Cooperative extremum seeking for a one-dimensional non-linear map using a two-robot formation

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Abstract— This paper presents a cooperative extremum seeking control scheme that uses a linear formation of two robots to find a maximum of an unknown one-dimensional non-linear map. A consensus control algorithm is used to drive the centre of the formation and an integral feedback control loop is proposed to drive the gradient of an unknown field function to zero. Simulation results are provided to evaluate the performance of the proposed method.

I. INTRODUCTION

Extremum seeking (ES) control naturally has found its application in the design of source seeking algorithms for autonomous mobile robots, where the aim is to find a source that emits a signal of interest and the strength of the signal is a function of distance from the source. In this type of problem, the field is unknown and only measurements of the field value at current robots' locations are available.

Various ES control techniques are used in the literature for coordination of a single and multiple autonomous robots in source seeking applications. For the single agent case the auxiliary movements are needed to estimate the gradient of the field, while in multi agent schemes the gradient estimation is performed using cooperative field sensing and information sharing. The authors of [2] propose an algorithm for a group of double integrators. Assuming that the robots are able to keep a record of previous measurements, the authors combine gradient projection on the vehicle's trajectory with a spring inter-vehicle potential function to produce a control input for each robot. In the subsequent work [15], the algorithm is modified for a group of robots moving in a formation that follows a virtual leader; gradient value at the leader's location is estimated by combining the field values across all platforms. A gradient climbing algorithm is presented in [3], where the local field gradient and Hessian matrices are estimated by a finite-difference method and the Newton's optimization method is used to determine the next reference position for the formation leader.

Randomized optimization based ES algorithms for particle swarming and bacteria foraging are presented in [11],[18]. A hybrid ES controller for a single integrator is described by [13]. The vehicle first travels in discrete steps along the

straight line until the minimum in the current direction is found and the *Recursive Smith-Powell Algorithm* is then used to choose a new search direction.

Sliding mode control is used for ES with a single unicycle in [12], where the angular velocity is being tuned while the linear velocity is constant. Another example of a sliding mode controller for a unicycle like vehicle is presented in [8] where the robot aims to track a moving target. Sliding mode algorithms do not require high computational capabilities, are easy to implement and has been showed to be robust to measurement noise. However, the main disadvantage is that the controller switches between two extreme control points causing the so-called "chattering effect" that wears out the actuators.

A perturbation based ES control (PESC) method which employs a dither signal to estimate the gradient of an unknown function is introduced in [1]. An extensive research in PESC was conducted by Krstic and co-workers. In [20], [19], [7] algorithms for agents with single and double integrator and unicycle dynamics are discussed, where either angular or forward velocity is tuned using PESC. One of the last works on the stochastic PESC is presented in [10], where the authors use white noise as the perturbation signal and both forward and angular velocities are being tuned.

The authors of [17] propose a switching ES control scheme, where single integrator agents are carrying out individual exploration approximating field gradient using time differences and switching to cooperative exploration based on signal to noise ratio. Approach is tested a field with circular level sets and shows to be robust to measurement noise. A dither-free approach to ES control is proposed in [9] that uses history data of the field measurements to estimate its gradient by the first-order least squares fit and the optimizer is just an integrator with an adaptation gain.

The series of works [14], [5], [4], [6] present an ES approach with a circular formation of agents, where the field gradient is estimated as an average of the measurements across all agents weighted by a displacement of the corresponding agent from the formation centre and the estimate is used as a reference velocity value for the formation centre, then a consensus control algorithm for the formation control. Stability is analyzed for circular and elliptical level sets. While [14] assumes all-to-all

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communication, [5] only considers communication between neighboring agents and additional consensus algorithm is used to agree on estimated gradient direction. In [4] the authors keep gradient estimation and formation control the same but propose gradient-ascent algorithm to drive the formation centre. An extension to the problem of moving target tracking is provided in [6].

In this paper we propose an ES control scheme for a two robot formation that employs consensus control and negative integral feedback to drive a formation of two robots to a maximum of a one-dimensional field. Without making an assumption of circular or elliptical level sets of the unknown field we show that for a field function with a bounded second derivative, the formation successfully converges to a bounded region around maximum of the field and stops there.

In Section II the proposed control loop is described and its stability is analysed. Section III presents the proposed cooperative extremum seeking algorithm for a two-robot formation driven by a consensus algorithm. Section IV provides simulation results, and in Section V we give a conclusion and a perspective on future work.

II. PROBLEM STATEMENT

A. Stabilization to zero of a non-linear function using integral feedback control

Consider a problem where we need to control an input u so that a non-linear system output $y = f(u)$ is driven to zero using a negative integral feedback. The problem is shown in Figure 1, where u_0 is a zero of the function $f(u)$ and K is a feedback gain, that is a design parameter. We want our control to guarantee that as $t \rightarrow \infty$, $u \rightarrow u_0$ and thus $y(t) \rightarrow 0$.

Fig. 1. The proposed feedback loop.

To analyze stability of the system in Figure 1, let z denote the output of the integrator. Then we can write:

$\dot{z} = y = f(u)$ and $u = u_0 - Kz$, that gives us:

$$\dot{z} = f(u_0 - Kz), \quad (1)$$

Linearization of equation (1) using the first order Taylor expansion around $z = 0$ gives us:

$$\dot{z} \approx f(u_0) + f'(u_0)(u_0 - Kz - u_0) = -K\alpha z, \quad (2)$$

where $\alpha = f'(u_0)$. The solution of the equation (2) is $z(t) = e^{-K\alpha t} z(0)$. Hence,

$$y(t) = -K\alpha e^{-K\alpha t} z(0) \quad (3)$$

From equation (3) we can see that $y(t) \rightarrow 0$ as $t \rightarrow \infty$, given that $K \neq 0$, $K\alpha > 0$ and the initial condition $z(0) = \frac{u_0 - u(0)}{K}$ is sufficiently small for linearization errors to be negligible.

B. Extension to the Extremum Seeking Control

Inspired by the technique described in the previous section, we extend the proposed method to the extremum seeking control problem for a group of mobile robots. Now we think of the non-linear map whose zero is to be found as the *gradient* of a one-dimensional field of interest and of an input u as a location of the robot formation centre.

Consider a linear formation of two mobile robots with single integrator kinematics:

$$\dot{x} = v \quad (4)$$

where x and v are the robot's position and linear velocity respectively. The problem is depicted in figure 2. The two robots (marked as 1 and 2) are moving along the X -axis, taking measurements of the function $f(x)$ along the way. Our goal is to drive the centre of the formation (marked with the yellow star) to a position x_{max} that corresponds to the maximum of the function.

Fig. 2. The two robot setup for one-dimensional extremum seeking.

III. EXTREMUM SEEKING SCHEME WITH CONSENSUS CONTROL

To adopt the technique from the previous section we add the robots' kinematics and velocity control in the loop.

A. Consensus control setup

We propose to use consensus control to drive the formation centre to the extremum and a simple proportional control for robots' velocity inputs. The information state for the consensus control algorithm is the location of the formation centre. We start with the dynamic consensus tracking algorithm involving n agents from [16], where information state of the i^{th} robot evolves as:

$$\dot{x}_i(t) = - \sum_{j=1}^n a_{ij}(t)[x_i(t) - x_j(t)] + e(t), \quad (5)$$

where $a_{ij}(t)$ is an entry of the adjacency matrix, $x_i(t)$ is the location of the formation centre as perceived by the i^{th} robot and $e(t)$ is a driving signal generated by the feedback loop, that is an approximation of the gradient of the field and for a single integrator dynamics is equivalent to a reference velocity direction. The consensus algorithm in (5) aim to ensure that all agents track the formation centre. To derive control inputs for each robot, we add desired state deviations into the consensus algorithm, so we modify (5) as follows:

$$\dot{x}_i(t) = -\sum_{j=1}^n a_{ij}(t)[x_i(t) - x_j(t) - (\delta_i - \delta_j)] + e(t), \quad (6)$$

where $\delta_{1,2} = \mp h$ are desired distances of the robots from the formation centre. The proposed control loop diagram is presented in Figure 3. Now in the system in Figure 3 and equation (6) x_1 and x_2 are positions of the robots, and the $\pm 2h$ offset is introduced to ensure the robots maintain the desired formation which is to keep them $2h$ meters apart. We also introduce two adaptation gains K_p in the velocity control loop and K in the feedback loop to be able to regulate formation velocity and strength of the driving signal $e(t)$ in the consensus algorithm.

Fig. 3. Basic loop extension to consensus control.

B. Convergence analysis

To analyze the stability of the proposed control loop we write system equations expanding (6) for x_1 and x_2 , with $\delta_{1,2} = \mp h$ and adjacency matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and adding proportional gain K_p into the velocity control loop.

$$\begin{aligned} \dot{x}_1 &= K_p(-e - x_1 + x_2 - 2h), \\ \dot{x}_2 &= K_p(-e - x_2 + x_1 + 2h) \end{aligned} \quad (7)$$

$$u = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (8)$$

where the input $e(t) = Kg(u)$ is coming from the feedback loop and the output is $u(t) = x_{centre}(t)$. From (8) and (7) we can write:

$$\dot{u} = \frac{1}{2}(\dot{x}_1 + \dot{x}_2) = -K_p e \quad (9)$$

whose transfer function $H(s)$ is given in (10) below.

$$H(s) = -\frac{K_p}{s} \quad (10)$$

After these transformations, the system in Figure 3 reduces to the system in Figure 4, since the dynamics of $x_2 - x_1$ are decoupled from the dynamics of the formation centre. We can see that the equivalent transfer function of the consensus control already contains an integrator so we do not need to include one in the feedback loop. The equivalent block diagram is the same as the basic control loop presented in Figure 1 and is stable as long as $K < 0$ so that $K\alpha > 0$, as we have shown in Section II.A. Here, $\alpha = f''(x_{\text{maximum}}) < 0$ for a maximum of the field function.

Fig. 4. Equivalent diagram the the consensus loop.

C. Linearized approximated gradient algorithm

A gradient approximation must be used to enable the implementation the proposed control scheme as we do not have the possibility to measure the gradient value directly. For a linear formation of two robots the gradient value at the centre of the formation is approximated using the finite difference formula:

$$g(u) = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} \quad (11)$$

where $f(x_1), f(x_2)$ are measurements of the field taken by the robots at their corresponding locations. Now using (11) for the diagram in Figure 3 we can write:

$$e = K \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} \quad (12)$$

K is a constant. When $x_1 = x_2$, we assume by continuity that $\frac{f(x_2) - f(x_1)}{(x_2 - x_1)} = f'(x_1) = f'(x_2)$.

Let f be a function $\mathbb{R}^1 \rightarrow \mathbb{R}^1$ which has the following properties:

- (i) The function f and its derivatives f', f'' are continuous;
- (ii) The function f has a single maximum, i.e., there is one and only one point x^* such that $f'(x^*) = 0$.
- (iii) The second derivative $f''(x)$ is uniformly bounded on $\mathbb{R}^1$, $|f''(x)| \leq M$ for some $M > 0$.
- (iv) At the maximum point x^* , $f''(x^*) < 0$.
- (v) For arbitrary x_1, x_2 ,

$$|f''(x_1) - f''(x_2)| \leq L. \quad (13)$$

Let $\delta = x_2 - x_1 - 2h$, then it follows from (7) that

$$\dot{\delta} = -2K_p\delta, \quad \delta(0) = x_2(0) - x_1(0) - 2h. \quad (14)$$

Thus, $\delta(t) = e^{-2K_p t}\delta(0)$, and if $K_p > 0$, then $\lim_{t \rightarrow \infty} \delta(t) = 0$.

Next, we observe from (9) and (12) that

$$\dot{u} = -KK_p \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad (15)$$

Let us consider the expression on the right hand side. Expand $f(x_1)$, $f(x_2)$ in the vicinity of $u = \frac{x_1+x_2}{2}$ using the Taylor expansion formula with the Lagrange reminder,

$$\begin{aligned} f(x_1) &= f(u) + f'(u)\frac{x_1 - x_2}{2} + \frac{1}{2}f''(\xi_1)\frac{(x_1 - x_2)^2}{4}, \\ f(x_2) &= f(u) + f'(u)\frac{x_2 - x_1}{2} + \frac{1}{2}f''(\xi_2)\frac{(x_2 - x_1)^2}{4}, \end{aligned} \quad (16)$$

where ξ_1, ξ_2 are some points between x_1 and u and between x_2 and u , respectively. Then from (15),

$$\dot{u} = -KK_p f'(u) - \frac{KK_p}{8}(f''(\xi_2) - f''(\xi_1))(\delta + 2h). \quad (17)$$

Using the notation $z = u - x^*$, the Taylor expansion of $f'(u)$ can be written as:

$$\begin{aligned} f'(u) &= f'(x^*) + f''(\xi_3)z = f'(x^*) + f''(x^*)z \\ &\quad + (f''(\xi_3) - f''(x^*))z \end{aligned}$$

where ξ_3 is a point between u and x^* . Note that this is the exact identity. Recall that $f'(x^*) = 0$ and substitute into (17)

$$\begin{aligned} \dot{z} &= -KK_p f''(x^*)z - KK_p(f''(\xi_3) - f''(x^*))z \\ &\quad - \gamma(t)(\delta(t) + 2h) = -KK_p f''(x^*)z - \beta(t)z \\ &\quad - \gamma(t)\delta(t) - 2\gamma(t)h, \end{aligned} \quad (18)$$

where

$$\begin{aligned} \beta(t) &= KK_p(f''(\xi_3(t)) - f''(x^*)), \\ \gamma(t) &= \frac{KK_p}{8}(f''(\xi_2(t)) - f''(\xi_1(t))). \end{aligned} \quad (19)$$

Now combine (18) and (14). Let $\chi = \begin{pmatrix} \delta \\ z \end{pmatrix}$, then

$$\begin{aligned} \dot{\chi} &= \underbrace{\begin{bmatrix} -2K_p & 0 \\ 0 & -KK_p f''(x^*) \end{bmatrix}}_A \chi + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_B \phi_1(t, \zeta_1) \\ &\quad + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \phi_2(t, \zeta_2) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \phi_3(t) \\ \phi_1(t, \zeta_1) &= \gamma(t)\zeta_1 \quad \zeta_1 = \delta = \underbrace{[1 \ 0]}_{=C_1} \chi, \\ \phi_2(t, \zeta_2) &= \beta(t)\zeta_2 \quad \zeta_2 = z = \underbrace{[0 \ 1]}_{=C_2} \chi, \\ \phi_3(t) &= 2\gamma(t)h. \end{aligned} \quad (20)$$

According to (v),

$$|\phi_1(t, \zeta_1)| < \alpha_1 |\zeta_1|, \quad |\phi_2(t, \zeta_2)| < \alpha_2 |\zeta_2|, \quad |\phi_3(t)| < 2\alpha_1 h, \quad (21)$$

where $\alpha_1 = \frac{|K|K_p L}{8}$, $\alpha_2 = |K|K_p L$. That is the system (20) can be considered as an uncertain system with two 'norm-bounded' uncertainty inputs ϕ_1, ϕ_2 subject to the constraints (21), and a bounded disturbance ϕ_3 .

Asymptotic properties of the system (20) can be established using the S-procedure method. Choose $V(\chi) = \chi' P \chi$ as a candidate Lyapunov function. Let $\tau_1, \tau_2 > 0$ and $\varepsilon > 0$ be three constants. Consider

$$\begin{aligned} \dot{V} &- \tau_1(|\phi_1|^2 - \alpha_1^2|\zeta_1|^2) - \tau_2(|\phi_2|^2 - \alpha_2^2|\zeta_2|^2) \\ &= 2\chi' P(A\chi + B\phi_1 + B\phi_2 + B\phi_3) - \tau_1(|\phi_1|^2 - \alpha_1^2|C_1\chi|^2) \\ &\quad - \tau_2(|\phi_2|^2 - \alpha_2^2|C_2\chi|^2) = \chi'(PA + A'P + \tau_1\alpha_1^2 C_1' C_1 \\ &\quad + \tau_2\alpha_2^2 C_2' C_2 + \frac{1}{\tau_1}PBB'P + \frac{1}{\tau_2}PBB'P + \frac{1}{\varepsilon}PBB'P)\chi \\ &\quad - \tau_1|\phi_1 - \frac{1}{\tau_1}B'P\chi|^2 - \tau_2|\phi_2 - \frac{1}{\tau_2}B'P\chi|^2 + \varepsilon|\phi_3|^2 \end{aligned} \quad (22)$$

Now suppose that for some $\lambda > 0$, the Linear Matrix Inequality

$$\begin{bmatrix} PA + A'P + \tau_1\alpha_1^2 C_1' C_1 & PB & PB & PB \\ + \tau_2\alpha_2^2 C_2' C_2 + \lambda P & & & \\ B'P & -\tau_1 & 0 & 0 \\ B'P & 0 & -\tau_2 & 0 \\ B'P & 0 & 0 & -\varepsilon \end{bmatrix} < 0 \quad (23)$$

admits a solution $P = P' > 0$, $\tau_1 > 0$, $\tau_2 > 0$, $\varepsilon > 0$. Then for all ϕ_1, ϕ_2 , and ϕ_3

$$\begin{aligned} \dot{V} + \lambda V &< \tau_1(|\phi_1|^2 - \alpha_1^2|C_1\chi|^2) \\ &\quad + \tau_2(|\phi_2|^2 - \alpha_2^2|C_2\chi|^2) + \varepsilon|\phi_3|^2, \end{aligned} \quad (24)$$

and taking into account (21), we obtain that for all admissible ϕ_1, ϕ_2, ϕ_3 ,

$$\dot{V} + \lambda V < 4\varepsilon\alpha_1^2 h^2 = \frac{\varepsilon K_p^2 K^2 L^2 h^2}{16}, \quad (25)$$

This implies that for any initial condition $\chi(0)$ the trajectories of the system (20), (21) are bounded as follows

$$\begin{aligned} |\delta(t)|^2 + |z(t)|^2 &\leq \frac{1}{\sigma_{\min}(P)} e^{-\lambda t} (|\delta(0)|^2 + |z(0)|^2) \\ &\quad + \frac{\varepsilon K_p^2 K^2 L^2 h^2}{16\lambda\sigma_{\min}(P)}. \end{aligned} \quad (26)$$

Note that $\delta(t) \rightarrow 0$, therefore we have the following theorem as our main result.

Theorem 1 Suppose the function f satisfies conditions (i)-(v). Also, suppose that there exist K, K_p and $\lambda > 0$ such that the LMI (23) is feasible; i.e., there exists a matrix $P > 0$ and constants $\tau_1 > 0, \tau_2 > 0, \varepsilon > 0$ such that (23) holds. Then the approximated gradient algorithm converges in the sense that $\lim_{t \rightarrow \infty} |x_1 - x_2| = 2h$ and

$$\limsup_{t \rightarrow \infty} \left| \frac{x_1 + x_2}{2} - x^* \right| \leq \frac{\varepsilon K_p^2 K^2 L^2 h^2}{16\lambda\sigma_{\min}(P)} \quad (27)$$

IV. SIMULATION RESULTS

To evaluate the performance of the scheme proposed in the previous section, we use the Matlab Simulink environment. The nonlinear field function for our simulations is of the form:

$$f(x) = 40 \sin(2\pi * 0.002(x - 100)) + 10 \sin(2\pi * 0.0055(x - 100) + 3\frac{\pi}{4})$$

where x is in meters. On the interval $x \in [0, 400]$ the function has a single maximum point at $x = u_0 = 247.178$ and $\dot{f}(u_0) < 0$. A plot of the function and its gradient is shown in Figure 5.

Fig. 5. Nonlinear function used for simulations.

Simulation results for two different initial locations are presented in Figures 6, 7. These simulations show that the robots converge to the desired formation (distance offset) and the formation converges to the neighborhood of the maximum point of the field and stops there.

For an implementation on a real hardware, we need to consider actuator saturation. Even though we did not conduct a stability analysis with saturation non-linearity in the loop, we carried out simulations with the robots' maximum velocity capped at 1.5 m/sec and have shown that the formation still reaches the extremum and stops there. Simulation results are presented in Figure 8.

V. CONCLUSION AND FUTURE WORK

We have proposed a cooperative extremum seeking control scheme that uses consensus control and an integral feedback loop to drive a two-robot formation to the maximum of an unknown non-linear map. We have found that the consensus algorithm itself gives us an integrator so we do not need to include one in the feedback loop. We have analysed the stability of the proposed loop and demonstrated convergence to the neighborhood of the extremum point using simulations. The simulations have shown that the proposed scheme works well even when a velocity saturation is accounted

Fig. 6. Simulation results for initial robot locations $x_1 = 50$, $x_2 = 55$

for in the loop, that is a very important aspect for physical implementation.

In future work we will consider a nonlinear stability analysis that would allow us to determine what types of nonlinear field functions our control scheme can handle. We also are aiming to extend our work to handle two-dimensional extremum seeking on a plane, and use a more accurate gradient estimation technique for a formation of multiple (three or more) robots.

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Fig. 7. Simulation results for initial robot locations $x_1 = 400$, $x_2 = 410$

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