Optimal Robust Control for Uncertain Impulsive Systems*

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Abstract: This paper aims to study the optimal robust control problem for uncertain impulsive systems. By using the Lyapunov function method and algebraic inequalities, conditions are derived under which not only the uncertain impulsive system is robustly asymptotically stable and the optimal bound of the hybrid performance functional can be estimated. The results are then specialized to interval linear impulsive systems with quadratic hybrid performance functional. Riccati inequality conditions are used to design feedback controllers to stabilize asymptotically the interval linear impulsive system and the optimal bound of the quadratic hybrid performance functional are derived.

Key Words: Uncertain impulsive systems, Hybrid performance functional, Robust stability, Riccati inequality, Interval matrix

1 INTRODUCTION

Impulsive systems arise in many applied fields such as control technology, communication networks, and biological management. Since impulsive systems provide a natural framework for mathematical modelling of many physical phenomena, their study is assuming a greater importance. The stability theory of impulsive systems and impulsive control have been developed quite rapidly in recent years. See [1-4, 7, 10-13, 16-19] and references cited therein.

The stability of a system may be affected by many factors including modelling uncertainty. With respect to uncertainty in the system, robust stability analysis is a very important research topic. Many results have been published to investigate the robust stability problem for uncertain continuous systems, (see e.g. [5]). However, there are few stability results reported for impulsive systems with uncertainties and disturbance inputs, (see e.g. [6], [9-10, 14-15]).

In this paper, we study the problem of robust stability and optimal feedback control for uncertain impulsive systems with a hybrid performance functional. Since there exist uncertainties in the system, it is difficult to calculate explicitly the value of the hybrid performance functional. Hence, it is meaningful to estimate the bound of the performance function. By employing the results developed in the [9], we derive some criteria under which the uncertain impulsive system is robustly asymptotically stable and the bounds of the hybrid performance functional can be estimated. These criteria are shown to be less conservative than those in [6].

The rest of this paper is organized as follows. In Section 2, we make some preliminaries and introduce some notations. We state and prove our main results in Sections 3. In Section 4, we discuss the problem for interval linear impulsive systems with quadratic hybrid performance functional.

2 PRELIMINARIES

Let $\mathbf{R}^n$ denote the n -dimensional Euclidean space. Let $\mathbf{R}_+ = [0, +\infty)$, $\mathbf{N} = \{1, 2, \dots\}$. Denote by $\mathcal{K}$ the class of functions $\phi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, which are continuous, strictly increasing and $\phi(0) = 0$, $\mathcal{K}_0$ the class of continuous functions $\psi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ such that $\psi(s) = 0$ if and only if

$s = 0$, and PC the class of functions $\lambda : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, where λ is continuous everywhere except $t_k (k \in \mathbf{N})$ at which λ is left continuous and the right limit $\lambda(t_k^+)$ exists. Let $\|\cdot\|$ stand for the Euclidean norm in $\mathbf{R}^n$. Let $\mathbf{P}^{n \times n}$ denote the set of $n \times n$ positive definite matrices, and $\mathbf{N}^{n \times n}$ the set of nonnegative definite matrices.

Consider the general uncertain impulsive controlled systems in form

$$\begin{aligned} \dot{x}(t) &= \tilde{f}_c(t, x(t), u_c(t)), \quad t \in (t_{k-1}, t_k] \\ \Delta x &= \tilde{f}_d(t, x(t), u_d(t)), \quad t = t_k, \quad k \in \mathbf{N} \\ x(t_0^+) &= x_0 \end{aligned} \quad (1)$$

where $0 < t_0 < t_1 < t_2 < \dots < t_k < \dots, t_k \rightarrow \infty$ as $k \rightarrow \infty$, and $x \in \mathbf{R}^n$, $\Delta x = x(t_k^+) - x(t_k^-) = x(t_k^+) - x(t_k)$; $u_c : \mathbf{R}_+ \rightarrow \mathbf{U}_c \subset \mathbf{R}^{n_c}$, $u_d : \mathbf{R}_+ \rightarrow \mathbf{U}_d \subset \mathbf{R}^{n_d}$, with $u_c(0) = 0, u_d(0) = 0$, here $(\mathbf{U}_c, \mathbf{U}_d) \subset \mathbf{R}^{n_c} \times \mathbf{R}^{n_d}$ is the class of admissible hybrid control inputs; $\tilde{f}_c \in \tilde{F}_c = \{\tilde{f}_c : \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^{n_c} \rightarrow \mathbf{R}^n, \tilde{f}_c(t, 0, 0) \equiv 0\}$, where $\tilde{f}_c(t, x, y)$ is Lipschitz continuous in (x, y) , $\tilde{f}_d \in \tilde{F}_d = \{\tilde{f}_d : \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^{n_d} \rightarrow \mathbf{R}^n, \tilde{f}_d(t, 0, 0) \equiv 0\}$, where $\tilde{f}_d$ is continuous. $\tilde{F}_c$ and $\tilde{F}_d$ denote the set of nonlinear uncertain functions with $\tilde{f}_{c0} \in \tilde{F}_c$, and $\tilde{f}_{d0} \in \tilde{F}_d$ defining the nominal nonlinear functions of $\tilde{f}_c \in \tilde{F}_c$ and $\tilde{f}_d \in \tilde{F}_d$, respectively. Denote $\tilde{F} = \tilde{F}_c \times \tilde{F}_d$.

The hybrid performance functional of system (1) can be described as

$$\begin{aligned} J_{(\tilde{f}_c, \tilde{f}_d)}^{t_f}(x_0, u_c, u_d) &= \int_{t_0}^{t_f} L_c(t, x(t), u_c(t)) dt \\ &+ \sum_{t_0 \leq t_k < t_f} L_d(t_k, x_k, u_d(t_k)) \end{aligned} \quad (2)$$

where $x_k = x(t_k)$, $x(t_f) = x_f$, where $t_0 < t_f < \infty$ or $t_f = \infty$; $L_c : \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^{n_c} \rightarrow \mathbf{R}$ and $L_d : \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^{n_d} \rightarrow \mathbf{R}$ are given.

The optimal robust control problem of (1) with hybrid performance functional (2) is to design the state feedback control law $(u_c(t), u_d(t_k)) = (\phi_c(x(t), t), \phi_d(x_k, t_k))$, where $\phi_c : \mathbf{R}^n \rightarrow \mathbf{R}^{n_c}$, $\phi_d : \mathbf{R}^n \rightarrow \mathbf{R}^{n_d}$ with $\phi_c(0) = 0, \phi_d(0) = 0$, such that the closed-loop system is robustly asymptotically

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stable, and the bounds of $J_{(\tilde{f}_c, \tilde{f}_d)}^{t_{\tilde{f}}}(x_0, \phi_c(x), \phi_d(x))$ can be estimated. Moreover, $(u_c(t), u_d(t_k)) = (\phi_c(x(t)), \phi_d(x_k))$ can minimizes the $J_{(\tilde{f}_{c0}, \tilde{f}_{d0k})}^{t_{\tilde{f}}}(x_0, u_c, u_d)$, which is the hybrid performance functional of the nominal system of (1). To proceed, some more preliminaries are needed.

Definition 2.1 (1) $X(t) \in \mathbf{R}^{n \times n}$ is said to be a positive definite matrix function if for any $t \in \mathbf{R}_+$, $X(t)$ is a positive definite matrix.

(2) $X(t) \in \mathbf{R}^{n \times n}$ is said to be a positive definite matrix function bounded from above if $X(t) \in \mathbf{R}^{n \times n}$ is a positive definite matrix function and there exists a positive real number $M > 0$ such that

$$\lambda_{\max}(X(t)) \leq M, t \in \mathbf{R}_+ \quad (3)$$

where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue.

(3) $X(t) \in \mathbf{R}^{n \times n}$ is said to be a uniformly positive definite matrix function if $X(t) \in \mathbf{R}^{n \times n}$ is a positive definite matrix function and there exists a positive real number $m > 0$ such that

$$\lambda_{\min}(X(t)) \geq m, t \in \mathbf{R}_+ \quad (4)$$

where $\lambda_{\min}(\cdot)$ is the minimum eigenvalue of matrix $(\cdot)$.

Definition 2.2 Let $V : \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}_+$, then V is said to belong to class ν_0 if

1) V is continuous in $(t_{k-1}, t_k] \times \mathbf{R}^n$ and for each $x \in \mathbf{R}^n, t \in (t_{k-1}, t_k], k \in \mathbf{N}$,

$$\lim_{(t,y) \rightarrow (t_{k-1}^+, x)} V(t, y) = V(t_{k-1}^+, x) \quad (5)$$

exists;

2) V is locally Lipschitzian in x .

Definition 2.4 Under hybrid control $(u_c(t), u_d(t_k)) = (\phi_c(x(t)), \phi_d(x_k))$, the uncertain impulsive system (1) is called robustly stable, robustly asymptotically stable, and robustly exponentially stable, respectively, if for any $\tilde{f}_c \in \tilde{F}_c, \tilde{f}_d \in \tilde{F}_d$, the equilibrium $x = 0$ of (1) is stable, asymptotic stable, and exponential stable, respectively.

3 MAIN RESULTS

In this section, we apply the results developed in [9] to analyze the hybrid performance functional $J_{(\tilde{f}_c, \tilde{f}_d)}^{t_{\tilde{f}}}(x_0)$ and robust stability criteria for system (1).

Theorem 3.1 Assume that there exist functions $\Gamma_c : \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^{n_c} \rightarrow \mathbf{R}, \Gamma_d : \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^{n_d} \rightarrow \mathbf{R}, V \in \nu_0$, where V is C^1 function, and $\phi_c : \mathbf{R}^n \rightarrow \mathbf{R}^{n_c}, \phi_d : \mathbf{R}^n \rightarrow \mathbf{R}^{n_d}$ with $\phi_c(0) = 0, \phi_d(0) = 0$, such that

(i) there exist $a, b \in \mathcal{K}$, such that for $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$,

$$b(\|x\|) \leq V(t, x) \leq a(\|x\|) \quad (6)$$

(ii) for all $t \in (t_{k-1}, t_k]$,

$$\begin{aligned} V_x(t, x) \tilde{f}_c(t, x, \phi_c(x)) &\leq V_x(t, x) \tilde{f}_{c0}(t, x, \phi_c(x)) \\ + \Gamma_c(t, x, \phi_c(x)) \end{aligned} \quad (7)$$

(iii) there exist $c \in \mathcal{K}, p \in PC$ such that for all $t \in (t_{k-1}, t_k]$,

$$\begin{aligned} V_t(t, x) + V_x(t, x) \tilde{f}_{c0}(t, x, \phi_c(x)) \\ + \Gamma_c(t, x, \phi_c(x)) \leq p(t)c(V(t, x(t))) \end{aligned} \quad (8)$$

(iv) there exist $\psi_k \in \mathcal{K}_0, k \in \mathbf{N}$, such that

$$\begin{aligned} V(t_k, x_k + \tilde{f}_d(t_k, x_k, \phi_d(x_k))) \\ \leq V(t_k, x_k + \tilde{f}_{d0}(t_k, x_k, \phi_d(x_k))) \\ + \Gamma_d(t_k, x_k, \phi_d(x_k)) \leq \psi_k(V(t_k, x_k)) \end{aligned} \quad (9)$$

(v) there exists a constant $\sigma > 0$ such that for all $z \in (0, \sigma)$

$$\int_{t_k}^{t_{k+1}} p(s)ds + \int_z^{\psi_k(z)} \frac{ds}{c(s)} \leq -r_k \quad (10)$$

where $r_k \geq 0$ and $\sum_{k=1}^{\infty} r_k = +\infty$;

(vi) for any $k \in \mathbf{N}$,

$$H_c(t, x, \phi_c(x)) = 0, \text{ for all } t \in (t_{k-1}, t_k] \quad (11)$$

$$H_d(t_k, x_k, \phi_d(x_k)) = 0, k \in \mathbf{N} \quad (12)$$

$$H_c(t, x, u_c) \geq 0, \text{ for all } u_c \in U_c \quad (13)$$

$$H_d(t, x, u_d) \geq 0, \text{ for all } u_d \in U_d \quad (14)$$

where

$$\begin{aligned} H_c(t, x, u_c) &= L_d(t, x, u_c) + V_t(t, x) \\ + V_x(t, x) \tilde{f}_{c0}(t, x, u_c) + \Gamma_c(t, x, u_c) \end{aligned} \quad (15)$$

$$\begin{aligned} H_d(t, x, u_d) &= V(t, x + \tilde{f}_{d0}(t, x, u_d)) \\ + L_d(t, x, u_d) - V(t, x) + \Gamma_d(t, x, u_d) \end{aligned} \quad (16)$$

Then, with the state feedback control law $(u_c(t), u_d(t_k)) = (\phi_c(x(t)), \phi_d(x_k)), t \geq t_0, k \in \mathbf{N}$, the zero solution $x(t) = 0$ of the closed-loop system (1) is robustly asymptotically stable for all $(\tilde{f}_c, \tilde{f}_d) \in \tilde{F}$, and

$$\begin{aligned} \sup_{(\tilde{f}_c, \tilde{f}_d) \in \tilde{F}} J_{(\tilde{f}_c, \tilde{f}_d)}^{t_{\tilde{f}}}(x_0, \phi_c(x), \phi_d(x)) \\ \leq I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^{t_{\tilde{f}}}(x_0, \phi_c(x), \phi_d(x)) \leq V(t_0, x_0) \end{aligned} \quad (17)$$

Specially,

$$\begin{aligned} \sup_{(\tilde{f}_c, \tilde{f}_d) \in F} J_{(\tilde{f}_c, \tilde{f}_d)}^{\infty}(x_0, \phi_c(x), \phi_d(x)) \\ \leq I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^{\infty}(x_0, \phi_c(x), \phi_d(x)) = V(t_0, x_0) \end{aligned} \quad (18)$$

where

$$\begin{aligned} I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^{t_{\tilde{f}}}(x_0, u_c, u_d) &= \int_{t_0}^{t_{\tilde{f}}} [L_c(t, x(t), u_c(t)) \\ + \Gamma_c(t, x(t), u_c(t))] dt + \sum_{t_0 \leq t_k < t_{\tilde{f}}} [L_d(t_k, \\ x(t_k), u_d(t_k)) + \Gamma_d(t_k, x_k, u_d(t_k))] \end{aligned} \quad (19)$$

where $t_{\tilde{f}} < \infty$ or $t_{\tilde{f}} = \infty$, and in (18)-(19), $x(t), t \neq t_0$, is a solution to system (1) with $(\tilde{f}_c, \tilde{f}_d) = (\tilde{f}_{c0}, \tilde{f}_{d0})$.

Moreover, the hybrid feedback control law $(u_c(t), u_d(t_k)) = (\phi_c(x(t)), \phi_d(x_k)), t \geq t_0, k \in \mathbf{N}$, minimizes $I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^{\infty}(x_0, u_c, u_d)$ in sense that

$$\begin{aligned} I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^{\infty}(x_0, \phi_c(x), \phi_d(x)) \\ = \min_{(u_c, u_d) \in (U_c, U_d)} I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^{\infty}(x_0, u_c, u_d) \end{aligned} \quad (20)$$

Proof At first, we show system (1) is robustly asymptotically stable. Let $x(t), t \neq t_0$ be the solution to system (1). Then, by conditions (ii)-(iii), we get

$$\begin{aligned} D^+V(t, x(t)) &= V_t(t, x) + V_x(t, x)f_c(t, x, \phi_c(x)) \\ &\leq V_t(t, x) + V_x(t, x)f_{c0}(t, x, \phi_c(x)) \\ &\quad + \Gamma_c(t, x, \phi_c(x)) \\ &\leq p(t)c(V(t, x(t))), \quad t \in (t_{k-1}, t_k], \quad k \in \mathbf{N} \end{aligned} \quad (21)$$

and from condition (iv),

$$V(t_k, x_k + f_d(t_k, x_k, \phi_d(x_k))) \leq \psi_k(V(t_k, x_k)) \quad (22)$$

Thus, from (21)-(22) and conditions (i) and (v), we obtain that all conditions of Theorem 3.1 in [9] are satisfied and hence, under the hybrid control law $(u_c(t), u_d(t_k)) = (\phi_c(x(t)), \phi_d(x_k)), t \geq t_0, k \in \mathbf{N}$, (1) is robustly asymptotically stable for all $(\tilde{f}_c, \tilde{f}_d) \in \tilde{F}$.

Secondly, we show estimations (17)-(18) hold.

By conditions (ii) and (vi) ((11)), we get

$$\begin{aligned} L_c(t, x, \phi_c(x)) &= -D^+V(t, x(t)) + L_c(t, x, \phi_c(x)) \\ &\quad + V_t(t, x) + V_x(t, x)\tilde{f}_c(t, x, \phi_c(x)) \\ &\leq -D^+V(t, x(t)) + H_c(t, x, \phi_c(x)) \\ &= -D^+V(t, x(t)) \end{aligned} \quad (23)$$

By conditions (iv) and (vi)((12)), we have

$$\begin{aligned} L_d(t_k, x_k, \phi_d(x_k)) &= -\Delta V(t_k, x_k) + L_d(t_k, x_k, \phi_d(x_k)) \\ &\quad + V(t_k, x_k + \tilde{f}_d(t_k, x_k, \phi_d(x_k))) - V(t_k, x_k) \\ &\leq -\Delta V(t_k, x_k) + H_d(t_k, x_k, \phi_d(x_k)) \\ &= -\Delta V(t_k, x_k) \end{aligned} \quad (24)$$

Now, integrating over the interval $[t_0, t_f]$ with $\{k \in \mathbf{N} : t_0 \leq t_k < t_f\} = \{1, 2, \dots, m\}$, by (23) and (24), we have

$$\begin{aligned} J_{(\tilde{f}_c, \tilde{f}_d)}^{t_f}(x_0, \phi_c, \phi_d) &= \sum_{i=0}^{m-1} \int_{t_i^+}^{t_{i+1}} L_c(t, x(t), \phi_c(x(t))) dt \\ &\quad + \int_{t_m^+}^{t_f} L_c(t, x, \phi_c(x)) dt + \sum_{i=1}^m L_d(t_i, x_i, \phi_d(x_i)) \\ &\leq -\sum_{i=0}^{m-1} \int_{t_i^+}^{t_{i+1}} D^+V(t, x) dt - \int_{t_m^+}^{t_f} D^+V(t, x) dt \\ &\quad - \sum_{i=1}^m \Delta V(t_i, x_i) \\ &= -(V(t_1, x_1) - V(t_0, x_0) + V(t_2, x_2) \\ &\quad - V(t_1, x_1 + f_d(t_1, x_1)) + \dots + V(t_m, x_m) \\ &\quad - V(t_{m-1}, x_{m-1} + f_d(t_{m-1}, x_{m-1})) \\ &\quad - (V(t_f, x_f) - V(t_m, x_m + f_d(t_m, x_m))) \\ &\quad - \sum_{i=1}^m \Delta V(t_i, x_i) = V(t_0, x_0) - V(t_f, x_f) \end{aligned} \quad (25)$$

Moreover, if $x(t)$ is a solution to (1) with $(\tilde{f}_c, \tilde{f}_d) = (\tilde{f}_{c0}, \tilde{f}_{d0})$, then, by condition (ii) and (11), for all $t \in$

$(t_{k-1}, t_k]$, we get

$$\begin{aligned} L_c(t, x, \phi_c(x)) + \Gamma_c(t, x(t), \phi_c(x)) \\ = -D^+V(t, x(t)) \end{aligned} \quad (26)$$

Similarly, the condition (iv) and (12) implies that

$$\begin{aligned} L_d(t_k, x_k, \phi_d(x_k)) + \Gamma_d(t_k, x_k, \phi_d(x_k)) \\ = -\Delta V(t_k, x_k) \end{aligned} \quad (27)$$

Hence, integrating over the interval $[t_0, t_f]$ with $\{k \in \mathbf{N} : t_0 \leq t_k < t_f\} = \{1, 2, \dots, m\}$, by (26)-(27), we have

$$\begin{aligned} I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^{t_f}(x_0, \phi_c(x), \phi_d(x)) &= \sum_{i=0}^{m-1} \int_{t_i^+}^{t_{i+1}} [L_c(t, x, \phi_c(x)) + \Gamma_c(t, x, \phi_c(x))] dt \\ &\quad + \int_{t_m^+}^{t_f} [L_c(t, x, \phi_c(x)) + \Gamma_c(t, x, \phi_c(x))] dt \\ &\quad + \sum_{i=1}^m [L_d(t_i, x(t_i), \phi_d(x_i)) + \Gamma_d(t_i, x_i, \phi_d(x_i))] \\ &= -\sum_{i=0}^{m-1} \int_{t_i^+}^{t_{i+1}} D^+V(t, x(t)) dt \\ &\quad - \int_{t_m^+}^{t_f} D^+V(t, x(t)) dt - \sum_{i=1}^m \Delta V(t_i, x_i) \\ &= V(t_0, x_0) - V(t_f, x_f) \end{aligned} \quad (28)$$

Thus, by $V(t_f, x_f) \geq 0$ and (25) and (28), we obtain that (17) holds. Moreover, by the asymptotic stability, which means that $\lim_{t_f \rightarrow \infty} V(t_f, x_f) = 0$, we get that (18) also holds.

Finally, we need to prove that the hybrid control law $(u_c(t), u_d(t_k)) = (\phi_c(x(t)), \phi_d(x(t_k))), t \geq t_0, k \in \mathbf{N}$, minimizes $I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^\infty(x_0, u_c, u_d)$ in sense of (20).

Let $(\tilde{f}_c, \tilde{f}_d) = (\tilde{f}_{c0}, \tilde{f}_{d0})$, then, by (15), for all $t \in (t_{k-1}, t_k], k \in \mathbf{N}$,

$$\begin{aligned} L_c(t, x(t), u_c(t)) + \Gamma_c(t, x(t), u_c(t)) \\ = -D^+V(t, x(t)) + H_c(t, x(t), u_c(t)) \end{aligned} \quad (29)$$

and by (16), for any $k \in \mathbf{N}$,

$$\begin{aligned} L_d(t_k, x_k, u_d(t_k)) + \Gamma_d(t_k, x_k, u_d(t_k)) \\ = -\Delta V(t_k, x_k) + H_d(t_k, x_k, u_d(t_k)) \end{aligned} \quad (30)$$

Thus, using (29) and (30), and the fact that system (1) with $(\tilde{f}_c, \tilde{f}_d) = (\tilde{f}_{c0}, \tilde{f}_{d0})$ is asymptotic stable, we have

$$\begin{aligned} I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^\infty(x_0, u_c, u_d) &= \int_{t_0}^\infty [-D^+V(t, x(t)) \\ &\quad + H_c(t, x(t), u_c(t))] dt \\ &\quad + \sum_{k=1}^\infty [-\Delta V(t_k, x_k) + H_d(t_k, x_k, u_d(t_k))] \\ &= -\lim_{t \rightarrow \infty} V(t, x(t)) + V(t_0, x_0) \\ &\quad + \int_{t_0}^\infty H_c(t, x(t), u_c(t)) dt + \sum_{k=1}^\infty H_d(t_k, x_k, u_d(t_k)) \\ &\geq V(t_0, x_0) = I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^\infty(x_0, \phi_c(x), \phi_d(x)) \end{aligned} \quad (31)$$

Hence, by (31), the hybrid feedback control law $(u_c(t), u_d(t_k)) = (\phi_c(x(t)), \phi_d(x_k))$, $t \geq t_0$, $k \in \mathbf{N}$, minimizes $I_{(\tilde{f}_{c0}, \tilde{f}_{d0})}^\infty(x_0, u_c, u_d)$ in sense (20). The proof is complete.

Remark 3.1 (1') By Theorem 3.2 in [9], in Theorem 3.1, if $p(t)$ is replaced by $-p(t)$, then the results of Theorem 3.1 still hold.

(2') By (11)-(12), if nonnegative functions $H_c(t, x, u_c)$, $H_d(t, x, u_d)$ are continuously differentiable in u_c and u_d , respectively, then,

$$\left. \frac{\partial H_c(t, x, u_c)}{\partial u_c} \right|_{u_c=\phi_c(x)} = 0, \left. \frac{\partial H_d(t, x, u_d)}{\partial u_d} \right|_{u_d=\phi_d(x)} = 0$$

which can be used to derive the hybrid state feedback control law $(u_c, u_d) = (\phi_c(x), \phi_d(x))$.

(3') From the proof of Theorem 3.1, we get that $D^+V(t, x) \leq p(t)c(V(t, x))$ for any $t \neq t_k$. While in [6], it needs that $D^+V(t, x) \leq 0$ for any $t \neq t_k$. Since $p(t) \geq 0$, the results of Theorem 3.1 is less conservative than that in [6].

4 SPECIALIZATION TO LINEAR IMPULSIVE SYSTEMS

In this section, we specialize Theorem 3.1 to interval linear impulsive system.

Consider interval linear impulsive system in form:

$$\begin{aligned} \dot{x} &= (A_c(t) + B_c(t))x + D_c(t)u_c(t), \quad t \neq t_k \\ \Delta x &= (A_{d_k}(t) + B_{d_k}(t))x + D_{d_k}u_d(t), \quad t = t_k \\ x(t_0^+) &= x_0, \quad k \in \mathbf{N} \end{aligned} \quad (32)$$

with the hybrid quadratic performance functional:

$$\begin{aligned} J_{(\tilde{A}_c, \tilde{A}_{d_k})}^{t_f}(x_0, u_c, u_d) &= \int_{t_0}^{t_f} [x^T(t)R_c(t)x(t) \\ &+ u_c^T(t)S_c(t)u_c(t)]dt + \sum_{t_0 \leq t_k < t_f} [x_k^T R_d x_k \\ &+ u_d^T(t_k)S_d(t_k)u_d(t_k)] \end{aligned} \quad (33)$$

where $A_c(t), A_{d_k}(t) \in \mathbf{R}^{n \times n}$, $D_c(t) \in \mathbf{R}^{n \times n_c}$, $D_{d_k}(t) \in \mathbf{R}^{n \times n_d}$, $S_c(t) \in \mathbf{P}^{n_c \times n_c}$, $S_d(t) \in \mathbf{N}^{n_d \times n_d}$, $R_c(t) \in \mathbf{P}^{n \times n}$, and $R_d \in \mathbf{N}^{n \times n}$, are known matrices, while $B_c(t), B_{d_k}(t), k \in \mathbf{N}$ are interval matrices: $B_c(t) \in N[P_c(t), Q_c(t)] = \{B_c(t) \in \mathbf{R}^{n \times n} : B_c(t) = (b_c(t)_{ij})_{n \times n}, p_c(t)_{ij} \leq b_c(t)_{ij} \leq q_c(t)_{ij}; i, j = 1, 2, \dots, n\}$, $B_{d_k}(t) \in N[P_{d_k}(t), Q_{d_k}(t)]$, where $P_c(t) = (p_c(t)_{ij})_{n \times n}$, $P_{d_k}(t) = (p_{d_k}(t)_{ij})_{n \times n}$, $Q_c(t) = (q_c(t)_{ij})_{n \times n}$, $Q_{d_k}(t) = (q_{d_k}(t)_{ij})_{n \times n}$, $k \in \mathbf{N}$, are known matrices.

By Lemma 3.1 in [9], for any interval matrix $B(t) \in N[P(t), Q(t)]$, where $P = (p(t)_{ij})_{n \times n}$ and $Q = (q(t)_{ij})_{n \times n}$ are known matrices, then, we have the representation of $B(t)$ as follows:

$$B(t) = B(t)_0 + E(t)\Sigma(t)F(t) \quad (34)$$

where $B(t)_0 = \frac{1}{2}(P(t) + Q(t))$, $H(t) = (h(t)_{ij})_{n \times n} = \frac{1}{2}(Q(t) - P(t))$, $\Sigma(t) \in \Sigma^* = \{\Sigma(t) \in \mathbf{R}^{n^2 \times n^2} : \Sigma(t) = \text{diag}(\varepsilon(t)_{11}, \dots, \varepsilon(t)_{1n}, \dots, \varepsilon(t)_{n1}, \dots,$

$\varepsilon(t)_{nn})$, $|\varepsilon(t)_{ij}| \leq 1; i, j = 1, 2, \dots, n\}$, and $E(t)E(t)^T = \text{diag}(\sum_{j=1}^n h(t)_{1j} \cdots \sum_{j=1}^n h(t)_{nj})$, $F(t)^T F(t) = \text{diag}(\sum_{j=1}^n h(t)_{j1} \cdots \sum_{j=1}^n h(t)_{jn})$. In the following, denote: $A_c(t) + B_c(t) = A_{c0}(t) + E_c(t)\Sigma_c(t)F_c(t)$ and $A_{d_k}(t) + B_{d_k}(t) = A_{d0_k}(t) + E_{d_k}(t)\Sigma_{d_k}(t)F_{d_k}(t)$, $\Sigma_{d_k} \in \Sigma^*$, $k \in \mathbf{N}$.

Theorem 4.1 Assume that there exist scalar functions $\lambda_k(t) > 0, \alpha(t) > 0, k \in \mathbf{N}$, and a uniformly positive definite matrix function bounded from above $X(t)$ such that (i) $X(t)$ is differentiable at $t \neq t_k$ and for all $t \in (t_{k-1}, t_k]$, $k \in \mathbf{N}$, the Riccati inequality holds:

$$\begin{aligned} \dot{X} + X(A_{c0} + D_c K_c) + (A_{c0} + D_c K_c)^T X \\ + 2\lambda_k^2 X E_c E_c^T X + \frac{1}{2\lambda_k^2} F_c^T F_c < -\alpha X \end{aligned} \quad (35)$$

where $K_c = K_c(t) = -S_c^{-1}(t)D_c^T(t)X(t)$, $\lambda_k = \lambda_k(t)$, and $\alpha = \alpha(t)$;

(ii)

$$\begin{aligned} R_c + K_c^T S_c K_c + \dot{X} + X(A_{c0} + D_c K_c) + (A_{c0} + \\ D_c K_c)^T X + 2\lambda_k^2 X E_c E_c^T X + \frac{1}{2\lambda_k^2} F_c^T F_c = 0 \end{aligned} \quad (36)$$

(iii) there exist some $r_k \geq 0$ with $\sum_{k=1}^n r_k = +\infty$, and $\epsilon_k > 0, k \in \mathbf{N}$ such that for all $k \in \mathbf{N}$,

$$-\int_{t_k}^{t_{k+1}} \alpha(s)ds + \ln \beta_k \leq -r_k \quad (37)$$

where

$$\begin{aligned} \beta_k = \lambda_{\max} \left\{ X_k^{-1} \left[(I + A_{d0_k} + D_{d_k} K_{d_k})^T [X_k + \right. \right. \\ \left. \left. \epsilon_k^{-1} X_k E_{d_k} E_{d_k}^T X_k] (I + A_{d0_k} + D_{d_k} K_{d_k}) + [\epsilon_k + \right. \right. \\ \left. \left. \lambda_{\max}(E_{d_k}^T X_k E_{d_k})] F_{d_k}^T F_{d_k} \right] \right\} \end{aligned}$$

and where, $X_k = X(t_k)$, $E_{d_k} = E_{d_k}(t_k)$, $F_{d_k} = F_{d_k}(t_k)$, $S_{d_k} = S_d(t_k)$, and $D_{d_k} = D_{d_k}(t_k)$, $K_{d_k} = K_d(t_k) = -S_{d_k}^{-1} D_{d_k}^T X_k (I + A_{d0_k})$

(iv)

$$\begin{aligned} R_d + K_{d_k}^T S_{d_k} K_{d_k} - X_k + (I + A_{d0_k} + D_{d_k} K_{d_k})^T \\ \cdot [X_k + \epsilon_k^{-1} X_k E_{d_k} E_{d_k}^T X_k] (I + A_{d0_k} + D_{d_k} K_{d_k}) \\ + [\epsilon_k + \lambda_{\max}(E_{d_k}^T X_k E_{d_k})] F_{d_k}^T F_{d_k} = 0 \end{aligned} \quad (38)$$

Then, with the state feedback control law

$$(u_c, u_d) = (\phi_c(x), \phi_d(x)) = (K_c x, K_d x) \quad (39)$$

the zero solution $x(t) = 0$ of the closed-loop system (32) and (39) is robustly asymptotically stable for all $(B_c, B_{d_k}) \in \Sigma = \Sigma_c^* \times \Sigma_d^*$, and

$$\begin{aligned} \sup_{(\Sigma_c, \Sigma_{d_k}) \in \Sigma} J_{(\tilde{A}_c, \tilde{A}_{d_k})}^{t_f}(x_0, u_c, u_d) \\ \leq I_{(A_{c0}, A_{d0_k})}^{t_f}(x_0, \phi_c(x), \phi_d(x)) \leq x_0^T X_0 x_0 \end{aligned} \quad (40)$$

Specially,

$$\begin{aligned} \sup_{(\Sigma_c, \Sigma_{d_k}) \in \Sigma} J_{(\tilde{A}_c, \tilde{A}_{d_k})}^\infty(x_0, u_c, u_d) \\ \leq I_{(A_{c0}, A_{d0_k})}^\infty(x_0, \phi_c(x), \phi_d(x)) = x_0^T X_0 x_0 \end{aligned} \quad (41)$$

where $X_0 = X(t_0)$, and,

$$\begin{aligned}
I_{(A_{c0}, A_{d0k})}^{t_f}(x_0, u_c, u_d) &= \int_{t_0}^{t_f} \left(x^T(t) [R_c(t) \right. \\
&+ 2\lambda_k^2(t) X(t) E_c(t) E_c^T(t) X(t) \\
&+ \frac{1}{2\lambda_k^2(t)} F_c^T(t) F_c(t)] x(t) + u_c^T(t) S_c(t) u_c(t) \Big) dt \\
&+ \sum_{t_0 \leq t_k < t_f} \left\{ x_k \left[R_d + \epsilon_k^{-1} (I + A_{d0k})^T X_k E_{d_k} \right. \right. \\
&\cdot E_{d_k}^T X_k (I + A_{d0k}) + [\epsilon_k + \lambda_{\max}(E_{d_k}^T X_k E_{d_k})] \\
&\cdot F_{d_k}^T F_{d_k}] x_k + u_d^T(t_k) S_d(t_k) u_d(t_k) \Big\} \quad (42)
\end{aligned}$$

where $t_f < \infty$ or $t_f = \infty$, and in (42), $x(t)$, $t \neq t_0$, is a solution to system (32) with $(B_c, B_{d_k}) = (0, 0)$, $k \in \mathbf{N}$. Moreover,

$$\begin{aligned}
I_{(A_{c0}, A_{d0k})}^{\infty}(x_0, \phi_c(x), \phi_d(x)) \\
= \min_{(u_c, u_d) \in (U_c, U_d)} I_{(A_{c0}, A_{d0k})}^{\infty}(x_0, u_c, u_d) \quad (43)
\end{aligned}$$

Proof Let $V(t, x) = x^T X(t) x$. Then, the results of this theorem are the direct consequence of Theorem 3.1 and Remark 3.1(1'). The detailed is omitted here.

Remark 4.1 By Theorem 3.1 and remark 3.1(1'), if $\alpha(t)$ is replaced by $-\alpha(t)$ in Theorem 4.1, the results are still true.

5 CONCLUSIONS

In this paper, we proposed the optimal robust control problem for uncertain impulsive systems with hybrid performance functional. By employing the Lyapunov function method and algebraic inequalities, we have derived robust stability conditions and the optimal bound of the hybrid performance functional. As the applications, we have investigated robust optimal control problem for the interval linear impulsive systems with quadratic hybrid performance functional. We have obtained the Riccati inequality conditions under which feedback control law can be designed to stabilize asymptotically the interval linear impulsive system and optimal bound of quadratic hybrid performance functional are derived.

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