omega \frac{U-a}{U}, \\ n_2 &= \frac{N}{U^2} \phi_2 + i\omega \frac{N}{U^2} \frac{2\Phi}{a}, \end{aligned} \quad (27)$$

$$\frac{\partial \phi_2}{\partial z^2} - \left(\exp(\Phi) - \frac{N}{U^2} \right) \phi_2 = -i\omega \frac{N}{U^2} \frac{2\Phi}{a}. \quad (28)$$

Again an exact solutions can be written down:

$$\begin{aligned} \phi_2 &= i\omega \Phi_a, \\ u_2 &= -i\omega \frac{\Phi_a}{U} + i\omega \frac{U-a}{U}, \\ n_2 &= i\omega \frac{N}{U^2} \Phi_a + i\omega \frac{N}{U^2} \frac{2\Phi}{a}. \end{aligned} \quad (29)$$

In order to calculate a dispersion relation $\omega(k)$, we need to expand to the third order which introduces a term $\sim k^2$. The third order equations are:

$$\begin{aligned} \frac{\partial}{\partial z}(Un_3 + Nu_3) \\ = i\omega \frac{N}{U^2} \Phi_z = -\omega^2 \frac{N}{U^2} \left(\Phi_a + \frac{2\Phi}{a} \right), \end{aligned} \quad (30)$$

$$\frac{\partial}{\partial z}(Uu_3 + \phi_3) = -\frac{\omega^2}{U} (U-a-\Phi_a), \quad (31)$$

$$\frac{\partial^2 \phi_3}{\partial z^2} - \exp(\Phi) \phi_3 + n_3 = -k^2 \Phi_z. \quad (32)$$

At this order, solutions can be written in integral forms:

$$u_3 = -\frac{\phi_3}{U} + \frac{\omega^2}{U} \int_z^\infty \frac{U-a-\Phi_a}{U} dz, \quad (33)$$

$$\begin{aligned} n_3 &= \frac{N}{U^2} \phi_3 - \frac{N\omega^2}{U^2} \int_z^\infty \frac{U-a-\Phi_a}{U} dz \\ &\quad + \frac{\omega^2}{U} \int_z^\infty \frac{N}{U^2} \left(\Phi_a + \frac{2\Phi}{a} \right) dz, \end{aligned} \quad (34)$$

$$\frac{\partial^2 \phi_3}{\partial z^2} - \left(\exp(\Phi) - \frac{N}{U^2} \right) \phi_3 = -k^2 \Phi_z + h(z), \quad (35)$$

where

$$\begin{aligned} h(z) &= +\frac{N\omega^2}{U^2} \int_z^\infty \frac{U-a-\Phi_a}{U} dz \\ &\quad - \frac{\omega^2}{U} \int_z^\infty \frac{N}{U^2} \left(\Phi_a + \frac{2\Phi}{a} \right) dz. \end{aligned} \quad (36)$$

Earlier we identified Φ_z as a solution of the homogeneous part of Eq. (35). If we substitute the ansatz $\phi_3 = \Phi_z Z$ into (35), we obtain

$$\Phi_z Z_{zz} + 2\Phi_{zz} Z_z = h(z) - k^2 \Phi_z, \quad (37)$$

or

$$\frac{d}{dz} (\Phi_z^2 Z_z) = h(z) \Phi_z - k^2 \Phi_z^2. \quad (38)$$

After integration, solving for Z and multiplying by Φ_z , we can write down a general solution to Eq. (35)

$$\begin{aligned} \phi_3 &= C_1 \Phi_z + C_2 \Phi_z \int_z^\infty \Phi_z^{-2} dz \\ &\quad - \Phi_z \int_z^\infty \Phi_z'^{-2} dz' \int_{z'}^\infty (h(z'') \Phi_{z''} - k^2 \Phi_{z''}^2) dz''. \end{aligned} \quad (39)$$

5. Stability of transverse oscillations in simple sheath

In this section, we outline the steps of the stability analysis, which is done in analogy to the one performed within the Kadomtsev–Petviashvili equation [7,15]. Eq. (35) can be written as

$$L\phi_3 = -k^2 \Phi_z + h(\Phi). \quad (40)$$

This equation is inhomogeneous and, in general, does not have finite solutions for arbitrary terms on the

right-hand side. After multiplying by Φ_z —a solution to the left-hand side—and integrating over z , the left-hand side vanishes. The right-hand side yields a dispersion relation for small transverse oscillations:

$$\int_0^\infty dz (-k^2 \Phi_z + h(z)) \Phi_z = 0, \quad (41)$$

which can be transformed into an integral in terms of Φ with the substitution $d\Phi = \Phi_z dz$ and $\Phi(\infty) = 0$ and $\Phi(0) = \phi_0$,

$$\int_{\phi_0}^0 d\Phi (-k^2 \Phi_z + h(\Phi)) = 0, \quad (42)$$

where $h(\Phi)$ is proportional to ω^2 (Eq. (36)). The dispersion equation describes transverse eigenmodes of the sheath. For the parameters considered in the Letter, these eigenmodes are stable. Unstable modes are in general possible and may lead to sheath turbulence [13]. Note that in this Letter we do not consider the longitudinal oscillations and stability of the sheath, i.e., modes in z -direction, therefore assuming that such modes are stable for simple sheath structures considered here. The longitudinal modes can be destabilized in more complex cases, such as sheaths with strong ion or electron flows [14,21], but these are outside of scope of the present Letter.

5.1. Solving for $\omega(k)$

To determine $\omega(k)$, we first numerically solve Poisson's equation for the stationary potential Φ . The boundary conditions used for the numerical solution are $\Phi(z=0) = \Phi_0$ and $\Phi(z=z_{\max}) = 0$ where $z_{\max}$ is a value chosen sufficiently large so that variation of it does not change the solution significantly. Then Eq. (42) is solved for $v^2 = \omega^2/k^2$.

It turns out that for positive ions the dispersion is always positive and therefore stable. Fig. 2 shows the phase velocity as a function of the Mach number a . The solution suggests that in this simple configuration the phase velocity of transverse oscillations is approximately proportional to a^2 , i.e., $\omega/k \propto a^2$, and always stable.

Fig. 2. Phase velocity in Mach numbers for single ion species and single electron species.

6. Double layer with two Boltzmann species

As a simple extension of the model a second electron species is introduced into the system. In an ionospheric context, these may originate from higher regions, travelling down earth magnetic field lines into the regions with colder electrons [18,19]. It may also be a simple approximation to the situation in laser heated plasmas where a small group of very fast electrons, created by parametric excitation of electron waves, travel ahead of the heat front into colder plasma regions [20].

With the second electron species, which can also be expressed by a Boltzmann distribution, two additional parameters are added to the system. One is the relative temperature T between the two electron species and the other is the density n with respect to the ion background density so as to maintain quasineutrality. The ion fluid equations remain the same and the only change is to Poisson's equation

$$\Delta \Phi = n \exp(\Phi) + (1 - n) \exp\left(\frac{\Phi}{T}\right) - N. \quad (43)$$

The normalization of the variables is chosen in the usual manner. An effective temperature T_{eff} is defined such that a global Debye length Λ_D^2 is given by

$$\frac{1}{\Lambda_D^2} = \frac{1}{\lambda_{D1}^2} + \frac{1}{\lambda_{D2}^2}, \quad (44)$$

Fig. 3. For two Boltzmann species supersonic double layer become possible with $a = 4.12$, $T = 19.45$.

Fig. 4. The transition to double layer $a = 4$, 4.0002, 4.0004 (solid, crosses, squares). Double layer solution for $a = 4$.

where λ_{D1} and λ_{D2} are the respective Debye lengths of the two electron species. Velocities are normalized to the linear phase velocity $C_s = \sqrt{\frac{ZT_{\text{eff}}}{M_i}}$, and the potential is measured in units of T_{eff}/e . The change to the Sagdeev potential due to the addition of the second electron species follows directly. For certain combinations of the parameters a, n, T so-called double layer solutions are possible. The Sagdeev potential for a specific double layer is shown in Fig. 3.

If we invoke the analogy with a particle (Φ is a distance) moving in the potential Ψ again the following solution emerges. Φ starts at zero in the bulk and moves to a negative potential ($\Phi = -1$) at the other boundary of the system where it remains because of zero derivative in Ψ . This is known as the double-

Fig. 5. Phase velocities for density at which double layer occurs at $a = 4$ (lower line) and twice that density of colder electrons (upper line).

layer. With the aforementioned change in Poisson's equation the procedure for the stability analysis can be repeated analogously to the single electron species case. It should be noted that for a double layer potential to occur large sheath regions have to be used. A typical sheath for a single electron species is just a few λ_D wide whereas a double layer only occurs if we allow for a transition region of several tens to few hundreds of λ_D . As a increases the solution smoothly transforms into a normal sheath potential. This occurs within a fractional change of a . Fig. 4 shows the solution to Poisson's equation for the stationary potential for three values of a . The value $a = 4$ corresponds to the exact double layer solution. We can see that the transition to a simple sheath happens for a fractional change in a of only 10^{-4} .

6.1. Dispersion for two Boltzmann species

The dispersion relation is again computed numerically for different values of a . For comparison it is solved for a different relation of the two electron species where only simple sheath solutions exist. Fig. 5 shows the phase velocities for the two cases. It should be noted that for two electron species also the Bohm criterion for the occurrence of a sheath is modified [16,23]. For the case of a double layer at $a = 4$ no solution exists for $a < 4$ due to that Bohm criterion.

Close to the double layer solution $a = 4$ stable transverse oscillations persist, but with phase velocities much slower than the typical ion sound velocity in the bulk. This indicates the possibility of oscillation phenomena that occur on time scales even slower than the ion sound waves.

7. Summary

The transverse stability of strongly nonlinear ion sound structures, sheath and double layer, was investigated. We have extended the stability analysis developed earlier for solitary waves to other regimes in the general Sagdeev potential. We have found the dispersion relation for transverse ion oscillations in a plasma with one and two electron species. The case of a single electron and ion species forming a plasma sheath gives stable transverse ion sound oscillations with dispersion $\omega/k \propto a^2$. By introducing a second Boltzmann electron species with well-chosen density and temperature we create a double-layer potential. With the potential approaching the double-layer form the phase velocity of transverse ion oscillations approaches zero.

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