QUANTUM GEOMETRY OF 3-DIMENSIONAL LATTICES AND TETRAHEDRON EQUATION[†]

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Abstract

We study geometric consistency relations between angles of 3-dimensional (3D) circular quadrilateral lattices — lattices whose faces are planar quadrilaterals inscribable into a circle. We show that these relations generate canonical transformations of a remarkable "ultra-local" Poisson bracket algebra defined on discrete 2D surfaces consisting of circular quadrilaterals. Quantization of this structure allowed us to obtain new solutions of the tetrahedron equation (the 3D analog of the Yang-Baxter equation) as well as reproduce all those that were previously known. These solutions generate an infinite number of non-trivial solutions of the Yang-Baxter equation and also define integrable 3D models of statistical mechanics and quantum field theory. The latter can be thought of as describing quantum fluctuations of lattice geometry.

Keywords: Quantum geometry, discrete differential geometry, integrable quantum systems, Yang-Baxter equation, tetrahedron equation, quadrilateral and circular 3D lattices.

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1 Introduction

Quantum integrability is traditionally understood as a *purely algebraic* phenomenon. It stems from the Yang-Baxter equation [1, 2] and other algebraic structures such as the affine quantum groups [3, 4] (also called the quantized Kac-Moody algebras), the Virasoro algebra [5] and their representation theory. It is, therefore, quite interesting to learn that these algebraic structures also have remarkable geometric origins [6], which will be reviewed here.

Our approach [6] is based on connections between integrable *three-dimensional* (3D) quantum systems and integrable models of 3D discrete differential geometry. The analog of the Yang-Baxter equation for integrable quantum systems in 3D is called the *tetrahedron equation.* It was introduced by Zamolodchikov in [7, 8] (see also [9–16] for further important results in this field). Similarly to the Yang-Baxter equation the tetrahedron equation provides local integrability conditions which are not related to the size of the lattice. Therefore the same solution of the tetrahedron equation defines different integrable models on lattices of different size, e.g., for finite periodic cubic lattices. Obviously, any such three-dimensional model can be viewed as a two-dimensional integrable model on a square lattice, where the additional third dimension is treated as an internal degree of freedom. Therefore every solution of the tetrahedron equation provides an infinite sequence of integrable 2D models differing by the size of this "hidden third dimension" [9]. Then a natural question arises whether known 2D integrable models can be obtained in this way. Although a complete answer to this question is unknown, a few non-trivial examples of such correspondence have already been constructed. The first example [11] reveals the 3D structure of the generalized chiral Potts model [17, 18]. Another example [16] reveals 3D structure of all two-dimensional solvable models associated with finite-dimensional highest weight representations for quantized affine algebras $U_q(\widehat{sl}_n)$, $n = 2, 3, \ldots, \infty$ (where n coincides with the size of the hidden dimension).

Here we unravel yet another remarkable property of the same solutions of the tetrahedron equation (in addition to the hidden 3D structure of the Yang-Baxter equation and quantum groups). We show that these solutions can be obtained from quantization of geometric integrability conditions for the 3D *circular lattices* — lattices whose faces are planar quadrilaterals inscribable into a circle.

The 3D circular lattices were introduced [19] as a discretization of orthogonal coordinate systems, originating from classical works of Lamé [20] and Darboux [21]. In the continuous case such coordinate systems are described by integrable partial differential equations (they are connected with the classical soliton theory [22, 23]). Likewise, the quadrilateral and circular lattices are described by integrable difference equations. The key idea of the geometric approach [19, 24–31] to integrability of discrete classical systems is to utilize various consistency conditions [32] arising from geometric relations between elements of the lattice. It is quite remarkable that these conditions ultimately reduce to certain incidence theorems of elementary geometry. For instance, the integrability conditions for the quadrilateral lattices merely reflect the fact of existence of the 4D Euclidean cube [25]. In Sect.2 we present these conditions algebraically in a standard form of the *functional tetrahedron equation* [15]. The latter serves as the classical analog of the quantum tetrahedron equation, discussed above, and provides a connecting link to integrable quantum systems.

In Sect. 3.1 we study relations between edge angles on the 3D circular quadrilateral lattices and show that these relations describe symplectic transformations of a remarkable "ultra-local" Poisson algebra on quadrilateral surfaces (see Eq.(21)). Quantization of this structure allows one to obtain *all currently known solutions* of the tetrahedron equation. They are presented in Sect. 4, namely[1],

(I) Zamolodchikov-Bazhanov-Baxter solution [8, 11, 13, 34]

(II) Bazhanov-Sergeev solution [16]

(III) Bazhanov-Mangazeev-Sergeev solution [6]

including their interaction-round-a-cube and vertex forms. Additional details on the corresponding solvable 3D models, in particular, on their quasi-classical limit and connections with geometry, can be found in [6].

2 Discrete differential geometry: "Existence as integrability"

In this section we consider classical discrete integrable systems associated with the quadrilateral lattices. There are several ways to extract algebraic integrable systems from the geometry of these lattices. One approach, developed in [25, 27, 35–37], leads to discrete analogs of the Kadomtsev-Petviashvili integrable hierarchy. Here we present a different approach exploiting the angle geometry of the 3D quadrilateral lattices.

2.1 Quadrilateral lattices

Consider three-dimensional lattices, obtained by embeddings of the integer cubic lattice $\mathbb{Z}^3$ into the N-dimensional Euclidean space $\mathbb{R}^N$, with $N \geq 3$. Let $\boldsymbol{x}(m) \in \mathbb{R}^N$, denote coordinates of the lattice vertices, labeled by the 3-dimensional integer vector $m = m_1 e_1 + m_2 e_2 + m_3 e_3 \in \mathbb{Z}^3$, where $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Further, for any given lattice vertex $\boldsymbol{x}_0 = \boldsymbol{x}(m)$, the symbols $\boldsymbol{x}_i = \boldsymbol{x}(m + e_i)$, $\boldsymbol{x}_{ij} = \boldsymbol{x}(m + e_i + e_j)$, etc., will denote neighboring lattice vertices. The lattice is called *quadrilateral* if all its faces $(\boldsymbol{x}_0, \boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_{ij})$ are planar quadrilaterals. The existence of these lattices is based on the following elementary geometry fact (see Fig. 1) [25],

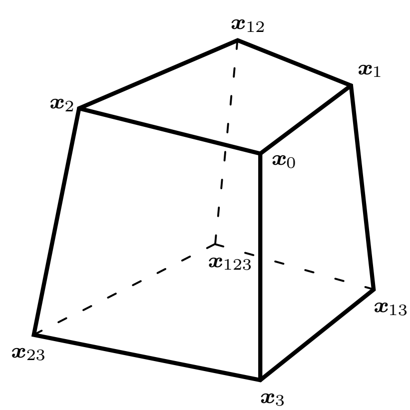

Figure 1: An elementary hexahedron of a cubic quadrilateral lattice.

Consider four points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ in general position in $\mathbb{R}^N$, $N \geq 3$. On each of the three planes $(\boldsymbol{x}_0, \boldsymbol{x}_i, \boldsymbol{x}_j)$, $1 \leq i < j \leq 3$ choose an extra point $\boldsymbol{x}_{ij}$ not lying

[1] The other known solutions, previously found by Hietarinta [33] and Korepanov [14], were shown to be special cases of [34].

on the lines $(\boldsymbol{x}_0, \boldsymbol{x}_i)$*,* $(\boldsymbol{x}_0, \boldsymbol{x}_j)$ *and* $(\boldsymbol{x}_i, \boldsymbol{x}_j)$*. Then there exist a unique point* $\boldsymbol{x}_{123}$ *which simultaneously belongs to the three planes* $(\boldsymbol{x}_1, \boldsymbol{x}_{12}, \boldsymbol{x}_{13})$*,* $(\boldsymbol{x}_2, \boldsymbol{x}_{12}, \boldsymbol{x}_{23})$ *and* $(\boldsymbol{x}_3, \boldsymbol{x}_{13}, \boldsymbol{x}_{23})$*.*

The six planes, referred to above, obviously lie in the same 3D subspace of the target space. They define a hexahedron with quadrilateral faces, shown in Fig. 1. It has the topology of the cube, so we will call it "cube", for brevity. Let us study elementary geometry relations among the angles of this cube. Denote the angles between the edges as in Fig. 2. Altogether we have

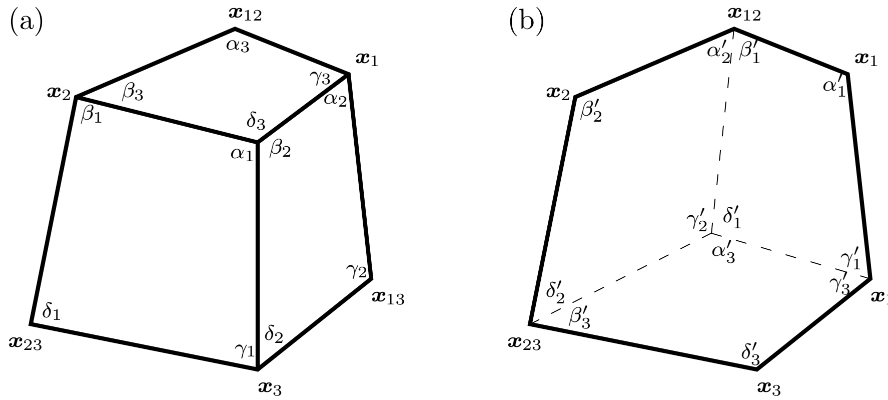

Figure 2: The "front" (a) and "back" (b) faces of the cube in Fig. 1 and their angles.

$6 \times 4 = 24$ angles, connected by six linear relations

$$\alpha_j + \beta_j + \gamma_j + \delta_j = 2\pi, \qquad \alpha'_j + \beta'_j + \gamma'_j + \delta'_j = 2\pi, \qquad j = 1, 2, 3, \tag{1}$$

which can be immediately solved for all "δ's". This leaves 18 angles, but only nine of them are independent. Indeed, a mutual arrangement (up to an overall rotation) of unit normal vectors to six planes in the 3D-space is determined by nine angles only. Once this arrangement is fixed all other angles can be calculated. Thus the nine independent angles of the three "front" faces of the cube, shown in Fig.2a, completely determine the angles on the three "back" faces, shown in Fig.2b, and vice versa. So the geometry of our cube provides an invertible map for three triples of independent variables

$$\mathcal{R}_{123}: \qquad \{\alpha_j, \beta_j, \gamma_j\} \to \{\alpha'_j, \beta'_j, \gamma'_j\}, \qquad j = 1, 2, 3. \tag{2}$$

Suppose now that all angles are known. To completely define the cube one also needs to specify lengths of its three edges. All the remaining edges can be then determined from simple linear relations. Indeed, the four sides of every quadrilateral are constrained by two relations, which can be conveniently presented in the matrix form

$$\begin{pmatrix} \ell'_p \\ \ell'_q \end{pmatrix} = X \begin{pmatrix} \ell_p \\ \ell_q \end{pmatrix}, \qquad X = \begin{pmatrix} A(\mathcal{A}) & B(\mathcal{A}) \\ C(\mathcal{A}) & D(\mathcal{A}) \end{pmatrix} = \begin{pmatrix} \frac{\sin\gamma}{\sin\delta} & \frac{\sin(\delta+\beta)}{\sin\delta} \\ \frac{\sin(\delta+\gamma)}{\sin\delta} & \frac{\sin\beta}{\sin\delta} \end{pmatrix} \tag{3}$$

where $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$ denotes the set of angles and $\ell_p, \ell_q, \ell_{p'}, \ell_{q'}$ denote the edge lengths, arranged as in Fig.3. Note that due to (1) the entries of the two by two matrix in (3) satisfy the relation

$$AD - BC = (AB - CD)/(DB - AC). \tag{4}$$

Assume that the lengths ℓ_p, ℓ_q, ℓ_r, on one side of the two pictures in Fig.4 are given. Let us find the other three lengths $\ell_{p'}, \ell_{q'}, \ell_{r'}$ on their opposite side, by iterating the relation (3). Obviously, this can be done in two different ways: either using the front three faces, or the back ones — the results must be the same. This is exactly where the geometry gets into play. The results must be consistent due to the very existence of the cube in Fig. 1 as a geometric body. However, they will be consistent only if all geometric relations between the two sets of angles

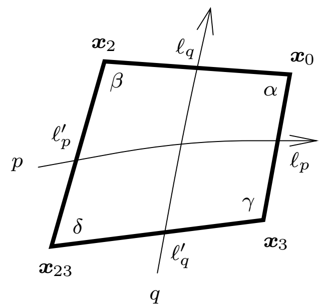

Figure 3: The angles $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$ and sides $\ell_p, \ell_q, \ell_{p'}, \ell_{q'}$ of a quadrilateral and the oriented rapidity lines.

in the front and back faces of the cube are taken into account. To write these relations in a convenient form we need to introduce additional notations. Note, that Fig.3 shows two thin lines, labeled by the symbols "p" and "q". Each line crosses a pairs of opposite edges, which we call "corresponding" (in the sense that they correspond to the same thin line). Eq.(3) relates the lengths (ℓ_p, ℓ_q) of two adjacent edges with the corresponding lengths (ℓ'_p, ℓ'_q) on the opposite side of the quadrilateral.

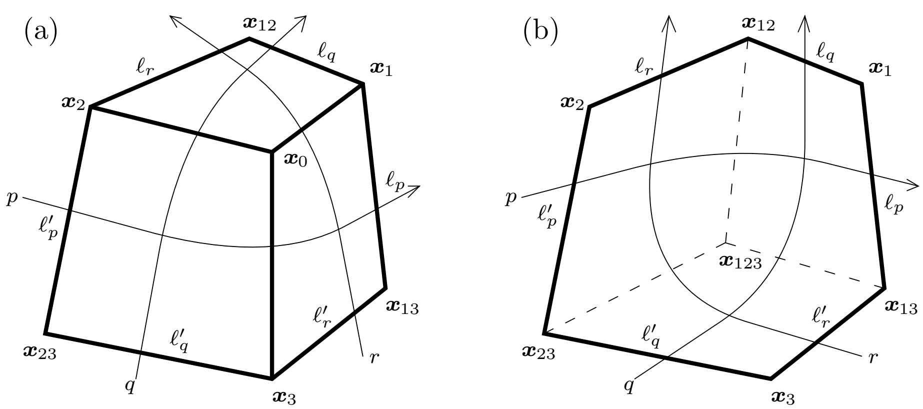

Figure 4: The "front" (a) and "back" (b) faces of the cube in Fig. 1 and "rapidity" lines.

Consider now Fig.4a which contains three directed thin lines connecting corresponding edges of the three quadrilateral faces. By the analogy with the 2D Yang-Baxter equation, where similar arrangements occur, we call them "rapidity" lines[2]. We will now apply (3) three times starting from the top face and moving against the directions of the arrows. Introduce the following three

[2]However, at the moment we do not assume any further meaning for these lines apart from using them as a convenient way of labeling to the corresponding (opposite) edges of quadrilaterals.

by three matrices

$$X_{pq}(\mathcal{A}) = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_{pr}(\mathcal{A}) = \begin{pmatrix} A & 0 & B \\ 0 & 1 & 0 \\ C & 0 & D \end{pmatrix}, \quad X_{qr}(\mathcal{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix}, \qquad (5)$$

where A, B, C, D are defined in (3) and their dependence on the angles $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$ is implicitly understood. It follows that

$$(\ell_p', \ell_q', \ell_r')^t \; = \; X_{pq}(\mathcal{A}_1)\, X_{pr}(\mathcal{A}_2)\, X_{qr}(\mathcal{A}_3)\; (\ell_p, \ell_q, \ell_r)^t \qquad (6)$$

where

$$\mathcal{A}_j = \{\alpha_j, \beta_j, \gamma_j, \delta_j\}, \quad j = 1, 2, 3, \qquad (7)$$

the lengths $\ell_p, \ell_q, \ldots$ are defined as in Fig.4, and the superscript "t" denotes the matrix transposition. Performing similar calculations for the back faces in Fig.4b and equating the resulting three by three matrices, one obtains

$$X_{pq}(\mathcal{A}_1)\, X_{pr}(\mathcal{A}_2)\, X_{qr}(\mathcal{A}_3) \; = \; X_{qr}(\mathcal{A}_3')\, X_{pr}(\mathcal{A}_2')\, X_{pq}(\mathcal{A}_1')\ . \qquad (8)$$

where

$$\mathcal{A}_j' = \{\alpha_j', \beta_j', \gamma_j', \delta_j'\}, \quad j = 1, 2, 3\ . \qquad (9)$$

This matrix relation contains exactly nine scalar equations where the LHS only depends on the front angles (7), while the RHS only depends on the back angles (9). Solving these equations one can obtain explicit form of the map (2). The resulting expressions are rather complicated and not particularly useful. However the mere fact that the map (2) satisfy a very special Eq.(8) is extremely important. Indeed, rewrite this equation as

$$X_{pq}(\mathcal{A}_1)\, X_{pr}(\mathcal{A}_2)\, X_{qr}(\mathcal{A}_3) \; = \; \mathcal{R}_{123}\Big(X_{qr}(\mathcal{A}_3)\, X_{pr}(\mathcal{A}_2)\, X_{pq}(\mathcal{A}_1)\Big) \qquad (10)$$

where $\mathcal{R}_{123}$ is an operator acting as the substitution (2) for any function $F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ of the angles,

$$\mathcal{R}_{123}\Big(F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)\Big) = F(\mathcal{A}_1', \mathcal{A}_2', \mathcal{A}_3') \qquad (11)$$

Then, following the arguments of [14], one can show that the map (2) satisfies the *functional tetrahedron equation* [15]

$$\mathcal{R}_{123} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{356} \; = \; \mathcal{R}_{356} \cdot \mathcal{R}_{246} \cdot \mathcal{R}_{145} \cdot \mathcal{R}_{123}\ , \qquad (12)$$

where both sides are compositions of the maps (2), involving six different sets of angles. Algebraically, this equation arises as an associativity condition for the cubic algebra (10). To discuss its geometric meaning we need to introduce *discrete evolution systems* associated with the map (2).

2.2 Discrete evolution systems: "Existence as integrability"

Consider a sub-lattice L of the 3D quadrilateral lattice, which only includes points $\boldsymbol{x}(m)$ with $m_1, m_2, m_3 \geq 0$. The boundary of this sub-lattice is a 2D discrete surface formed by quadrilaterals with the vertices $\boldsymbol{x}(m)$ having at least one of their integer coordinates m_1, m_2, m_3 equal to zero and the other two non-negative. Assume that all quadrilateral angles on this surface are known, and consider them as initial data. Then repeatedly applying the map (2) one can calculate angles on all faces of the sub-lattice L, defined above (one has to start from the corner $\boldsymbol{x}(0)$). The process can be visualized as an evolution of the initial data surface where every transformation (2) corresponds to a "flip" between the front and back faces (Fig. 2) of some cube adjacent to the surface. This makes the surface looking as a 3D "staircase" (or a pile of cubes) in the intersection corner of the three coordinate planes, see Fig. 5 showing two stages of this process. Note, that the corresponding evolution equations can be written in a covariant form for an arbitrary lattice cube (see Eq.(23) below for an example). It is also useful to have in mind that the above evolution can be defined purely geometrically as a *ruler-and-compass* type construction. Indeed the construction of the point $\boldsymbol{x}_{123}$ in Fig. 1 from the points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_{12}, \boldsymbol{x}_{13}, \boldsymbol{x}_{23}$ (and that is what is necessary for flipping a cube) only requires a *2D-ruler* which allows to draw planes through any three non-collinear points in the Euclidean space.

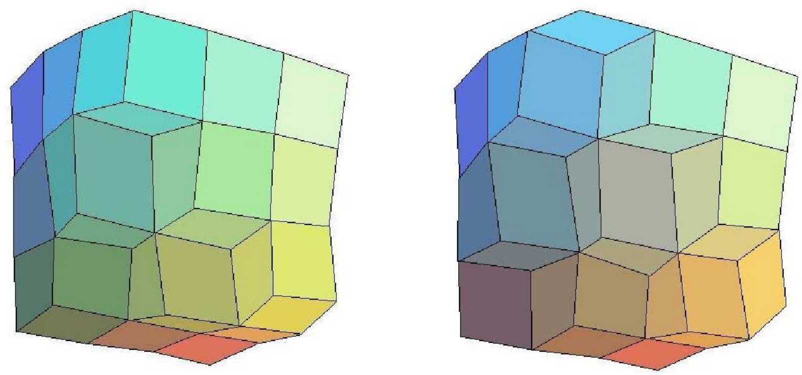

Figure 5: Visualization of the 3D "staircase" evolution.

Similar evolution systems can be defined for other quadrilateral lattices instead of the 3D cubic lattice considered above. Since the evolution is local (only one cube is flipped at a time) one could consider finite lattices as well. For example, consider six adjacent quadrilateral faces covering the front surface of the rhombic dodecahedron[3] shown in Fig. 6. Suppose that all angles on these faces are given and consider them as initial data. Now apply a sequence of four maps (2) and calculate angles on the back surface of the rhombic dodecahedron. This can be done in two alternative ways, corresponding to the two different dissection of the rhombic dodecahedron into four cubes shown in Fig.6. The functional tetrahedron equation (12) states that the results will be the same. Thereby it gives an algebraic proof for the equivalence of two "ruler-and-compass" type constructions of the back surface of the dodecahedron in Fig. 6. Can we also prove this equivalence geometrically? Although from the first sight this does not look trivial, it could be easily done from the point of view of the 4D geometry. The required statement follows just from the fact of existence of the quadrilateral lattice with the topology of the 4D cube [25]. The latter is defined by eight intersecting 3-planes in a general position in the 4-space. The two rhombic dodecahedra shown in Fig. 6 are obtained by a dissection of the 3-surface of this 4-cube, along its 2-faces, so these dodecahedra must have exactly the same quadrilateral 2-surface. Thus the functional tetrahedron equation (12), which plays the role of integrability condition for the discrete evolution system associated with the map (2), simply follows from the mere fact of existence of the 4-cube, which is the simplest 4D quadrilateral lattice. For a further

[3]It is worth noting that the most general rhombic dodecahedron with quadrilateral faces can only be embedded into (at least) the 4D Euclidean space.

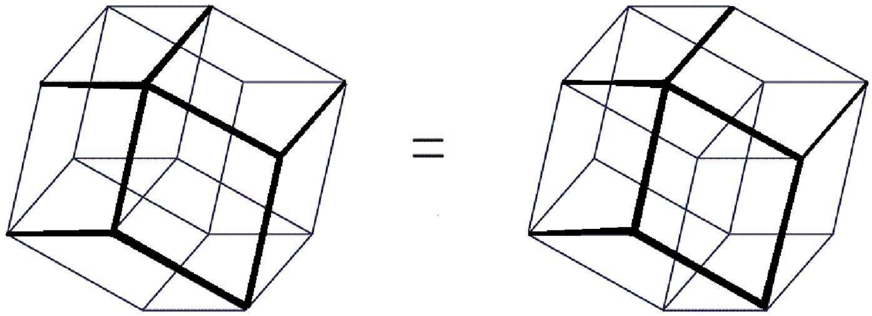

Figure 6: Two dissections of the rhombic dodecahedron into four quadrilateral hexahedra.

discussion of a relationship between the geometric consistency and integrability see [32].

3 Quantization of the 3D circular lattices

3.1 Poisson structure of circular lattices

The 3D *circular lattice* [19, 28, 29] is a special 3D quadrilateral lattice where all faces are circular quadrilaterals (i.e., quadrilaterals which can be inscribed into a circle). The existence of these lattices is established by the following beautiful geometry theorem due to Miquel [38] (see Fig. 7)

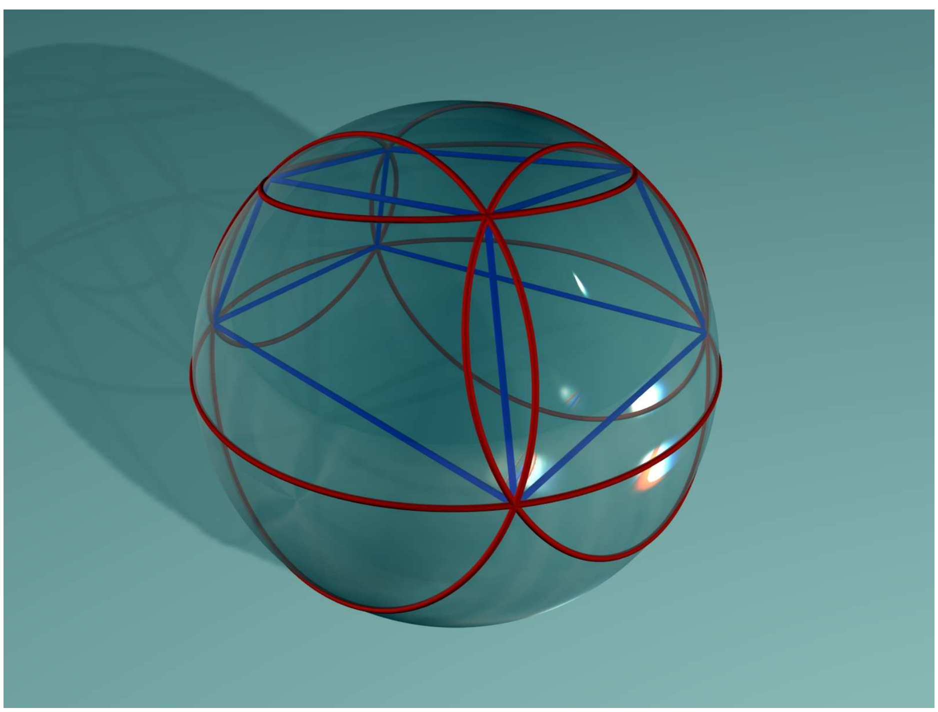

Figure 7: Miquel configuration of circles in 3D space, an elementary hexahedron and its circumsphere.

Miquel theorem. *Consider four points $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ in general position in $\mathbb{R}^N$, $N \geq 3$. On each of the three circles $c(\boldsymbol{x}_0, \boldsymbol{x}_i, \boldsymbol{x}_j)$, $1 \leq i < j \leq 3$ choose an additional new point $\boldsymbol{x}_{ij}$. Then there exist a unique point $\boldsymbol{x}_{123}$ which simultaneously belongs to the three circles $c(\boldsymbol{x}_1, \boldsymbol{x}_{12}, \boldsymbol{x}_{13})$, $c(\boldsymbol{x}_2, \boldsymbol{x}_{12}, \boldsymbol{x}_{23})$ and $c(\boldsymbol{x}_3, \boldsymbol{x}_{13}, \boldsymbol{x}_{23})$.*

It is easy to see that the above six circles lie on the same sphere. It follows then that every elementary "cube" on a circular lattice (whose vertices are at the circle intersection points) is inscribable into a sphere, see Fig. 7. The general formulae of the previous subsection can be readily specialized for the circular lattices. A circular quadrilateral has only two independent angles. In the notation of Fig. 3 one has

$$\gamma = \pi - \beta, \qquad \delta = \pi - \alpha \ . \tag{13}$$

Due to the Miquel theorem we can simply impose these restrictions on all faces of the lattice without running to any contradictions. The two by two matrix in (3) takes the form

$$X \ = \ \begin{pmatrix} k & a^* \\ -a & k \end{pmatrix}, \qquad \det X = 1 \ , \tag{14}$$

where we have introduced new variables

$$k \ = \ (\csc\alpha)\sin\beta \ , \quad a = (\csc\alpha)\sin(\alpha+\beta) \ , \quad a^* = (\csc\alpha)\sin(\alpha-\beta), \tag{15}$$

instead of the two angles $\{\alpha, \beta\}$. Note that the new variables are constrained by the relation

$$aa^* = 1 - k^2 \ . \tag{16}$$

Conversely, one has

$$\cos\alpha = \frac{a-a^*}{2k} \ , \quad \cos\beta = \frac{a+a^*}{2} \ . \tag{17}$$

Let the variables $\{k_j, a_j, a_j^*\}$, $\{k_j', a_j', a_j^{*\prime}\}$, $j = 1, 2, 3$, correspond to the front and back faces of the cube. The map (2) then read explicitly

$$\mathcal{R}_{123}: \qquad \begin{cases} (k_2 a_1^*)' = \ k_3 a_1^* - \varepsilon k_1 a_2^* a_3, & (k_2 a_1)' = k_3 a_1 - \varepsilon k_1 a_2 a_3^*, \\[2ex] (a_2^*)' = a_1^* a_3^* + \varepsilon k_1 k_3 a_2^*, & (a_2)' = a_1 a_3 + \varepsilon k_1 k_3 a_2, \\[2ex] (k_2 a_3^*)' = k_1 a_3^* - \varepsilon k_3 a_1 a_2^*, & (k_2 a_3)' = k_1 a_3 - \varepsilon k_3 a_1^* a_2, \end{cases} \tag{18}$$

where $\varepsilon = +1$ and

$$k_2' = \sqrt{1 - a_2' a_2^{*\prime}} \ . \tag{19}$$

At this point we note that exactly the same map together with the corresponding equations (8) and (12) were previously obtained in [16]. Moreover, it was discovered that this map is a canonical transformation preserving the Poisson algebra

$$\{a_i, a_j^*\} \ = \ 2\,\delta_{ij}\,k_i^2 \ , \quad \{k_i, a_j\} = \delta_{ij}\,k_i\,a_i \ , \quad \{k_i, a_j^*\} = -\,\delta_{ij}\,k_i\,a_i^*, \qquad i, j = 1, 2, 3 \ , \tag{20}$$

where $k_i^2 = 1 - a_i a_i^*$. Note that variables k, a, a^* on different quadrilaterals are in involution. The same Poisson algebra in terms of angle variables reads

$$\{\alpha_i, \beta_j\} = \delta_{ij} \ , \quad \{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 0 \ . \tag{21}$$

This "ultra-local" symplectic structure trivially extends to any circular quad-surface of initial data, discussed above. To resolve an apparent ambiguity in naming of the angles, this surface must be equipped with oriented rapidity lines, similar to those in Fig. 4[4]. In addition, the angles for each quadrilateral should be arranged as in Fig. 3. Then one can assume that the indices i, j in (21) refer to all quadrilaterals on this surface.

Thus, the evolution defined by the map (18) is a symplectic transformation. The corresponding equations of motion for the whole lattice (the analog of the Hamilton-Jacobi equations) can be written in a "covariant" form. For every cube define

$$A_{32} = a_1, \;\; A_{23} = a_1^*, \;\; A_{31} = a_2', \;\; A_{13} = a_2'^*, \;\; A_{21} = a_3, \;\; A_{12} = a_3^* \tag{22}$$

where A_{jk} stands for $A_{jk}(m)$, where m is such that $\boldsymbol{x}(m)$ coincides with the coordinates of the top front corner of the cube (vertex $\boldsymbol{x}_0$ in Fig.1). Let T_k be the shift operator $T_k\, A_{ij}(m) = A_{ij}(m + e_k)$. Then

$$\widetilde{T}_k A_{ij} \;=\; \frac{A_{ij} - A_{ik}A_{kj}}{K_{ik}K_{kj}}\;, \quad K_{ij} = K_{ji} = \sqrt{1 - A_{ij}A_{ji}}\;, \tag{23}$$

where (i, j, k) is an arbitrary permutation of $(1, 2, 3)$ and

$$\widetilde{T}_1 = T_1\;, \quad \widetilde{T}_2 = T_2^{-1}\;, \quad \widetilde{T}_3 = T_3\;. \tag{24}$$

Note that Eq.(23) also imply

$$(\widetilde{T}_k K_{ij})K_{kj} = (\widetilde{T}_i K_{kj})K_{ij}\;. \tag{25}$$

Remarks. The equations (23) have been previously obtained in [29], see Eq.(7.20) therein. The quantities A_{ij} in (23) should be identified with the *rotation coefficients* denoted as $\tilde{\beta}_{ij}$ in [29]. The same equations (23) are discussed in §3.1 of [32], where one can also find a detailed bibliography on the circular lattices (we are indebted to A.I.Bobenko for these important remarks).

3.2 Quantization and the tetrahedron equation

In the next section we consider different quantizations of the map (18) and obtain several solutions of the full quantum tetrahedron equation (see Eq.(33) below). In all cases we start with the canonical quantization of the Poisson algebra (21),

$$[\alpha_i, \beta_j] = \xi\, \hbar\, \delta_{ij}\;, \qquad [\alpha_i, \alpha_j] = 0\;, \qquad [\beta_i, \beta_j] = 0\;, \tag{26}$$

where $\hbar$ is the quantum parameter (the Planck constant) and ξ is a numerical coefficient, introduced for a further convenience. The indices i, j label the faces of the "surface of initial data" discussed above. Since the commutation relations (26) are ultra-local (in the sense that the angle variables on different faces commute with each other), let us concentrate on the local Heisenberg algebra,

$$\mathsf{H}: \qquad\qquad [\alpha, \beta] = \xi\, \hbar\;, \tag{27}$$

for a single lattice face (remind that the angles shown in Fig. 3 are related by (13)). The map (18) contains the quantities k, a, a^*, defined in (15), which now become operators. For

[4] We refer the reader to our previous paper [39] where the relationship between the rapidity graphs and quadrilateral lattices is thoroughly discussed.

definiteness, assume that the non-commuting factors in (15) are ordered exactly as written. Then the definitions (15) give

$$\begin{aligned} k &= (U - U^{-1})^{-1}\,(V - V^{-1}), \\ a &= q^{-\frac{1}{2}}\,(U - U^{-1})^{-1}\,(U\,V - U^{-1}\,V^{-1}), \\ a^* &= q^{+\frac{1}{2}}\,(U - U^{-1})^{-1}\,(U\,V^{-1} - U^{-1}\,V), \end{aligned} \tag{28}$$

where the elements U and V generate the Weyl algebra,

$$U\,V = q\,V\,U, \qquad U = e^{i\alpha}, \qquad V = e^{i\beta}, \qquad q = e^{-\xi\hbar}\ . \tag{29}$$

The operators (28) obey the commutation relations of the q-oscillator algebra,

$$\mathsf{Osc}_q : \qquad \begin{cases} q\,a^*\,a - q^{-1}\,a\,a^* = q - q^{-1}, \qquad k\,a^* = q\,a^*\,k, \qquad k\,a = q^{-1}\,a\,k\ , \\ k^2 = q\,(1 - a^*\,a) = q^{-1}\,(1 - a\,a^*)\ . \end{cases} \tag{30}$$

where the element k is assumed to be invertible. This algebra is, obviously, a quantum counterpart of the Poisson algebra (20). In the previous Section we have already mentioned the result of [16] that

(i) the map (18) is an automorphism of the tensor cube of the Poisson algebra (20) (remind that the relation (16) should be taken into account in (18)).

In the same paper [16] it was also shown that

(ii) there exists a quantum version of the map (18), which acts as an automorphism of the tensor cube of the q-oscillator algebra (30). The formulae (18) for the quantum map stay exactly the same, but the relation (16) should be replaced by either of the two relations on the second line of (30), for instance, $k^2 = q\,(1 - a^*a)$. In particular, (19) should be replaced with

$$(k_2')^2 = q\,(1 - a_2^{*\prime} a_2{}')\ . \tag{31}$$

(iii) the quantum version of the map (18), defined in (ii) above, satisfies the functional tetrahedron equation (12).

For any irreducible representations of the q-oscillator algebra (30) the formulae (18) and (31) uniquely determine the $\mathcal{R}_{123}$ as an internal automorphism,

$$\mathcal{R}_{123}\big(F\big) \;=\; R_{123}\;F\;R_{123}^{-1}, \qquad F \in \mathsf{Osc}_q \otimes \mathsf{Osc}_q \otimes \mathsf{Osc}_q\ . \tag{32}$$

It follows then from (12) that the linear operator R_{123} satisfies the quantum tetrahedron equation

$$R_{123}\,R_{145}\,R_{246}\,R_{356} \;=\; R_{356}\,R_{246}\,R_{145}\,R_{123}\ , \tag{33}$$

where each of the operators R_{123}, R_{145}, R_{246} and R_{356} act as (32) in the three factors (indicated by the subscripts) of a tensor product of six q-oscillator algebras and act as the unit operator in the remaining three factors.

4 Solutions of the tetrahedron equation

Here we show that all known solutions of the tetrahedron equation can be obtained by solving (32) for the linear operator R_{123} with different irreducible representations of the q-oscillator algebra (30).

4.1 Fock representation solution

In this subsection we set $\xi = 1$ in (26) and $q = e^{-\hbar}$. Define the Fock representation of a single q-oscillator algebra (30),

$$a\,|0\rangle = 0, \quad a\,|n+1\rangle = |n\rangle, \quad a^*\,|n\rangle = (1 - q^{2+2n})\,|n+1\rangle, \quad k\,|n\rangle = q^{n+1/2}\,|n\rangle, \tag{34}$$

spanned by the vectors $|n\rangle$, $n = 0, 1, \ldots, \infty$. Then using (18), (30), (31) and (32) one can show that

$$\begin{aligned}\langle n_1, n_2, n_3 | R\,|n'_1, \quad n'_2, n_3\rangle = \delta_{n_1+n_2, n'_1+n'_2}\delta_{n_2+n_3, n'_2+n'_3}\ (-1)^{n_2}\, q^{(n'_1-n_2)(n'_3-n_2)} \\ \times \binom{n_3}{n'_2}_{q^2}\ {}_2\phi_1(q^{-2n'_2}, q^{2(1+n'_3)}, q^{2(1-n'_2+n_3)}; q^2, q^{2(1+n_1)})\ ,\end{aligned} \tag{35}$$

where

$$(x; q^2)_n \ = \ \prod_{j=0}^{n-1}(1 - q^{2j}x)\ , \quad \binom{n}{k}_{q^2} \ = \ \frac{(q^2; q^2)_n}{(q^2; q^2)_k (q^2; q^2)_{n-k}}\ , \tag{36}$$

and

$${}_2\phi_1(a, b, c; q^2, z) \ = \ \sum_{n=0}^{\infty} \frac{(a; q^2)_n (b; q^2)_n}{(q^2; q^2)_n (c; q^2)_n} z^n \tag{37}$$

is the q-deformed Gauss hypergeometric series. This 3D R-matrix satisfies the constant tetrahedron equation (33). In matrix form this equation reads

$$\begin{aligned}\sum_{n'_j=0}^{\infty} R_{n_1,n_2,n_3}^{n'_1,n'_2,n'_3} \quad R_{n'_1,n_4,n_5}^{n''_1,n'_4,n'_5}\, R_{n'_2,n'_4,n_6}^{n''_2,n''_4,n'_6}\, R_{n'_3,n'_5,n'_6}^{n''_3,n''_5,n''_6} = \\ = \sum_{n'_j=0}^{\infty} R_{n_3,n_5,n_6}^{n'_3,n'_5,n'_6}\, R_{n_2,n_4,n'_6}^{n'_2,n'_4,n''_6}\, R_{n_1,n'_4,n'_5}^{n'_1,n''_4,n''_5}\, R_{n'_1,n'_2,n'_3}^{n''_1,n''_2,n''_3}\end{aligned} \tag{38}$$

where the sum is taken over six indices $n'_1, n'_2, n'_3, n'_4, n'_5, n'_6$ and

$$R_{n_1,n_2,n_3}^{n'_1,n'_2,n'_3} = \langle n_1, n_2, n_3 | R\,|n'_1, n'_2, n'_3\rangle\ . \tag{39}$$

Note that Eq.(38) does not contain any spectral parameters. Originally the R-matrix (35) was obtained in [16] in terms of a solution of some recurrence relation, which was subsequently reduced to the q-hypergemetric function in [6].

4.2 Modular double solution

In this subsection we set in (26)

$$\xi = -i, \qquad \hbar = \pi\, b^2, \qquad q = e^{i\pi b^2} \tag{40}$$

where b is a free parameter, $\mathrm{Re}\, b \neq 0$. Here it will be more convenient to work with a slightly modified version[5] of the map (18), with the value $\varepsilon = -1$.

Consider a non-compact representation [40] of the q-oscillator algebra (30) in the space of functions $f(\sigma) \in \mathbb{L}^2(\mathbb{R})$ on the real line admitting an analytical continuation into an appropriate horizontal strip, containing the real axis in the complex σ-plane (see [40] for further details). Such representation essentially reduces to that of the Weyl algebra

$$\mathsf{W}_q : \qquad k\, w = q\, w\, k, \qquad q = e^{i\pi b^2}, \tag{41}$$

realized as multiplication and shift operators

$$k\,|\sigma\rangle = \mathsf{i} e^{\pi\sigma b}\,|\sigma\rangle\ , \qquad w\,|\sigma\rangle = |\sigma - \mathsf{i} b\rangle. \tag{42}$$

The generators a, a^* in (30) are expressed as

$$a = w^{-1}\ , \quad a^* = (1 - q^{-1}k^2)\, w\ . \tag{43}$$

As explained in [41] the representation (42) is not, in general, irreducible. Therefore, the relation (32) alone does not unambiguously define the linear operator R_{123} in this case. Following the idea of [41] consider the modular dual of the algebra (41),

$$\mathsf{W}_{\tilde{q}} : \qquad \tilde{k}\, \tilde{w} = \tilde{q}\, \tilde{w}\, \tilde{k}, \qquad \tilde{q} = e^{-i\pi b^{-2}}, \tag{44}$$

acting in the same representation space

$$\tilde{k}\,|\sigma\rangle = -\mathsf{i} e^{\pi\sigma b^{-1}}\,|\sigma\rangle\ , \qquad \tilde{w}\,|\sigma\rangle = |\sigma + \mathsf{i} b^{-1}\rangle. \tag{45}$$

We found that if the relation (32) is complemented by its modular dual

$$\tilde{\mathcal{R}}_{123}\big(\tilde{F}\big) \;=\; R_{123}\; \tilde{F}\; R_{123}^{-1}, \qquad \tilde{F} \in \mathsf{Osc}_{\tilde{q}} \otimes \mathsf{Osc}_{\tilde{q}} \otimes \mathsf{Osc}_{\tilde{q}}\ , \tag{46}$$

then the pair of relations (32) and (46) determine the operator R_{123} uniquely[6]. The dual q-oscillator algebra $\mathsf{Osc}_{\tilde{q}}$ is realized through the dual Weyl pair (44) and the relations

$$\tilde{a} = \tilde{w}^{-1}\ , \quad \tilde{a}^* = (1 - \tilde{q}^{-1}\tilde{k}^2)\, \tilde{w}\ . \tag{47}$$

The dual version of the map $\tilde{\mathcal{R}}_{123}$ is defined by the same formulae (18), where quantities k_j, a_j, a_j^*, $j = 1, 2, 3$ are replaced by their "tilded" counterparts $\tilde{k}_j, \tilde{a}_j, \tilde{a}_j^*$. The value of q does not, actually, enter the map (18), but needs to be taken into account in the relations between the generators of the q-oscillator algebra. Thus, the linear operator R_{123} in this case simultaneously provides the two maps $\mathcal{R}_{123}$ and $\tilde{\mathcal{R}}_{123}$ (given by (18) with $\varepsilon = -1$). The explicit form of this operator is given below.

Denote

$$\eta = \frac{b + b^{-1}}{2}, \tag{48}$$

and define a special function

$${}_2\Psi_2\left(\begin{matrix} c_1, c_2 \\ c_3, c_4 \end{matrix}\middle|\, c_0\right) \;=\; \int_{\mathbb{R}} dz\, e^{2\pi \mathsf{i} z(-c_0 - \mathsf{i}\eta)}\, \frac{\varphi(z + \frac{c_1 + \mathsf{i}\eta}{2})\varphi(z + \frac{c_2 + \mathsf{i}\eta}{2})}{\varphi(z + \frac{c_3 - \mathsf{i}\eta}{2})\varphi(z + \frac{c_4 - \mathsf{i}\eta}{2})}\ , \tag{49}$$

[5]Note that Eq.(18) is a particular case a general map considered in [16]; see also Eq.(59) below.

[6]It is worth mentioning similar phenomena in the construction of the R-matrix [42] for the modular double of the quantum group $U_q(sl_2)$ and the representation theory of $U_q(sl_2, \mathbb{R})$ [43].

where φ is the non-compact quantum dilogarithm [44]

$$\varphi(z) = \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0} \frac{e^{-2\mathrm{i}zx}}{\sinh(xb)\sinh(x/b)}\,\frac{dx}{x}\right) . \tag{50}$$

The values of c_1, c_2, c_3, c_4 are assumed to be such that poles of numerator in the integrand of (49) lie above the real axis, while the zeroes of the denominator lie below the real axis. For other values of c_j the integral (49) is defined by an analytic continuation. Note that for $\operatorname{Im} b^2 > 0$ the integral ${}_2\Psi_2$ can be evaluated by closing the integration contour in the upper half plane (see eq.(78) in [6]), which is very convenient for numerical calculations.

The matrix elements of R-matrix solving the pair of the equations (32) and (46) are given by

$$\begin{aligned}\langle \sigma_1, \quad \sigma_2, \sigma_3 | R\, | \sigma_1', \sigma_2', \sigma_3'\rangle &= \delta_{\sigma_1+\sigma_2, \sigma_1'+\sigma_2'}\delta_{\sigma_2+\sigma_3, \sigma_2'+\sigma_3'} \times \\ &\times e^{\mathrm{i}\pi(\sigma_1'\sigma_3' + \mathrm{i}\eta(\sigma_1'+\sigma_3'-\sigma_2))}\ \ {}_2\Psi_2\left(\begin{matrix}\sigma_1-\sigma_3, -\sigma_1+\sigma_3 \\ \sigma_1+\sigma_3, -\sigma_1'-\sigma_3'\end{matrix}\middle|\sigma_2\right) .\end{aligned} \tag{51}$$

This R-matrix satisfies the constant tetrahedron equation (33). Its matrix form is given by (38) where $R_{n_i,n_j,n_k}^{n_i',n_j',n_k'}$ is substituted by $R_{\sigma_i,\sigma_j,\sigma_k}^{\sigma_i',\sigma_j',\sigma_k'} = \langle\sigma_i,\sigma_j,\sigma_k|R\,|\sigma_i',\sigma_j',\sigma_k'\rangle$, $i,j,k = 1,2,\ldots,6$, and the sums are replaced by the integrals over $\sigma_1', \sigma_2', \sigma_3', \sigma_4', \sigma_5', \sigma_6'$ along the real lines $-\infty < \sigma_i' < \infty$. One can verify that these integrals converge. The solution (51) was obtained in [6].

4.2.1 The "interaction-round-a-cube" formulation of the modular double solution

Note that due to the presence of two delta-functions in (51) the edge spins are constrained by two relations $\sigma_1 + \sigma_3 = \sigma_1' + \sigma_3$ and $\sigma_2 + \sigma_3 = \sigma_2' + \sigma_3'$ at each vertex of the lattice. Here we re-formulate this solution in terms of unconstrained *corner* spins, which also take continuous values on the real line. Figure 8 shows an elementary cube of the lattice with the corner spins "a,b,c,d,e,f,g,h" arranged in the same way as in [10]. The corresponding Boltzmann weight reads [6]

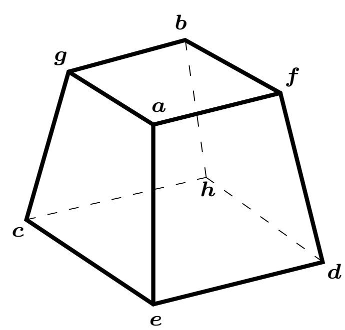

Figure 8: The arrangement of corner spins around a cube.

$$\begin{aligned}W(a|e,f,g|b,c,d|h;\mathsf{T}_1,\mathsf{T}_2,\mathsf{T}_3) &= \\ &= e^{\mathrm{i}\pi(\sigma_1'\sigma_3' + \mathrm{i}\eta(\sigma_1'+\sigma_3'-\sigma_2))}\ \ {}_2\Psi_2\left(\begin{matrix}\sigma_1-\sigma_3, -\sigma_1+\sigma_3 \\ \sigma_1+\sigma_3, -\sigma_1'-\sigma_3'\end{matrix}\middle|\sigma_2\right) ,\end{aligned} \tag{52}$$

where

$$\begin{aligned}&\sigma_1 = g+f-a-b-\mathsf{T}_1, \quad \sigma_2 = a+c-e-g+\mathsf{T}_2, \quad \sigma_3 = e+f-a-d-\mathsf{T}_3, \\ &\sigma_1' = c+d-e-h-\mathsf{T}_1, \quad \sigma_2' = f+h-b-d+\mathsf{T}_2, \quad \sigma_3' = b+c-g-h-\mathsf{T}_3\end{aligned} \tag{53}$$

Note, that the variables σ_i and σ'_i automatically satisfy the delta function constrains in (51). The weight functions (52) contains three arbitrary (spectral) parameters $\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3$. It satisfies

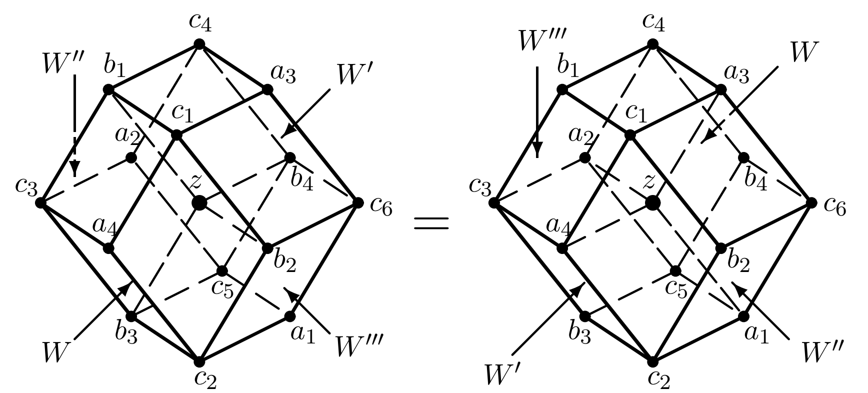

Figure 9: Graphical representation for the tetrahedron equations for interaction-round-a-cube models.

the tetrahedron equation in the "interaction-round-a-cube" form [10] (see Fig. 9),

$$\begin{aligned}
&\int_{\mathbb{R}} dz\, W(a_4|c_1, c_3, c_2|b_3, b_2, b_1|z)\, W'(c_1|a_3, b_1, b_2|z, c_6, c_4|b_4) \\
&\qquad \times W''(b_1|c_4, c_3, z|b_3, b_4, a_2|c_5)\, W'''(z|b_4, b_3, b_2|c_2, c_6, c_5|a_1) = \\
&\qquad\quad = \int_{\mathbb{R}} dz\, W'''(b_1|c_4, c_3, c_1, |a_4, a_3, a_2|z)\, W''(c_1|a_3, a_4, b_2|c_2, c_6, z|a_1) \\
&\qquad\qquad\qquad \times W'(a_4|z, c_3, c_2|b_3, a_1, a_2|c_5)\, W(z|a_3, a_2, a_1|c_5, c_6, c_4|b_4),
\end{aligned} \tag{54}$$

where the four sets of the spectral parameters are constrained as

$$\mathsf{T}'_1 = \mathsf{T}_1, \quad \mathsf{T}''_1 = -\mathsf{T}_2, \quad \mathsf{T}'''_1 = \mathsf{T}_3, \quad \mathsf{T}''_2 = \mathsf{T}'_2, \quad \mathsf{T}'''_2 = -\mathsf{T}'_3, \quad \mathsf{T}'''_3 = \mathsf{T}''_3\ . \tag{55}$$

Note that the parameters $\mathsf{T}_1, \mathsf{T}_2, \mathsf{T}_3$ are similar to those in the Zamolodchikov model [8] and its generalization for an arbitrary number of spin states [11], which is considered in Sect. 4.3 below. One can relate them as $\mathsf{T}_j = \log[\tan(\theta_j/2)]$ to the angles $\theta_1, \theta_2, \theta_3$ of the spherical triangle in Sect. 4.3.2. In total, there are twelve parameters $\mathsf{T}_j, \mathsf{T}'_j, \mathsf{T}''_j, \mathsf{T}'''_j$, $j = 1, 2, 3$ (three for each weight function) constrained by six relations (55). Thus, the tetrahedron equation (54) contains six independent parameters (in contract to only five parameters in the cyclic case; see the text after Eq.(72) below).

The above solution can be simply generalized by multiplying the weight (52) by an "external field" factor

$$W(\ldots) \to \exp\Big[\sum_{j=1}^{3} f_j(\sigma_j + \sigma'_j)\Big]\, W(\ldots)\ . \tag{56}$$

This substitution does not affect the validity of the tetrahedron equation (54), provided the field parameters f_j for different W's are constrained as

$$f''_3 = f'_3 - f_3, \quad f'''_1 = f''_1 - f'_1, \quad f'''_2 = f''_2 + f_1, \quad f'''_3 = f_2 - f'_2. \tag{57}$$

Similar generalizations apply for the solutions (35) and (51). The solution (52) was previously obtained in [6].

4.3 Cyclic representation solution

4.3.1 Generalized form of the q-oscillator map

Here, we will solve equation (32) with a more general form of the map (18), considered in [16]. Let

$$\mathcal{L}(\mathcal{H}_j)=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \lambda_j \boldsymbol{k}_j & \boldsymbol{a}_j^{+} & 0\\ 0 & \lambda_j\mu_j \boldsymbol{a}_j^{-} & -\mu_j \boldsymbol{k}_j & 0\\ 0 & 0 & 0 & \lambda_j\mu_j \end{pmatrix} \qquad j=1,2,3, \tag{58}$$

be an operator-valued matrix acting in the direct product of two vector spaces $\mathbb{C}^2\otimes\mathbb{C}^2$, whose elements depends on a set of generators $\mathcal{H}_j=\{k_j,a_j,a_j^*\}$ of the q-oscillator algebra (30) and continuous parameters λ_j, and μ_j, where $j=1,2,3$. The RHS of (58) is understood as a block matrix with two-dimensional blocks where matrix indices of the second space $\mathbb{C}^2$ numerate the blocks, while those for first space numerate the elements inside the blocks.

The required generalization of (32) reads [16][7]

$$L_{12}(\mathcal{H}_1)\, L_{13}(\mathcal{H}_2) L_{23}\,(\mathcal{H}_3) = R_{123}\, L_{23}(\mathcal{H}_3)\, L_{13}(\mathcal{H}_2)\, L_{12}(\mathcal{H}_1)\, R_{123}^{-1} \tag{59}$$

Here $\mathcal{L}_{12}$ denote the matrix which acts as (58) in the first and second component of the tensor product $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ and coincides with the unit operator in the third component ($\mathcal{L}_{13}$ and $\mathcal{L}_{23}$ are defined similarly). It is important to note that the additional parameters λ_j and μ_j enter (59) only in three combinations λ_2/λ_3, $\lambda_1\mu_3$ and μ_1/μ_2 (explicit formulae for the corresponding generalization of the map (18) are given in Eqs.(24-26) of [16]).

4.3.2 Cyclic representations of the q-oscillator algebra

Let $N\geq 3$ be an odd integer,

$$q=-\mathrm{e}^{i\pi/N},\quad q^N=1,\quad N=\text{odd},\quad N\geq 3. \tag{60}$$

and V^N be an N-dimensional vector space, spanned by the vectors $|n\rangle,\ n\in\mathbb{Z}_N=\{0,1,\ldots,N-1\}$. Define N by N matrices (the index n is treated *modulo* N)

$$X\,|n\rangle = q^n\,|n\rangle,\qquad Z\,|n\rangle=|n+1\rangle,\qquad X^N=Z^N=1, \tag{61}$$

and their embedding into a direct product $V^N\otimes V^N\otimes V^N$,

$$X_j = 1\otimes\cdots\otimes \underset{j\text{-th}}{X}\otimes\cdots\otimes 1,\quad Z_j = 1\otimes\cdots\otimes \underset{j\text{-th}}{Z}\otimes\cdots\otimes 1,\quad j=1,2,3. \tag{62}$$

Equation (59) involves the direct product of three copies of the q-oscillator algebra (30), labeled by the subscript $j=1,2,3$. The most general cyclic representation for this product

$$\boldsymbol{k}_j=\varkappa_j\,X_j\;,\quad \boldsymbol{a}_j^*=\frac{1}{\rho_j}(1-q^{-1}\varkappa_j^2\,X_j^2)\,Z_j\;,\quad \boldsymbol{a}_j\ =\rho_j\,Z_j^{-1},\quad j=1,2,3, \tag{63}$$

contains two continuous parameters $\varkappa_j,\rho_j$, for each factor, so there six arbitrary parameters altogether. Detailed inspection of (59) shows, however, that ρ_1,ρ_2,ρ_3 only enter through a ratio $\rho_1\rho_3/\rho_2$, so there are only four essential parameters.

[7] Equation (59) reduces to (32) when $\lambda_1=\lambda_2=\lambda_3=1$ and $\mu_1=\mu_2=\mu_3=-\varepsilon$, where ε enters the map (18).

Let $\theta_1, \theta_2, \theta_3$ and a_1, a_2, a_3 be angles and sides of a spherical triangle, respectively. Define

$$2\beta_1 = a_2 + a_3 - a_1, \quad 2\beta_2 = a_1 + a_3 - a_2, \quad 2\beta_3 = a_1 + a_2 - a_3, \quad \beta_0 = \pi - \beta_1 - \beta_2 - \beta_3. \tag{64}$$

and set

$$\varkappa_1 = \mathrm{i}\sqrt[N]{\tan\frac{\theta_1}{2}}, \quad \varkappa_2 = \mathrm{i}\sqrt[N]{\cot\frac{\theta_2}{2}}, \quad \varkappa_3 = \mathrm{i}\sqrt[N]{\tan\frac{\theta_3}{2}}\ . \tag{65}$$

Then Eq.(59) implies

$$\rho_1\rho_3/\rho_2 \;=\; \mathrm{e}^{-\mathrm{i}\beta_2/N}\sqrt[N]{\sin a_2/\sin\beta_0}, \tag{66}$$

and

$$\lambda_3/\lambda_2 = \mathrm{e}^{\mathrm{i}a_1/N}\ , \quad \lambda_1\mu_3 = \mathrm{e}^{-\mathrm{i}a_2/N}\ , \quad \mu_1/\mu_2 = \mathrm{e}^{\mathrm{i}a_3/N}\ . \tag{67}$$

4.3.3 Solution of the tetrahedron equation

Let p denote a point on Fermat curve

$$p \stackrel{\text{def}}{=} (x, y \mid x^N + y^N = 1). \tag{68}$$

Introduce a function $\varphi_p(n)$, $n \in \mathbb{Z}_N$, (cyclic analog of the quantum dilogarithm) [11]

$$\varphi_p(0) = 1\ , \quad \frac{\varphi_p(n-1)}{\varphi_p(n)} \;=\; \frac{1 - xq^{2n}}{y}, \quad \varphi_p(n+N) = \varphi_p(n) \tag{69}$$

The R-matrix solving (59) for the cyclic representation (63) is given by

$$\begin{aligned}\langle n|R|n'\rangle \;=\; & \delta_{n_1+n_2,n_1'+n_2'}\delta_{n_2+n_3,n_2'+n_3'} \times \\ & q^{n_1n_3-n_2'(n_1+n_3)} \sum_{n\in\mathbb{Z}_N} q^{-2nn_2'}\,\frac{\varphi_{p_1}(n+n_1+n_3')\varphi_{p_2}(n)}{\varphi_{p_3}(n+n_1)\varphi_{p_4}(n+n_3)}\ ,\end{aligned} \tag{70}$$

where the points p_j on the curve (68) are defined by

$$\begin{aligned} x_1 &= \mathrm{e}^{-\mathrm{i}a_2/N}\sqrt[N]{\sin\beta_2/\sin\beta_0}\ , & y_1 &= \mathrm{e}^{\mathrm{i}\beta_2/N}\sqrt[N]{\sin a_2/\sin\beta_0}\ , \\ x_2 &= \mathrm{e}^{-\mathrm{i}a_2/N}\sqrt[N]{\sin\beta_0/\sin\beta_2}\ , & y_2 &= \mathrm{e}^{\mathrm{i}\beta_0/N}\sqrt[N]{\sin a_2/\sin\beta_2}\ , \\ x_3 &= \mathrm{e}^{-\mathrm{i}(a_2+\pi)/N}\sqrt[N]{\sin\beta_3/\sin\beta_1}\ , & y_3 &= \mathrm{e}^{-\mathrm{i}\beta_3/N}\sqrt[N]{\sin a_2/\sin\beta_1}\ , \\ x_4 &= \mathrm{e}^{-\mathrm{i}(a_2+\pi)/N}\sqrt[N]{\sin\beta_1/\sin\beta_3}\ , & y_4 &= \mathrm{e}^{-\mathrm{i}\beta_1/N}\sqrt[N]{\sin a_2/\sin\beta_3}\ . \end{aligned} \tag{71}$$

We write this R-matrix as $R_{123}(\theta_1, \theta_2, \theta_3)$ to indicate its dependence on the angles $\theta_1, \theta_2, \theta_3$. It satisfies the tetrahedron equation (33), provided the R-matrices therein are parametrized by four triples of dihedral angles at four vertices of an Euclidean tetrahedron

$$\begin{aligned} R_{123} &= R_{123}(\theta_1, \theta_2, \theta_3), & R_{145} &= R_{145}(\theta_1, \pi-\theta_4, \pi-\theta_5), \\ R_{246} &= R_{246}(\pi-\theta_2, \pi-\theta_4, \theta_6), & R_{356} &= R_{356}(\theta_3, \theta_5, \theta_4)\ . \end{aligned} \tag{72}$$

Here θ_j, $j = 1, 2, \ldots, 6$ are *inner* dihedral angles of the tetrahedron. These angles are constrained by one relation, therefore Eq.(33) in this case contains five arbitrary parameters. Finally, the substitution

$$\begin{aligned} n_1 &= g + f - a - b, & n_2 &= a + c - e - g, & n_3 &= e + f - a - d, \\ n_1' &= c + d - e - h, & n_2' &= f + h - b - d, & n_3' &= b + c - g - h \end{aligned} \tag{73}$$

and some trivial equivalence transformations bring (70) into its "interaction-round-a-cube" form

$$W(a|e,f,g|b,c,d|h) \;=\; \sum_{n\in\mathbb{Z}_N} q^{2n(b+d-f-h)}\,\frac{\varphi_{p_1}(n-h+c)\varphi_{p_2}(n-f+a)}{\varphi_{p_3}(n-b+g)\varphi_{p_4}(n-d+e)}\;, \tag{74}$$

which satisfies (54) where the integration is replaced by the summation over $\mathbb{Z}_N$, while the angle parametrization remains the same as in (72). Although our considerations in the cyclic case were restricted to odd values of N in (60), the final result (74) as valid for even values of N as well. The solution (74) was previously obtained in [11]; for $N=2$ it coincides with the Baxter's form [10] for the Boltzmann weights the Zamolodchikov model [8].

5 Conclusion

In this paper we have exposed various connections between discrete differential geometry and statistical mechanics, displaying geometric origins of algebraic structures underlying integrability of quantum systems.

We have shown that the 3D circular lattices are associated with an integrable discrete Hamiltonian system and constructed three different quantizations of this system. In this way in Sect. 4 we have obtained all previously known solutions of the tetrahedron equation. They are given by Eqs. (35), (51), (52), (70). The resulting 3D integrable models can be thought of as describing quantum fluctuations of the lattice geometry. The classical geometry of the 3D circular lattices reveals itself [6] as a stationary configuration giving the leading contribution to the partition function of the quantum system in the quasi-classical limit.

The solutions of the tetrahedron equation discussed here possess a remarkable property: they can be used to obtain [11,16] infinite number of two-dimensional solvable models related to various representations of quantized affine algebras $U_q(\widehat{sl}_n)$, $n=2,3,\ldots,\infty$ (where n coincides with the size of the "hidden third dimension"). Apparently, a similar 3D interpretation, originating from other simple geometrical models, also exists for the trigonometric solutions of the Yang-Baxter equation, related with all other infinite series of quantized affine algebras [45,46] and super-algebras [47] (see [48,49] for recent results in this direction). Therefore, it might very well be that not only the phenomenon of quantum integrability but the quantized algebras themselves are deeply connected with geometry.

There are many important outstanding questions remain, in particular, the geometric meaning of the Poisson algebra (21) and connections of the 3D circular lattices with the 2D circle patterns [50] on the plane or the sphere. It would also be interesting to understand underlying reasons of a "persistent" appearance of the q-oscillator algebra (30) as a primary algebraic structure in many other important aspects of the theory of integrable systems, such as, for example, the construction of Baxter's $\mathbf{Q}$-operators [51] and the calculation of correlation functions of the XXZ model [52]. It would also be interesting to explore connections of our results with the invariants of the 3D manifolds [53–55], the link invariants [56–58], quantization of the Techmueller space [59,60] and the representation theory of $U_q(sl(2|\mathbb{R})$ [61]. So, there are many interesting questions about the quantum integrability still remain unanswered, but one thing is getting is more and more clear: it is not just connected with geometry, it is geometry itself! (though the Quantum Geometry).

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