ONE POSITIVE AND TWO NEGATIVE RESULTS FOR DERIVED CATEGORIES OF ALGEBRAIC STACKS

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Abstract. Let X be a quasi-compact and quasi-separated scheme. There are two fundamental and pervasive facts about the unbounded derived category of X: (1) $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by perfect complexes and (2) if X is noetherian or has affine diagonal, then the functor $\Psi_X \colon \mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$ is an equivalence. Our main results are that for algebraic stacks in positive characteristic, the assertions (1) and (2) are typically false.

1. Introduction

Fix a field k and an algebraic group G over k. Ben-Zvi posed the following question [BZ09]: if k has positive characteristic, then is the unbounded derived category of representations of G compactly generated?

The second author recently answered Ben-Zvi's question negatively in the case of $\mathbb{G}_a$ [Nee14, Rem. 4.2]. We establish a much stronger version of this result: in the unbounded derived category of representations of $\mathbb{G}_a$ in positive characteristic, there are no compact objects besides 0 (Proposition 3.1).

We say that G is *poor* if k has positive characteristic and $\overline{G} = G \otimes_k \overline{k}$ has a subgroup isomorphic to $\mathbb{G}_a$, or, equivalently, if $\overline{G}^0_{\mathrm{red}}$ is not semi-abelian (Lemma 4.2). Examples of poor groups are $\mathbb{G}_a$ and GL_n. The results of this article imply that in positive characteristic, the derived category of representations of G is not compactly generated if G is poor. Conversely, when G is not poor, the first and third author showed that its derived category of representations is compactly generated [HR15, Thm. A]. Ben-Zvi's question is thus completely resolved.

A somewhat subtle point that we have suppressed so far is that there are two potential ways to look at the unbounded derived category of representations of G. First, there is $\mathsf{D}(\mathsf{Rep}(G))$; second, there is $\mathsf{D}_{\mathrm{qc}}(BG)$, the unbounded derived category of lisse-étale $\mathcal{O}_{BG}$-modules with quasi-coherent cohomology. There is a natural functor $\mathsf{D}(\mathsf{Rep}(G)) \to \mathsf{D}_{\mathrm{qc}}(BG)$ and if G is affine, then this functor induces an equivalence on bounded below derived categories.

In the present article, we will show that in positive characteristic if G is affine and poor, then this functor is not full. We also prove that if G is poor, then neither $\mathsf{D}(\mathsf{Rep}(G))$ nor $\mathsf{D}_{\mathrm{qc}}(BG)$ is compactly generated.

The results above are actually special cases of some general results for unbounded derived categories of quasi-coherent sheaves on algebraic stacks. We say that an algebraic stack is *poorly stabilized* (see §4) if it has a point with poor stabilizer group. Our first main result is the following.

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Theorem 1.1. *Let X be an algebraic stack that is quasi-compact, quasi-separated and poorly stabilized.*

(1) *The triangulated category $\mathsf{D}_{\mathrm{qc}}(X)$ is not compactly generated.*
(2) *Assume in addition that X has affine diagonal or is noetherian. If X is of global type, then $\mathsf{D}(\mathsf{QCoh}(X))$ is not compactly generated.*

An algebraic stack X is of *global type* if there exists a quasi-compact, étale, representable, and surjective morphism $[U/\mathrm{GL}_n] \to X$, where U is a quasi-affine scheme [Ryd15, §2]. More colloquially, X is of global type if it has affine stabilizers and étale-locally has the resolution property [Tot04, Gro17]. By Sumihiro's Theorem—and its recent generalization due to Brion [Sum74, Bri15]—many quotient stacks are of global type [HR17, Prop. 9.1]. So too are stacks admitting good moduli spaces and, more generally, those with linearly reductive stabilizers at closed points [AHR15].

We wish to point out that Theorem 1.1 is counter to the prevailing wisdom. Indeed, let X be a quasi-compact and quasi-separated algebraic stack. If X is a scheme, then it is well-known that $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by perfect complexes [BB03, Thm. 3.1.1(b)]. More generally, recent work of Krishna [Kri09, Lem. 4.8], Ben-Zvi–Francis–Nadler [BZFN10, §3.3], Toën [Toë12, Cor. 5.2], and the first and third authors [HR17], has shown that the unbounded derived category $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated by perfect complexes if X is a Deligne–Mumford stack with separated diagonal or is of equicharacteristic zero and of s-global type.

Also recall that if X is a scheme that is either quasi-compact with affine diagonal or noetherian, then the functor $\Psi_X \colon \mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$ is an equivalence of triangulated categories—see [BN93, Cor. 5.5] for the separated case (the argument adapts trivially to the case of affine diagonal) and [Stacks, Tags 08H1 & 09TN] in the setting of algebraic spaces. Our second main result is a partial extension of this to algebraic stacks.

Theorem 1.2. *Let X be an algebraic stack that is quasi-compact with affine diagonal or noetherian and affine-pointed. If $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated, then the functor $\Psi_X \colon \mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$ is an equivalence of categories.*

An algebraic stack X is *affine-pointed* if every morphism $\operatorname{Spec} k \to X$, where k is a field, is affine. If X has quasi-affine or quasi-finite diagonal, then X is affine-pointed [HR14, Lem. 4.5].

In particular, Ψ_X is an equivalence for every Deligne–Mumford stack with affine diagonal, every noetherian Deligne–Mumford stack with separated diagonal, and every stack in characteristic zero with affine diagonal that étale-locally has the resolution property [HR17]. This is a vast extension of work of Lieblich [Lie04, Prop. 2.2.4.6] and Krishna [Kri09, Cor. 3.7]. Lieblich gives a sketch of the proof of the equivalence of Ψ_X when X is an Artin stack with affine diagonal, the resolution property, and a good moduli space which is a scheme. Krishna treats the special case when X is a Deligne–Mumford stack that is separated, of finite type over a field of characteristic 0, has the resolution property and whose coarse moduli space is a scheme.

It is natural to ask whether Ψ_X is always an equivalence of categories. On the positive side, we prove that the restricted functor $\Psi_X^+ \colon \mathsf{D}^+(\mathsf{QCoh}(X)) \to \mathsf{D}^+_{\mathrm{qc}}(X)$ is an equivalence of triangulated categories when either X is quasi-compact with affine diagonal or noetherian and affine-pointed (Theorem C.1, also see [Lur04, Thm. 3.8] and [SGA6, Prop. II.3.5]). On the negative side, we have the following result.

Theorem 1.3. *Let X be an algebraic stack that is quasi-compact with affine diagonal or noetherian and affine-pointed. If X is poorly stabilized, then the functor $\Psi_X \colon \mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$ is not full.*

We were unable to determine whether the functor Ψ_X in Theorem 1.3 is faithful or not. For stacks with non-affine stabilizer groups the situation is even worse: if $X = BE$, where E is an elliptic curve over $\mathbb{C}$, then the functor $\Psi_X^b \colon \mathsf{D}^b(\mathsf{Coh}(X)) \to \mathsf{D}^b_{\mathrm{Coh}}(X)$ is neither essentially surjective nor full.

Note that when X has affine diagonal or is noetherian and affine-pointed, the first claim in Theorem 1.1 is a trivial consequence of Theorems 1.2 and 1.3.

Left-completeness. In the course of proving Theorem 1.3, we will prove that the triangulated category $\mathsf{D}(\mathsf{QCoh}(X))$ is not left-complete whenever X is poorly stabilized with affine diagonal. This generalizes an example of Neeman [Nee11] and amplifies some observations of Drinfeld–Gaitsgory [DG13, Rem. 1.2.10].

In Appendix B, we will prove that $\mathsf{D}_{\mathrm{qc}}(X)$ is left-complete for all algebraic stacks X. An analogous assertion in the context of derived algebraic geometry has been addressed by Drinfeld–Gaitsgory [DG13, Lem. 1.2.8]. In the Stacks Project [Stacks, Tag 08IY] a similar result has been proved, albeit in a different context.

As remarked to us by Bhatt [Bha12] and a reviewer, if X is quasi-compact with affine diagonal or noetherian and affine-pointed, then $\mathsf{D}_{\mathrm{qc}}(X)$ can be identified with the left-completion of $\mathsf{D}(\mathsf{QCoh}(X))$ (in the sense of [HA, §1.2.1])—see Remark C.4.

Well generation. In Appendix A we show that if $\mathcal{A}$ is a Grothendieck abelian category and $\mathcal{M} \subseteq \mathcal{A}$ is a weak Serre subcategory that is closed under coproducts and is Grothendieck abelian, then $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is a well generated triangulated category—a result we expect to be of independent interest. We prove this using the Gabriel–Popescu Theorem.

Since the inclusion $\mathsf{QCoh}(X) \subseteq \mathsf{Mod}(X)$ has these properties, this establishes that the triangulated category $\mathsf{D}_{\mathrm{qc}}(X)$ is well generated. This result is applied extensively in the article. It is used in the construction of adjoint functors (e.g., the derived quasi-coherator) and infinite products.

As remarked by a reviewer, the well generation of $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ also follows from some general results in the theory of presentable ∞-categories (Remark A.4). We also wish to point out that while [KS06, Prop. 14.2.4] is quite general, it does not apply in our situation. Indeed, they require that the embedding $\mathcal{M} \subseteq \mathcal{A}$ is closed under $\mathcal{A}$-subquotients (i.e., $\mathcal{M}$ is a Serre subcategory of $\mathcal{A}$), which is not the case for $\mathsf{QCoh}(X) \subseteq \mathsf{Mod}(X)$.

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2. Preliminaries

Let $\phi \colon X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Then the restriction of the functor $(\phi_{\text{lis-ét}})_* \colon \mathsf{Mod}(X) \to \mathsf{Mod}(Y)$ to $\mathsf{QCoh}(X)$ factors through $\mathsf{QCoh}(Y)$ [Ols07, Lem. 6.5(i)], giving rise to a functor $(\phi_{\mathsf{QCoh}})_* \colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$. Since the categories $\mathsf{Mod}(X)$ and $\mathsf{QCoh}(X)$ are Grothendieck abelian [Stacks, Tag 0781], the unbounded derived functors of $(\phi_{\text{lis-ét}})_*$ and $(\phi_{\mathsf{QCoh}})_*$ exist [Stacks, Tags 079P & 070K], and we denote these as $\mathsf{R}(\phi_{\text{lis-ét}})_*$ and $\mathsf{R}(\phi_{\mathsf{QCoh}})_*$, respectively. By [Ols07, Lem. 6.20], the restriction of $\mathsf{R}(\phi_{\text{lis-ét}})_*$ to $\mathsf{D}^+_{\mathrm{qc}}(X)$ factors uniquely through $\mathsf{D}^+_{\mathrm{qc}}(Y)$. If, in addition, ϕ is concentrated (e.g., representable), then the restriction of $\mathsf{R}(\phi_{\text{lis-ét}})_*$ to $\mathsf{D}_{\mathrm{qc}}(X)$ factors through $\mathsf{D}_{\mathrm{qc}}(Y)$ (see [Hal14, Lem. 2.1] for the representable case and [HR17, Thm. 2.6(ii)] in general).

For an algebraic stack W let $\Psi_W\colon \mathsf{D}(\mathsf{QCoh}(W)) \to \mathsf{D}_{\mathrm{qc}}(W)$ denote the natural functor. The universal properties of right-derived functors provide a diagram:

$$\begin{array}{ccc} \mathsf{D}(\mathsf{QCoh}(X)) & \xrightarrow{\mathsf{R}(\phi_{\mathsf{QCoh}})_*} & \mathsf{D}(\mathsf{QCoh}(Y)) \\ \downarrow & & \downarrow \\ \mathsf{D}(X) & \xrightarrow{\mathsf{R}(\phi_{\text{lis-ét}})_*} & \mathsf{D}(Y), \end{array}$$

together with a natural transformation of functors:

$$\epsilon_\phi\colon \Psi_Y \circ \mathsf{R}(\phi_{\mathsf{QCoh}})_* \Rightarrow \mathsf{R}(\phi_{\text{lis-ét}})_* \circ \Psi_X. \tag{2.1}$$

The following result, for schemes, is well-known [TT90, B.8]; for algebraic spaces, see [Stacks, Tags 09TH & 08GX].

Proposition 2.1. *Let $\phi\colon X \to Y$ be a morphism of algebraic stacks. Suppose that both X and Y are quasi-compact with affine diagonal or noetherian and affine-pointed. If $M \in \mathsf{D}^+(\mathsf{QCoh}(X))$, then the morphism induced by (2.1):*

$$\epsilon_\phi(M)\colon \Psi_Y \circ \mathsf{R}(\phi_{\mathsf{QCoh}})_*(M) \to \mathsf{R}(\phi_{\text{lis-ét}})_* \circ \Psi_X(M)$$

is an isomorphism. In particular, since $\Psi_Y^+\colon \mathsf{D}^+(\mathsf{QCoh}(Y)) \to \mathsf{D}^+_{\mathrm{qc}}(Y)$ is an equivalence (Theorem C.1), it follows that there is a natural isomorphism for each $M \in \mathsf{D}^+(\mathsf{QCoh}(X))$:

$$\mathsf{R}(\phi_{\mathsf{QCoh}})_*(M) \to (\Psi_Y^+)^{-1} \circ \mathsf{R}(\phi_{\text{lis-ét}})_* \circ \Psi_X^+(M).$$

Proof. The functors $(\phi_{\mathsf{QCoh}})_*$ and $(\phi_{\text{lis-ét}})_*$ are left-exact, thus the functors $\mathsf{R}(\phi_{\mathsf{QCoh}})_*$ and $\mathsf{R}(\phi_{\text{lis-ét}})_*$ are bounded below. Since M is assumed to belong to the bounded below derived category, standard "way-out" arguments show that it is sufficient to prove the result in the case when $M \simeq N[0]$, where $N \in \mathsf{QCoh}(X)$. The isomorphism, in this case, reduces to proving that if $N \in \mathsf{QCoh}(X)$, then the natural morphism $\mathsf{R}^i(\phi_{\mathsf{QCoh}})_*N \to \mathsf{R}^i(\phi_{\text{lis-ét}})_*N$ is an isomorphism for all integers $i \geq 0$, where $\mathsf{R}^i(\phi_{\mathsf{QCoh}})_*$ (resp. $\mathsf{R}^i(\phi_{\text{lis-ét}})_*$) denotes the ith right-derived functor of $(\phi_{\mathsf{QCoh}})_*$ (resp. $(\phi_{\text{lis-ét}})_*$). A standard δ-functor argument shows that it is sufficient to prove that $\mathsf{R}^i(\phi_{\text{lis-ét}})_*I = 0$ for every $i > 0$ and injective I of $\mathsf{QCoh}(X)$. But X and Y are assumed to be either quasi-compact with affine diagonal or noetherian and affine-pointed, so this vanishing claim follows from Lemma C.3(2). □

We briefly recall the following definitions from [HR17, §2]. Let $\phi\colon X \to Y$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Then ϕ has *finite cohomological dimension* if there exists an integer $n > 0$ such that $\mathsf{R}^m(\phi_{\text{lis-ét}})_*M = 0$ for all $m \geq n$ and $M \in \mathsf{QCoh}(X)$. If for every morphism of algebraic stacks $Z \to Y$, where Z is quasi-compact and quasi-separated, the morphism $X \times_Y Z \to Z$ has finite cohomological dimension, then we say that ϕ is *concentrated*.

If Y is quasi-compact with quasi-affine diagonal, then finite cohomological dimension is equivalent to concentrated [HR17, Lem. 2.5(v)]. Also, if ϕ is representable, then it is concentrated [HR17, Lem. 2.5(iii)].

Concentrated morphisms are the natural ones to consider for unbounded derived categories of quasi-coherent sheaves. Indeed, if ϕ is concentrated, then $\mathsf{R}(\phi_{\text{lis-ét}})_*$ sends $\mathsf{D}_{\mathrm{qc}}(X)$ to $\mathsf{D}_{\mathrm{qc}}(Y)$, is compatible with flat base change and preserves small coproducts [HR17, Thm 2.6]. In the next corollary, we see that this is also often the case for $\mathsf{R}(\phi_{\mathsf{QCoh}})_*$.

Corollary 2.2. *Let $\phi\colon X \to Y$ be a concentrated morphism of algebraic stacks. If X and Y are quasi-compact with affine diagonal or noetherian and affine-pointed,* *then there exists an integer $r \geq 0$ such that for all $M \in \mathsf{D}(\mathsf{QCoh}(X))$ and integers n the natural map:*

$$\tau^{\geq n}\mathsf{R}(\phi_{\mathsf{QCoh}})_*M \to \tau^{\geq n}\mathsf{R}(\phi_{\mathsf{QCoh}})_*\tau^{\geq n-r}M$$

is a quasi-isomorphism. It follows that

(1) *$\mathsf{R}(\phi_{\mathsf{QCoh}})_*$ preserves small coproducts;*
(2) *for all $M \in \mathsf{D}(\mathsf{QCoh}(X))$ the natural morphism induced by (2.1):*

$$\epsilon_\phi(M)\colon \Psi_Y \circ \mathsf{R}(\phi_{\mathsf{QCoh}})_*M \to \mathsf{R}(\phi_{\text{lis-ét}})_* \circ \Psi_X(M)$$

is an isomorphism;
(3) *the formation of $\mathsf{R}(\phi_{\mathsf{QCoh}})_*$ is compatible with flat base change on Y; and*
(4) *if a left adjoint $\mathsf{L}\phi^*_{\mathsf{QCoh}}$ to $\mathsf{R}(\phi_{\mathsf{QCoh}})_*$ exists (see Lemma 4.3, e.g., ϕ is flat), then $\mathsf{L}\phi^*_{\mathsf{QCoh}}$ sends compacts to compacts.*

Proof. Since ϕ is a concentrated morphism and Y is quasi-compact and quasi-separated, there exists an integer $r \geq 0$ such that if $N \in \mathsf{QCoh}(X)$, then $\mathsf{R}^i(\phi_{\text{lis-ét}})_*N = 0$ for all $i > r$. By Proposition 2.1 it follows that $\mathsf{R}^i(\phi_{\mathsf{QCoh}})_*N = 0$ for all $i > r$ too. The claim now follows from [Stacks, Tag 07K7]. The claims (1)–(3) are all simple consequences of the main claim, Proposition 2.1 and [HR17, Thm 2.6]. Finally, (4) follows from (1) and [HR17, Ex. 3.8]. □

Corollary 2.3. *Let X be an algebraic stack that is quasi-compact with affine diagonal or noetherian and affine-pointed. If C is a compact object of either $\mathsf{D}(\mathsf{QCoh}(X))$ or $\mathsf{D}_{\mathrm{qc}}(X)$, then C is perfect. Moreover if X is noetherian, then C is quasi-isomorphic to a bounded complex of coherent sheaves on X.*

Proof. Let C be a compact object of $\mathsf{D}_{\mathrm{qc}}(X)$. By [HR17, Lem. 4.4(i)], C is a perfect complex and in particular belongs to $\mathsf{D}^b_{\mathrm{qc}}(X) \subseteq \mathsf{D}^+_{\mathrm{qc}}(X)$. By Theorem C.1, it follows that $C \simeq \Psi_X(\tilde{C})$ for some $\tilde{C} \in \mathsf{D}^b(\mathsf{QCoh}(X))$. If X is noetherian, $\tilde{C}$ even belongs to $\mathsf{D}^b_{\mathsf{Coh}(X)}(\mathsf{QCoh}(X))$. Combining [LMB, Prop. 15.4] with [SGA6, II.2.2], we deduce that C belongs to the image of $\mathsf{D}(\mathsf{Coh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$.

Now let C be a compact object of $\mathsf{D}(\mathsf{QCoh}(X))$. Let $p\colon U \to X$ be a smooth surjection from an affine scheme U. By Corollary 2.2(4), $\mathsf{L}p^*_{\mathsf{QCoh}}C \in \mathsf{D}(\mathsf{QCoh}(U))$ is compact. Since $U = \operatorname{Spec} A$ is affine, it follows that $\mathsf{QCoh}(U) \cong \mathsf{Mod}(A)$ and so $\mathsf{L}p^*_{\mathsf{QCoh}}C$ is a perfect complex [Stacks, Tag 07LT]. If X is noetherian, then $C \in \mathsf{D}^b_{\mathsf{Coh}(X)}(\mathsf{QCoh}(X))$. Arguing as before, we deduce that C belongs to the image of $\mathsf{D}(\mathsf{Coh}(X)) \to \mathsf{D}(\mathsf{QCoh}(X))$. □

In the following Lemma we will give a sufficient condition for compactness of a perfect object in $\mathsf{D}(\mathsf{QCoh}(X))$. We do not know if this condition is necessary. The analogous condition in $\mathsf{D}_{\mathrm{qc}}(X)$ is necessary [HR17, Lem. 4.5].

Lemma 2.4. *Let X be an algebraic stack that is quasi-compact with affine diagonal or noetherian and affine-pointed. Let $P \in \mathsf{D}(\mathsf{QCoh}(X))$ be a perfect complex. Consider the following conditions*

(1) *P is a compact object of $\mathsf{D}(\mathsf{QCoh}(X))$.*
(2) *There exists an integer $r \geq 0$ such that $\operatorname{Hom}_{\mathcal{O}_X}(P, N[i]) = 0$ for all $N \in \mathsf{QCoh}(X)$ and $i > r$.*
(3) *There exists an integer $r \geq 0$ such that the natural map*

$$\tau^{\geq j}\mathsf{R}\operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}(X))}(P, M) \to \tau^{\geq j}\mathsf{R}\operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}(X))}(P, \tau^{\geq j-r}M)$$

is a quasi-isomorphism for all $M \in \mathsf{D}(\mathsf{QCoh}(X))$ and integers j.

Then (2) and (3) are equivalent and imply (1).

Proof. Condition (2) is a special case of (3): let $M = N[i]$ and $j = 0$.

Conversely, assume that condition (2) holds and let $M \in \mathsf{D}(\mathsf{QCoh}(X))$. Since the category $\mathsf{QCoh}(X)$ is Grothendieck abelian, there is a quasi-isomorphism $M \to I^\bullet$ in $\mathsf{D}(\mathsf{QCoh}(X))$, where $I^\bullet$ is K-injective and I^j is injective for every integer j [Ser03].

Let $p \geq r+1$ be an integer with the property that $P \in \mathsf{D}^{\geq -p+1}(\mathsf{QCoh}(X))$. Then the natural morphism of chain complexes:

$$\tau^{\geq j} \operatorname{Hom}^\bullet_{\mathsf{K}(\mathsf{QCoh}(X))}(P, I^\bullet) \to \tau^{\geq j} \operatorname{Hom}^\bullet_{\mathsf{K}(\mathsf{QCoh}(X))}(P, \sigma^{\geq j-p} I^\bullet), \tag{2.2}$$

where σ is the brutal truncation, is a quasi-isomorphism. For every integer j there is also a morphism $s_j \colon \sigma^{\geq j} I^\bullet \to \tau^{\geq j} I^\bullet$. If $C^\bullet_j$ is the mapping cone of s_j, then $C^\bullet_j \simeq d(I^{j-1})[-(j-1)]$. Thus, by condition (2), it follows that for every integer j

$$\tau^{\geq j+r} \mathsf{R}\mathrm{Hom}_{\mathsf{D}(\mathsf{QCoh}(X))}(P, C^\bullet_j) \simeq 0.$$

Since there is also a distinguished triangle in $\mathsf{D}(\mathsf{QCoh}(X))$ for every integer j:

$$\mathsf{R}\mathrm{Hom}(P, \sigma^{\geq j-p} I^\bullet) \longrightarrow \mathsf{R}\mathrm{Hom}(P, \tau^{\geq j-p} I^\bullet) \longrightarrow \mathsf{R}\mathrm{Hom}(P, C^\bullet_{j-p}),$$

it follows that for every integer j there is a quasi-isomorphism:

$$\tau^{\geq j} \mathsf{R}\mathrm{Hom}(P, \sigma^{\geq j-p} I^\bullet) \simeq \tau^{\geq j} \mathsf{R}\mathrm{Hom}(P, \tau^{\geq j-p} M). \tag{2.3}$$

For every integer j, we also have a distinguished triangle

$$H^{j-r-1}(M)[-(j-r-1)] \longrightarrow \tau^{\geq j-r-1} M \longrightarrow \tau^{\geq j-r} M.$$

As before, it follows that $\tau^{\geq j} \mathsf{R}\mathrm{Hom}(P, \tau^{\geq j-r-1} M) \simeq \tau^{\geq j} \mathsf{R}\mathrm{Hom}(P, \tau^{\geq j-r} M)$ and thus by induction a quasi-isomorphism:

$$\tau^{\geq j} \mathsf{R}\mathrm{Hom}(P, \tau^{\geq j-p} M) \simeq \tau^{\geq j} \mathsf{R}\mathrm{Hom}(P, \tau^{\geq j-r} M). \tag{2.4}$$

Combining the quasi-isomorphisms (2.2)–(2.4) gives (3).

For (3) implies (1): this follows from Theorem C.1 and [HR17, Lem. 1.2(iii)]. □

We now relate compact generation in $\mathsf{D}(\mathsf{QCoh}(X))$ with compact generation in $\mathsf{D}_{\mathrm{qc}}(X)$.

Lemma 2.5. *Let X be an algebraic stack that is quasi-compact with affine diagonal or noetherian and affine-pointed.*

1. *If $P \in \mathsf{D}(\mathsf{QCoh}(X))$ is a perfect complex such that $\Psi(P)$ is compact in $\mathsf{D}_{\mathrm{qc}}(X)$, then P is compact in $\mathsf{D}(\mathsf{QCoh}(X))$.*
2. *If X has finite cohomological dimension, then every perfect complex is compact in both $\mathsf{D}(\mathsf{QCoh}(X))$ and $\mathsf{D}_{\mathrm{qc}}(X)$.*
3. *If a set of objects $\{P_i\}$ of $\mathsf{D}(\mathsf{QCoh}(X))$ has the property that $\{\Psi(P_i)\}$ compactly generates $\mathsf{D}_{\mathrm{qc}}(X)$, then $\{P_i\}$ compactly generates $\mathsf{D}(\mathsf{QCoh}(X))$.*

Proof. For (1), by [HR17, Lem. 4.5], since $\Psi(P)$ is compact, there exists an integer r such that if $i > r$ and $N \in \mathsf{QCoh}(X)$, then $\operatorname{Hom}_{\mathcal{O}_X}(\Psi(P), N[i]) = 0$. The functor Ψ^+ is an equivalence (Theorem C.1), so $\operatorname{Hom}_{\mathcal{O}_X}(P, N[i]) = 0$ for all $i > r$ and $N \in \mathsf{QCoh}(X)$. It follows that P is compact by Lemma 2.4.

Statement (2) is a direct consequence of (1) and [HR17, Lem. 4.4(iii)].

For (3), let $M \in \mathsf{D}(\mathsf{QCoh}(X))$. If P is perfect and $\Psi(P)$ is compact, then $\mathsf{R}\mathrm{Hom}(P, M) = \mathsf{R}\mathrm{Hom}(\Psi(P), \Psi(M))$. Indeed, there exists an integer r such that for all integers j

$$\begin{aligned}\tau^{\geq j} \mathsf{R}\mathrm{Hom}(P, M) &\simeq \tau^{\geq j} \mathsf{R}\mathrm{Hom}(P, \tau^{\geq j-r} M) \\ &\simeq \tau^{\geq j} \mathsf{R}\mathrm{Hom}(\Psi(P), \tau^{\geq j-r} \Psi(M)) \simeq \tau^{\geq j} \mathsf{R}\mathrm{Hom}(\Psi(P), \Psi(M)),\end{aligned}$$

by Lemma 2.4 and [HR17, Lem. 4.5] since Ψ^+ is an equivalence of triangulated categories (Theorem C.1) and Ψ is t-exact. Thus, if $\operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}(X))}(P_i[l], M) = 0$

for all i and integers l, then $\mathrm{Hom}_{\mathcal{O}_X}(\Psi(P_i)[l], \Psi(M)) = 0$ for all i and l. It follows that $\Psi(M) = 0$ and, since Ψ is conservative, that $M = 0$. □

The following lemma, while technical, gives an explicit description of an adjunction that is useful in the article.

Lemma 2.6. *Let X be an algebraic stack and let $M \in \mathsf{D}(\mathsf{QCoh}(X))$.*

(1) *The functor $\Psi_X : \mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$ admits a right adjoint $\Phi_X : \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathsf{QCoh}(X))$.*
(2) *If X is quasi-compact with affine diagonal or noetherian and affine-pointed, then there exists a compatible quasi-isomorphism:*

$$\Phi_X\Psi_X(M) \simeq \operatorname*{holim}_n \tau^{\geq -n} M.$$

Proof. We suppress the subscript X from Ψ and Φ throughout. Since Ψ preserves small coproducts and $\mathsf{D}(\mathsf{QCoh}(X))$ is well generated [Nee01a, Thm. 0.2], Ψ admits a right adjoint $\Phi\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathsf{QCoh}(X))$ [Nee01b, Prop. 1.20]. This proves (1).

To prove (2), by left-completeness of $\mathsf{D}_{\mathrm{qc}}(X)$ (Theorem B.1),

$$\Phi\Psi(M) \to \Phi(\operatorname*{holim}_n \tau^{\geq -n}\Psi(M))$$

is a quasi-isomorphism. Since Φ is a right adjoint, it preserves homotopy limits. Also, Ψ is t-exact. Hence, there is a quasi-isomorphism

$$\Phi(\operatorname*{holim}_n \tau^{\geq -n}\Psi(M)) \simeq \operatorname*{holim}_n \Phi\Psi(\tau^{\geq -n}M).$$

By Theorem C.1, however, $\tau^{\geq -n}M \simeq \Phi\Psi(\tau^{\geq -n}M)$. This proves the claim. □

Remark 2.7. From Lemma 2.6(2) it is immediate that when X is quasi-compact with affine diagonal or is noetherian and affine-pointed, the left-completeness of $\mathsf{D}(\mathsf{QCoh}(X))$ is equivalent to Ψ_X being fully faithful.

We now prove Theorem 1.2 using an argument similar to [BIK11, Lem. 4.5].

Proof of Theorem 1.2. By Lemma 2.5(3), both $\mathsf{D}(\mathsf{QCoh}(X))$ and $\mathsf{D}_{\mathrm{qc}}(X)$ are compactly generated and Ψ takes a set of compact generators to a set of compact generators. In particular, the right adjoint $\Phi\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(\mathsf{QCoh}(X))$ of Ψ preserves small coproducts [Nee96, Thm. 5.1].

Consider the unit $\eta_M : M \to \Phi\Psi(M))$ and the counit $\epsilon_M : \Psi\Phi(M) \to M$ of the adjunction. Since Ψ^+ is an equivalence, we have that η_P and ϵ_P are isomorphisms for every compact object P. Since Ψ and Φ preserve small coproducts and $\mathsf{D}_{\mathrm{qc}}(X)$ and $\mathsf{D}(\mathsf{QCoh}(X))$ are compactly generated, it follows that η and ϵ are equivalences. We conclude that Ψ is an equivalence. □

3. The case of $B_k\mathbb{G}_a$ in positive characteristic

Throughout this section we let k denote a field of characteristic $p > 0$. Let $B_k\mathbb{G}_a$ be the algebraic stack classifying $\mathbb{G}_a$-torsors over k. We remind ourselves that the category of quasi-coherent sheaves on $B_k\mathbb{G}_a$ is the category of $\mathbb{G}_a$-modules, which is equivalent to the category of locally small modules over a certain ring R. In fact R is the ring

$$R = \frac{k[x_1, x_2, x_3, \ldots]}{(x_1^p, x_2^p, x_3^p, \ldots)}$$

and a module is locally small if every element is annihilated by all but finitely many x_i [DG70, II.2.2.6(b)]. Let us write $\mathsf{D}(R^{\mathrm{ls}})$ for the derived category of the category of locally small R-modules, and observe that $\mathsf{D}(R^{\mathrm{ls}}) \cong \mathsf{D}(\mathsf{QCoh}(B_k\mathbb{G}_a))$.

Proposition 3.1. *The only compact objects, in either $\mathsf{D}(\mathsf{QCoh}(B_k\mathbb{G}_a))$ or $\mathsf{D}_{\mathrm{qc}}(B_k\mathbb{G}_a)$, are the zero objects.*

Proof. The algebraic stack $B_k\mathbb{G}_a$ is noetherian with affine diagonal and so, by Corollary 2.3, every compact object is the image of a bounded complex of coherent sheaves. Let C be a compact object; we need to show that C vanishes.

Our compact object C is the image of a finite complex of finitely generated modules in $\mathsf{D}(R^{\mathrm{ls}})$. In particular, there exists an integer $n > 1$ such that x_i annihilates C for all $i \geq n$. Let us put this slightly differently: consider the ring homomorphisms $S \xrightarrow{\alpha} T \xrightarrow{\beta} R \xrightarrow{\gamma} T$ where

$$S = k[x_n]/(x_n^p), \qquad T = \frac{k[x_1, x_2, \ldots, x_{n-1}, x_n]}{(x_1^p, x_2^p, \ldots, x_{n-1}^p, x_n^p)}$$

where the maps $S \xrightarrow{\alpha} T \xrightarrow{\beta} R$ are the natural inclusions, and where $\gamma \colon R \to T$ is defined by

$$\gamma(x_i) = \begin{cases} x_i & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

Note that $\gamma\beta = \mathrm{id}$. Restriction of scalars gives induced maps of derived categories, which we write as $\mathsf{D}(T) \xrightarrow{\gamma_*} \mathsf{D}(R^{\mathrm{ls}}) \xrightarrow{\beta_*} \mathsf{D}(T) \xrightarrow{\alpha_*} \mathsf{D}(S)$, and $\beta_*\gamma_* = \mathrm{id}$. Our complex C, which is a bounded complex annihilated by x_i for all $i \geq n$, is of the form $\gamma_* B$ where $B \in \mathsf{D}^b(T)$ is a bounded complex of finite T-modules. And the fact that x_n annihilates C translates to saying that $\alpha_* B$ is a complex of modules annihilated by x_n, that is a complex of k-vector spaces. We wish to show that $C = 0$ or, equivalently, that $\alpha_* B$ is acyclic. We will show that if C is non-zero, then this gives rise to a contradiction.

Thus, assume that the cohomology of $\alpha_* B$ is non-trivial: in $\mathsf{D}(S)$ the complex $\alpha_* B$ is isomorphic to a non-zero sum of suspensions $k[\ell]$ of k. Then there are infinitely many integers m and non-zero maps in $\mathsf{D}(S)$ of the form $\alpha_* B \to k[m]$. Indeed, $\mathrm{Ext}^m_S(k,k) \neq 0$ for all $m \geq 0$. But α_* has a right adjoint $\alpha^\times = \mathsf{R}\mathrm{Hom}_S(T, -)$, and we deduce infinitely many non-zero maps in $\mathsf{D}(T)$ of the form $B \to \alpha^\times k[m] = \mathrm{Hom}_S(T,k)[m]$ (this is because T is a finite flat S-algebra). Since $\mathsf{D}(T)$ is left-complete, these combine to a map in $\mathsf{D}(T)$

$$\Psi \colon B \to \prod_m \mathrm{Hom}_S(T,k)[m] \cong \coprod_m \mathrm{Hom}_S(T,k)[m]$$

for which the composites

$$B \xrightarrow{\Psi} \coprod_m \mathrm{Hom}_S(T,k)[m] \xrightarrow{\pi_m} \mathrm{Hom}_S(T,k)[m]$$

are non-zero. Applying γ_*, which preserves coproducts, we deduce maps

$$\gamma_* B \xrightarrow{\gamma_*\Psi} \coprod_m \gamma_* \mathrm{Hom}_S(T,k)[m] \xrightarrow{\gamma_*\pi_m} \gamma_* \mathrm{Hom}_S(T,k)[m]$$

whose composites cannot vanish in $\mathsf{D}(R^{\mathrm{ls}})$, since β_* takes them to non-zero maps. The equivalence $\mathsf{D}(R^{\mathrm{ls}}) \cong \mathsf{D}(\mathsf{QCoh}(B_k\mathbb{G}_a))$ gives us that the composites in $\mathsf{D}(\mathsf{QCoh}(B_k\mathbb{G}_a))$ do not vanish. Furthermore, the composites lie in $\mathsf{D}^+(\mathsf{QCoh}(B_k\mathbb{G}_a)) \subseteq \mathsf{D}(\mathsf{QCoh}(B_k\mathbb{G}_a))$, and on $\mathsf{D}^+(\mathsf{QCoh}(B_k\mathbb{G}_a))$ the map to $\mathsf{D}_{\mathrm{qc}}(B_k\mathbb{G}_a)$ is fully faithful [Lur04, Thm. 3.8]. Hence the images of the composites are non-zero in $\mathsf{D}_{\mathrm{qc}}(B_k\mathbb{G}_a)$ as well. But this contradicts the compactness of $C = \gamma_* B$. □

4. The general case

In this section we extend the results of the previous section and show that the presence of $\mathbb{G}_a$ in the stabilizer groups of an algebraic stack X is an obstruction to compact generation in positive characteristic. The existence of finite unipotent subgroups such as $\mathbb{Z}/p\mathbb{Z}$ and $\boldsymbol{\alpha}_p$ is an obstruction to the compactness of the structure sheaf $\mathcal{O}_X$ but does not rule out compact generation [HR15]. The only connected groups in characteristic p without unipotent subgroups are the groups of multiplicative type. The following well-known lemma characterizes the groups without $\mathbb{G}_a$'s.

Lemma 4.1. *Let G be a group scheme of finite type over an algebraically closed field k. Then the following are equivalent:*

(1) *G^0_{red} is semiabelian, that is, a torus or the extension of an abelian variety by a torus;*
(2) *there is no subgroup $\mathbb{G}_a \hookrightarrow G$.*

Proof. By Chevalley's Theorem [Con02, Thm. 1.1] there is an extension $1 \to H \to G^0_{\mathrm{red}} \to A \to 1$ where H is smooth, affine and connected and A is an abelian variety. A subgroup $\mathbb{G}_a \hookrightarrow G$ would have to be contained in H which implies that H is not a torus. Conversely, recall that $H(k)$ is generated by its semi-simple and unipotent elements by the Jordan Decomposition Theorem [Bor91, Thm. 4.4]. If H is not a torus, then there exist non-trivial unipotent elements in $H(k)$. But any non-trivial unipotent element of $H(k)$ lies in a subgroup $\mathbb{G}_a \hookrightarrow G$. The result follows. □

If k is of positive characteristic, then we say that G is poor if G^0_{red} is not semiabelian. We say that an algebraic stack X is *poorly stabilized* if there exists a geometric point x of X whose residue field $\kappa(x)$ is of characteristic $p > 0$ and stabilizer group scheme G_x is poor. In particular, the algebraic stacks $B_k\mathbb{G}_a$ and $B_k\mathrm{GL}_n$ for $n > 1$ are poorly stabilized in positive characteristic. The following characterization of poorly stabilized algebraic stacks will be useful.

Lemma 4.2. *Let X be a quasi-separated algebraic stack.*

(1) *The stack X is poorly stabilized if and only if there exists a field k of characteristic $p > 0$ and a representable morphism $\phi\colon B_k\mathbb{G}_a \to X$.*
(2) *If X has affine stabilizers, then every representable morphism $\phi\colon B_k\mathbb{G}_a \to X$ is quasi-affine.*
(3) *Let $X' \to X$ be a quasi-finite, representable and surjective morphism of algebraic stacks. If X is poorly stabilized, then so too is X'.*

Proof. We first prove (1). Let k be an algebraically closed field and let $x\colon \operatorname{Spec} k \to X$ be a geometric point with stabilizer group scheme G. This induces a representable morphism $BG \to X$. If X is poorly stabilized, then there exists a point x such that G^0_{red} is not semiabelian. By the previous lemma, there is a subgroup $\mathbb{G}_a \hookrightarrow G$ and hence a representable morphism $B\mathbb{G}_a \to BG$.

Conversely, given a representable morphism $\phi\colon B_k\mathbb{G}_a \to X$, there is an induced representable morphism $\psi\colon B_k\mathbb{G}_a \to B_kG$. The morphism ψ is induced by some subgroup $\mathbb{G}_a \hookrightarrow G$ (unique up to conjugation) so X is poorly stabilized.

We now treat (2). The structure morphism $\iota_x\colon \mathcal{G}_x \hookrightarrow X$ of the residual gerbe $\mathcal{G}_x$ at x is quasi-affine [Ryd11, Thm. B.2] and $\phi = \iota_x \circ \rho \circ \psi$ where $\rho\colon B_kG \to \mathcal{G}_x$ is affine. If X has affine stabilizers, then G is affine and it follows that the quotient $G/\mathbb{G}_a$ is quasi-affine since $\mathbb{G}_a$ is unipotent [Ros61, Thm. 3]. We conclude that the morphism $\psi\colon B_k\mathbb{G}_a \to B_kG$, as well as ϕ, is quasi-affine.

For (3), using (1) we obtain a morphism $\phi\colon B_k\mathbb{G}_a \to X$, where k is a field of characteristic $p > 0$. Let $W = X' \times_X B_k\mathbb{G}_a$. The resulting projection $W \to B_k\mathbb{G}_a$ is quasi-finite, representable and surjective. In particular, $W \neq \emptyset$. Let $w\colon \operatorname{Spec} l \to W$ be a point, where l is an algebraically closed field of characteristic $p > 0$; then the stabilizer G_w of w is a subgroup scheme of $(\mathbb{G}_a)_l$ of finite index. But $(\mathbb{G}_a)_l$ is connected, so $G_w = (\mathbb{G}_a)_l$. It follows from (1) that X' is also poorly stabilized. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 4.2, there exists a field of characteristic $p > 0$ and a quasi-compact, quasi-separated and representable morphism $\phi\colon B_k\mathbb{G}_a \to X$.

If $\mathsf{D}_{\mathrm{qc}}(X)$ is compactly generated, then there is a compact object $M \in \mathsf{D}_{\mathrm{qc}}(X)$ and a non-zero map $M \to \mathsf{R}(\phi_{\text{lis-ét}})_*\mathcal{O}_{B_k\mathbb{G}_a}$. Indeed, $\mathsf{R}(\phi_{\text{lis-ét}})_*\mathcal{O}_{B_k\mathbb{G}_a} \in \mathsf{D}_{\mathrm{qc}}(X)$ and is non-zero. By adjunction, there is a non-zero map $\mathsf{L}\phi^*M \to \mathcal{O}_{B_k\mathbb{G}_a}$. But the functor $\mathsf{L}\phi^*$ sends compact objects of $\mathsf{D}_{\mathrm{qc}}(X)$ to compact objects of $\mathsf{D}_{\mathrm{qc}}(B_k\mathbb{G}_a)$ [HR17, Ex. 3.8 & Thm. 2.6(iii)]. By Proposition 3.1, it follows that $\mathsf{L}\phi^*M \simeq 0$ and we have a contradiction. Hence $\mathsf{D}_{\mathrm{qc}}(X)$ is not compactly generated.

Now assume that X is of global type, is noetherian or has affine diagonal, and $\mathsf{D}(\mathsf{QCoh}(X))$ is compactly generated. It follows that there is a compact object $M \in \mathsf{D}(\mathsf{QCoh}(X))$ and a non-zero map $M \to \mathsf{R}(\phi_{\mathsf{QCoh}})_*\mathcal{O}_{B_k\mathbb{G}_a}$. By assumption, there is an étale covering $p\colon X' \to X$ such that X' has affine diagonal and the resolution property. By Lemma 4.2(3), we may assume that ϕ factors through a map $\phi'\colon B_k\mathbb{G}_a \to X'$. Since X is of global type, if it is noetherian, then it is affine-pointed. It follows immediately from Proposition 2.1 that

$$\mathsf{R}(\phi_{\mathsf{QCoh}})_*\mathcal{O}_{B_k\mathbb{G}_a} \simeq \mathsf{R}(p_{\mathsf{QCoh}})_*\mathsf{R}(\phi'_{\mathsf{QCoh}})_*\mathcal{O}_{B_k\mathbb{G}_a}.$$

Since p is flat, Lemma 4.3 implies that a left adjoint $\mathsf{L}p^*_{\mathsf{QCoh}}$ to $\mathsf{R}(p_{\mathsf{QCoh}})_*$ exists. Moreover, p is concentrated, so $\mathsf{L}p^*_{\mathsf{QCoh}}$ takes compacts to compacts (Corollary 2.2(4)). Also, X' has affine diagonal and the resolution property, so Lemma 4.3 implies that a left adjoint $\mathsf{L}(\phi'_{\mathsf{QCoh}})_*$ to $\mathsf{R}(\phi'_{\mathsf{QCoh}})_*$ exists and we also see that $\mathsf{L}\phi'^*_{\mathsf{QCoh}}$ takes compacts to compacts. Adjunction produces a non-zero morphism $\mathsf{L}\phi'^*_{\mathsf{QCoh}}\mathsf{L}p^*_{\mathsf{QCoh}}M \to \mathcal{O}_{B_k\mathbb{G}_a}$ from a compact object of $\mathsf{D}(\mathsf{QCoh}(B_k\mathbb{G}_a))$. This contradicts Proposition 3.1. □

To prove Theorem 1.1(2), we required the following Lemma.

Lemma 4.3. *Let $p\colon X' \to X$ be a morphism of algebraic stacks. A left adjoint to* $\mathsf{R}(p_{\mathsf{QCoh}})_*\colon \mathsf{D}(\mathsf{QCoh}(X')) \to \mathsf{D}(\mathsf{QCoh}(X))$ *exists if*

(1) *p is flat; or*
(2) *both X' and X are quasi-compact with affine diagonal and either:*
   (a) *X has the resolution property; or*
   (b) $\mathsf{QCoh}(X)$ *has enough flats; or*
   (c) $\mathsf{K}(\mathsf{QCoh}(X))$, *the homotopy category of unbounded complexes in* $\mathsf{QCoh}(X)$, *has enough* K*-flats.*

Proof. By deriving the adjunction between p^* and p_*, it is sufficient to prove that $\mathsf{L}p^*_{\mathsf{QCoh}}$ exists, that is, p^* admits a left derived functor, under one of the four additional conditions. If p is flat, then p^* is exact and this is clear. If X has the resolution property, then $\mathsf{QCoh}(X)$ obviously has enough flats (i.e., every quasi-coherent sheaf is the quotient of a flat quasi-coherent sheaf). Also, if $\mathsf{QCoh}(X)$ has enough flats, then $\mathsf{K}(\mathsf{QCoh}(X))$ has enough K-flats [Spa88, Thm. 3.4]. Hence, it is sufficient to prove the existence of $\mathsf{L}p^*_{\mathsf{QCoh}}$ when $\mathsf{K}(\mathsf{QCoh}(X))$ has enough K-flats. It is sufficient to prove: if $M \in \mathsf{K}(\mathsf{QCoh}(X))$ is K-flat and acyclic, then p^*M is acyclic. Now the acyclicity of p^*M is local on X', so we may assume that X' is an affine scheme. Since X has affine diagonal, it follows that p is now affine. In particular, it is sufficient to prove that p_*p^*M is acyclic. But $(p_*\mathcal{O}_{X'}) \otimes_{\mathcal{O}_X} M \cong p_*p^*M$ and M is K-flat, so p_*p^*M is acyclic. □

Remark 4.4. The proof of Theorem 1.1(2) can be varied to prove the following assertion: if X is a poorly stabilized algebraic stack that is quasi-compact with affine diagonal or noetherian and affine-pointed, then there exists a closed substack $i\colon Z \hookrightarrow X$ such that Z is poorly stabilized and $\mathsf{D}(\mathsf{QCoh}(Z))$ is not compactly generated. The only obstruction to taking i to be the identity morphism in this level of generality is that we do not know when a left adjoint to $\mathsf{R}(i_{\mathsf{QCoh}})_*$ exists.

Proof of Theorem 1.3. By Lemma 4.2, there exists a field of characteristic $p > 0$ and a quasi-affine morphism $\phi\colon B_k\mathbb{G}_a \to X$. By Corollary 2.2, there exists an integer $n \geq 1$ such that if $N \in \mathsf{QCoh}(B_k\mathbb{G}_a)$, then $\mathsf{R}(\phi_{\mathsf{QCoh}})_*N \in \mathsf{D}^{[0,n-1]}(\mathsf{QCoh}(X))$. By [Nee11, Thm. 1.1], there exists $M \in \mathsf{QCoh}(B_k\mathbb{G}_a)$ such that the natural map in $\mathsf{D}(\mathsf{QCoh}(B_k\mathbb{G}_a))$:

$$\bigoplus_{i\geq 0} M[in] \to \prod_{i\geq 0} M[in]$$

is not a quasi-isomorphism—note that while [Nee11, Thm. 1.1] only proves the above assertion in the case where $n = 1$, a simple argument by induction on n gives the claim above. Lemma 4.5 now implies that the natural map:

$$\bigoplus_{i\geq 0} \mathsf{R}(\phi_{\mathsf{QCoh}})_*M[in] \to \prod_{i\geq 0} \mathsf{R}(\phi_{\mathsf{QCoh}})_*M[in]$$

is not a quasi-isomorphism. Since $\mathsf{R}(\phi_{\mathsf{QCoh}})_*M \in \mathsf{D}^{[0,n-1]}(\mathsf{QCoh}(X))$, it follows that $\mathsf{D}(\mathsf{QCoh}(X))$ is not left-complete. By Remark 2.7, we have established that Ψ_X is not fully faithful. To prove that Ψ_X is not full, we will have to argue further.

Let $L = \mathsf{R}(\phi_{\mathsf{QCoh}})_*M$, $S = \oplus_{i\geq 0}L[in]$, and $P = \prod_{i\geq 0} L[in]$. Also, $\Phi_X\Psi_X(S) \simeq P$ (Lemma 2.6(2)). If Ψ_X is full, then there exists a map $P \to S$ such that the induced map $P \to S \to \Phi_X\Psi_X(S) \simeq P$ is the identity morphism. That is, P is a direct summand of S. Since $\prod_{i\geq 0} M[in]$ is not bounded above [Nee11, Rem. 1.2] and ϕ is quasi-affine, it follows that P is not bounded above. But S is bounded above, so P cannot be a direct summand of S; hence, we have a contradiction and Ψ_X is not full. □

In the proof we used the following lemma in the special case where X and Y have affine diagonals or are noetherian and affine-pointed. Then it is a direct consequence of [HR17, Cor. 2.8] and Corollary 2.2.

Lemma 4.5. *If $\phi\colon X \to Y$ is a quasi-affine morphism of algebraic stacks, then $\mathsf{R}(\phi_{\mathsf{QCoh}})_*\colon \mathsf{D}(\mathsf{QCoh}(X)) \to \mathsf{D}(\mathsf{QCoh}(Y))$ is conservative.*

Proof. Since ϕ is quasi-affine, there is a factorization of ϕ as $X \xrightarrow{j} \overline{X} \xrightarrow{\overline{\phi}} Y$, where j is a quasi-compact open immersion and $\overline{\phi}$ is affine. Let $I \in \mathsf{K}(\mathsf{QCoh}(X))$ be a K-injective complex. Since $j_*\colon \mathsf{QCoh}(X) \to \mathsf{QCoh}(\overline{X})$ has an exact left adjoint j^*, it follows immediately that j_*I is K-injective in $\mathsf{K}(\mathsf{QCoh}(\overline{X}))$. In particular since $\overline{\phi}$ is affine, it follows that $\mathsf{R}(\phi_{\mathsf{QCoh}})_* \simeq \mathsf{R}(\overline{\phi}_{\mathsf{QCoh}})_* \circ \mathsf{R}(j_{\mathsf{QCoh}})_*$. Hence to prove the conservativity of $\mathsf{R}(\phi_{\mathsf{QCoh}})_*$, we may assume that either ϕ is a quasi-compact open immersion or affine. The affine case is trivial. And when ϕ is a quasi-compact open immersion, we simply observe that $\phi^*\phi_*I \simeq I$. □

Appendix A. $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is well generated

We begin with a general lemma.

Lemma A.1. *Let $\mathcal{T}$ be a well generated triangulated category and let $\mathcal{S} \subseteq \mathcal{T}$ be a localizing subcategory. The category $\mathcal{S}$ is well generated if and only if there is a set of generators in $\mathcal{S}$. That is: $\mathcal{S}$ is well generated if and only if there is a set of objects $S \subseteq \mathcal{S}$ such that any nonzero object $y \in \mathcal{S}$ admits a nonzero map $s \to y$, with $s \in S$.*

Proof. If $\mathcal{S}$ is well generated then it has a set of generators S satisfying a bunch of properties, one of which is that S detects nonzero objects—see the definitions in [Nee01b, pp. 273-274]. What needs proof is the reverse implication.

Suppose therefore that $\mathcal{S}$ contains a set of objects S as in the Lemma, that is every nonzero object $y \in \mathcal{S}$ admits a nonzero map $s \to y$ with $s \in S$. By [Nee01b, Prop. 8.4.2] the set S is contained in $\mathcal{T}^\alpha$ for some regular cardinal α. If $\mathcal{L} = \mathrm{Loc}(S)$

is the localizing subcategory generated by S then [Nee01b, Thm. 4.4.9] informs us that $\mathcal{L}$ is well generated. Since $S \subseteq \mathcal{S}$ and $\mathcal{S}$ is localizing it follows that $\mathcal{L} \subseteq \mathcal{S}$.

We know that $\mathcal{L}$ is well generated; to finish the proof it suffices to show that the inclusion $\mathcal{L} \subseteq \mathcal{S}$ is an equality. In any case the inclusion is a coproduct-preserving functor from the well generated category $\mathcal{L}$ and must have a right adjoint. For every object $y \in \mathcal{S}$, [Nee01b, Prop. 9.1.8] tells us that there is a triangle in $\mathcal{S}$

$$x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x$$

with $x \in \mathcal{L}$ and $z \in \mathcal{L}^{\perp}$. Since $z \in \mathcal{L}^{\perp} \subseteq S^{\perp}$ we have that the morphisms $s \to z$, with $s \in S$, all vanish. By the hypothesis of the Lemma it follows that $z = 0$, and hence $y \cong x$ belongs to $\mathcal{L}$. □

Remark A.2. We specialize Lemma A.1 to the situation where $\mathcal{T} = \mathsf{D}(\mathcal{A})$ is the derived category of a Grothendieck abelian category $\mathcal{A}$; by [Nee01a, Thm. 0.2] we know that $\mathcal{T}$ is well generated, and Lemma A.1 informs us that a localizing subcategory of $\mathsf{D}(\mathcal{A})$ is well generated if and only if it has a set of generators.

Let $\mathcal{A}$ be an abelian category and fix a fully faithful subcategory $\mathcal{C} \subseteq \mathcal{A}$. Following [Stacks, Tag 02MO] we say that

(1) $\mathcal{C}$ is a *Serre* subcategory if it is non-empty and if $C_1 \to A \to C_2$ is an exact sequence in $\mathcal{A}$ with $C_1, C_2 \in \mathcal{C}$, then $A \in \mathcal{C}$;
(2) $\mathcal{C}$ is a *weak Serre* subcategory if it is non-empty and if

$$C_1 \longrightarrow C_2 \longrightarrow A \longrightarrow C_3 \longrightarrow C_4,$$

is an exact sequence in $\mathcal{A}$, where the $C_i \in \mathcal{C}$ and $A \in \mathcal{A}$, then $A \in \mathcal{C}$.

Clearly, Serre subcategories are weak Serre subcategories. Also, weak Serre subcategories are automatically abelian and the inclusion $\mathcal{C} \subseteq \mathcal{A}$ is exact [Stacks, Tags 02MP & 0754]. Moreover, the subcategory $\mathsf{D}_{\mathcal{C}}(\mathcal{A})$ of $\mathsf{D}(\mathcal{A})$, consisting of complexes in $\mathcal{A}$ with cohomology in $\mathcal{C}$, is triangulated [Stacks, Tag 06UQ].

The main result of this appendix is

Theorem A.3. *Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathcal{M} \subseteq \mathcal{A}$ be a weak Serre subcategory closed under coproducts. If $\mathcal{M}$ is Grothendieck abelian, then $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is well generated.*

The example we have in mind is where X is an algebraic stack, $\mathcal{A}$ is the category of lisse-étale sheaves of $\mathcal{O}_X$-modules, and $\mathcal{M}$ is the subcategory of quasi-coherent sheaves.

Remark A.4. Note that in the ∞-category of ∞-categories, we have the following homotopy cartesian diagram:

$$\begin{array}{ccc} \mathcal{D}_{\mathcal{M}}(\mathcal{A}) & \longrightarrow & \mathcal{D}(\mathcal{A}) \\ {\scriptstyle \prod_n \mathcal{H}^n(-)} \downarrow & & \downarrow {\scriptstyle \prod_n \mathcal{H}^n(-)} \\ \prod_{n\in\mathbb{Z}} \mathrm{N}(\mathcal{M}) & \longrightarrow & \prod_{n\in\mathbb{Z}} \mathrm{N}(\mathcal{A}), \end{array}$$

where $\mathcal{D}(\mathcal{A})$ denotes the derived ∞-category of $\mathcal{A}$ [HA, Defn. 1.3.5.8] and $\mathcal{D}_{\mathcal{M}}(\mathcal{A})$ denotes the ∞-subcategory with homology in $\mathcal{M}$. This is immediate because the replete full subcategory $\mathcal{M} \subseteq \mathcal{A}$ gives rise to a categorical fibration $\mathrm{N}(\mathcal{M}) \subseteq \mathrm{N}(\mathcal{A})$ of ∞-categories. In particular, $\mathcal{D}_{\mathcal{M}}(\mathcal{A})$ is a presentable and stable ∞-category (combine [HTT, Prop. 5.5.3.12] with [HA, Prop. 1.3.5.21]). Since the homotopy category of $\mathcal{D}_{\mathcal{M}}(\mathcal{A})$ is $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$, the derived category we are interested in is well-generated [HA, Cor. 1.4.4.2].

If λ is a cardinal and $\mathcal{B}$ is a cocomplete category, then we let $\mathcal{B}^\lambda$ denote the subcategory of λ-presentable objects. If $\mathcal{B}$ is locally presentable, then $\mathcal{B}^\lambda$ is always a set.

It is clear that $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is a localizing subcategory of the well generated triangulated category $\mathsf{D}(\mathcal{A})$; Remark A.2 tells us that to prove Theorem A.3 it suffices to exhibit a set S of generators for $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$. The idea is simple enough: we will find a cardinal λ such that $S = \mathsf{D}^-_{\mathcal{M}^\lambda}(\mathcal{A}^\lambda) \subseteq \mathsf{D}_{\mathcal{M}}(\mathcal{A})$, which is obviously essentially small, suffices. Thus, the problem becomes to better understand the category of λ-presentable objects in $\mathcal{A}$. The results below are easy to extract from [Nee14], but for the reader's convenience we give a self-contained treatment.

Remark A.5. Let $\mathcal{A}$ be a Grothendieck abelian category. By the Gabriel–Popescu theorem, there exists a ring R and a pair of adjoint additive functors

$$F\colon \mathsf{Mod}(R) \rightleftarrows \mathcal{A} : G$$

such that F is exact and $FG \simeq \mathrm{id}$. Let μ be an infinite cardinal $\geq$ to the cardinality of R.

Lemma A.6. *With notation as in Remark A.5, let $\lambda > \mu$ be a regular cardinal. Then the λ-presentable objects of $\mathcal{A}$ are precisely the objects of $\mathcal{A}$ isomorphic to some FN, where N is an R-module of cardinality $< \lambda$.*

Proof. Let us first prove that, if N is an R-module of cardinality $< \lambda$, then FN is λ-presentable. Suppose $\{x_i,\, i \in I\}$ is a λ-filtered system in $\mathcal{A}$, and suppose that in the category $\mathcal{A}$ we are given a map $\phi\colon FN \longrightarrow \operatorname{colim} x_i$. We need to show that ϕ factors through some x_i. In the category of R-modules, there is a natural map

$$\operatorname{colim} Gx_i \xrightarrow{\ \rho\ } G\big[\operatorname{colim} x_i\big].$$

Since F respects colimits and $FG \simeq \mathrm{id}$, the map $F\rho$ is an isomorphism. As F is exact it must annihilate the kernel and cokernel of ρ. Form the pullback square

$$\begin{array}{ccc} P & \xrightarrow{\ \phi\ } & N \\ \downarrow & & \downarrow{\scriptstyle \theta} \\ \operatorname{colim} Gx_i & \xrightarrow{\ \rho\ } & G\big[\operatorname{colim} x_i\big]. \end{array}$$

The image of ϕ is a submodule of N, hence has cardinality $< \lambda$. If we lift every element of $\mathrm{Image}(\phi)$ arbitrarily to an element of P, the lifts generate a submodule $M \subseteq P$ of cardinality $< \lambda$. The kernel (respectively cokernel) of the map $M \to N$ is a submodule of $\mathrm{Kernel}(\rho)$ (respectively $\mathrm{Coker}(\rho)$), and hence both are annihilated by F. Summarizing: we have produced a morphism $M \to N$ of R-modules, with M of cardinality $< \lambda$ and $FM \to FN$ an isomorphism in $\mathcal{A}$, and such that the composite $M \to N \longrightarrow G\big[\operatorname{colim} x_i\big]$ factors through $\operatorname{colim} Gx_i$. But $\{Gx_i,\, i \in I\}$ is a λ-filtered system in $\mathsf{Mod}(R)$ and M is of cardinality $< \lambda$, and hence the map factors as $M \to Gx_i$ for some $i \in I$.

It remains to prove the converse: suppose $a \in \mathcal{A}$ is a λ-presentable object. Then Ga is an R-module, hence it is the λ-filtered colimit of all its submodules N_i of cardinality $< \lambda$. But then the identity map $a \to a$ is a map from the λ-presentable object a to the λ-filtered colimit of the FN_i, and therefore factors through some FN_i. Thus a is a direct summand of an object FN_i where the cardinality of N_i is $< \lambda$. On the other hand the map $N_i \to Ga$ is injective, hence so is $FN_i \to FGa = a$. Thus $a \cong FN_i$. □

Lemma A.7. *Let $\mathcal{A}$ be a Grothendieck abelian category. There is an infinite cardinal ν with the following properties: if $\lambda \geq \nu$ is a regular cardinal, then*

(1) *$\mathcal{A}^\lambda$ is a Serre subcategory of $\mathcal{A}$;*
(2) *every object of $\mathcal{A}$ is a λ-filtered colimit of subobjects belonging to $\mathcal{A}^\lambda$;*
(3) *an object belongs to $\mathcal{A}^\lambda$ if and only if it is the quotient of a coproduct of $< \lambda$ objects of $\mathcal{A}^\nu$; and*
(4) *any pair of morphisms $x \to y \leftarrow n$ in $\mathcal{A}$, where $x \to y$ is epi and $n \in \mathcal{A}^\lambda$, may be completed to a commutative square*

$$\begin{array}{ccc} m & \twoheadrightarrow & n \\ \downarrow & & \downarrow \\ x & \twoheadrightarrow & y \end{array}$$

with $m \in \mathcal{A}^\lambda$ and $m \to n$ epi. Moreover, if $n \to y$ is mono, then $m \to x$ can be chosen to be mono.

Proof. Let $\nu = \mu + 1$ be the successor of the infinite cardinal μ of Remark A.5. By Lemma A.6 the objects of $\mathcal{A}^\lambda$ are precisely the ones isomorphic to FM where M is of cardinality $< \lambda$.

For (1), it is readily verified that a subobject (resp. a quotient) of FM can be expressed as FN where N is a submodule (resp. a quotient) of M. This shows that $\mathcal{A}^\lambda$ is closed under taking subobjects and quotients; we will later see that it is also closed under extensions.

For (2), if a is an object of $\mathcal{A}$ then Ga is the λ-filtered colimit of all the submodules $M_i \subseteq Ga$ of cardinality $< \lambda$, and $a \cong FGa$ is the λ-filtered colimit of $FM_i \in \mathcal{A}^\lambda$.

For (3), observe that any coproduct of $< \lambda$ objects in $\mathcal{A}^\lambda$ belongs to $\mathcal{A}^\lambda$, and if M is a module of cardinality $< \lambda$ then M is a quotient of the free module on all its elements, which is a coproduct of $< \lambda$ copies of R. Thus FM is the quotient of a coproduct of $< \lambda$ copies of $FR \in \mathcal{A}^\nu$.

For (4), let $x \to y \leftarrow n$ be a pair of morphisms in $\mathcal{A}$, with $x \to y$ epi and $n \in \mathcal{A}^\lambda$. Let $\tilde{m}$ be the pullback of $n \to y$ along $x \to y$. It is sufficient to find a subobject m of $\tilde{m}$ belonging to $\mathcal{A}^\lambda$ such that $m \to \tilde{m} \to n$ is epi. By (2), we may express $\tilde{m} = \operatorname{colim} m_i$ as a λ-filtered colimit of subobjects belonging $\mathcal{A}^\lambda$. If $n_i \subseteq n$ is the image of m_i in n, then (1) implies that $n_i \in \mathcal{A}^\lambda$. Since $n \in \mathcal{A}^\lambda$, there is an i such that $n_i = n$. Taking $m = m_i$ does the job. By construction, if $n \to y$ is mono, then $m \to x$ is mono.

Finally, to show that $\mathcal{A}^\lambda$ is closed under extensions, we note that if $0 \to k \to x \to n \to 0$ is an exact sequence with $k,\ n \in \mathcal{A}^\lambda$, then (4) implies that there is a subobject m of x such that $m \in \mathcal{A}^\lambda$ and $m \to n$ is epi. It follows that $k \oplus m \to x$ is epi and consequently, $x \in \mathcal{A}^\lambda$, as required. □

Proof of Theorem A.3. Because $\mathcal{M}$ and $\mathcal{A}$ are both Grothendieck abelian categories we may choose regular cardinals ν for $\mathcal{M}$ and ν' for $\mathcal{A}$ as in Lemma A.7. The category $\mathcal{M}^\nu$ is an essentially small subcategory of $\mathcal{A}$, hence must be contained in $\mathcal{A}^\beta$ for some regular cardinal β. Let λ be a regular cardinal $> \max(\beta, \nu, \nu')$. By construction $\mathcal{M}^\nu \subseteq \mathcal{A}^\lambda$, and by Lemma A.7 every object in $\mathcal{M}^\lambda$ is the quotient of a coproduct of $< \lambda$ objects in $\mathcal{M}^\nu$. Hence $\mathcal{M}^\lambda \subseteq \mathcal{A}^\lambda$. But since every object in $\mathcal{M} \cap \mathcal{A}^\lambda$ is λ-presentable in $\mathcal{A}$ it must be λ-presentable in the smaller $\mathcal{M}$, and we conclude that $\mathcal{M} \cap \mathcal{A}^\lambda = \mathcal{M}^\lambda$.

We have now made our choice of λ and we let $\mathcal{B} = \mathcal{A}^\lambda$. By Remark A.2 it suffices to show that, given any non-zero object $Z \in \mathsf{D}_{\mathcal{M}}(\mathcal{A})$, there is an object $N \in \mathsf{D}^-_{\mathcal{B}\cap\mathcal{M}}(\mathcal{B})$ and a non-zero map $N \to Z$. If Z is the chain complex

$$\cdots \longrightarrow Z^{i-1} \xrightarrow{\ \partial\ } Z^i \xrightarrow{\ \partial\ } Z^{i+1} \longrightarrow \cdots,$$

we let $Y^i \subseteq Z^i$ be the cycles, in other words the kernel of $\partial\colon Z^i \to Z^{i+1}$, and $X^i \subseteq Y^i$ be the boundaries, that is the image of $\partial\colon Z^{i-1} \to Z^i$. We are assuming that $Z \in \mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is non-zero, meaning its cohomology is not all zero; without loss of generality we may assume $H^0(Z) \neq 0$. Thus Y^0/X^0 is a non-zero object of $\mathcal{M}$.

By Lemma A.7, applied to $\mathcal{B} \cap \mathcal{M} = \mathcal{M}^\lambda \subseteq \mathcal{M}$, the object $Y^0/X^0 \in \mathcal{M}$ is a λ-filtered colimit of its subobjects belonging to $\mathcal{B} \cap \mathcal{M}$; since $Y^0/X^0 \neq 0$ we may choose a subobject $M \subseteq Y^0/X^0$, with $M \in \mathcal{B} \cap \mathcal{M}$ and $M \neq 0$. By Lemma A.7, applied to the pair of maps $Y^0 \to Y^0/X^0 \leftarrow M$ in $\mathcal{A}$, we may complete to a commutative square

$$\begin{array}{ccc} N^0 & \xrightarrow{\ \phi\ } & M \\ \downarrow & & \downarrow \\ Y^0 & \twoheadrightarrow & Y^0/X^0 \end{array}$$

with $N^0 \in \mathcal{B}$. Since Y^0 is the kernel of $Z^0 \to Z^1$ this gives us a commutative square

$$\begin{array}{ccc} N^0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Z^0 & \longrightarrow & Z^1 \end{array}$$

such that the image of the map $N^0 \to Y^0/X^0 = H^0(Z)$ is non-zero and belongs to $\mathcal{B} \cap \mathcal{M}$.

We propose to inductively extend this to the left. We will define a commutative diagram

$$\begin{array}{cccccccccccccc} & & & & N^i & \to & N^{i+1} & \to \cdots \to & N^{-1} & \to & N^0 & \to & 0 & \to \cdots \\ & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots & \to & Z^{i-1} & \to & Z^i & \to & Z^{i+1} & \to \cdots \to & Z^{-1} & \to & Z^0 & \to & Z^1 & \to \cdots \end{array}$$

where

(1) The subobjects $N^j \subseteq Z^j$ belong to $\mathcal{B}$.
(2) For $j > i$ the cohomology of $N^{j-1} \to N^j \to N^{j+1}$ belongs to $\mathcal{B} \cap \mathcal{M}$.
(3) Let K^i be the kernel of the map $N^i \to N^{i+1}$. Then the image of the natural map $K^i \to H^i(Z)$ belongs to $\mathcal{B} \cap \mathcal{M}$.

Since we have constructed N^0 we only need to prove the inductive step. Let us therefore suppose we have constructed the diagram as far as i; we need to extend it to $i-1$. We first form the pullback square

$$\begin{array}{ccc} L^i & \longrightarrow & K^i \\ \downarrow & & \downarrow \\ X^i & \longrightarrow & Y^i \end{array}$$

Since $X^i \to Y^i$ and $K^i \to Y^i$ are monomorphisms so are $L^i \to K^i$ and $L^i \to X^i$. Since N^i belongs to $\mathcal{B}$ so do its subobjects $L^i \subseteq K^i$. The cokernel of $L^i \to K^i$ is the image of $K^i \to Y^i/X^i = H^i(Z)$, and belongs to $\mathcal{B} \cap \mathcal{M}$ by (3). Next we apply Lemma A.7 to the pair of maps $Z^{i-1}/X^{i-1} \to X^i \leftarrow L^i$ in $\mathcal{A}$, completing to a commutative square

$$\begin{array}{ccc} M^i & \twoheadrightarrow & L^i \\ \downarrow & & \downarrow \\ Z^{i-1}/X^{i-1} & \twoheadrightarrow & X^i \end{array}$$

with $M^i \in \mathcal{B}$. Form the pullback

$$\begin{array}{ccc}
\widetilde{M}^i & \longrightarrow & M^i \\
\downarrow & & \downarrow \\
Y^{i-1}/X^{i-1} & \longrightarrow & Z^{i-1}/X^{i-1}
\end{array}$$

Since $Y^{i-1}/X^{i-1} \to Z^{i-1}/X^{i-1}$ is injective so is $\widetilde{M}^i \to M^i$, making $\widetilde{M}^i$ a subobject of $M^i \in \mathcal{B}$. Hence $\widetilde{M}^i$ belongs to $\mathcal{B}$. But now the map $\widetilde{M}^i \to Y^{i-1}/X^{i-1} = H^{i-1}(Z)$ is a morphism from the λ-presentable object $\widetilde{M}^i \in \mathcal{B} = \mathcal{A}^\lambda$ to the object $H^{i-1}(Z) \in \mathcal{M}$, which by Lemma A.7 is a λ-filtered colimit of its subobjects in $\mathcal{M}^\lambda = \mathcal{B} \cap \mathcal{M}$. Hence the map $\widetilde{M}^i \to Y^{i-1}/X^{i-1}$ factors as $\widetilde{M}^i \to P^i \to Y^{i-1}/X^{i-1}$ with $P^i \in \mathcal{B} \cap \mathcal{M}$ a subobject of Y^{i-1}/X^{i-1}. Form the pushout square in $\mathcal{B}$

$$\begin{array}{ccc}
\widetilde{M}^i & \longrightarrow & M^i \\
\downarrow & & \downarrow \\
P^i & \longrightarrow & Q^i
\end{array}$$

and let $Q^i \to Z^{i-1}/X^{i-1}$ be the natural map. We have a commutative square

$$\begin{array}{ccc}
Q^i & \overset{\phi}{\twoheadrightarrow} & L^i \\
\downarrow & & \downarrow \\
Z^{i-1}/X^{i-1} & \twoheadrightarrow & X^i
\end{array}$$

and the kernel of ϕ maps isomorphically to the subobject $P^i \subseteq H^{i-1}(Z)$, with $P^i \in \mathcal{B} \cap \mathcal{M}$. Finally apply Lemma A.7 to the pair of maps $Z^{i-1} \to Z^{i-1}/X^{i-1} \leftarrow Q^i$ to complete to a square

$$\begin{array}{ccc}
N^{i-1} & \twoheadrightarrow & Q^i \\
\downarrow & & \downarrow \\
Z^{i-1} & \twoheadrightarrow & Z^{i-1}/X^{i-1}
\end{array}$$

with $N^i \in \mathcal{B}$. We leave it to the reader to check that the diagram

$$\begin{array}{ccccccccccccccc}
 & & N^{i-1} & \to & N^i & \to & \cdots & \to & N^{-1} & \to & N^0 & \to & 0 & \to & \cdots \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & Z^{i-2} & \to Z^{i-1} & \to & Z^i & \to & \cdots & \to & Z^{-1} & \to & Z^0 & \to & Z^1 & \to & \cdots
\end{array}$$

satisfies hypotheses (1), (2) and (3) of our induction. □

Appendix B. $\mathsf{D}_{\mathrm{qc}}(X)$ is left-complete

Let $\mathcal{T}$ be a triangulated category with a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$. If $\mathcal{T}$ admits countable projects, then we say that the t-structure is *left-complete* if the map

$$M \to \operatorname*{holim}_n \tau^{\geq -n} M$$

is an isomorphism for every $M \in \mathcal{T}$. A thorough discussion of t-structures using the language of stable ∞-categories is available in [HA, §1.2.1].

If $\mathcal{A}$ is a Grothendieck abelian category, then the standard t-structure on $\mathsf{D}(\mathcal{A})$ is $(\mathsf{D}^{\leq 0}(\mathcal{A}), \mathsf{D}^{\geq 0}(\mathcal{A}))$. If $\mathcal{M} \subseteq \mathcal{A}$ is a Grothendieck abelian weak Serre subcategory, then the standard t-structure on $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is the one induced by restriction from $\mathsf{D}(\mathcal{A})$. If no t-structure on $\mathsf{D}_{\mathcal{M}}(\mathcal{A})$ is specified, then we will always mean the standard one.

In this section we prove the following Theorem.

Theorem B.1. *If X is an algebraic stack, then $\mathsf{D}_{\mathrm{qc}}(X)$ is well generated. In particular, it admits small products. Moreover, $\mathsf{D}_{\mathrm{qc}}(X)$ is left-complete.*

Proof. The subcategory $\mathsf{QCoh}(X) \subseteq \mathsf{Mod}(X_{\text{lis-ét}})$ is weak Serre and the inclusion is coproduct preserving. Since $\mathsf{QCoh}(X)$ and $\mathsf{Mod}(X_{\text{lis-ét}})$ are Grothendieck abelian categories [Stacks, Tags 07A5 & 0781], it follows that $\mathsf{D}_{\mathrm{qc}}(X)$ is well generated (Theorem A.3). In particular, $\mathsf{D}_{\mathrm{qc}}(X)$ admits small products [Nee01b, Cor. 1.18].

It remains to prove that $\mathsf{D}_{\mathrm{qc}}(X)$ is left-complete, which we accomplish by extracting from [Stacks, Tag 08U3] a useful special case (which was communicated to us by Bhatt [Bha12]). So the inclusion $\omega\colon \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}(X)$ is exact and coproduct preserving; thus, the functor ω admits a right adjoint λ [Nee01b, Prop. 1.20]. Because the functor ω is fully faithful, the adjunction $\mathrm{id} \Rightarrow \lambda \circ \omega$ is an isomorphism of functors.

Note that because λ is a right adjoint, it preserves products and so homotopy limits. In particular, it remains to prove that if $K \in \mathsf{D}_{\mathrm{qc}}(X)$, then a naturally induced map:

$$c\colon \omega(K) \to \operatorname*{holim}_n \tau^{\geq -n}\omega(K)$$

is a quasi-isomorphism in $\mathsf{D}(X)$ (where we also take the homotopy limit in $\mathsf{D}(X)$). Indeed, this follows from the observation that $\tau^{\geq -n}\omega(K) \simeq \omega(\tau^{\geq -n}K)$ for all integers n and $K \to \lambda \circ \omega(K)$ is an isomorphism.

To see that c is a quasi-isomorphism in $\mathsf{D}(X)$, it is sufficient to prove that $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(p_!\mathcal{O}_U, c)$ is a quasi-isomorphism for every smooth morphism $p\colon U \to X$, where U is an affine scheme. Observe that

$$\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(p_!\mathcal{O}_U, \tau^{\geq -n}K) \simeq \mathsf{R}\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \tau^{\geq -n}p^{-1}K) \simeq \tau^{\geq -n}K(U),$$

with the final quasi-isomorphism because U is affine and K has quasi-coherent cohomology [Stacks, Tags 01XB & 0756]. But $\mathsf{R}\mathrm{Hom}_{\mathcal{O}_X}(p_!\mathcal{O}_U, -)$ commutes with homotopy limits, so it is sufficient to prove that we have the following quasi-isomorphism of abelian groups:

$$K(U) \to \operatorname*{holim}_n \tau^{\geq -n}K(U),$$

which is well-known, because the products in the category of abelian groups are exact. For details, see for example [Stacks, Tag 07KC]. □

Appendix C. The bounded below derived category

In this section, we prove an analog of [Har66, Cor. II.7.19] for noetherian algebraic stacks that are affine-pointed, cf. [Lie04, Rem. 2.2.4.7]. Essentially for free, we will also establish Lurie's result [Lur04, Thm. 3.8].

Theorem C.1. *Let X be an algebraic stack. If X is either quasi-compact with affine diagonal or noetherian and affine-pointed, then the natural functor*

$$\Psi_X^+\colon \mathsf{D}^+(\mathsf{QCoh}(X)) \to \mathsf{D}^+_{\mathrm{qc}}(X)$$

is an equivalence.

The conditions on X are essentially sharp: Ψ_X^b can fail to be fully faithful if:

(1) X is a non-noetherian quasi-compact and quasi-separated scheme with non-affine diagonal [SGA6, Exp. II, App. I].
(2) X is noetherian with non-affine stabilizers, e.g., if X is the classifying stack of an elliptic curve.

For noetherian algebraic spaces, a version of Theorem C.1 for the unbounded derived category was proved in [Stacks, Tag 09TN] and we will closely follow this approach. The following two lemmas do most of the work.

Lemma C.2 (cf. [Stacks, Tag 09TJ])**.** *Let X be a quasi-compact and quasi-separated algebraic stack and let I be an injective object of $\mathsf{QCoh}(X)$.*

(1) *Then I is a direct summand of p_*J, where $p\colon \operatorname{Spec} A \to X$ is smooth and surjective and J is an injective A-module.*
(2) *If X is noetherian, then I is a direct summand of a filtered colimit $\operatorname{colim}_i F_i$ of quasi-coherent sheaves of the form $F_i = \gamma_{i,*}G_i$, where $\gamma_i\colon Z_i \to X$ is a morphism from an artinian scheme Z_i and $G_i \in \mathsf{Coh}(Z_i)$.*

Proof. Let $p\colon U \to X$ be a smooth and surjective morphism, where $U = \operatorname{Spec} A$ is an affine scheme. Let I be an injective object of $\mathsf{QCoh}(X)$. Choose an injective object J of $\mathsf{QCoh}(U)$ and an injection $p^*I \subseteq J$. By adjunction, we have an inclusion $I \subseteq p_*J$. Since p^* is exact, p_*J is injective in $\mathsf{QCoh}(X)$ and I is a direct summand of p_*J. This proves (1). For (2): we may now reduce to the case where $X = U$. The result is now well-known (e.g., [Stacks, Tag 09TI]). □

Lemma C.3 (cf. [Stacks, Tag 09TL])**.** *Let X be an algebraic stack and let I be an injective object of $\mathsf{QCoh}(X)$. If X is quasi-compact with affine diagonal (resp. noetherian and affine-pointed), then*

(1) *$H^q(U_{\text{lis-ét}}, I) = 0$ for every $q > 0$ and smooth morphism $u\colon U \to X$ that is affine (resp. has affine fibers);*
(2) *for any morphism $f\colon X \to Y$ of algebraic stacks, where Y has affine diagonal (resp. Y is affine-pointed) we have $\mathsf{R}^q(f_{\text{lis-ét}})_*I = 0$ for $q > 0$.*

Proof. Let W be an affine (resp. artinian) scheme and let $M \in \mathsf{QCoh}(W)$ be injective (resp. $M \in \mathsf{Coh}(W)$). Let $w\colon W \to X$ be a smooth and surjective morphism (resp. a morphism). By Lemma C.2, it is sufficient to prove the result for $I = w_*M$. Since X has affine diagonal (resp. X is affine-pointed), w is affine. In particular, the natural map $(w_*M)[0] \to \mathsf{R}(w_{\text{lis-ét}})_*M$ is a quasi-isomorphism.

We now prove (1). Let $u_W\colon W_U \to W$ be the pull back of u along w and let $w_U\colon W_U \to U$ be the pull back of w along u. In both cases, u_W is smooth and affine and w_U is affine; in particular, W_U is an affine scheme. Since u is smooth,

$$\begin{aligned}\mathsf{R}\Gamma(U_{\text{lis-ét}}, I) &\simeq \mathsf{R}\Gamma(U_{\text{lis-ét}}, u^*\mathsf{R}(w_{\text{lis-ét}})_*M) \simeq \mathsf{R}\Gamma(U_{\text{lis-ét}}, \mathsf{R}((w_U)_{\text{lis-ét}})_*(u_W^*M))\\ &\simeq \mathsf{R}\Gamma((W_U)_{\text{lis-ét}}, M).\end{aligned}$$

The result now follows from the affine case (e.g., [EGA, III.1.3.1]).

For (2): let $v\colon V \to Y$ be a smooth morphism, where V is an affine scheme. Since Y has affine diagonal (resp. is affine-pointed), v is affine (resp. has affine fibers). By (1), $H^q((V \times_Y X)_{\text{lis-ét}}, I) = 0$. But $\mathsf{R}^q(f_{\text{lis-ét}})_*I$ is the sheafification of the presheaf $V \mapsto H^q((V \times_Y X)_{\text{lis-ét}}, I)$; the result follows. □

Proof of Theorem C.1. We first establish that Ψ_X^+ is fully faithful: given F, $G \in \mathsf{D}^+(\mathsf{QCoh}(X))$ we wish to prove that the natural map

$$\operatorname{Hom}_{\mathsf{D}(\mathsf{QCoh}(X))}(F, G) \to \operatorname{Hom}_{\mathsf{D}(X)}(F, G)$$

is an isomorphism. A standard way-out argument shows that it is sufficient to prove that the natural map

$$\operatorname{Ext}^q_{\mathsf{QCoh}(X)}(N, M) \to \operatorname{Ext}^q_{\mathcal{O}_X}(N, M)$$

is an isomorphism for every $q \in \mathbb{Z}$ and M, $N \in \mathsf{QCoh}(X)$. For $q < 0$ both sides vanish and for $q = 0$ we clearly have an isomorphism. For $q > 0$, since every M

embeds in a quasi-coherent injective I, a standard δ-functor argument shows that it is sufficient to prove that if I is an injective object of $\mathsf{QCoh}(X)$, then

$$\mathrm{Ext}^q_{\mathcal{O}_X}(N, I) = 0$$

for all $q > 0$ and $N \in \mathsf{QCoh}(X)$. To see this we note that by Lemma C.2(1), I is a direct summand of $(p_{\mathsf{QCoh}})_* J$, where $p\colon \operatorname{Spec} A \to X$ is smooth and surjective and J is an injective A-module. Thus, it suffices to prove the result when $I = (p_{\mathsf{QCoh}})_* J$. By Lemma C.3(2), the natural map $((p_{\mathsf{QCoh}})_* J)[0] \to \mathsf{R}(p_{\text{lis-ét}})_* J$ is a quasi-isomorphism. Hence, there are natural isomorphisms:

$$\mathrm{Ext}^q_{\mathcal{O}_X}(N, (p_{\mathsf{QCoh}})_* J) \cong \mathrm{Ext}^q_{\mathcal{O}_X}(N, \mathsf{R}(p_{\text{lis-ét}})_* J) \cong \mathrm{Ext}^q_{\mathcal{O}_{\operatorname{Spec} A}}(p^* N, J).$$

We are now reduced to the affine case, which is well-known (e.g., [BN93, Lem. 5.4]).

For the essential surjectivity, we argue as follows: by induction and using the full faithfulness, one easily sees that $\mathsf{D}^b(\mathsf{QCoh}(X)) \simeq \mathsf{D}^b_{\mathrm{qc}}(X)$. Passing to homotopy colimits, we obtain the claim. □

The following observation was made by Bhatt [Bha12] and a reviewer.

Remark C.4. Let X be an algebraic stack that is either quasi-compact with affine diagonal or noetherian and affine-pointed. Since $\mathsf{D}_{\mathrm{qc}}(X)$ is left-complete, Ψ_X factors uniquely through the left-completion functor $\mathsf{D}(\mathsf{QCoh}(X)) \to \widehat{\mathsf{D}}(\mathsf{QCoh}(X))$ [HA, §1.2.1]. But $\widehat{\mathsf{D}}(\mathsf{QCoh}(X))$ is also the left completion of $\mathsf{D}^+(\mathsf{QCoh}(X))$ and Ψ^+_X is an equivalence. Hence, $\widehat{\mathsf{D}}(\mathsf{QCoh}(X)) \to \mathsf{D}_{\mathrm{qc}}(X)$ is an equivalence.

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