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Invariant subspaces of submarkovian semigroups

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Abstract. We characterize the invariance under a submarkovian semigroup of a measurable subset by capacity conditions on its boundary.

1. Introduction

Let S be a submarkovian semigroup acting on $L_2(X\,;\mu)$ where X is a locally compact σ-compact metric space and μ a Radon measure with $\operatorname{supp}\mu = X$ (see [7] and [3]). If Ω is a measurable subset of X we give characterizations of the S-invariance of $L_2(\Omega)$, i.e. the property $S_t L_2(\Omega) \subseteq L_2(\Omega)$ for all $t > 0$. This corresponds to S-invariance of the set Ω in the terminology of [7], Section 1.6.

If one interprets S as describing a dissipative evolution then the invariance of Ω corresponds to impenetrability of the boundary $\partial\Omega$. Therefore one would expect the invariance to be characterized by properties of the boundary. Our main result establishes that this is indeed the case; invariance of Ω is equivalent to a capacity condition on the boundary $\partial\Omega$. In this respect it differs from the standard characterizations of invariance. These are either in terms of local properties of the generator of S (see, for example, [7], Section 1.6 and [10], Section 2.1) or, in the case of irreducibility of the semigroup, in terms of spectral properties or algebraic properties (see [11], Section XIII.12, and [12], Proposition 2.1).

In order to formulate our main result we must first introduce some basic definitions and notation.

Let H denote the positive self-adjoint generator of S and h the corresponding Dirichlet form (see [7] and [3]). We assume that h is regular in the sense of [7], i.e. $D(h) \cap C_c(X)$ is dense in $D(h)$ with respect to the graph norm $\varphi \mapsto \|\varphi\|_{D(h)} = (h(\varphi) + \|\varphi\|_2^2)^{1/2}$ and also dense in $C_0(X)$ with respect to the uniform norm. Moreover, we define h to be local if $h(\varphi, \psi) = 0$ for all $\varphi, \psi \in D(h)$ with $\varphi\,\psi = 0$. This notion appears slightly stronger than locality as defined in [7] but it is in fact equivalent by a result of Schmuland [13]. Nevertheless, it is weaker than the form of locality introduced in [3]. Next if Ω is a subset of X and $A \subseteq \overline{\Omega}$ then we define $\operatorname{cap}_\Omega(A) \in [0, \infty]$ by

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$$\mathrm{cap}_\Omega(A) \;=\; \inf\Big\{\|\varphi\|^2_{D(h)} : \varphi\in D(h) \text{ and there exists an open } V\subset X \text{ such that } A\subset V \text{ and } \varphi\geq 1 \text{ a.e. on } V\cap\Omega\Big\}.$$

If there is any possible ambiguity then we use the notation $\mathrm{cap}_{\Omega,h}(A)$ instead of $\mathrm{cap}_\Omega(A)$. Note that $\Omega\mapsto \mathrm{cap}_\Omega(A)$ is a monotonically increasing function for each fixed A. Moreover, $\mathrm{cap}_X(A) = \mathrm{cap}(A)$ where $\mathrm{cap}(A)$ is the usual capacity of A with respect to h as defined in [7], Section 2.1, or [3], Section 1.8. A similar definition of relative capacity is introduced in [1] [2] and in special cases the two definitions coincide, e.g. if H is the Laplacian on $\mathbf{R}^d$ and Ω is a bounded set with Lipschitz boundary. We do not know, however, whether the definitions coincide in general or whether our results are valid when reformulated in terms of this alternative definition.

The main result of this note is the following statement.

THEOREM 1.1. *Let Ω be a measurable subset of X and suppose that the regular Dirichlet form h is local. The following conditions are equivalent.*

I. $S_t L_2(\Omega)\subseteq L_2(\Omega)$ *for all* $t>0$.
II. *There exist* $A_1, A_2\subseteq\partial\Omega$ *such that* $\partial\Omega = A_1\cup A_2$ *and* $\mathrm{cap}_\Omega(A_1)=0=\mathrm{cap}_{\Omega^c}(A_2)$.

The proof of the implication I⇒II in Theorem 1.1 does not require locality of h but this property is used in proving the converse. In the course of the proof we also derive several alternative characterizations of the S-invariance of $L_2(\Omega)$. Another characterization of S-invariance for general measurable subsets Ω is given in [5], Theorem 1.3.

Theorem 1.1 extends to local regular Dirichlet forms a result of [12] for forms defined by degenerate elliptic operators on $\mathbf{R}^d$ and open subsets Ω. Theorem 1.1 in [12] states that if the coefficients of the degenerate operator are Lipschitz continuous and Ω is a Lipschitz domain then S-invariance of $L_2(\Omega)$ is equivalent to $\mathrm{cap}(\partial\Omega)=0$. Nevertheless it is straightforward to construct examples of S-invariant subspaces $L_2(\Omega)$ with $|\partial\Omega|>0$. But then $\mathrm{cap}(\partial\Omega)\geq|\partial\Omega|>0$. So the condition $\mathrm{cap}(\partial\Omega)=0$ does not give a characterization of S-invariance in general. The current results show that S-invariance of the subspace $L_2(\Omega)$ can indeed be characterized by conditions of small capacity of sets close to the boundary of Ω without any detailed assumptions of regularity.

In the last section we illustrate our results with various examples of degenerate elliptic operators. Note that for degenerate operators a general sufficient criterion for the existence of invariant subspaces was given in [6], Lemma 6.4 and Proposition 6.10.

2. Invariance criteria

We begin with a useful corollary of a theorem of Ouhabaz valid for 'local' accretive closed sesquilinear forms.

PROPOSITION 2.1. *Let* $\mathfrak{a}$ *be an accretive closed sesquilinear form in a Hilbert space* $L_2(Y)$*, where* Y *is a measure space. Suppose that* $\mathfrak{a}(\varphi, \psi) = 0$ *for all* $\varphi, \psi \in D(\mathfrak{a})$ *with* $\varphi\,\psi = 0$*. Let* T *be the semigroup associated to* $\mathfrak{a}$ *and let* Ω *be a measurable subset of* Y*. Then the following are equivalent.*

I. $T_t L_2(\Omega) \subseteq L_2(\Omega)$ *for all* $t > 0$.
II. $\mathbb{1}_\Omega \varphi \in D(\mathfrak{a})$ *for all* $\varphi \in D(\mathfrak{a})$.
III. *There exists a core* D *for* $\mathfrak{a}$ *such that* $\mathbb{1}_\Omega \varphi \in D(\mathfrak{a})$ *for all* $\varphi \in D$.

Proof. Let P be the orthogonal projection of $L_2(Y)$ onto $L_2(\Omega)$. By [10], Theorem 2.2, the invariance of $L_2(\Omega)$ under T is equivalent to the statement(s) that there exists a core D for $\mathfrak{a}$ (or for every core) such that $P(D) \subseteq D(\mathfrak{a})$ and $\operatorname{Re} \mathfrak{a}(P\varphi, \varphi - P\varphi) \geq 0$ for all $\varphi \in D$. This implies that I⇒II⇒III. But $P\varphi = \mathbb{1}_\Omega \varphi$ and $\varphi - P\varphi = \mathbb{1}_{\Omega^c}\varphi$. So if $P(D) \subseteq D(\mathfrak{a})$ then by 'locality' $\mathfrak{a}(P\varphi, \varphi - P\varphi) = 0$ for all $\varphi \in D$. This proves the proposition. □

As a preliminary for the proof of Theorem 1.1 we derive some criteria for the S-invariance property. The first result is an implication of invariance which does not require locality of the form h.

PROPOSITION 2.2. *Let* Ω *be a measurable subset of* X*. If* $S_t\, L_2(\Omega) \subseteq L_2(\Omega)$ *for all* $t > 0$ *then for all* $\varepsilon > 0$ *there exist open sets* $U, V_1, V_2 \subseteq X$ *such that* $\operatorname{cap}(U) < \varepsilon$*,* $\partial\Omega \subseteq V_1 \cup V_2$*,* $|V_1 \cap \Omega \cap U^c| = 0$ *and* $|V_2 \cap \Omega^c \cap U^c| = 0$.

Proof. Let W be a relatively compact open subset of X. We first show that for all $\varepsilon > 0$ there exists an open set $U \subseteq X$ such that $\operatorname{cap}(U) < \varepsilon$ and for all $x \in W \cap \partial\Omega$ there exists a $\delta > 0$ such that $|B(x\,;\delta) \cap \Omega \cap U^c| = 0$ or $|B(x\,;\delta) \cap \Omega^c \cap U^c| = 0$.

Since h is regular there exists a $\varphi \in D(h) \cap C_c(X)$ with $0 \leq \varphi \leq 1$ and $\varphi|_{\overline{W}} = 1$. Set $\psi = \varphi\,\mathbb{1}_\Omega$. Then since $L_2(\Omega)$ is S-invariant it follows, by [7], Theorem 1.6.1, that $\psi \in D(h)$. Let $\widetilde{\psi}$ be a quasi-continuous representative of ψ. Then there exists an open set $U \subseteq X$ such that $\operatorname{cap}(U) < \varepsilon$ and $\widetilde{\psi}|_{U^c}$ is continuous. Then

$$\widetilde{\psi}|_{W\cap\Omega\cap U^c} = \psi|_{W\cap\Omega\cap U^c} = 1 \text{ a.e.}$$

and

$$\widetilde{\psi}|_{W\cap\Omega^c\cap U^c} = \psi|_{W\cap\Omega^c\cap U^c} = 0 \text{ a.e.}$$

by construction of φ and ψ.

Let $x \in W \cap \partial\Omega$. If $x \in U$ then clearly there exists a $\delta > 0$ such that $B(x\,;\delta) \subseteq U$. Hence $|B(x\,;\delta) \cap \Omega \cap U^c| = 0$. So suppose that $x \in U^c$. Since $\widetilde{\psi}|_{U^c}$ is continuous at x there exists a $\delta > 0$ such that

$$|\widetilde{\psi}(y) - \widetilde{\psi}(x)| < \frac{1}{2} \tag{2.1}$$

for all $y \in B(x\,;\delta) \cap U^{\rm c}$. We may assume that $B(x\,;\delta) \subseteq W$. Since $B(x\,;\delta) \cap \Omega \cap U^{\rm c} \subseteq W \cap \Omega \cap U^{\rm c}$ it follows that $\widetilde{\psi}(y) = 1$ for a.e. $y \in B(x\,;\delta) \cap \Omega \cap U^{\rm c}$. Similarly $\widetilde{\psi}(y) = 0$ for a.e. $y \in B(x\,;\delta) \cap \Omega^{\rm c} \cap U^{\rm c}$. Hence if $|B(x\,;\delta) \cap \Omega \cap U^{\rm c}| > 0$ then it follows from (2.1) that $|B(x\,;\delta) \cap \Omega^{\rm c} \cap U^{\rm c}| = 0$.

Since X is σ-compact, for all $\varepsilon > 0$ there exists an open set $U \subseteq X$ such that $\operatorname{cap}(U) < \varepsilon$ and for all $x \in \partial\Omega$ there exists $\delta_x > 0$ with $|B(x\,;\delta_x) \cap \Omega \cap U^{\rm c}| = 0$ or $|B(x\,;\delta_x) \cap \Omega^{\rm c} \cap U^{\rm c}| = 0$. Define

$$A_1 = \{x \in \partial\Omega : |B(x\,;\delta_x) \cap \Omega \cap U^{\rm c}| = 0\}$$

and

$$A_2 = \{x \in \partial\Omega : |B(x\,;\delta_x) \cap \Omega^{\rm c} \cap U^{\rm c}| = 0\}.$$

By the basic covering theorem, [8] Theorem 1.2, there exists a subset $A_1' \subseteq A_1$ such that $\bigcup_{x \in A_1} B(x\,;5^{-1}\delta_x) \subseteq \bigcup_{x \in A_1'} B(x\,;\delta_x)$ and the sets $B(x\,;5^{-1}\delta_x)$ with $x \in A_1'$ are pairwise disjoint. Since X is σ-compact, it is separable. Therefore A_1' is countable. Set $V_1 = \bigcup_{x \in A_1'} B(x\,;\delta_x)$. Then V_1 is open in X and $A_1 \subseteq \bigcup_{x \in A_1} B(x\,;5^{-1}\delta_x) \subseteq V_1$. Moreover,

$$|V_1 \cap \Omega \cap U^{\rm c}| \leq \sum_{x \in A_1'} |B(x\,;\delta_x) \cap \Omega \cap U^{\rm c}| = 0.$$

Similarly there exists an open $V_2 \subseteq X$ such that $A_2 \subseteq V_2$ and $|V_2 \cap \Omega^{\rm c} \cap U^{\rm c}| = 0$. Finally, $\partial\Omega = A_1 \cup A_2 \subseteq V_1 \cup V_2$. □

The implication I⇒II in Theorem 1.1 follows from Proposition 2.2 and the next proposition. We emphasize that the following proposition does not require locality of the form h.

PROPOSITION 2.3. *Let Ω be a measurable subset of X. If for all $\varepsilon > 0$ there exist open sets $U, V_1, V_2 \subseteq X$ such that* $\operatorname{cap}(U) < \varepsilon$*, $\partial\Omega \subseteq V_1 \cup V_2$, $|V_1 \cap \Omega \cap U^{\rm c}| = 0$ and $|V_2 \cap \Omega^{\rm c} \cap U^{\rm c}| = 0$, then there exist $A_1, A_2 \subseteq \partial\Omega$ such that $\partial\Omega = A_1 \cup A_2$ and* $\operatorname{cap}_\Omega(A_1) = 0 = \operatorname{cap}_{\Omega^{\rm c}}(A_2)$.

Proof. By assumption for all $n \in \mathbf{N}$ there exist open sets $U_n, V_{1n}, V_{2n} \subseteq X$ such that $\operatorname{cap}(U_n) < 2^{-n}$, $\partial\Omega \subseteq V_{1n} \cup V_{2n}$, $|V_{1n} \cap \Omega \cap U_n^{\rm c}| = 0$ and $|V_{2n} \cap \Omega^{\rm c} \cap U_n^{\rm c}| = 0$. Then $\operatorname{cap}(\bigcup_{k=n}^\infty U_k) < 2^{-n+1}$,

$$\left| \left(\bigcup_{k=n}^\infty V_{1k} \right) \cap \Omega \cap \left(\bigcup_{k=n}^\infty U_k \right)^{\rm c} \right| \leq \sum_{k=n}^\infty |V_{1k} \cap \Omega \cap U_k^{\rm c}| = 0$$

and a similar expression involving the V_{2k}, for all $n \in \mathbf{N}$. So without loss of generality we may assume that $U_1 \supseteq U_2 \supseteq \ldots$, $V_{11} \supseteq V_{12} \supseteq \ldots$ and $V_{21} \supseteq V_{22} \supseteq \ldots$. Define

$$A_1 = \bigcap_{n=1}^\infty V_{1n} \cap \partial\Omega \text{ and } A_2 = \bigcap_{n=1}^\infty V_{2n} \cap \partial\Omega.$$

Let $x \in \partial\Omega$ and suppose that $x \notin A_1$. Then there exists an $n \in \mathbf{N}$ such that $x \notin V_{1n}$. Therefore $x \notin V_{1k}$ for all $k \geq n$. Since $\partial\Omega \subseteq V_{1k} \cup V_{2k}$ for all $k \in \mathbf{N}$ it follows that $x \in V_{2\,k}$ for all $k \geq n$ and $x \in A_2$. Therefore $\partial\Omega = A_1 \cup A_2$.

Finally, let $n \in \mathbf{N}$. Since $\operatorname{cap}(U_n) < 2^{-n}$ there exists a $\varphi \in D(h)$ such that $\varphi \geq 1$ a.e. on U_n and $\|\varphi\|^2_{D(h)} < 2^{-n}$. Because $|V_{1n} \cap \Omega \cap U_n^{\rm c}| = 0$ it follows that $\varphi \geq 1$ a.e. on $V_{1n} \cap \Omega$. So $\operatorname{cap}_\Omega(A_1) = 0$. Similarly, $\operatorname{cap}_{\Omega^{\rm c}}(A_2) = 0$. □

If the form h is local then there is also a converse of Propositions 2.2 and 2.3. In fact there is even a seemingly weaker version. The next theorem amplifies the characterizations of Theorem 1.1.

THEOREM 2.4. *Let Ω be a measurable subset of X and suppose that the regular Dirichlet form h is local. Then the following conditions are equivalent.*

- **I.** $S_t\, L_2(\Omega) \subseteq L_2(\Omega)$ *for all* $t > 0$.
- **II.** *For all* $\varepsilon > 0$ *there exist open sets* $U, V_1, V_2 \subseteq X$ *such that* $\operatorname{cap}(U) < \varepsilon$, $\partial\Omega \subseteq V_1 \cup V_2$, $|V_1 \cap \Omega \cap U^{\rm c}| = 0$ *and* $|V_2 \cap \Omega^{\rm c} \cap U^{\rm c}| = 0$.
- **III.** *There exist* $A_1, A_2 \subseteq \partial\Omega$ *such that* $\partial\Omega = A_1 \cup A_2$ *and* $\operatorname{cap}_\Omega(A_1) = 0 = \operatorname{cap}_{\Omega^{\rm c}}(A_2)$.
- **IV.** *There exists an* $M > 0$ *such that for all* $\varepsilon > 0$ *there exist open sets* $V_1, V_2 \subseteq X$ *and a function* $\psi \in D(h)$ *such that* $\partial\Omega \subseteq V_1 \cup V_2$, $h(\psi) \leq M$, $\|\psi\|_2 < \varepsilon$ *and* $\psi = 1$ *a.e. on* $(V_1 \cap \Omega) \cup (V_2 \cap \Omega^{\rm c})$.

Proof. The implication I⇒II follows from Proposition 2.2, the implication II⇒III by Proposition 2.3 and the implication III⇒IV from the estimate $\|\varphi \vee \psi\|^2_{D(h)} \leq \|\varphi\|^2_{D(h)} + \|\psi\|^2_{D(h)}$ for all $\varphi, \psi \in D(h)$ (see [3], Lemma 8.1.2.1).

'IV⇒I'. Let $\varphi \in D(h) \cap C_c(X)$ and let $M > 0$ be as in Condition 2.4. Let $n \in \mathbf{N}$. By assumption there exist open sets $V_{1n}, V_{2n} \subseteq X$ and a function $\psi_n \in D(h)$ such that $\partial\Omega \subseteq V_{1n} \cup V_{2n}$, $h(\psi_n) \leq M$, $\|\psi_n\|_2 \leq n^{-1}$ and $\psi_n = 1$ a.e. on $(V_{1n} \cap \Omega) \cup (V_{2n} \cap \Omega^{\rm c})$. We may assume that $0 \leq \psi_n \leq 1$. Set

$$\varphi_n = (\varphi - \varphi\,\psi_n)\mathbb{1}_\Omega.$$

We shall prove that $\varphi_n \in D(h)$.

Set $K = \operatorname{supp}\varphi$. Define

$$K_{10} = K \cap \Omega \cap V_{1n}^{\rm c},\ \ K_{20} = K \cap \Omega^{\rm c} \cap V_{2n}^{\rm c},$$

$K_1 = \overline{K_{10}}$ and $K_2 = \overline{K_{20}}$. Then K_1 and K_2 are compact. We next show that K_1 and K_2 are disjoint. Let $x \in K_1 \cap K_2$. Then $x \in \overline{\Omega} \cap \overline{\Omega^{\rm c}} = \partial\Omega \subseteq V_{1n} \cup V_{2n}$. Suppose $x \in V_{1n}$. Now $x \in K_1 = \overline{K_{10}}$, so $V_{1n} \cap K_{10} \neq \emptyset$. This is a contradiction. Similarly $x \in V_{2n}$ gives a contradiction. Hence K_1 and K_2 are disjoint. Since h is regular there exists a $\chi_n \in D(h) \cap C_c(X)$ such that $\chi_n|_{K_1} = 1$ and $\chi_n|_{K_2} = 0$. In particular $\chi_n(x) = 1$ for all $x \in K_{10}$ and $\chi_n(x) = 0$ for all $x \in K_{20}$. Note that

$$K \cap \Big((V_{1n} \cap \Omega) \cup (V_{2n} \cap \Omega^{\mathrm{c}}) \Big)^{\mathrm{c}} \cap \Omega = K_{10}$$

and

$$K \cap \Big((V_{1n} \cap \Omega) \cup (V_{2n} \cap \Omega^{\mathrm{c}}) \Big)^{\mathrm{c}} \cap \Omega^{\mathrm{c}} = K_{20}.$$

Hence it follows that $\varphi_n = (\varphi - \varphi\,\psi_n)\chi_n$ a.e. So $\varphi_n \in D(h)$.

Since $\varphi - \varphi\,\psi_n \in D(h)$ and $\varphi_n \in D(h)$ it follows that $(\varphi - \varphi\,\psi_n)\mathbb{1}_{\Omega^{\mathrm{c}}} \in D(h)$. Then, by locality of h,

$$h(\varphi - \varphi\,\psi_n) = h((\varphi - \varphi\,\psi_n)\mathbb{1}_\Omega) + h((\varphi - \varphi\,\psi_n)\mathbb{1}_{\Omega^{\mathrm{c}}}) \ge h(\varphi_n).$$

Moreover,

$$h(\varphi\,\psi_n)^{1/2} \le h(\varphi)^{1/2}\,\|\psi_n\|_\infty + \|\varphi\|_\infty\, h(\psi_n)^{1/2} \le h(\varphi)^{1/2} + M^{1/2}\,\|\varphi\|_\infty$$

for all $n \in \mathbf{N}$. So the sequence $\varphi_1, \varphi_2, \ldots$ is bounded in $D(h)$. Therefore this sequence has a weakly convergent subsequence. Passing to a subsequence, if necessary, there exists a $\hat{\varphi} \in D(h)$ such that $\lim_{n\to\infty} \varphi_n = \hat{\varphi}$ weakly in $D(h)$. Then $\lim_{n\to\infty} \varphi_n = \hat{\varphi}$ weakly in $L_2(X)$. But $\lim_{n\to\infty} \varphi\,\psi_n = 0$ in $L_2(X)$. So

$$\lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} (\varphi - \varphi\,\psi_n)\mathbb{1}_\Omega = \varphi\,\mathbb{1}_\Omega$$

in $L_2(X)$. Therefore $\varphi\,\mathbb{1}_\Omega = \hat{\varphi} \in D(h)$. Now the Statement I follows from Propostion 2.1 since $D(h) \cap C_c(X)$ is a core for h. □

A combination of Theorem 1.1 and Proposition 2.1 gives the following corollary.

COROLLARY 2.5. *Let Ω be a measurable subset of X and suppose that the regular Dirichlet form h is local. The following conditions are equivalent.*

I. $\mathbb{1}_\Omega \varphi \in D(h)$ *for all* $\varphi \in D(h)$.
II. *There exist* $A_1, A_2 \subseteq \partial\Omega$ *such that* $\partial\Omega = A_1 \cup A_2$ *and* $\mathrm{cap}_\Omega(A_1) = 0 = \mathrm{cap}_{\Omega^{\mathrm{c}}}(A_2)$.

We end this section with some remarks concerning comparison with the ordinary capacity.

COROLLARY 2.6. *Suppose the regular Dirichlet form h is local. If* $\mathrm{cap}(\partial\Omega) = 0$ *then* $L_2(\Omega)$ *is S-invariant.*

Proof. The condition $\mathrm{cap}(\partial\Omega) = 0$ clearly implies Condition II of Theorem 1.1. □

This corollary extends to regular local Dirichlet forms an earlier result in [12] valid for forms defined by degenerate elliptic operators on $\mathbf{R}^d$. In fact Theorem 2.4 gives the following improvement.

COROLLARY 2.7. *Let* Ω *be a measurable subset of* X *and suppose that the regular Dirichlet form* h *is local. Let* $M \in \mathbf{R}$. *Suppose for all* $\varepsilon > 0$ *there exists a* $\psi \in D(h)$ *such that* $h(\psi) \leq M$, $\|\psi\|_2 < \varepsilon$ *and* $\psi \geq 1$ *on a neighbourhood of* $\partial\Omega$. *Then* $S_t\, L_2(\Omega) \subseteq L_2(\Omega)$ *for all* $t > 0$.

The space $L_2(\Omega)$ is clearly unchanged if Ω is modified by a set of measure zero. Therefore the S-invariance, Condition I of Theorem 1.1, is not sensitive to modifications of this type. Alternatively, if Ω_1 and Ω_2 are two measurable subsets of X with $\overline{\Omega_1} = \overline{\Omega_2}$ and $|\Omega_1 \Delta \Omega_2| = 0$ then $\mathrm{cap}_{\Omega_1}(A) = \mathrm{cap}_{\Omega_2}(A)$ for all $A \subseteq \overline{\Omega_1}$. This follows immediately from the definition. On the other hand, the boundaries of Ω_1 and Ω_2 may be different. The capacity assumptions of Condition II of the theorem are therefore more sensitive to variations of Ω since there are sets of measure zero which have strictly positive capacity. Nevertheless these assumptions do not depend necessarily on the entire boundary of Ω.

An open set Ω is called topologically regular if Ω is equal to the interior of its closure, i.e. if $\Omega = \overset{\circ}{\overline{\Omega}}$ or, equivalently, if $\partial\Omega = \partial(\overline{\Omega}^{\mathrm{c}})$. For any set Ω the set $\overset{\circ}{\overline{\Omega}}$ is topologically regular. If Ω is a measurable subset of X with $|\partial\Omega| = 0$ then $\partial\overset{\circ}{\overline{\Omega}} \subseteq \partial\overline{\Omega} \subseteq \partial\Omega$, so $|\partial\overset{\circ}{\overline{\Omega}}| = |\partial\overline{\Omega}| = 0$. Hence $L_2(\Omega) = L_2(\overline{\Omega}) = L_2(\overset{\circ}{\overline{\Omega}})$. Therefore $L_2(\Omega)$ is S-invariant if and only if $L_2(\overset{\circ}{\overline{\Omega}})$ is also S-invariant. Next, note that if in addition Ω is open then $\Omega \subseteq \overset{\circ}{\overline{\Omega}}$ and

$$\partial\Omega = \partial\overset{\circ}{\overline{\Omega}} \cup \left(\overset{\circ}{\overline{\Omega}} \setminus \Omega\right).$$

Moreover, by definition of $\mathrm{cap}_{\Omega^{\mathrm{c}}}(A)$ it follows (with $V = \overset{\circ}{\overline{\Omega}}$ and $\varphi = 0$) that $\mathrm{cap}_{\Omega^{\mathrm{c}}}(\overset{\circ}{\overline{\Omega}} \setminus \Omega) = 0$, although in general $\mathrm{cap}(\overset{\circ}{\overline{\Omega}} \setminus \Omega) \neq 0$. (See Example 3.2) Consequently one has the following extension of Theorem 1.1 for open sets Ω.

COROLLARY 2.8. *Let* Ω *be an open subset of* X *with* $|\partial\Omega| = 0$ *and suppose that the regular Dirichlet form* h *is local. The following conditions are equivalent.*

I. $S_t L_2(\Omega) \subseteq L_2(\Omega)$ *for all* $t > 0$.

II. *There exist* $A_1, A_2 \subseteq \partial\Omega$ *such that* $A_1 \cup A_2 = \partial\overset{\circ}{\overline{\Omega}}$ *and* $\mathrm{cap}_\Omega(A_1) = 0 = \mathrm{cap}_{\overline{\Omega}^{\mathrm{c}}}(A_2)$.

Note that in Condition II the open set $\overline{\Omega}^{\mathrm{c}}$ is used instead of Ω^{c}.

The sets A_1 and A_2 in Theorem 1.1 might have a non-empty intersection and we next prove that the capacity of $A_1 \cap A_2$ vanishes. This is a consequence of a more general statement which again does not require locality of the Dirichlet form h. Note that a condition on $|A \setminus (\Omega_1 \cup \Omega_2)|$ is necessary in the next lemma since $\mathrm{cap}(B) \geq |B|$ for any measurable set.

LEMMA 2.9. *Let* Ω_1, Ω_2 *be two subsets of* X *and let* $A \subseteq \overline{\Omega_1} \cap \overline{\Omega_2}$. *Suppose that* $\mathrm{cap}_{\Omega_1}(A) = \mathrm{cap}_{\Omega_2}(A) = 0$. *Moreover, suppose that there exists an open set* U *in* X *such that* $A \subseteq U$ *and* $|U \setminus (\Omega_1 \cup \Omega_2)| = 0$. *Then* $\mathrm{cap}(A) = 0$.

Proof. Let $\varepsilon > 0$. There exist open V_1, V_2 in X and $\varphi_1, \varphi_2 \in D(h)$ such that $\|\varphi_1\|^2_{D(h)} < \varepsilon$, $\|\varphi_2\|^2_{D(h)} < \varepsilon$, $A \subseteq V_1$, $A \subseteq V_2$, $\varphi_1 \geq 1$ a.e. on $V_1 \cap \Omega_1$ and $\varphi_2 \geq 1$ a.e. on $V_2 \cap \Omega_2$. Then $\|\varphi_1 \vee \varphi_2\|^2_{D(h)} < 2\varepsilon$, $V_1 \cap V_2 \cap U$ is open, $A \subseteq V_1 \cap V_2 \cap U$ and $\varphi_1 \vee \varphi_2 \geq 1$ a.e. on $(V_1 \cap \Omega_1) \cup (V_2 \cap \Omega_2)$. Clearly $\varphi_1 \vee \varphi_2 \geq 1$ a.e. on the null set $U \setminus (\Omega_1 \cup \Omega_2)$. But

$$V_1 \cap V_2 \cap U \subseteq (V_1 \cap \Omega_1) \cup (V_2 \cap \Omega_2) \cup \Big(U \setminus (\Omega_1 \cup \Omega_2)\Big).$$

Therefore $\varphi_1 \vee \varphi_2 \geq 1$ a.e. on $V_1 \cap V_2 \cap U$. Consequently $\mathrm{cap}(V_1 \cap V_2 \cap U) < 2\varepsilon$. Hence $\mathrm{cap}(A) = 0$. □

COROLLARY 2.10. *Let* Ω *be a subset of* X *and let* $A \subseteq \partial\Omega$ *be such that* $\mathrm{cap}_{\Omega}(A) = 0 = \mathrm{cap}_{\Omega^c}(A)$. *Then* $\mathrm{cap}(A) = 0$.

Finally we establish a type of domination for invariant subspaces.

COROLLARY 2.11. *Let* h, k *be two regular Dirichlet forms on* X *with* $k \leq h$ *and* k *local. Further let* S, T *be the associated semigroups on* $L_2(X)$ *and* Ω *a measurable subset of* X. *If* $L_2(\Omega)$ *is* S*-invariant then* $L_2(\Omega)$ *is* T*-invariant.*

Proof. It follows by definition of the order relation for quadratic forms that $D(h) \subseteq D(k)$ and $k(\varphi) \leq h(\varphi)$ for all $\varphi \in D(h)$. Therefore if W is a subset of X and $A \subseteq \overline{W}$ then $\mathrm{cap}_{W,k}(A) \leq \mathrm{cap}_{W,h}(A)$.

It follows from Propositions 2.2 and 2.3 that there exist $A_1, A_2 \subseteq \partial\Omega$ such that $\partial\Omega = A_1 \cup A_2$ and $\mathrm{cap}_{\Omega,h}(A_1) = 0 = \mathrm{cap}_{\Omega^c,h}(A_2)$. Therefore $\mathrm{cap}_{\Omega,k}(A_1) = 0 = \mathrm{cap}_{\Omega^c,k}(A_2)$. Hence the corollary follows from Theorem 1.1. □

A similar conclusion can be drawn from Theorem 1.6.1 in [7] if $D(h)$ is a core for $D(k)$. (See also [7], Corollary 4.6.4.)

3. Examples

We present several examples of degenerate elliptic operators.

EXAMPLE 3.1. Let $X = \mathbf{R}$, $\Omega = \langle 0, \infty\rangle$, $D(h) = W^{1,2}(\Omega) \oplus L_2(\Omega^c)$ and $h(\varphi) = \int_\Omega |\varphi'|^2$, where φ' is the distributional derivative. Then Ω is topologically regular, $|\partial\Omega| = 0$ and h is a regular local Dirichlet form. Moreover, $\mathrm{cap}_{\Omega^c}(\partial\Omega) = \mathrm{cap}_{\langle -\infty, 0\rangle}(\{0\}) = 0$, so $L_2(\Omega)$ is S-invariant. But $\mathrm{cap}(\partial\Omega) = \mathrm{cap}_{\Omega}(\partial\Omega) = \mathrm{cap}_{\langle 0,\infty\rangle}(\{0\}) \neq 0$.

EXAMPLE 3.2. Let $X = \mathbf{R}$, $D(h) = W^{1,2}(\mathbf{R})$ and $h(\varphi) = \int_{\mathbf{R}} |\varphi'|^2$. The corresponding operator is the one-dimensional Laplacian. If $\Omega = \mathbf{R} \setminus \{0\}$ then $|\partial\Omega| = 0$ and h is a regular local Dirichlet form. Since $\overline{\Omega}^{\mathrm{c}} = \emptyset$ one has $\mathrm{cap}_{\overline{\Omega}^{\mathrm{c}}}(\partial\Omega) = 0$ and $L_2(\Omega)$ is S-invariant. The set Ω is not topologically regular and $\mathrm{cap}(\Omega) \neq 0$. Nevertheless, H is a strongly elliptic operator with constant, and therefore Lipschitz continuous, coefficients. This example shows that a converse of Corollary 2.6 is not valid for degenerate elliptic operators on $\mathbf{R}^d$ with coefficients in $W^{1,\infty}(\mathbf{R}^d)$.

The latter remark is of interest since it is proved in [12] that if h is the local regular Dirichlet form obtained as the closure of the form

$$\varphi \mapsto \sum \int c_{ij}\, \partial_i\varphi\, \partial_j\varphi \ (\varphi \in W^{1,2}(\mathbf{R}^d))$$

with $c_{ij} \in W^{1,\infty}(\mathbf{R}^d)$, $c_{ij} = c_{ji}$ and $(c_{ij}(x)) \geq 0$ for all $x \in \mathbf{R}^d$, and if Ω is a Lipschitz domain in $\mathbf{R}^d$ such that $L_2(\Omega)$ is S-invariant, then $\mathrm{cap}(\partial\Omega) = 0$. It is unclear whether the Lipschitz property of Ω could be replaced by the assumption that Ω is regular in topology.

EXAMPLE 3.3. Let $X = \mathbf{R}$, $V = \langle -\infty, -1\rangle \cup \langle 0, 1\rangle$, $D(h) = W^{1,2}(V) \oplus L_2(V^{\mathrm{c}})$ and $h(\varphi) = \int_V |\varphi'|^2$. Then h is a local regular Dirichlet form. Choose $\Omega = \langle -1, 1\rangle$. Then Ω is topologically regular. Moreover, $\mathrm{cap}_\Omega(\{-1\}) = 0$ and $\mathrm{cap}_{\overline{\Omega}^{\mathrm{c}}}(\{1\}) = 0$. Hence $L_2(\Omega)$ is S-invariant. But $\mathrm{cap}_\Omega(\partial\Omega) = \mathrm{cap}_\Omega(\{1\}) \neq 0$ and $\mathrm{cap}_{\overline{\Omega}^{\mathrm{c}}}(\partial\Omega) = \mathrm{cap}_{\overline{\Omega}^{\mathrm{c}}}(\{-1\}) \neq 0$.

EXAMPLE 3.4. Let $X = \mathbf{R}^2$, $\Omega = \langle 0, \infty\rangle \times \mathbf{R}$ the right half-plane and $V = (\langle 0, \infty\rangle \times \langle 0, \infty\rangle) \cup (\langle -\infty, 0\rangle \times \langle -\infty, 0\rangle)$ the union of the first and third quadrants. Let $D(h) = W^{1,2}(V) \oplus L_2(V^{\mathrm{c}})$ and $h(\varphi) = \int_V |\nabla\varphi|^2$. Then h is a local regular Dirichlet form. It is easy to verify that there exists an open set U such that the requirements in Condition II of Theorem 2.4 are valid for all $x \in \partial\Omega \setminus \{(0, 0)\}$. But if l is the form associated to the Laplacian on $\mathbf{R}^2$ then the capacity of the set $\{(0, 0)\}$ with respect to the form l is zero. Hence $\mathrm{cap}(\{(0, 0)\}) = 0$ where the capacity is with respect to the form h. Therefore the requirements in Condition II of Theorem 2.4 are also valid for $(0, 0)$ by adjusting the open set U around $(0, 0)$. So $L_2(\Omega)$ is S-invariant. Alternatively, Condition II of Theorem 1.1 is satisfied with $A_1 = \{0\} \times \langle -\infty, 0]$ and $A_2 = \{0\} \times [0, \infty\rangle$.

The next example is a multi-dimensional elliptic operator.

EXAMPLE 3.5. Let $X = \mathbf{R}^d$ and h the relaxation, or viscosity closure, of the elliptic form

$$\varphi \mapsto \sum_{i,j=1}^{d} \int_{\mathbf{R}^d} c_{ij}(\partial_i\varphi)(\partial_j\varphi) \ (\varphi \in C_c^\infty(\mathbf{R}^d)),$$

where the coefficients $c_{ij} = c_{ji} \in L_\infty(\mathbf{R}^d)$ are real and the matrix $C = (c_{ij})$ is positive-definite almost everywhere in $\mathbf{R}^d$. (The relaxation of a quadratic form is described in [4] (see page 28) and the viscosity closure is defined in [6], Section 2.) Then h is again a local regular Dirichlet form (see [5], Theorem 1.1). Assume that $c_{1i} = 0$ for all $i \in \{2, \ldots, d\}$. Moreover, assume that there exists a positive function $\tilde{c} \in C_b(\mathbf{R})$ such that $c_{11}(x) \leq \tilde{c}(x_1)$ for almost all $x = (x_1, \ldots, x_n) \in \mathbf{R}^d$. Let k be the closure of the form $\psi \mapsto \int_{\mathbf{R}} \tilde{c}\, |\psi'|^2$, with $\psi \in C_c^\infty(\mathbf{R}^d)$. Then k is a local regular Dirichlet form on $L_2(\mathbf{R})$. Assume that $\tilde{c}$ is such that $\mathrm{cap}_{\langle 0,\infty\rangle,k}(\{0\}) = 0$. If $\psi \in D(k)$ and $\tau \in C_c^\infty(\mathbf{R}^{d-1})$ then $\psi \otimes \tau \in D(h)$ and one has an estimate

$$\|\psi \otimes \tau\|_{D(h)} \leq M\, \|\varphi\|_{D(k)}\, \|\tau\|_{W^{1,2}(\mathbf{R}^{d-1})}$$

where $M = 1 + \max_{i,j} \|c_{ij}\|_\infty$. Therefore setting $\Omega = \langle 0, \infty\rangle \times \mathbf{R}^{d-1}$ one deduces that $\mathrm{cap}_\Omega(\partial\Omega) = \mathrm{cap}_\Omega(\{0\} \times \mathbf{R}^{d-1}) = 0$. Hence one can choose $A_1 = \partial\Omega$ in Condition II of Theorem 1.1 and $L_2(\Omega)$ is invariant under the semigroup corresponding to h.

One can construct more complicated examples by combination with the idea underlying Example 3.4.

Assume that there exist positive functions $\tilde{c}_1, \tilde{c}_2 \in C_b(\mathbf{R})$ such that

$$c_{11}(x) \leq \begin{cases} \tilde{c}_1(x_1) & \text{if } x_2 > 0 \\ \tilde{c}_2(x_1) & \text{if } x_2 < 0 \end{cases}$$

for almost all $x \in \mathbf{R}^d$. Let k_1 and k_2 denote the Dirichlet forms on $L_2(\mathbf{R})$ with coefficients $\tilde{c}_1$ and $\tilde{c}_2$, respectively. Suppose $\mathrm{cap}_{\langle 0,\infty\rangle,k_1}(\{0\}) = 0$ and $\mathrm{cap}_{\langle -\infty,0\rangle,k_2}(\{0\}) = 0$. Then one may choose $A_1 = \{0\} \times [0, \infty\rangle \times \mathbf{R}^{d-2}$ and $A_2 = \{0\} \times \langle -\infty, 0] \times \mathbf{R}^{d-2}$ in Condition II of Theorem 1.1, and $L_2(\Omega)$ is again invariant under the semigroup corresponding to h.

Finally we note that these examples can all be extended by application of the domination principle given by Corollary 2.11.

The regularity of the form k in Corollary 2.11 is essential as the next example shows.

EXAMPLE 3.6. Let $X = \langle 0, 1\rangle \cup \langle 2, 3\rangle$, $D(h) = W_0^{1,2}(X)$ and $h(\varphi) = \int_X |\varphi'|^2$. Define the form k by

$$D(k) = \{\varphi \in W^{1,2}(X) : \varphi(0) = \varphi(3) \text{ and } \varphi(1) = \varphi(2)\}$$

and $k(\varphi) = \int_X |\varphi'|^2$. Then $h \leq k$. Let S, T be the associated semigroups on $L_2(X)$. Choose $\Omega = \langle 0, 1\rangle$. Then Ω is S-invariant, but not T-invariant.

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