A Distributed Observer for a Discrete-Time Linear System

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***Abstract*— A simply structured distributed observer is described for estimating the state of a discrete-time, jointly observable, input-free, linear system whose sensed outputs are distributed across a time-varying network. It is explained how to construct the local estimators which comprise the observer so that their state estimation errors all converge exponentially fast to zero at a fixed, but arbitrarily chosen rate provided the network's graph is strongly connected for all time. This is accomplished by exploiting several well-known properties of invariant subspaces plus several kinds of suitably defined matrix norms.**

I. INTRODUCTION

With the growing interest in sensor networks and multi-agent systems, the problem of estimating the state of a dynamical system whose measured outputs are distributed across a network has been under study in one form or another for a number of years [1]–[7]. Despite this, only quite recently have provably correct distributed state estimators begun to emerge which solve this problem under reasonably non-restrictive assumptions [8]–[15].

In its simplest form, the discrete-time version of the distributed state estimation problem starts with a network of $m > 1$ agents labeled $1, 2, \ldots, m$ which are able to receive information from their neighbors. Neighbor relations are characterized by a directed graph $\mathbb{N}$, which may or may not depend on time, whose vertices correspond to agents and whose arcs depict neighbor relations. Each agent i senses a signal $y_i \in \mathbb{R}^{s_i},\ i \in \mathbf{m} = \{1, 2, \ldots, m\}$ generated by a discrete-time system of the form $x(\tau+1) = Ax(\tau),\ y_i(\tau) = C_i x(\tau),\ \ i \in \mathbf{m}$ and $x \in \mathbb{R}^n$. It is typically assumed that $\mathbb{N}$ is strongly connected and that the system is jointly observable. It is invariably assumed that each agent receives certain real-time signals from its neighbors although what is received can vary from one problem formulation to the next. In all formulations, the goal is to devise local estimators, one for each agent, whose outputs are all asymptotically correct estimates of x. The local estimator dynamics for agent i is typically assumed to depend only on the pair (C_i, A) and certain properties of $\mathbb{N}$. The problem is basically the same in continuous time, except that rather than the discrete-time model just described, the continuous-time model $\dot{x} = Ax,\ y_i(t) = C_i x,\ i \in \mathbf{m}$ is considered instead.

One way to try to address the estimation problem is to recast it as a discrete-time classical decentralized control problem [16] as was done in [8]. Following this approach, it is possible to devise a provable correct procedure for crafting a distributed linear filter with a prescribed spectrum which solves the continuous-time version of the problem assuming $\mathbb{N}$ is a constant strongly connected graph [9]; the same procedure is easily modified to deal with the discrete-time version of the problem. Prompted by work in [11], an entirely different and simpler approach to the continuous-time version of the estimation problem was developed in [12]. The same approach was simplified still further in [15] by exploiting certain well-known properties of invariant subspaces. There are however two distinct limitations of the types of estimators discussed in [11], [12], [15]. First, as they stand these estimators cannot deal with time-varying neighbor graphs. Second, there does not appear to be a way to easily modify these estimators to address the discrete-time state estimation problem; this is because the continuous-time estimators rely on a "high gain" concept for which there is no discrete-time counterpart. Despite these limitations, there is a very useful idea in these papers, stemming from the work in [11], which can be used to advantage in developing a discrete-time solution to the problem. Roughly speaking, the idea is to using the invariance of the unobservable spaces of the the pairs (C_i, A) to "split" the estimators into two parts - one for which conventional spectrum assignment tools can be used to control convergence rate and the other for which convergence rate can be controlled by switching and averaging.

This paper is organized as follows. Certain basic properties of invariant subspaces are reviewed in §I-A. The specific problem to be addressed is then formulated in §II. In §III the observer which solves this problem is described. The error model needed to analyze the observer is developed in §IV. Several techniques are outlined for picking the number of switches required between "event times" in order to achieve a prescribed convergence rate in §V. Finally in §VI, numerical examples are provided to illustrate how to pick the parameters of the observer.

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A. Invariant Subspaces

Throughout this paper certain basic and well-known algebraic properties of invariant subspaces will be exploited. To understand what they are, let A be any square matrix, and suppose $\mathcal{V}$ is an A-invariant subspace. Let Q be any full row rank matrix whose kernel is $\mathcal{V}$ and suppose that V is any "basis matrix" for $\mathcal{V}$; i.e., a matrix whose columns form a basis for $\mathcal{V}$. Then the linear equations

$$QA = \bar{A}_V Q \quad \text{and} \quad AV = VA_V$$

have unique solutions $\bar{A}_V$ and A_V respectively. Let V^{-1} be any left inverse of V and let Q^{-1} be that right inverse of Q for which $V^{-1}Q^{-1} = 0$. Then

$$A = H^{-1}\begin{bmatrix} \bar{A}_V & 0 \\ \widehat{A}_V & A_V \end{bmatrix} H, \quad \text{where} \quad H = \begin{bmatrix} Q \\ \\ V^{-1} \end{bmatrix}$$

and $\widehat{A}_V = V^{-1}AQ^{-1}$. Use will be made of these simple algebraic facts in the sequel.

II. Problem

We are interested in a time-varying network of $m > 1$ agents labeled $1, 2, \ldots, m$ which are able to receive information from their neighbors where by a *neighbor* of agent i is meant any agent in agent i's reception range. We write $\mathcal{N}_i(t)$ for the set of labels of agent i's neighbors at real time t and take agent i to be a neighbor of itself for all t. Relations between neighbors are characterized by a directed graph $\mathbb{N}(t)$ with m vertices and a set of arcs defined so that there is an arc from vertex j to vertex i whenever agent j is a neighbor of agent i. Each agent i can sense a discrete-time signal $y_i(\tau) \in \mathbb{R}^{s_i}$ at *event times* τT, $\tau = 0, 1, 2, \ldots$ where T is a positive constant; for $i \in \mathbf{m} \stackrel{\Delta}{=} \{1, 2, \ldots, m\}$ and $\tau = 0, 1, 2, \ldots$

$$y_i(\tau) = C_i x(\tau), \qquad x(\tau + 1) = Ax(\tau) \tag{1}$$

and $x \in \mathbb{R}^n$. We assume throughout that $\mathbb{N}(t)$ is strongly connected and that the system defined by (1) is *jointly observable*; i.e., with $C = \begin{bmatrix} C_1' & C_2' & \cdots & C_m' \end{bmatrix}'$, the matrix pair (C, A) is observable. Joint observability is equivalent to the requirement that

$$\bigcap_{i \in \mathbf{m}} \mathcal{V}_i = 0$$

where $\mathcal{V}_i$ is the *unobservable space* of (C_i, A); i.e. $\mathcal{V}_i = \ker \begin{bmatrix} C_i' & (C_iA)' & \cdots & (C_iA^{n-1})' \end{bmatrix}'$. As is well known, $\mathcal{V}_i$ is the largest A-invariant subspace contained in the kernel of C_i.

Each agent i is to estimate x using a dynamical system whose output $x_i(\tau) \in \mathbb{R}^n$ is to be an asymptotically correct estimate of $x(\tau)$ in the sense that the estimation error $x_i(\tau) - x(\tau)$ converges to zero as $\tau \to \infty$ as fast as λ^τ does, where λ is an arbitrarily chosen but fixed positive number[1] less than 1. To accomplish this it is assumed that the information agent i can receive from neighbor j at event time τT is $x_j(\tau)$. It is further assumed that agent i can also receive certain additional information from its neighbors at a finite number of times between each successive pair of event times; what this information is will be specified below.

III. The Observer

In this paper it will be assumed that each agent's neighbors do not change between event times. In other words, for $i \in \mathbf{m}$,

$$\mathcal{N}_i(t) = \mathcal{N}_i(\tau T), \quad t \in [\tau T, (\tau+1)T), \quad \tau = 0, 1, 2, \ldots$$

With this assumption, the observer to be considered consists of m private estimators, one for each agent. The estimator for agent i is of the form

$$x_i(\tau + 1) = (A + K_iC_i)\bar{x}_i(\tau) - K_i y_i(\tau) \tag{2}$$

where $\bar{x}_i(\tau)$ is an "averaged state" computed recursively during the real time interval $[\tau T,\ (\tau+1)T)$ using the update equations

$$\begin{aligned} z_i(0, \tau) &= x_i(\tau) && (3) \\ z_i(k, \tau) &= (I - P_i)z_i(k-1, \tau) \\ &+ \frac{1}{m_i(\tau)} P_i \sum_{j \in \mathcal{N}_i(\tau T)} z_j(k-1, \tau), \quad k \in \mathbf{q} && (4) \\ \bar{x}_i(t) &= z_i(q, \tau) && (5) \end{aligned}$$

Here $m_i(\tau)$ is the number of labels in $\mathcal{N}_i(\tau T)$, q is a suitably defined positive integer, $\mathbf{q} \stackrel{\Delta}{=} \{1, 2, \ldots, q\}$, and P_i is the orthogonal projection on the unobservable space of (C_i, A). Each matrix K_i is defined as follows.

For fixed $i \in \mathbf{m}$, write Q_i for any full rank matrix whose kernel is the unobservable space of (C_i, A), and let $\bar{C}_i$ and $\bar{A}_i$ be the unique solutions to $\bar{C}_iQ_i = C_i$ and $Q_iA = \bar{A}_iQ_i$ respectively. Then the matrix pair $(\bar{C}_i, \bar{A}_i)$ is observable. Thus by using a standard spectrum assignment algorithm, a matrix $\bar{K}_i$ can be chosen to ensure that the convergence of $(\bar{A}_i + \bar{K}_i\bar{C}_i)^\tau$ to zero as $\tau \to \infty$ is as fast as the convergence to zero of λ^τ is. Having chosen such $\bar{K}_i$, K_i is then defined to be $K_i = Q_i^{-1}\bar{K}_i$ where Q_i^{-1} is a right inverse for Q_i. The definition implies that $Q_i(A + K_iC_i) = (\bar{A}_i + \bar{K}_i\bar{C}_i)Q_i$ and that $(A + K_iC_i)\mathcal{V}_i \subset \mathcal{V}_i$. The latter, in turn, implies that there is a unique matrix A_i which satisfies $(A + K_iC_i)V_i = V_iA_i$ where V_i is a basis matrix[2] for $\mathcal{V}_i$. To explain what needs to be considered in choosing q it is necessary to describe the structure of the "error model" of the overall observer. This will be done next.

[1]For the type of observer to be developed, finite-time convergence is not possible.

[2]For simplicity, we assume that the columns of V_i constitute an orthonormal basic for $\mathcal{V}_i$ in which case $P_i = V_iV_i'$.

IV. The Error Model

For $i \in \mathbf{m}$, write $e_i(\tau)$ for the *state estimation error* $e_i(\tau) = x_i(\tau) - x(\tau)$. In view of (2),

$$e_i(\tau+1) = (A + K_iC_i)\bar{e}_i(\tau)$$

where $\bar{e}_i(\tau) = \bar{x}_i(\tau) - x(\tau)$. Moreover if $\epsilon_i(k,\tau) \stackrel{\Delta}{=} z_i(k,\tau) - x(\tau),\ k \in \{0, 1, \ldots, q\}$ then

$$\begin{aligned}
\epsilon_i(0,\tau) &= e_i(\tau) & (6)\\
\epsilon_i(k,\tau) &= (I - P_i)\epsilon_i(k-1,\tau)\\
&+ \frac{1}{m_i(\tau)}P_i \sum_{j \in \mathcal{N}_i(\tau T)} \epsilon_j(k-1,\tau), \quad k \in \mathbf{q} & (7)\\
\bar{e}_i(\tau) &= \epsilon_i(q,\tau) & (8)
\end{aligned}$$

because of (3) – (5). It is possible to combine these m subsystems into a single system. For this let e = column $\{e_1, e_2, \ldots, e_m\}$, define $\bar{A}$ = block diagonal $\{A + K_1C_1, A + K_2C_2, \ldots, A + K_mC_m\}$, P = block diagonal $\{P_1, P_2, \ldots, P_m\}$ and write $S(\tau)$ for the stochastic matrix $S(\tau) = D^{-1}_{\mathbb{N}(\tau T)}A'_{\mathbb{N}(\tau T)}$ where $A_{\mathbb{N}(\tau T)}$ is the adjacency matrix of $\mathbb{N}(\tau T)$ and $D_{\mathbb{N}(\tau T)}$ is the diagonal matrix whose ith diagonal entry is the in-degree of $\mathbb{N}(\tau T)$'s ith vertex. Note that $\mathbb{N}(\tau T)$ is the graph[3] of $S'(\tau)$ and that the diagonal entries of $S'(\tau)$ are all positive because each agent is a neighbor of itself.

Let $\bar{e}(\tau)$ = column $\{\bar{e}_1(\tau), \bar{e}_2(\tau), \ldots, \bar{e}_m(\tau)\}$ and $\epsilon(k,\tau) = \text{column}\{\epsilon_1(k,\tau), \epsilon_2(k,\tau), \ldots, \epsilon_m(k,\tau)\}$. Then

$$e(\tau+1) = \bar{A}\bar{e}(\tau)$$

and

$$\begin{aligned}
\epsilon(0,\tau) &= e(\tau)\\
\epsilon(k,\tau) &= (I_{mn} - P(I_{mn} - \bar{S}(\tau)))\epsilon(k-1,\tau),\ k \in \mathbf{q}\\
\bar{e}(\tau) &= \epsilon(q,\tau)
\end{aligned}$$

where $\bar{S}(\tau) = S(\tau) \otimes I_n$; here $\otimes$ denotes Kronecker product, and I_n and I_{mn} are the $n \times n$ and $mn \times mn$ identity matrices respectively. Clearly

$$\bar{e}(\tau) = (I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q e(\tau)$$

so

$$e(\tau+1) = \bar{A}(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q e(\tau) \qquad (9)$$

Our aim is to explain why for q sufficiently large, the time-varying matrix $\bar{A}(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q$ appearing in (9) is a discrete-time stability matrix for which the product

$$\Phi(\tau) = \prod_{s=1}^{\tau} \bar{A}(I_{mn} - P(I_{mn} - \bar{S}(s)))^q \qquad (10)$$

converges to zero as $\tau \to \infty$ as fast as λ^τ does. As a first step towards this end, note that the subspace $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_m$ is $\bar{A}$ - invariant because $(A + K_iC_i)\mathcal{V}_i \subset \mathcal{V}_i,\ i \in \mathbf{m}$. Next, let Q = block diagonal $\{Q_1, Q_2, \ldots, Q_m\}$ and V = block diagonal $\{V_1, V_2, \ldots, V_m\}$ in which case Q is a full rank matrix whose kernel is $\mathcal{V}$ and V is a basis matrix for $\mathcal{V}$ whose columns form an orthonormal set. It follows that $P = VV'$, and that

$$\begin{aligned}
Q\bar{A} &= \bar{A}_V Q & (11)\\
\bar{A}V &= V\tilde{A} & (12)
\end{aligned}$$

where

$$\bar{A}_V = \text{block diagonal } \{\bar{A}_1 + \bar{K}_1\bar{C}_1, \ldots, \bar{A}_m + \bar{K}_m\bar{C}_m\} \qquad (13)$$

and

$$\tilde{A} = \text{block diagonal } \{A_1, A_2, \ldots, A_m\};$$

as before, $(A + K_iC_i)V_i = V_iA_i$. Moreover

$$\begin{aligned}
Q(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q &= Q & (14)\\
(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q V &= V(V'\bar{S}(\tau)V)^q & (15)
\end{aligned}$$

Note that (14) holds because $QP = 0$. To understand why (15) is true, note first that $(I_{mn} - P(I_{mn} - \bar{S}(\tau)))V = V(I_{\bar{n}} - V'(I_{mn} - \bar{S}(\tau))V)$ because $P = VV'$; here $\bar{n} = \dim(\mathcal{V})$. But $I_{\bar{n}} - V'(I_{mn} - \bar{S}(\tau))V = V'\bar{S}(\tau)V$ because $V'V = I_{\bar{n}}$. Thus (15) holds for $q = 1$; it follows by induction that (15) holds for any positive integer q.

Using (11) – (15), one obtains the equations

$$\begin{aligned}
Q\bar{A}(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q &= \bar{A}_V Q & (16)\\
\bar{A}(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q V &= VA_V(\tau) & (17)
\end{aligned}$$

where

$$A_V(\tau) = \tilde{A}(V'\bar{S}(\tau)V)^q \qquad (18)$$

These equations imply that

$$\bar{A}(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q = H^{-1}\begin{bmatrix} \bar{A}_V & 0 \\ \hat{A}_V(\tau) & A_V(\tau) \end{bmatrix} H \qquad (19)$$

where

$$H = \begin{bmatrix} Q \\ V^{-1} \end{bmatrix}$$

and $\widehat{A}_V(\tau) = V^{-1}\bar{A}(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q Q^{-1}$.

Since the spectrum of each $\bar{A}_i + \bar{K}_i\bar{C}_i,\ \ i \in \mathbf{m}$, is assignable with $\bar{K}_i$, and $\widehat{A}_V(\tau)$ is a bounded matrix, to show that for suitably defined $\bar{K}_i$ and q sufficiently large, the matrix $\Phi(\tau)$ defined in (10) converges to zero as fast as λ^τ does, it is sufficient to show that for q sufficiently large, $A_V(\tau)$ is a discrete-time stability matrix whose state-transition matrix converges to zero as fast as λ^τ does. To accomplish this, use will be made of the following results.

Lemma 1: Let M be an $m \times m$ row stochastic matrix whose transpose has a strongly connected graph. There exists a diagonal matrix Π_M whose diagonal entries are positive for which the matrix $L_M = \Pi_M - M'\Pi_M M$ is positive semi-definite; moreover $L_M\mathbf{1} = 0$ where $\mathbf{1}$ is the m-vector of 1s.

[3]The *graph* of an $n \times n$ matrix M is that directed graph on n vertices possessing a directed arc from vertex i to vertex j if $m_{ij} \neq 0$ {p. 357, [17].}

If, in addition, the diagonal entries of M are all positive, then the kernel of L_M is one-dimensional.

Proof of Lemma 1: Since M is a stochastic matrix, it must have a spectral radius of 1 and an eigenvalue at 1 as must M'. Moreover, since the graph of M' is strongly connected, M' is irreducible {Theorem 6.2.24, [17]}. Thus by the Perron-Frobenius Theorem there must be a positive vector π such that $M'\pi = \pi$. Without loss of generality, assume π is normalized so that the sum of its entries equals 1; i.e., π is a probability vector. Let Π_M be that diagonal matrix whose diagonal entries are the entries of π. Then $\Pi_M \mathbf{1} = \pi$.

Since $M\mathbf{1} = \mathbf{1}$, $\Pi_M\mathbf{1} = \pi$, and $M'\pi = \pi$, it must be true that $M'\Pi_M M\mathbf{1} = \pi$ and thus that $L_M\mathbf{1} = 0$. To show that L_M is positive-semidefinite note first that L_M can also be written as $L_M = D - \hat{A}$ where D is a diagonal matrix whose diagonal entries are the diagonal entries of L_M and $\hat{A}$ is the nonnegative matrix $\hat{A} = D - L_M$. As such, L_M is the generalized Laplacian [18] of that simple undirected graph $\mathbb{G}$ whose adjacency matrix is the matrix which results when the nonzero entries a_{ij} in $\hat{A}$ are replaced by ones. Since L_M can also be written as

$$L_M = \sum_{(i,j)\in\mathcal{E}} a_{ij}(e_i - e_j)(e_i - e_j)'$$

where e_i is the ith unit vector and $\mathcal{E}$ is the edge set of $\mathbb{G}$, L_M is positive semi-definite as claimed.

Now suppose that the diagonal entries of M are all positive. Then the diagonal entries of $M'\Pi_M$ must also all be positive. It follows that every arc in the graph of M' must be an arc in the graph of $M'\Pi_M M$ so the graph of $M'\Pi_M M$ must be strongly connected. Since $I - \Pi_M$ is a nonnegative matrix, the graph of $M'\Pi_M M$ must be a spanning subgraph of the graph of $I - \Pi_M + M'\Pi_M M$. Since $I - L_M = I - \Pi_M + M'\Pi_M M$ and the graph of $M'\Pi_M M$ is strongly connected, the graph of $I - L_M$ must be strongly connected as well. But $I - L_M$ is a nonnegative matrix so it must be irreducible. In addition, since $(I - L_M)\mathbf{1} = \mathbf{1}$, the row sums of $(I - L_M)$ all equal one. Therefore the infinity norm of $I - L_M$ is one so its spectral radius is no greater than 1. Moreover 1 is an eigenvalue of $I - L_M$. Thus by the Perron-Frobenius Theorem, the geometric multiplicity of this eigenvalue is one. It follows that the geometric multiplicity of the eigenvalue of L_M at 0 is also one; ie, the dimension of the kernel of L_M is one as claimed. ■

Proposition 1: For each fixed value of τ,

$$(V'\bar{S}(\tau)V)'R(\tau)(V'\bar{S}(\tau)V) - R(\tau) < 0 \qquad (20)$$

where $R(\tau)$ is the positive definite matrix, $R(\tau) = V'(\Pi_{S(\tau)} \otimes I_n)V$.

Note that (20) shows that for each fixed τ, $x'R(\tau)x$ a discrete-time Lyapunov function for the equation $w(k+1) = V'\bar{S}(\tau)Vw(k)$. Thus for fixed τ, $V'\bar{S}(\tau)V$ is a discrete-time stability matrix.

Proof of Proposition 1: Fix τ and write S for $S(\tau)$ and $\bar{S}$ for $\bar{S}(\tau)$. Note that the graph of S', namely $\mathbb{N}$, is strongly connected. In view of Lemma 1, the matrix $L = \Pi_S - S'\Pi_S S$ is positive semi-definite and $L\mathbf{1} = 0$. Moreover, since the diagonal entries of S and thus S' are all positive, the kernel of L is one-dimensional.

Write R for $R(\tau)$. To prove the proposition it is enough to show that the matrix

$$Q = R - (V'\bar{S}'V)R(V'\bar{S}V) \qquad (21)$$

is positive definite.

To proceed, set $\bar{L} = L \otimes I_n$ in which case $\bar{L}$ is positive semi-definite because L is. Moreover, $\bar{L} = \bar{\Pi} - \bar{S}'\bar{\Pi}\bar{S}$ where $\bar{\Pi} = \Pi_S \otimes I_n$. Note that that $VRV' = P\bar{\Pi}P$ where P is the orthogonal projection matrix $P = VV'$. Clearly $VRV' = P\bar{\Pi}^{\frac{1}{2}}\bar{\Pi}^{\frac{1}{2}}P$. Note that both P and $\bar{\Pi}^{\frac{1}{2}}$ are block diagonal matrices with corresponding diagonal blocks of the same size. Because of this and the fact that each diagonal block in $\bar{\Pi}^{\frac{1}{2}}$ is a scalar times and identity matrix, it must be true that P and $\bar{\Pi}^{\frac{1}{2}}$ commute; thus $P\bar{\Pi}^{\frac{1}{2}} = \bar{\Pi}^{\frac{1}{2}}P$. From this and the fact that P is idempotent, it follows that $VRV' = \bar{\Pi}^{\frac{1}{2}}P\bar{\Pi}^{\frac{1}{2}}$. Clearly $\bar{\Pi}^{\frac{1}{2}}P\bar{\Pi}^{\frac{1}{2}} \leq \bar{\Pi}^{\frac{1}{2}}\bar{\Pi}^{\frac{1}{2}}$ so $VRV' \leq \bar{\Pi}$. It follows using (21) that $Q \geq R - V'\bar{S}'\bar{\Pi}\bar{S}V = R + V'\bar{L}V - V'\bar{\Pi}V$. Therefore

$$Q \geq V'\bar{L}V \qquad (22)$$

In view of this, to complete the proof it is enough to show that $V'\bar{L}V$ is positive definite.

Since $\bar{L}$ is positive semi-definite, so is $V'\bar{L}V$. To show that $V'\bar{L}V$ is positive definite, let $z = \text{column}\{z_1, z_2, \ldots, z_m\}$ be any vector such that $z'V'\bar{L}Vz = 0$. Then $\bar{L}Vz = 0$. Since the kernel of L is spanned $\mathbf{1}$, the kernel of $\bar{L}$ must be spanned by $\mathbf{1}\otimes I_n$. It follows that $V_i z_i = V_j z_j,\ i, j \in \mathbf{m}$. But because of joint observability, $\bigcap_{i\in\mathbf{m}} \mathcal{V}_i = 0$ so $V_i z_i = 0,\ i \in \mathbf{m}$. Thus $z_i = 0,\ i \in \mathbf{m}$ so $z = 0$. Therefore $V'\bar{L}V$ is positive definite. Therefore Q is positive definite because of (22). From this and (21) it follows that (20) is true. ■

V. Choosing q

In what follows it will be assumed that each $\bar{K}_i$ has been selected so that the matrix $\bar{A}_V$ defined by (13), is such that $\bar{A}_V^\tau$ converges to zero as $\tau \to \infty$ as fast as λ^τ does. This can be done using standard spectrum assignment techniques to make the spectral radius of $\bar{A}_V$ at least as small as λ. In view of (19), it is clear that to assign the convergence rate of the state transition matrix of $\bar{A}(I_{mn} - P(I_{mn} - \bar{S}(\tau)))^q$ it is necessary and sufficient to control the convergence rate of the state transition matrix of $A_V(\tau)$. This can be accomplished by choosing q sufficiently large. There are two different ways to do this, each utilizing a different matrix norm. Both approaches will be explained next using the abbreviated notation $B(\tau) = V'\bar{S}(\tau)V$; note that with this simplification, $A_V(\tau) = \tilde{A}B^q(\tau)$ because of (18).

A. Weighted Two-Norm

For each fixed τ and each appropriately-sized matrix M, write $\|M\|_{R(\tau)}$ for the matrix norm induced by the vector norm $\|x\|_{R(\tau)} \triangleq \sqrt{x'R(\tau)x}$. Note that $\|M\|_{R(\tau)}$ is the largest singular value of $R^{\frac{1}{2}}(\tau)MR^{-\frac{1}{2}}(\tau)$. Note in addition that

$$(R^{\frac{1}{2}}(\tau)B(\tau)R^{-\frac{1}{2}}(\tau))'(R^{\frac{1}{2}}(\tau)B(\tau)R^{-\frac{1}{2}}(\tau)) < I$$

because of (20). This shows that the largest singular value of $R^{\frac{1}{2}}(\tau)B(\tau)R^{-\frac{1}{2}}(\tau)$ is less than one. Therefore

$$\|B(\tau)\|_{R(\tau)} < 1 \tag{23}$$

1) $\mathbb{N}$ is constant: In this case both $B(\tau)$ and $R(\tau)$ are constant so it is sufficient to choose q so that $\|\tilde{A}B^q(\tau)\|_{R(\tau)} \leq \lambda$. Since $\|\cdot\|_{R(\tau)}$ is submultiplicative, this can be done by choosing q so that

$$\|B(\tau)^q\|_{R(\tau)} \leq \frac{\lambda}{\|\tilde{A}\|_{R(\tau)}} \tag{24}$$

This can always be accomplished because of (23).

2) $\mathbb{N}$ changes with time: In this case it is not possible to use the weighted two-norm $\|\cdot\|_{R(\tau)}$ because it is time-dependent. A simple fix, but perhaps not the most efficient one, would be to use the standard two-norm $\|\cdot\|_2$ instead since it does not depend on time. Using this approach, the first step would be to first choose, for each fixed τ, an integer $p_1(\tau)$ large enough so that $\|B^{p_1(\tau)}(\tau)\| < 1$. Such values of $p_1(\tau)$ must exist because each $B(\tau)$ is a discrete-time stability matrix or equivalently, a matrix with a spectral radius less than 1. Computing such a value amounts to looking at the largest singular value of $B^{p_1(\tau)}(\tau)$ for successively largest values of $p_1(\tau)$ until that singular value is less than 1. Having accomplished this, a number p can easily be computed so that $\|B^p(\tau)\| < 1 \; \forall\tau$ since there are only a finite number of distinct strongly connected graphs on m vertices and consequently only a finite number of distinct matrices $B(\tau)$ in the set $\mathcal{B} = \{B(\tau) : \tau \geq 0\}$. Choosing p to be the maximum of $p_1(\tau)$ with respect to τ is thus a finite computation. The next step would be to compute an integer $\bar{p}$ large enough so that each $\|\tilde{A}(B^p(\tau))^{\bar{p}}\|_2 \leq \lambda$. A value of q with the required property would then be $q = p\bar{p}$.

B. Mixed Matrix Norm

There is a different way to choose q which does not make use of either Lemma 1 or Proposition 1. The approach exploits the "mixed matrix norm" introduced in [19]. To define this norm requires several steps. To begin, let $\|\cdot\|_\infty$ denote the standard induced infinity norm and write $\mathbb{R}^{mn\times mn}$ for the vector space of all $m\times m$ block matrices $M = \begin{bmatrix} M_{ij}\end{bmatrix}$ whose ijth entry is a matrix $M_{ij} \in \mathbb{R}^{n\times n}$. With $n_i = \dim \mathcal{V}_i,\ i \in \mathbf{m}$, and $\bar{n} = n_1 + n_2 + \cdots n_m$, write $\mathbb{R}^{mn\times\bar{n}}$ for the vector space of all $m\times m$ block matrices $M = \begin{bmatrix} M_{ij}\end{bmatrix}$ whose ijth entry is a matrix $M_{ij} \in \mathbb{R}^{n\times n_j}$. Similarly write $\mathbb{R}^{\bar{n}\times mn}$ for the vector space of all $m\times m$ block matrices $M = \begin{bmatrix} M_{ij}\end{bmatrix}$ whose ijth entry is a matrix $M_{ij} \in \mathbb{R}^{n_i\times n}$. Finally write $\mathbb{R}^{\bar{n}\times\bar{n}}$ for the vector space of all $m\times m$ block matrices $M = \begin{bmatrix} M_{ij}\end{bmatrix}$ whose ijth entry is a matrix $M_{ij} \in \mathbb{R}^{n_i\times n_j}$.

Note that $B \in \mathbb{R}^{mn\times mn}$, $\tilde{A} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, $V \in \mathbb{R}^{mn\times\bar{n}}$, and $V' \in \mathbb{R}^{\bar{n}\times mn}$. For M in any one of these four spaces, the *mixed matrix norm* [19] of M, written $\|M\|$, is

$$\|M\| = \|\langle M\rangle\|_\infty \tag{25}$$

where $\langle M\rangle$ is the matrix in $\mathbb{R}^{m\times m}$ whose ijth entry is $\|M_{ij}\|_2$. It is very easy to verify that $\|\cdot\|$ is in fact a norm. It is even sub-multiplicative whenever matrix multiplication is defined. Note in addition that $\|V\| = 1$ and $\|V'\| = 1$ because the columns of each V_i form an orthonormal set.

Recall that $P = VV'$ is an orthogonal projection matrix. Using this, the definition of $B(\tau)$ and the fact that $PV = V$, it is easy to see that for any integer $p > 0$

$$B^p(\tau) = V'(P\bar{S}(\tau)P)^pV$$

Thus

$$\|B^p(\tau)\| \leq \|(P\bar{S}(\tau)P)^p\|$$

Using this and the fact that the graph of S' is strongly connected, one can conclude that

$$\|(P\bar{S}(\tau)P)^p\| < 1, \quad p \geq (m-1)^2$$

This is a direct consequence of Proposition 2 of [19]. Thus

$$\|B^p(\tau)\| < 1, \quad p \geq (m-1)^2 \tag{26}$$

1) $\mathbb{N}$ is constant: In this case $B(\tau)$ is constant so it is sufficient to choose q so that $\|\tilde{A}B^q(\tau)\| \leq \lambda$. This can be done by choosing $q = p\bar{p}$ where $p \geq (m-1)^2$ and $\bar{p}$ is such that

$$\|B^p(\tau)\|^{\bar{p}} \leq \frac{\lambda}{\|\tilde{A}\|} \tag{27}$$

This can always be accomplished because of (26).

2) $\mathbb{N}$ changes with time: Note that (26) holds for all τ. Assuming p is chosen so that $p \geq (m-1)^2$ it is thus possible to find, for each τ, a positive integer $\bar{p}(\tau)$, for which

$$\|B^p(\tau)\|^{\bar{p}(\tau)} \leq \frac{\lambda}{\|\tilde{A}\|} \tag{28}$$

Having accomplished this, a number $\bar{p}$ can easily be computed so that

$$\|B^p(\tau)\|^{\bar{p}} \leq \frac{\lambda}{\|\tilde{A}\|} \tag{29}$$

holds for all τ, since there there are only a finite number of distinct strongly connected graphs on m vertices and consequently only a finite number of distinct matrices $B(\tau)$ in the set $\mathcal{B}$ defined earlier. Choosing $\bar{p}$ to be the maximum of $\bar{p}(\tau)$ with respect to τ is thus a finite computation. A value of q with the required property would then be $q = p\bar{p}$.

VI. Numerical Examples

The following simulations are intended to illustrate how to pick parameter q of the observer. Consider the three channel, four-dimensional, discrete-time system described by the equations $x(\tau+1) = Ax(\tau),\ y_i = C_i x,\ i \in \{1,\ 2,\ 3\}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

and C_i is the ith unit row vector in $\mathbb{R}^{1\times 4}$. Note that A is a matrix with eigenvalues at ± 1.414, and ± 1.732. While the system is jointly observable, no single pair (C_i, A) is observable. The observer convergence rate is designed to be $\lambda = 0.5$. The first step is to design K_i as stated in §III.

For agent 1:

$$A_1 = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$V_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}', \quad K_1 = \begin{bmatrix} 0 & -1.75 & 0 & 0 \end{bmatrix}'$$

For agent 2:

$$A_2 = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$V_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}', \quad K_2 = \begin{bmatrix} -1.75 & 0 & 0 & 0 \end{bmatrix}'$$

For agent 3:

$$A_3 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$V_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}', \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & -2.75 \end{bmatrix}'$$

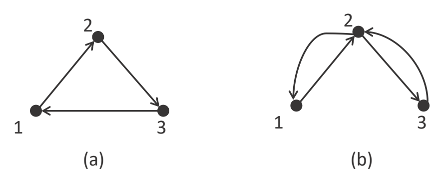

Fig. 1. $\mathbb{N}(\tau T)$

Consider the case when the neighbor graph $\mathbb{N}$ is constant which is Fig. 1 (a). With weighted two-norm, $\|\tilde{A}\|_{R(\tau)} = 2.30$ where $R(\tau) = \frac{1}{3}I_6$. $\|\tilde{A}\|_{R(\tau)}\|B(\tau)^5\|_{R(\tau)} \leq .5$ while $\|\tilde{A}\|_{R(\tau)}\|B(\tau)^4\|_{R(\tau)} > .5$. Thus by Eq. (24), $q = 5$. With mixed matrix norm, $\|\tilde{A}\| = 2.30$. First choose $p = 4$. $\|\tilde{A}\|\|B^p\|^2 \leq 0.5$, thus by Eq. (27) $q = 2p = 8$.

Consider the case when the neighbor graph is switching between Fig. 1 (a) and (b). With weighted two-norm for both cases $\|B(\tau)\|_2 < 1$. Choose $q = 6$ so that $\|\tilde{A}B(\tau)^6\|_2 \leq 0.5$. With mixed matrix norm, $\|\tilde{A}\| = 2.30$. First choose $p = 4$, then $\|\tilde{A}\|\|B^p\|^3 \leq 0.5$ for both cases. Thus $q = 3p = 12$.

VII. Concluding Remarks

The state estimator developed in this paper relies on an especially useful observation about distributed observer structure first noted in [11] and subsequently exploited in [12] and [15]. Just how much further this idea can be advanced remains to be seen. For sure, the synchronous switching upon which the local estimators in this paper depend, can be relaxed by judicious application of the mixed matrix norm discussed here. This generalization will be addressed in a future paper.

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