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Relationship between poles and zeros of input–output and chain-scattering systems

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Abstract

A system is frequently represented by transfer functions in an input–output characterization. However, such a system (under mild assumptions) can also be represented by transfer functions in a port characterization, frequently referred to as a chain-scattering representation. Due to its cascade properties, the chain-scattering representation is used throughout many fields of engineering. This paper studies the relationship between poles and zeros of input–output and chain-scattering representations of the same system.

Keywords: Input–output systems; Chain-scattering systems; Poles; Zeros; Two-port networks

1. Introduction

The chain-scattering representation is used extensively in various fields of engineering to represent the scattering properties of a physical system [9], especially in circuit theory where it has been widely used to deal with the cascade connection of circuits originating in analysis and synthesis problems [3,15,14]. In circuit theory, the chain-scattering representation is also called a scattering matrix of a two-port network [22]. Compared with the usual input–output (I/O) representation (Fig. 1), the chain-scattering representation (Fig. 2) is in fact an alternative way of representing a system. Cascade structure is the main property of the chain-scattering representation, which enables feedback in the I/O representation (Fig. 3) to be represented simply as a matrix multiplication in the chain-scattering representation (Fig. 4). Duality of transformation between the chain-scattering transformation and its inverse is its another useful property in the analysis of such systems [8,16]. Due to these features, Kimura [9] and others used the chain-scattering representation to provide a unified framework of cascade synthesis for H_∞ control theory [11–13,17]. Within this cascade framework, the H_∞ control problem is reduced to a factorization problem called a J-lossless factorization.

Pole-zero analysis is one of the most elementary tools of control theory to study the properties of a system [1,2,18]. It is consequently desirable to understand the connection between poles and zeros of the I/O representation with poles and zeros of the corresponding chain-scattering representation. For example, in deriving necessary and sufficient conditions for the solvability of the H_∞ control problem in terms of a J-lossless factorizations, one would typically impose certain conditions on the poles and zeros of the chain-scattering system [9]. It is natural to try to understand what these conditions correspond to in the I/O representation.

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Nomenclature

$\mathbb{R}[s]$	polynomial matrices with real coefficients
$\mathbb{R}_P(s)$	proper real rational transfer function matrices
$\mathbb{C}$	field of complex numbers
Ω	subset in $\mathbb{C}$
pole(G)	the set of all poles of $G(s) \in \mathbb{R}_P(s)$ including repeated poles[3]
zero(G)	the set of all transmission zeros of $G(s) \in \mathbb{R}_P(s)$ including repeated zeros[3]
$\{\Gamma_1, \Gamma_2\}$	the set of all elements of set Γ_1 and set Γ_2 including repetitions, e.g. if $\Gamma_1 = \{1, 1, 2\}$ and $\Gamma_2 = \{1, 3\}$, then $\{\Gamma_1, \Gamma_2\} = \{1, 1, 1, 2, 3\}$
$\mathbf{RH}_\infty$	the set of all stable proper real rational transfer function matrices
$\|G\|_\infty$	the H_∞-norm of $G(s) \in \mathbf{RH}_\infty$
$\mathbf{BH}_\infty$	a subset of $\mathbf{RH}_\infty$ containing all $G(s) \in \mathbf{RH}_\infty$ satisfying $\|G\|_\infty < 1$

Numerous papers have been written on poles and zeros of linear systems. Notable publications on zeros of multivariable systems in the period 1970–1987 are surveyed in [19]. [22] presents relationships between the transmission zeros of an impedance and the two-port impedance parameters $z_{ij}(s)(i, j = 1, 2)$ or the chain parameters of its Darlington equivalent. [4,5] investigate the pole/zero analysis of analog circuits. [20] studies a problem of robust pole placement design for a system with zeros located on the boundary of the stability region.

This paper will study the relationship between poles and zeros of I/O and chain-scattering representations. Firstly, the I/O and chain-scattering representations are presented. Secondly, explicit relationships between poles and zeros of I/O and chain-scattering representations are derived. Lastly, some application examples are given.

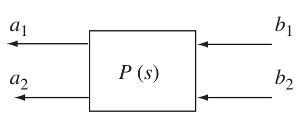

Fig. 1. Input–output representation.

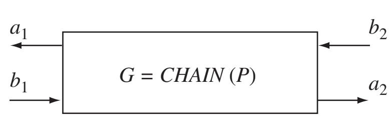

Fig. 2. Chain-scattering representation.

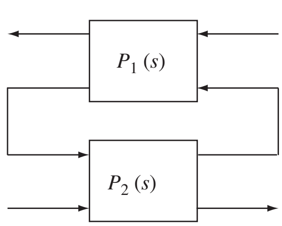

Fig. 3. Feedback connection in I/O representation.

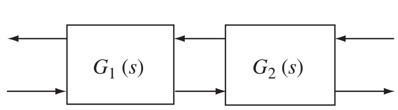

Fig. 4. Cascade connection in chain-scattering representation.

2. I/O and chain-scattering representations

Consider a multiple-input multiple-output (MIMO) system with two kinds of inputs (b_1, b_2) and two kinds of outputs (a_1, a_2), as shown in Fig. 1, represented as

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = P(s) \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{1}$$

The chain-scattering representation of $P(s)$, as shown in Fig. 2, is

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = G(s) \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix}, \tag{2}$$

where

$$\begin{aligned} G(s) &:= \mathrm{CHAIN}(P) \\ &= \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix} \end{aligned} \tag{3}$$

and exists if $P_{21}(s)$ is invertible in $\mathbb{R}_P(s)$.

[3] For example, pole$(G) = \{-1, -1-1, -2\}$ and zero$(G) = \{-2\}$ for $G(s) = \begin{bmatrix} \frac{1}{(s+1)^2(s+2)} & 0 \\ 0 & \frac{s+2}{s+1} \end{bmatrix}$, see [23] for details. Similarly for repeated zeros.

Then the mapping from chain-scattering representation to I/O representation is

$$
\begin{aligned}
P(s) &= \mathrm{CHAIN}^{-1}(G) \\
&= \begin{bmatrix} G_{12}G_{22}^{-1} & G_{11} - G_{12}G_{22}^{-1}G_{21} \\ G_{22}^{-1} & -G_{22}^{-1}G_{21} \end{bmatrix} \\
&= \begin{bmatrix} I & G_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22}^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -G_{21} \end{bmatrix}, \qquad (4)
\end{aligned}
$$

where $G_{22}(s) = P_{21}^{-1}(s)$ is invertible in $\mathbb{R}_P(s)$.

3. Pole-zero relations between I/O and chain-scattering systems

Poles and transmission zeros of any real rational transfer function matrix $G(s) \in \mathbb{R}_P(s)$ are obtained from its so-called McMillan form $U(s)G(s)V(s) = M(s)$ through some pre- and post-unimodular polynomial matrices $U(s), V(s) \in \mathbb{R}[s]$. Please refer to standard texts such as [6,23] for a McMillan decomposition of a real rational transfer function matrix and related definitions of poles and (transmission) zeros.

The following lemma studies the poles and transmission zeros of a cascade connection of MIMO systems.

Lemma 1. *Given a cascade connection $G(s) = G_1(s)G_2(s)$.*

(1) *If $G_1(s)$ has full column normal rank*[4] *or $G_2(s)$ has full row normal rank, then* $\mathrm{zero}(G) \subset \{\mathrm{zero}(G_1), \mathrm{zero}(G_2)\}$;
(2) $\mathrm{pole}(G) \subset \{\mathrm{pole}(G_1), \mathrm{pole}(G_2)\}$.

Proof. (1) Suppose $G_1(s)$ has full column normal rank. Then the McMillan decompositions [6,23] of $G_1(s)$ and $G_2(s)$ are given by

$$
G_1(s) = U_1(s) \begin{bmatrix} \wedge_1(s) \\ 0 \end{bmatrix} V_1(s),
$$

$$
G_2(s) = U_2(s) \begin{bmatrix} \wedge_2(s) & 0 \\ 0 & 0 \end{bmatrix} V_2(s),
$$

where $U_i(s),\ V_i(s)$ are unimodular polynomial matrices and $\wedge_i(s)$ are diagonal square transfer function matrices with full normal rank. Then,

$$
\begin{aligned}
G(s) &= G_1(s)G_2(s) \\
&= U_1(s) \begin{bmatrix} \wedge_1(s)V_1(s)U_2(s) \begin{bmatrix} \wedge_2(s) \\ 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \\
&\quad \times V_2(s). \qquad (5)
\end{aligned}
$$

It is clear that

$$
F(s) := \wedge_1(s)V_1(s)U_2(s) \begin{bmatrix} \wedge_2(s) \\ 0 \end{bmatrix}
$$

has full column normal rank. Suppose $z_0 \in \mathrm{zero}(G)$, which is equivalent to $z_0 \in \mathrm{zero}(F)$. Then there exists a $0 \neq u_0 \in \mathbb{C}^k$ such that $F(z_0)u_0 = 0$ [23]. If $z_0 \notin \mathrm{zero}(\wedge_2)$, then

$$
0 \neq V_1(z_0)U_2(z_0) \begin{bmatrix} \wedge_2(z_0) \\ 0 \end{bmatrix} u_0 \in \mathbb{C}^r.
$$

And thus $z_0 \in \mathrm{zero}(\wedge_1)$. Hence a transmission zero of $F(s)$ is a transmission zero of either $\wedge_1(s)$ or $\wedge_2(s)$. This is equivalent to the statement that a transmission zero of $G(s)$ is a transmission zero of either $G_1(s)$ or $G_2(s)$. That is $\mathrm{zero}(G) \subset \{\mathrm{zero}(G_1), \mathrm{zero}(G_2)\}$.

Similarly, a dual result can be proved that if $G_2(s)$ has full row normal rank, then $\mathrm{zero}(G) \subset \{\mathrm{zero}(G_1), \mathrm{zero}(G_2)\}$.

(2) It is trivial to show that $\mathrm{pole}(G) \subset \{\mathrm{pole}(G_1), \mathrm{pole}(G_2)\}$. □

Now, we are ready to give some pole-zero relations between I/O and chain-scattering representations in the following theorem.

Theorem 2. *The poles and transmission zeros of chain-scattering system $G(s) = \mathrm{CHAIN}(P)$ have the following relations with the poles and transmission zeros of I/O system $P(s)$:*

(1) $\mathrm{zero}(G) \subset \{\mathrm{pole}(P_{11}), \mathrm{zero}(P_{12}), \mathrm{pole}(P_{21}), \mathrm{pole}(P_{22})\}$;
(2) $\mathrm{pole}(G) \subset \{\mathrm{pole}(P_{11}), \mathrm{pole}(P_{12}), \mathrm{zero}(P_{21}), \mathrm{pole}(P_{22})\}$;
(3) $\mathrm{zero}(P_{21}) \subset \mathrm{pole}(G)$;
(4) $\mathrm{zero}(P_{12}) \subset \mathrm{zero}(G)$.

Proof. (1) In (3),

$$
G(s) = \begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}.
$$

Since both $\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}$ have full normal rank, using Lemma 1, we have

$$
\begin{aligned}
\mathrm{zero}(G) \subset \Bigg\{ & \mathrm{zero}\left(\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix} \right), \\
& \mathrm{zero}\left(\begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix} \right), \\
& \mathrm{zero}\left(\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix} \right) \Bigg\}.
\end{aligned}
$$

[4] The normal rank of $G(s)$ is the maximally possible rank of $G(s)$ for at least one $s \in \mathbb{C}$.

It is clear that

$$\begin{aligned}
\mathrm{zero}\left(\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}\right) &= \mathrm{pole}\left(\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}^{-1}\right) \\
&= \mathrm{pole}\left(\begin{bmatrix} I & -P_{11} \\ 0 & I \end{bmatrix}\right) \\
&= \mathrm{pole}(P_{11}), \\
\mathrm{zero}\left(\begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix}\right) &= \{\mathrm{zero}(P_{12}), \mathrm{pole}(P_{21})\}, \\
\mathrm{zero}\left(\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}\right) &= \mathrm{pole}\left(\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}^{-1}\right) \\
&= \mathrm{pole}\left(\begin{bmatrix} I & 0 \\ P_{22} & I \end{bmatrix}\right) \\
&= \mathrm{pole}(P_{22}).
\end{aligned}$$

Thus, $\mathrm{zero}(G) \subset \{\mathrm{pole}(P_{11}), \mathrm{zero}(P_{12}), \mathrm{pole}(P_{21}), \mathrm{pole}(P_{22})\}$.

(2) Using Lemma 1 and (3), we have

$$\begin{aligned}
\mathrm{pole}(G) \subset \Bigg\{ & \mathrm{pole}\left(\begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}\right), \\
& \mathrm{pole}\left(\begin{bmatrix} P_{12} & 0 \\ 0 & P_{21}^{-1} \end{bmatrix}\right), \\
& \mathrm{pole}\left(\begin{bmatrix} I & 0 \\ -P_{22} & I \end{bmatrix}\right)\Bigg\} \\
= \{ & \mathrm{pole}(P_{11}), \mathrm{pole}(P_{12}), \\
& \mathrm{zero}(P_{21}), \mathrm{pole}(P_{22})\}.
\end{aligned}$$

(3) In (3),

$$G(s) = \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix}.$$

It is easy to see that $\mathrm{zero}(P_{21}) = \mathrm{pole}(P_{21}^{-1}) \subset \mathrm{pole}(G)$.

(4) Perform a McMillan decomposition of $P_{11}(s)$ as $P_{11}(s) = U(s)N_\alpha(s)N_\beta^{-1}(s)V(s)$, where $U(s)$, $V(s)$ are unimodular polynomial matrices, and $N_\alpha(s)$, $N_\beta(s)$ are given by

$$N_\alpha(s) := \begin{bmatrix} \alpha_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \alpha_r(s) & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{m\times n},$$

$$N_\beta(s) := \begin{bmatrix} \beta_1(s) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \beta_r(s) & 0 \\ 0 & \cdots & 0 & I \end{bmatrix}_{n\times n},$$

where $\alpha_i(s)$, $\beta_i(s)$ are scalar polynomials.

Then from (3), we have

$$\begin{aligned}
G(s) &= \begin{bmatrix} I & P_{11} \\ 0 & I \end{bmatrix}\begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I & UN_\alpha N_\beta^{-1}V \\ 0 & I \end{bmatrix}\begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} U & 0 \\ 0 & V^{-1}N_\beta \end{bmatrix}\begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix} \\
&\quad \times \begin{bmatrix} U^{-1} & 0 \\ 0 & N_\beta^{-1}V \end{bmatrix}\begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix}. \qquad (6)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\begin{bmatrix} U^{-1} & 0 \\ 0 & N_\beta^{-1}V \end{bmatrix} G(s) \\
&\quad = \begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix}\begin{bmatrix} U^{-1} & 0 \\ 0 & N_\beta^{-1}V \end{bmatrix}\begin{bmatrix} P_{12} & 0 \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix} \\
&\quad = \begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix}\begin{bmatrix} U^{-1}P_{12} & 0 \\ -N_\beta^{-1}VP_{21}^{-1}P_{22} & N_\beta^{-1}VP_{21}^{-1} \end{bmatrix}. \qquad (7)
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathrm{zero}(P_{12}) \\
&\quad = \mathrm{zero}(U^{-1}P_{12}) \\
&\quad \subset \mathrm{zero}\left(\begin{bmatrix} U^{-1}P_{12} & 0 \\ -N_\beta^{-1}VP_{21}^{-1}P_{22} & N_\beta^{-1}VP_{21}^{-1} \end{bmatrix}\right) \\
&\quad = \mathrm{zero}\left(\begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix}\begin{bmatrix} U^{-1}P_{12} & 0 \\ -N_\beta^{-1}VP_{21}^{-1}P_{22} & N_\beta^{-1}VP_{21}^{-1} \end{bmatrix}\right) \\
&\quad = \mathrm{zero}\left(\begin{bmatrix} U^{-1} & 0 \\ 0 & N_\beta^{-1}V \end{bmatrix} G\right) \\
&\quad \subset \mathrm{zero}(G), \qquad (8)
\end{aligned}$$

since $\begin{bmatrix} I & N_\alpha \\ 0 & I \end{bmatrix}$ is also a unimodular polynomial matrix and $\begin{bmatrix} U^{-1} & 0 \\ 0 & N_\beta^{-1}V \end{bmatrix}$ has full normal rank and has no transmission zeros. That is $\mathrm{zero}(P_{12}) \subset \mathrm{zero}(G)$. □

In order to visualize the relationship between poles and zeros of I/O and chain-scattering representations, we will next analyze situations where $P(s)$ or $G(s)$ has no poles or zeros in some region in the complex plane $\mathbb{C}$. Suppose Ω is a subset of $\mathbb{C}$, as shown in Fig. 5, which can be any region of the s-plane. The following is a corollary to Theorem 2.

Corollary 3. *Suppose $P(s)$ has no poles in Ω. Then the following results hold*:

(1) *$G(s)$ has no transmission zeros in Ω if and only if $P_{12}(s)$ has no transmission zeros in Ω*;
(2) *$G(s)$has no poles in Ω if and only if $P_{21}(s)$ has no transmission zeros in Ω.*

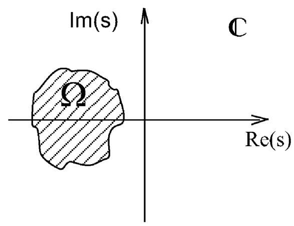

Fig. 5. Subset Ω in $\mathbb{C}$.

We now give a slightly different result that requires a milder assumption in the corollary statement. This result considers the situation where $G(s)$ has no poles nor zeros in Ω and gives a necessary and sufficient condition for the case.

Corollary 4. *Suppose $P_{21}(s)$ has no poles in Ω. Then $G(s)$ has no poles nor transmission zeros in Ω if and only if $P(s)$ has no poles in Ω and $P_{12}(s)$, $P_{21}(s)$ have no transmission zeros in Ω.*

Proof. ($\Leftarrow$) It is easy to prove using Corollary 3.

($\Rightarrow$) First, we will prove that $P_{21}(s)$ has no transmission zeros in Ω and $P(s)$ has no poles in Ω. From (3),

$$G(s) = \begin{bmatrix} P_{12} - P_{11}P_{21}^{-1}P_{22} & P_{11}P_{21}^{-1} \\ -P_{21}^{-1}P_{22} & P_{21}^{-1} \end{bmatrix}.$$

It is easy to see that if $G(s)$ has no poles in Ω, then none of P_{21}^{-1}, $P_{11}P_{21}^{-1}$, $P_{21}^{-1}P_{22}$ and $P_{12} - P_{11}P_{21}^{-1}P_{22}$ has poles in Ω. Since $P_{21}(s)$ is also assumed to have no poles in Ω, $P_{21}(s)$ has no poles nor zeros in Ω. Consequently, also $P_{11}(s)$, $P_{22}(s)$, $P_{12}(s)$ have no poles in Ω which in turn implies that $P(s)$ has no poles in Ω. Next, from result (4) of Theorem 2, if $G(s)$ has no transmission zeros in Ω, $P_{12}(s)$ has no transmission zeros in Ω. □

4. Dual results

This section contains dual results to Theorem 2 and Corollaries 3 and 4, given for completeness. Proofs are not given as they are similar to those given in the previous section.

Theorem 5. *The poles and transmission zeros of I/O system $P(s)$ have the following relations with the poles and transmission zeros of chain-scattering system $G(s)=\mathrm{CHAIN}(P)$:*

(1) $\mathrm{zero}(P)\subset\{\mathrm{zero}(G_{11}), \mathrm{pole}(G_{12}), \mathrm{pole}(G_{21}), \mathrm{pole}(G_{22})\}$;
(2) $\mathrm{pole}(P)\subset\{\mathrm{pole}(G_{11}), \mathrm{pole}(G_{12}), \mathrm{pole}(G_{21}), \mathrm{zero}(G_{22})\}$;
(3) $\mathrm{zero}(G_{22})\subset\mathrm{pole}(P)$;
(4) $\mathrm{zero}(G_{11})\subset\mathrm{zero}(P)$.

Corollary 6. *Suppose $G(s)$ has no poles in Ω. Then the following results hold*:

(1) *$P(s)$ has no transmission zeros in Ω if and only if $G_{11}(s)$ has no transmission zeros in Ω*;
(2) *$P(s)$ has no poles in Ω if and only if $G_{22}(s)$ has no transmission zeros in Ω.*

Corollary 7. *Suppose $G_{22}(s)$ has no poles in Ω. Then $P(s)$ has no poles nor transmission zeros in Ω if and only if $G(s)$ has no poles in Ω and $G_{11}(s)$, $G_{22}(s)$ have no transmission zeros in Ω.*

5. Application examples

In this section, we will use the above results in some application examples.

Consider a generalized plant $P(s) \in \mathbb{R}_P(s)$ described in the I/O representation (1). If a_2 is fed back to b_2 by

$$b_2(s) = K(s)a_2(s), \tag{9}$$

where $K(s) \in \mathbb{R}_P(s)$ is a controller, then the closed-loop transfer function $\Phi(s)$ from b_1 to a_1 is given by $a_1(s) = \Phi(s)b_1(s)$. This closed-loop transfer function $\Phi(s)$ is given in the following expression:

$$\begin{aligned}\Phi(s) &= \mathrm{LF}(P, K) \\ &:= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.\end{aligned} \tag{10}$$

$\mathrm{LF}(P, K)$ is called a linear fractional transformation (LFT) in the control literature. See [6,9] for extensive discussions on properties of LFTs.

The same relation can be described in terms of the chain-scattering representation (2). Substitution of (9) in (2) yields

$$\begin{aligned}\Phi(s) &= \mathrm{HM}(G, K) \\ &:= (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}.\end{aligned} \tag{11}$$

$\mathrm{HM}(G, K)$ is called a homographic transformation, which was used in classical circuit theory. Again, see [6,9] for extensive discussions on properties of homographic transformations. In classical circuit theory, (9) represents the "termination" of a port by a load. The "termination" of a chain-scattering representation is thus the same as feedback in an I/O representation of the same system.

The chain-scattering representation is for example used to provide a framework of cascade synthesis for H_∞ control theory. Within this cascade framework, the H_∞ control problem is reduced to a factorization problem called a J-lossless factorization. See [8,9] for a definition of a J-lossless factorization.

The "normalized H_∞ control problem" is to synthesize a stabilizing controller $K(s)$ such that the closed-loop transfer function $\Phi(s)$ given in (10) or (11) satisfies $\|\Phi\|_\infty < 1$. The following result has been established in [8,9].

Theorem 8. *Assume that the generalized plant $P(s)$ given in* (1) *has a chain-scattering representation $G(s)=\mathrm{CHAIN}(P)$ such that $G(s)$ is left invertible and has no poles nor zeros on the $j\omega$-axis. Then the normalized H_∞ control problem*

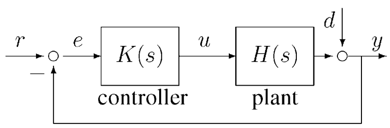

Fig. 6. Unity feedback scheme.

is solvable for $P(s)$ *if and only if* $G(s)$ *has a J-lossless factorization*[5]

$$G(s) = \mathrm{CHAIN}(P) = \Theta(s)\Pi(s),$$

where $\Theta(s)$ *is a J-lossless matrix*[5] *and* $\Pi(s)$ *is unit in* $\mathbf{RH}_\infty$.

In that case, $K(s)$ *is a desired controller if and only if*

$$K(s) = \mathrm{HM}(\Pi^{-1}, S) \quad \textit{for an } S(s) \in \mathbf{BH}_\infty. \tag{12}$$

From Corollary 4, we can see that $G(s) = \mathrm{CHAIN}(P)$ having no poles nor zeros on the $j\omega$-axis is a key in this theorem. Using the derived results in Section 3, we will understand what this assumption condition corresponds to in the I/O representation via some examples.

(1) *Sensitivity reduction problem*: Consider the feedback interconnection given in Fig. 6. In a sensitivity reduction problem, the designer is interested in synthesizing a $K(s)$ such that the transfer function $\widehat{\Phi}(s)$ from "*r*" to "*e*" is made as small as possible over a specified frequency range Ψ, thereby forcing "*y*" to closely follow "*r*". This transfer function $\widehat{\Phi}(s)$ is given by $\widehat{\Phi}(s) = (I + H(s)K(s))^{-1}$.

Choosing an appropriate (square) frequency weighting function $W(s)$ which is significant on $s = j\omega \in \Psi$, the problem is reduced to finding a controller $K(s)$ that stabilizes the closed-loop system of Fig. 6 and satisfies $\|W\widehat{\Phi}\|_\infty < 1$. It is clear (by inspection) that if we set the generalized plant

$$P(s) = \begin{bmatrix} W(s) & -W(s)H(s) \\ I & -H(s) \end{bmatrix}, \tag{13}$$

then $\Phi(s) := W(s)\widehat{\Phi}(s) = \mathrm{LF}(P, K)$.

Then, via (3),

$$G(s) = \mathrm{CHAIN}(P) = \begin{bmatrix} 0 & W(s) \\ H(s) & I \end{bmatrix}. \tag{14}$$

Hence the sensitivity reduction problem specified by $\|W\widehat{\Phi}\|_\infty < 1$ reduces to solving the normalized H_∞ control problem for the generalized plant given by (13). Since $P_{21}(s)$ has no poles on the $j\omega$-axis, we can use Corollary 4 to derive an equivalent condition to the assumption in Theorem 8 that $G(s)$ has no poles nor zeros on the $j\omega$-axis. In fact, this equivalent condition is that $P(s)$ has no poles on the $j\omega$-axis and $P_{12}(s)$, $P_{21}(s)$ have no zeros on the $j\omega$-axis. From (13), this reduces to $W(s)$ and $H(s)$ having no poles on the $j\omega$-axis and $W(s)H(s)$ having no transmission zeros on the $j\omega$-axis.

[5] Definitions and properties of a *J*-lossless matrix and a *J*-lossless factorization can be found in [9].

(2) *Robust stabilization problems* [7,21]: Now, let $H(s)$ in Fig. 6 be replaced by

$$H(s) = H_0(s) + \Delta(s)W(s), \tag{15}$$

where $H_0(s)$ is a given nominal plant, $W(s)$ is a given weighting function (square) and $\Delta(s)$ is an unknown transfer function that is only known to be stable and satisfies $\|\Delta\|_\infty < 1$ (i.e. $\Delta(s) \in \mathbf{BH}_\infty$). Consequently, we now are considering an uncertain plant class

$$\mathscr{H} = \{H(s) = H_0(s) + \Delta(s)W(s) : \ \Delta(s) \in \mathbf{BH}_\infty\}. \tag{16}$$

It is well known [23] that a controller $K(s)$ stabilizes the closed-loop system of Fig. 6 for all systems $H(s) \in \mathscr{H}$ if and only if $K(s)$ stabilizes $H_0(s)$ and satisfies

$$\|WK(I + H_0K)^{-1}\|_\infty < 1. \tag{17}$$

The problem can again be reduced to a normalized H_∞ control problem by choosing a generalized plant $P(s)$ for which $\Phi(s) := W(s)K(s)(I + H_0(s)K(s))^{-1} = \mathrm{LF}(P, K)$. Such a $P(s)$ is given by (by inspection) plant

$$P(s) = \begin{bmatrix} 0 & W(s) \\ I & -H_0(s) \end{bmatrix}. \tag{18}$$

Then, via (3),

$$G(s) = \mathrm{CHAIN}(P) = \begin{bmatrix} W(s) & 0 \\ H_0(s) & I \end{bmatrix}. \tag{19}$$

Since $P_{21}(s)$ has no poles on the $j\omega$-axis, we can use Corollary 4 to derive an equivalent condition to the assumption in Theorem 8 that $G(s)$ has no poles nor zeros on the $j\omega$-axis. From (18), the equivalent condition reduces to $W(s)$ having no poles nor zeros on the $j\omega$-axis and $H_0(s)$ having no poles on the $j\omega$-axis.

We will now attempt to motivate that the derived relationships between poles and zeros of chain-scattering representations and input–output representations also give control engineers information beyond just the simple interpretation of a technical supposition in a theorem.

From Theorem 8, note that when the normalized H_∞ control problem is solvable for $P(s)$, then

$$G(s) = \Theta(s)\Pi(s) \tag{20}$$

and

$$K(s) = \mathrm{HM}(\Pi^{-1}, S). \tag{21}$$

This is drawn in Fig. 7. It is easily seen from this figure that the unimodular (in $\mathbf{RH}_\infty$) portion $\Pi(s)$ of $G(s)$ is totally cancelled out by the controller and the resulting closed-loop mapping becomes $\mathrm{HM}(\Theta, S)$.

If $\Pi(s)$ contain lightly damped stable poles or zeros as depicted in Fig. 8, then these lightly damped poles/zeros are cancelled out by the controller. Such a cancellation is typically very dangerous in real-life systems, because

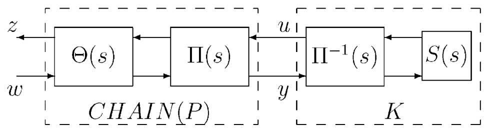

Fig. 7. Closed-loop structure of H_∞ control.

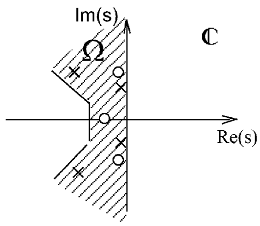

Fig. 8. Poles/zeros of $\Pi(s)$.

uncertainty in modelling may vary the frequencies of these lightly damped modes, thus cause poor closed-loop performance on the real system when such controllers are used. Similar issues are discussed in [10].

The derived results in this paper can hence assist the control engineer to determine what objects need to be "tweaked" in $P(s)$ such that $\Pi(s)$ is not too lightly damped.

6. Conclusions

This paper studies the relationship between poles and zeros of input–output and chain-scattering representations for systems whose $P_{21}(s)$ is invertible in $\mathbb{R}_P(s)$.

If $P_{12}(s)$ rather than $P_{21}(s)$ is invertible in $\mathbb{R}_P(s)$, a dual chain-scattering representation of $P(s)$ exists, denoted DCHAIN(P). Dual results on poles and zeros of I/O and dual chain-scattering systems can very easily be derived in the same way. Preliminary parts of this work were presented at IFAC world congress [24].

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